Volume 23, Number 1 July 2017
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fourteen times annually)

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Journal of Computational Analysis and Applications

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Golub-Kahan-Lanczos based preconditioner for least squares problems in overdetermined and underdetermined cases

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Abstract

We present an effective preconditioner for solving least squares problems in full ranked overdetermined and underdetermined cases. The preconditioner, generated from Golub-Kahan-Lanczos method, can approximately replace a few largest singular values by one without altering the rest. This property accelerates the convergence, thereby improves the efficiency of the algorithm for solving the least squares problems with ill-conditioned system matrix which is caused by large singular values. In this paper we focus on the overdetermined and the underdetermined cases.

Key words: Least squares problems; Preconditioner; Lanczos bidiagonalization process; Krylov subspace method; Golub-Kahan-Lanczos method

AMSC: 65K05; 65F08; 65F10

1 Introduction

In this paper, we assume that the least squares problems are in the form as

$$\min \|b - Ax\|_2, \tag{1}$$

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where $A_{m\times n}$ is a full-ranked coefficient matrix which is large and sparse.

In the situation that $m = n$, we can obtain an approximate solution by solving the linear system $Ax = b$ and minimize the residual in the sense of 2-norm. The minimal norm residual method, based on the iterative Krylov methods, is a suitable algorithm to obtain the optimal approximation, and full details can be found in [2]. We have superscript T denoted the transposition of a matrix, and use subscript to indicate the size of matrix. The overdetermined cases

$$\min \|b - Ax\|_2, A \in R_{m\times n}, m > n \tag{2}$$

and the underdetermined cases

$$\min \|b - Ax\|_2, A \in R_{m\times n}, m < n \tag{3}$$

are taken into consideration in the following.

In this paper, we take the preconditioner as a left preconditioner in both overdetermined and underdetermined cases. To the overdetermined system (2) in least squares problems, we generally translate the corresponding linear system

$$Ax = b, A \in R_{m\times n}, m > n, \tag{4}$$

into a normal equation by premultipling A^T on both sides. R is the set of real number here and in the following. Similarly, we translate the underdetermined system (3) into a normal equation in the same way in the corresponding linear system

$$Ax = b, A \in R_{m\times n}, m < n. \tag{5}$$

Thereby we have the normal equation in the following form

$$A^T Ax = A^T b. \tag{6}$$

We notice that the coefficient matrix in (6) is symmetric positive definite, so the normal equation can be solved by the CG method[16]. Thanks to previous researchers, many classic methods, such as CGNE [4] and CGLS[3], can be regarded as an extensions of the CG method and solve least squares problems efficiently. Similarly, the LSQR method[7] is an effective method for solving the least squares problems, so does the LSMR method[15].

For the symmetric positive definition (SPD) matrix, we know the convergence of iterative Krylov methods depends on the condition number κ of the coefficient matrix, in other word, the spectral distribution, where $\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$ with $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denoting the largest and the smallest eigenvalues of A, respectively. To discuss the spectral distribution of $A^T A$ in (6), we give the singular value decomposition of the original coefficient matrix A as follow. Notice that all the matrixes in this paper are full ranked.

We have the singular value decomposition of A in this form

$$A = \hat{U}_{m\times n} D \hat{V}^T_{n\times n}, D = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix}, \tag{7}$$

where $\hat{U}_{m\times n}$ and $\hat{V}_{n\times n}$ are both unitary matrices, σ_i denotes the singular value that $\sigma_1 > \sigma_2 > \cdots > \sigma_n$. From (7), we have

$$A^T A = \hat{V}_{n\times n} D^2 \hat{V}^T_{n\times n}, \tag{8}$$

which can be regarded as the eigenvalue decomposition of the coefficient matrix in the normal equation (6).

If we denote $\Sigma = diag\{\sigma_1^2, \sigma_2^2, \cdots, \sigma_r^2\}$, where $r = \min(m, n)$, it could be easily concluded that the spectral distribution of the coefficient matrix in (6) is Σ. Therefore, the condition numbers of linear systems can be presented as $\kappa(A^T A) = \frac{\sigma_1^2}{\sigma_r^2}$. To accelerate the convergence, thereby improve the algorithm, we expect the condition number to be as small as possible. Therefore, removing the smallest eigenvalue from the spectrum of the coefficient matrix is purpose of the preconditioner. Also, we leave the rest unchanged. Such kind of preconditioners and relevant applications can be located in [8], [9] and [10].

Also, when the property of ill-condition is caused by a few largest eigenvalues, we expect a preconditioner, from the similar point of view, to eliminate the largest eigenvalues from the spectrum in order to accelerate the convergence. A preconditioner formed by Lanczos bidiagonalization is formulated to change the largest singular values to one approximately without altering the others, so that the preconditioner change the corresponding eigenvalues in normal equations. In the ill-conditioned overdetermined case and the ill-conditioned underdetermined case, we utilize the preconditioner to speed up the convergence. To illustrate the effects of the preconditioners proposed in this paper, we utilize two methods to solve a series of the least squares problems. Of course, we divide every experiments into two parts, using preconditioner and not using it.

In the following sections, the process of Lanczos bidiagonalization will be stated in section 2; the preconditioners for solving overdetermined and underdetermined least squares problems (2) (3) will be defined in section 3; numerical examples are demonstrated in section 4; conclusions are presented in section 5 finally.

2 The process of Lanczos bidiagonalization

2.1 Standard Lanczos bidiagonalization

Lanczos biorthogonalization, which can be located in [6] [4], is an important process in methods like LSQR[7], BiCG[11] and BiCGSTAB[12]. A variation of Lanczos biorthogonalization, formed as

$$AV_n = U_{n+1}B, B = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_n & \alpha_n \\ & & & \beta_{n+1} \end{pmatrix}, \tag{9}$$

is denoted as Golub-Kahan-Lanczos method [5] , where V_n and U_{n+1} are both unitary matrices and we assume A is a matrix of size $n \times n$. One characteristic of decomposition (9) is that the lower bidiagonal matrix B shares the same singular values as A's. Furthermore, we have analyzed and concluded in the previous section that the singular values distribution of A directly reflects the spectral distribution of A^TA in problems (6). Hence we expect a preconditioner based on Lanczos bidiagonalization to optimize spectral distributions of system matrices in least squares problems. Some similar preconditioner based on the Golub-Kahan-Lanczos bidiagonalization for square coefficient matrixes has been proposed and applied. For example, inreference[13], the author optimized the spectral distribution of a ill-posed coefficient matrix by a Lanczos-based preconditioner.

However, limited by the dimension of the coefficient matrix in overdetermined and underdetermined cases, the algorithm will break down when maximal number of iteration is greater than both row dimension and column dimension. Therefore, in order to be applied to overdetermined and underdetermined cases, the standard form of Golub-Kahan-Lanczos method requires modification. To extend applications of the Lanczos-based preconditioner, we define variants of the preconditioner which can be utilized in overdetermined cases and underdetermined cases, thereby it is available for least squares problems. At first, we give the standard algorithm for Golub-Kahan-Lanczos method as stated in [5].

Algorithm 1 Standard Golub-Kahan-Lanczos bidiagonalization

1. $\beta_1 = \|b\|_2$, $u_1 = \frac{b}{\beta_1}$, $v_0 = 0$
2. for $i = 1, 2, ..., n$
3. $\quad p_k = A^T u_k - \beta_k v_{k-1}$
4. $\quad \alpha_k = \|p_k\|_2$
5. $\quad v_k = \frac{p_k}{\alpha_k}$
6. $\quad q_k = Av_k - \alpha k u_k$
7. $\quad \beta_{k+1} = \|q_k\|_2$
8. $\quad u_{k+1} = \frac{q_k}{\beta_{k+1}}$

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The $\alpha' s$ and $\beta' s$ generated in the above algorithm are equal to the ones in (9), also rows of V and U in (9) are obtained through Algorithm 1 as v_k and u_k respectively. Therefore, we could establish the Lanczos bidiagonalization form by a series of iterations performed according to Algorithm 1, when the coefficient matrix A is of size $n \times n$.

To define the Lanczos-based preconditioners in overdetermined cases and underdetermined cases, we have to modify algorithm 1, the standard Lanczos bidiagonalization process, in order to accommodate the situations that the coefficient matrices are *m-by-n* and $m \neq n$.

2.2 Modified Lanczos bidiagonalization

The main distinction between the overdetermined, or underdetermined, determined and square cases is the dimension of the coefficient matrix A. As stated before, the matrix B, generated by Lanczos bidiagonalization, and A in (9) share the same singular value distribution. We limit the steps of Lanczos bidiagonalizaion process under the minimal number between m and n where A is *m-by-n*. We utilize iterative Krylov subspace methods to solve the linear systems (6), with symmetric positive definite coefficient matrices. Therefore we conclude easily that the rank of B can not exceed the minimum of m and n. Then, a restrictive condition should be added to the corresponding Lanczos bidiagonalization process to terminate it in appropriate number of steps.

Different from (9), We set a termination rule that the maximal iteration in Golub-Kahan-Lanczos bidiagonalization is less or equal to the minimum between the row dimension and the column dimension toensure that the algorithm will terminate in appropriate number of steps. Following this rule, we have the bidiagonalization decomposition of A in overdetermined situation as

$$AV_{n\times n} = U_{m\times(n+1)}B_n, B_n = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_n & \alpha_n \\ & & & \beta_{n+1} \end{pmatrix}, \tag{10}$$

and the bidiagonalization decomposition of A in underdetermined situation as

$$AV_{n\times m} = U_{m\times(m+1)}B_m, B_m = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_m & \alpha_m \\ & & & \beta_{m+1} \end{pmatrix}. \tag{11}$$

Considering the computational cost of the Lanczos bidiagonalization process, we try to avoid bidiagonalizing A completely. The preconditioner, mentioned in

the previous section and defined in the next section, is structured for the purpose of changing the largest singular values to one, in order to optimize the condition numbers of normal equation (6). Hence, we stop the Lanczos dibiagonalization process when the current smallest singular value σ_k, generated in the kth step of Lanczos dibiagonalization process, is much smaller than the largest one σ_1. We set a scalar number δ to be the threshold of termination, i.e, terminates when $\sigma_k < \delta\sigma_1$. If the bidiagonalization process stops at the kth step, the bidiagonalization composition is of the form below

$$AV_{n\times k} = U_{m\times(k+1)}B_k, B_k = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_k & \alpha_k \\ & & & \beta_{k+1} \end{pmatrix}. \tag{12}$$

M Rezghi set the scalar number δ as the square root of machine precision in [13] while applying it in ill-conditioned systems derived from blurring images. Since δ is a scalar to judge whether we should terminate the Lanczos bidiagonalization process and the Lanczos bidiagonalization process aims to remove the largest singular values, the choice of δ has different effects in different numerical examples. We will present the influence caused the change of δ under different numerical examples and iterative methods in the section of experiments. In general ill-conditioned systems, we need not to set δ so small and some cases will be presented in the 4th section. Here we add the above two restrictive conditions to standard Lanczos bidiagonalization, then we have modified Lanczos bidiagonalization as following.

Algorithm 2 Modified Lanczos bidiagonalization

1. $\beta_1 = \|b\|_2$, $u_1 = \frac{b}{\beta_1}$, $v_0 = 0$, $r = min\{m, n\}$, δ
2. for $i = 1, 2, ..., r$
3. $p_k = A^T u_k - \beta_k v_{k-1}$
4. $\alpha_k = \|p_k\|_2$
5. $v_k = \frac{p_k}{\alpha_k}$
6. $q_k = Av_k - \alpha k u_k$
7. $\beta_{k+1} = \|q_k\|_2$
8. $u_{k+1} = \frac{q_k}{\beta_{k+1}}$
9. get singular values of B: $\sigma_1, \sigma_2, \cdots, \sigma_i$
10. if $\sigma_i < \delta\sigma_1$, break down.
11.end

In this section, we introduced the standard Lanczos bidiagonalization process in Algorithm 1, and defined the modified Lanczos bidiagonalization process in

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Algorithm 2, which is adapted to the overdetermined and the underdetermined situations. A preconditioner based on modified Lanczos bidiagonalization process will be introduced and defined in the next section.

3 Lanczos-based preconditioner for least squares problems

To solve the least squares problems formed as (2) and (3), we solve the corresponding linear systems (4) and (5) instead by translating them into normal equations (6) respectively. If we have the singular value decompositions of A which are structured as (7), and the singular value distributions are scattered and wide, that is the largest singular value is much greater than the smallest one, thereby the condition number of the normal equation (6) will be terribly greater according to analysis of (8). For the purpose of speeding up the convergence, we expect to optimize, or reduce, the condition number of $A^T A$. Since the condition number of normal equations (6) could be presented as $\kappa(A^T A) = \frac{\sigma_1^2}{\sigma_r^2}$ where σ_1 and σ_r denote the largest and the smallest singular value of A, enlargement or elimination of the smallest singular values, and decrease or elimination of the largest singular values are both effective methods to reduce the condition number. Deflation-based preconditioners, like the deflation preconditioner and the balancing preconditioner[8, 9, 10], have such characteristics and properties to eliminate smallest eigenvalues of system matrix. We do not pay much attention to the preconditioners based on deflation, but the preconditioners functioned for decreasing, or eliminating, the largest ones are what we concern. In the following, all the preconditioners based on Lanczos bidiagonalization are defined for the overdetermined cases (2) and the underdetermined cases (3).

First we shall discuss the situation of the underdetermined case. In linear system (5), the coefficient matrix A has the singular value decomposition illustrated as (7). We assume a diagonal matrix

$$D_k = diag\{\sigma_1, \sigma_2, \cdots, \sigma_k\},$$

where σ_i with $i = 1, 2, \cdots, k$, denotes the first k largest singular values of A. The Lanczos bidiagonalization process for underdetermined cases within k steps have been proposed as (12).

On the premise that B, which is structured by Lanczos bidiagonalization, shares the same singular values with A, we have the following conclusion that: the B_m derived from (11) has singular value decomposition form as

$$B_m = \tilde{U}_{(m+1)\times(m+1)} \begin{pmatrix} D \\ 0 \end{pmatrix}_{(m+1)\times m} \tilde{V}^T_{m\times m},$$

where D in the above equation is equal to the one in (7), with $\tilde{U}_{m+1}$ and $\tilde{V}_{m\times m}$ both unitary matrices. Similarly, the B_k derived from (12) has singular value decomposition form as

$$B_k = \tilde{U}_k \begin{pmatrix} D_k \\ 0 \end{pmatrix} \tilde{V}_k^T, \tag{13}$$

where D_k has been defined at the beginning in this section, with $\tilde{U}_k$ and $\tilde{V}_k$ both unitary matrices.

When we consider the underdetermined case (11), some deductions are stated as follow. We use singular value decomposition of B replacing the one in (11) and we have

$$AV_{n\times m} = U_{m\times(m+1)}\tilde{U}_{(m+1)\times(m+1)} \begin{pmatrix} D \\ 0 \end{pmatrix}_{(m+1)\times m} \tilde{V}_{m\times m}^T.$$

The dimension of matrices are denoted as subscripts in previous sections, and now the subscripts will be omitted for simplification. Then we postmultiply $\tilde{V}$ on both sides and we have

$$AV\tilde{V} = U\tilde{U} \begin{pmatrix} D \\ 0 \end{pmatrix},$$

Here we set $\bar{V} = V\tilde{V} = \{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_m\}$ and $\bar{U} = U\tilde{U} = \{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_{m+1}\}$. As for equation

$$A\bar{V} = \bar{U} \begin{pmatrix} D \\ 0 \end{pmatrix},$$

we regard it as a singular value decomposition of A, similar to (7), approximately. If we set $\bar{U}_m = \{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_m\}$, the first m columns of $U\tilde{U}$, we assume that

$$\begin{aligned} \bar{U}_m &= \hat{U} \\ \bar{V} &= \hat{V} \end{aligned}$$

where $\hat{U}$ and $\hat{V}$ are obtained from (7).

Now we focus on the formulation (8). If a matrix is structured as

$$P = \bar{V}D^{-2}\bar{V}^T,$$

then combining with the previous assumption($\bar{V} = \hat{V}$), it gives that

$$\begin{aligned} PA^TA &= \bar{V}D^{-2}\bar{V}^T\hat{V}D^2\hat{V}^T \\ &= \bar{V}I\bar{V}^T \\ &= I. \end{aligned}$$

It seems that we could have obtained solution directly through the application of such a preconditioner P. In view of computation, however, it is inadvisable for

the following reasons: 1. the preconditioner P is based on a complete Lanczos bidiagonalization, so this process has expensive computational cost even no less than direct methods.; 2. the $\bar{V}$ is approximately equal to $\hat{V}$ in practical implement, but we give the above deduction just in theory, without the consideration of computational errors. Although we can not utilize the preconditioner P in practical computation, a variant of P based on incomplete Lanczos bidiagonalization is defined as follow to solve underdetermined least squares problems.

Here we construct a preconditioner P which is similar with the one mentioned above with merely replacing B_m(from (11)) by B_k(from (12)). After simple deduction, we have

$$P = \bar{V} \begin{pmatrix} D_k^{-2} & 0 \\ 0 & I_{m-k} \end{pmatrix} \bar{V}^T,$$

where D_k is from (13). We set $\bar{V}_k = V\tilde{V}_k$ is the first k columns of $\bar{V}$, where $\tilde{V}_k$ is obviously the first k columns of $\tilde{V}$. Hence we set $\bar{V} = [\bar{V}_k, \bar{V}_{m-k}]$. Based on the definition of $\bar{V}$, we have

$$I = \bar{V}\bar{V}^T = \bar{V}_k\bar{V}_k^T + \bar{V}_{m-k}\bar{V}_{m-k}^T.$$

Analyzing the above information, it gives that

$$\begin{aligned} P &= \bar{V}_k D_k^{-2} \bar{V}_k^T + \bar{V}_{m-k}\bar{V}_{m-k}^T \\ &= V\tilde{V}_k D_k^{-2} \tilde{V}_k^T V^T + (I_{m\times m} - \bar{V}_k\bar{V}_k^T) \\ &= V(B_k^T B_k)^{-1} V^T + (I_{m\times m} - VV^T). \end{aligned}$$

where V and B_k can both be obtained through Algorithm 2. If we utilize P as a left preconditioner in normal equation (6) for underdetermined cases (5), we have

$$PA^TA = \hat{V} \begin{pmatrix} I_k & 0 \\ 0 & D_{m-k}^2 \end{pmatrix} \hat{V}^T,$$

where $D_{m-k} = diag\{\sigma_{k+1}, \sigma_{k+2}, \cdots, \sigma_m\}$ with σ_i's denoting the $m-k$ smallest singular values.

According to the statement above, we can conclude that the Laczos-based preconditioner has the property to change k largest singular values of coefficient matrix A, or k largest eigenvalues of the system matrix in normal equation (6) in other word, to one without touching the others. The preconditioner is able to optimize the condition number of normal equation (6) when the ill condition is caused by these large singular values. Since $k \ll m$, the computational cost is greatly reduced, so is the computational error. The conclusion, furthermore, is under the premise that the linear system corresponding to least squares problems is underdetermined, so that

$$P_{under} = V(B_k^T B_k)^{-1} V^T + (I_{n\times n} - VV^T) \tag{14}$$

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could be used as a left-preconditioner in underdetermined least squares problems. Next we consider the overdetermined cases.

In the overdetermined cases, we construct a Lanczos-based preconditioner that follows the same strategy as stated in the previous subsection. To solve the overdetermined system (4), we solve the normal equation (6) instead to obtain approximate solution. Considering the decomposition form (8) of A^TA, we expect to construct a preconditioner, similar to the underdetermined cases, presented as

$$P = \hat{V} \begin{pmatrix} D_k^{-2} & 0 \\ 0 & I_{n-k} \end{pmatrix} \hat{V}^T.$$

Through an analogical deduction to underdetermined cases, a preconditioner formed as

$$P_{over} = V(B_k^T B_k)^{-1} V^T + (I_{n\times n} - VV^T) \tag{15}$$

can be used as a left-preconditioner in overdetermined least squares problems. B_k and $V_{n\times k}$ can be obtained from Algorithm 2. Furthermore it is not computationally costly because of $k \ll n$.

From the above discussion, we can see that the forms of the Lanczos-based preconditioners in over- and under- determined cases are the same, although we deduced them in separate ways. Also, such a preconditioner for the linear system with a square coeffcient matrix has the same form. Therefore, we can conclude that we deduce the preconditioners, proposed in this paper, from the point of overdetermined and underdetermined cases and ultimately get a result similar to the one in square problems, which has been proposed in [13]. Of course, the result of this paper can also be regarded as the expansion of the application of the Lanczos-based preconditioner into the overdetermined and underdetermined least squares problems. Now we unify the preconditioner as follow

$$P = V(B_k^T B_k)^{-1} V^T + (I - VV^T), \tag{16}$$

which can be used as a left preconditioner in ordinary linear systems, overdetermined least squares problems and underdetermined least squares problems. The relevant numerical experiments are presented in the following section, from which we can see the effects of Lanczos-based preconditioners.

4 Numerical experiments

In this section, we will take a series of numerical examples to present the effect of the Lanczos-based preconditioner in the least squares problems. At first, we introduce two iterative methods as the basic algorithm for solving these underdetermined and overdetermined problems. Here, we choose an old and classic method as the first one for solving the least squares problems. It is the CGLS

method[3]. In this method, we first transform the least squares problems into symmetric positive definite(SPD) problems by the normal equations then solve it by the CG method[16]. Integrating the above ideas, we have the CGLS method. Now we present the preconditioned CGLS method algorithm 3, where we just consider the situation of left precondition.

Algorithm 3 Preconditioned CGLS method

```
select $x_0$ as the initial guess, $r_0 = b - Ax_0$ and $P$ as the preconditioner
initialization: we set $\bar{r}_0 = A^T r_0$, $\hat{r}_0 = P\bar{r}_0$, $f_0 = z_0$
for $i = 0, 1, 2, ...$
    $g_i = Af_i$
    $\alpha_i = (\hat{r}_i, \bar{r}_i)/\|g_i\|_2^2$
    $x_{i+1} = x_i + \alpha_i f_i$
    $r_{i+1} = r_i - \alpha_i g_i$
    $\bar{r}_{i+1} = A^T r_{i+1}$
    $\hat{r}_{i+1} = P\bar{r}_{i+1}$
    $\beta_i = (\hat{r}_{i+1}, \bar{r}_{i+1})/(\hat{r}_i, \bar{r}_i)$
    $f_{i+1} = \hat{r}_{i+1} + \beta_i f_i$
endfor
```

The second method to solve the least squares problems is the BAGMRES method[14], a variant of the GMRES method[1]. In this method, the least squares problems will be post-multiplied by a matrix B, an arbitrary nonsingular matrix. Now we give the BAGMRES method as Algorithm4.

Ex.	Group and name	id	#rows	#cols	Nonzeros	Problem kind
1	JGD_Forest/TF10	1944	99	107	622	Combinatorial
2	JGD_Forest/TF11	1945	216	236	1607	Combinatorial
3	HB/wm3	277	207	260	2948	Economic
4	Pajek/Sandi_sandi	1520	314	360	613	Bipartite graph
5	Meszaros/refine	1759	29	62	153	Linear programming
6	JGD_margulies/flower_4_1	2155	121	129	386	Combinatorial

Table 1: The structures of six test underdetermined problems

Algorithm 4 BA-GMRES with k restart

1. select x_0 as the initial guess, $r_0 = B(b - Ax_0)$ and $\nu_1 = r_0/\|r_0\|_2$
2. for $i = 1, 2, ..., m$
3. $\quad \omega_i = BA\nu_i$
4. $\quad$ for $j = 1, 2, ..., i$
5. $\qquad h_{j,i} = (\omega_i, \nu_j)$
6. $\qquad \omega_i = \omega_i - h_{j,i}\nu_j$
7. $\quad$ endfor
8. $\quad h_{i+1,i} = \|\omega_i\|_2$
9. $\quad \nu_{i+1} = \omega_i/h_{i+1,i}$
10. $\quad$ Compute y_m to minimize $\|\hat{r_i}\|_2 = \|\|\hat{r_0}\|_2 e_1 - \overline{H}_i y\|_2$
11. $\quad$ if $\|r_i\|_2 < \tau$
12. $\qquad x_i = x_0 + [\nu_1, ..., \nu_i]y_i$
13. $\qquad$ stop
14. $\quad$ endif
15. endfor
16. set $x_0 = x_k$ and return to line 2 until convergence

In the following numerical experiments, the examples all come from practical applications from [17].

All the required information about the underdetermined and overdetermined cases is contained in Table 1 and Table 2 respectively. They both consist of group, number of rows, columns and nonzero elements and the type of problem of each example.

In the next two subsections, we solve the above 12 problems by the PCGLS method and the BAGMRES method combined with the Lanczos-based preconditioners. Then we change the scalar δ, involving the termination rule of the modified Lanczos bidiagonalization, and show its influence on the iterative process. Because the preconditioner is designed to modify the singular values, the distributions of singular values under different scalar δ's will be presented as well.

Ex.	Group and name	id	#rows	#cols	Nonzeros	Problem kind
7	HB/abb313	5	313	176	1557	Least squares
8	JGD_margulies/cat_ears_3_1	2151	204	181	542	Combinatorial
9	JGD_margulies/cat_ears_4_1	2153	377	313	938	Combinatorial
10	JGD_margulies/flower_5_1	2157	211	201	602	Combinatorial
11	JGD_margulies/flower_7_1	2159	463	393	1178	Combinatorial
12	Pajek/Cities	1457	55	46	1342	Weighted bipartite graph

Table 2: The structures of six test overdetermined problems

4.1 The acceleration of iterative processes

To discuss the acceleration of iterative processes, we refer to the PCGLS method and the BAGMRES method in [14, 4]. For the BAGMRES method, we have the following relation between the initial residual and the one from the kth iteration in underdetermined cases,

$$\|Br_k\|_2 = \|CA^T r_k\| \leq 2(\frac{\sigma_1 - \sigma_m}{\sigma_1 + \sigma_m})^k \|Br_0\|_2, \tag{17}$$

where C is a nonsingular matrix, $\kappa(C)$ is the condition number of matrix C and σ's denote the singular values of BA. And we have the relation between r_0 and r_k as

$$\|Br_k\|_2 = \|CA^T r_k\| \leq 2\sqrt{\kappa(C)}(\frac{\sigma_1 - \sigma_n}{\sigma_1 + \sigma_n})^k \|Br_0\|_2, \tag{18}$$

where C is a nonsingular matrix, $\kappa(C)$ is the condition number of matrix C and σ's denote the singular values of BA. More information of the above conclusion can be found in [14]. Now we give the convergence analysis of the PCGLS method, that is

$$\|e_k\|_A \leq 2(\frac{\sigma_1 - \sigma_r}{\sigma_1 + \sigma_r})^k \|e_0\|_A, \tag{19}$$

where $r = \min(m, n)$ and σ's denoting the singular values of $PA^T A$.

Based on equation (17), (18) and (19), it is obvious that we can accelerate the convergence if the gap between the largest singular value of normal equations and the smallest one is narrowed. In this paper, the Lanczos-based preconditioner is just for resetting the largest singular values to one, which can be regarded as shrink of the singular value distribution. Now, the effect of the Lanczos-based preconditioner in underdetermined cases is shown from Figure 1 to Figure 6.

In the numerical experiments, we set the tolerance $tol = 10^{-12}$, the maximal number of iteration $max_i t = 1000$ and the restarted number in the BAGMRES method $restart = 600$. Furthermore, the scalar δ upon which to terminates the

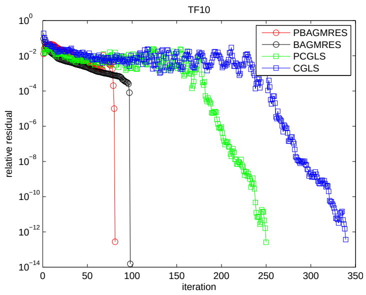

Figure 1: Relative residuals *vs* iterations in `TF10`

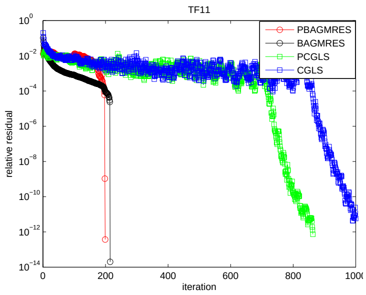

Figure 2: Relative residuals *vs* iterations in `TF11`

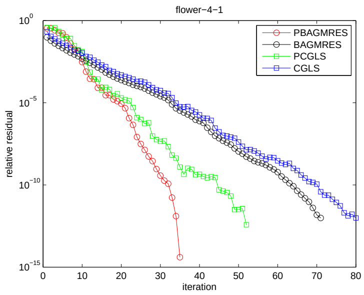

Figure 3: Relative residuals *vs* iterations in `wm3`

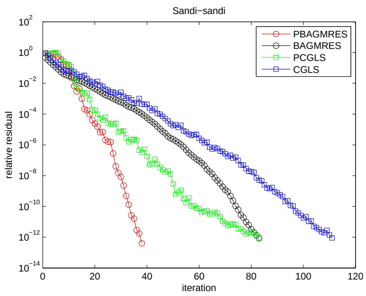

Figure 4: Relative residuals *vs* iterations in `Sandi_sandi`

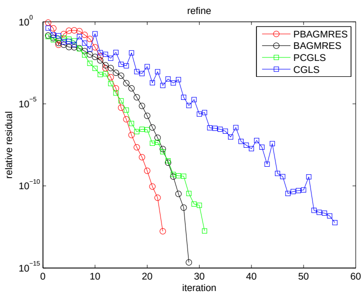

Figure 5: Relative residuals *vs* iterations in `refine`

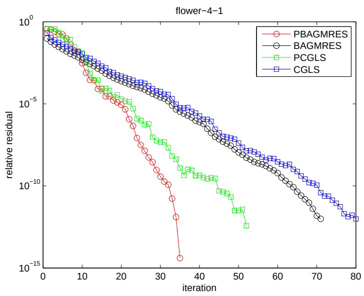

Figure 6: Relative residuals *vs* iterations in `flower_4_1`

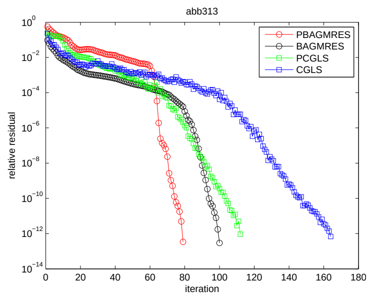

Figure 7: Relative residuals *vs* iterations in `abb313`

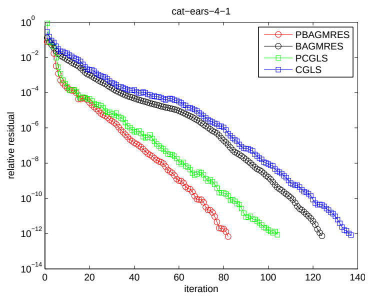

Figure 8: Relative residuals *vs* iterations in `itercat_ears_4_1`

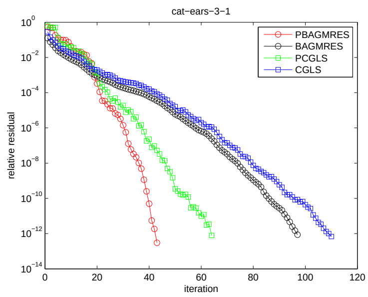

Figure 9: Relative residuals *vs* iterations in `itercat_ears_3_1`

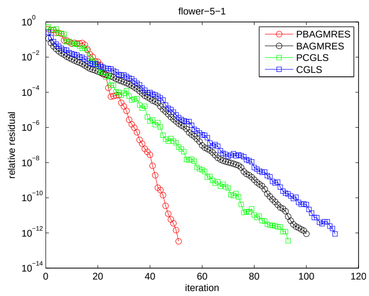

Figure 10: Relative residuals *vs* iterations in `iterflower_5_1`

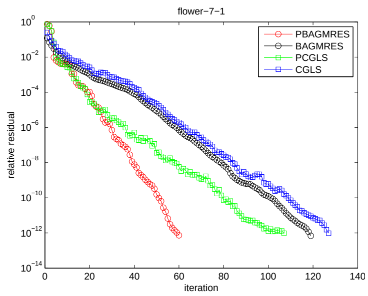

Figure 11: Relative residuals *vs* iterations in `iterflower_7_1`

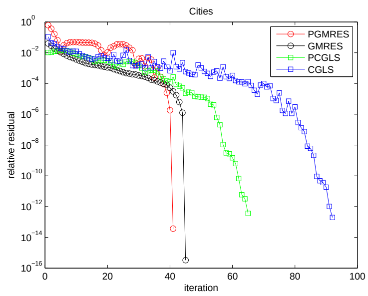

Figure 12: Relative residuals *vs* iterations in `itercities`

Lanczos bidiagonalization process is 0.05 in the test. We set $B = PA^T$ in the preconditioned BAGMRES method and $B = A^T$ in the nonpreconditioned BAGMRES method. From Figure 1 to Figure 6, we can see that both the BAGMRES method and the PCGLS method are accelerated by the Lanczos-based preconditioner as we expected. Next we show the iterative process while solving the overdetermined problems.

Figure 7 to Figure 12 present the results of experiments with the tolerance $tol = 10^{-12}$, the maximal number of iteration $max\_it = 1000$ and the restarted number in the BAGMRES method $restart = 600$. The scalar δ upon which to terminate the Lanczos bidiagonalization process is 0.05 in the test. Similarly, we set $B = PA^T$ in the preconditioned BAGMRES method and $B = A^T$ in the nonpreconditioned BAGMRES method. In Figure 7 to Figure 12, it is obvious that Lanczos-based preconditioners also accelerate the iterative processes in these overdetermined problems, so we think the proconditioner proposed in this paper is helpful to optimize the structure of coefficient matrix thereby accelerate the convergence. Moreover, all the numerical examples here are derived from practical applications. We believe, therefore, the Lanczos preconditioner has the result as

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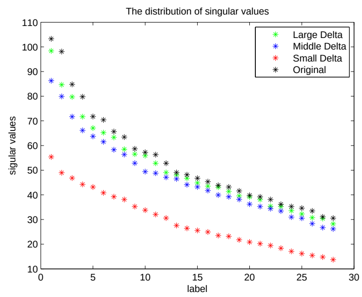

Figure 13: The distribution of singular values in `TF10`

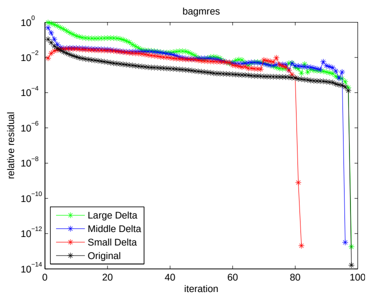

Figure 14: The iterative process of BAGMRE in `TF10`

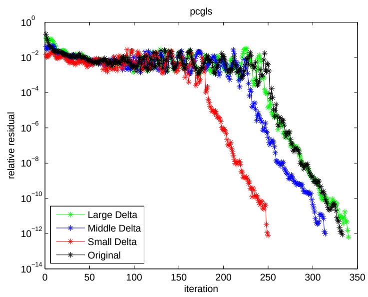

Figure 15: The iterative process of PCGLS in `TF10`

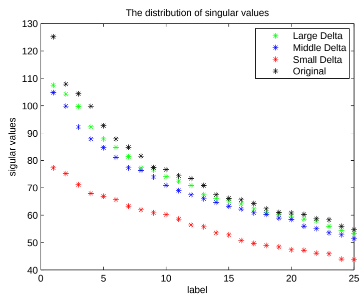

Figure 16: The distribution of singular values in `TF11`

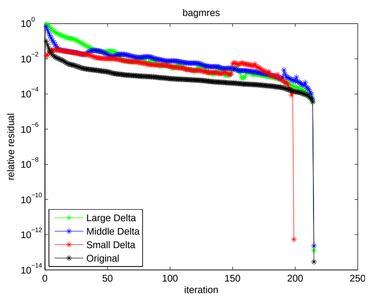

Figure 17: The iterative process of BAGMRE in `TF11`

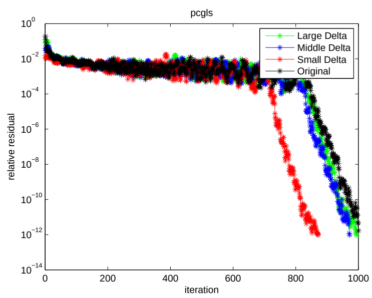

Figure 18: The iterative process of PCGLS in `TF11`

we expected.

4.2 The influence of the scalar δ

Referring to the illustration above, we have known that the scalar δ is used as a termination rule during the implementation of the Lanczos bidiagonalization process. By the definition of scalar δ, the smaller the δ is, the more large singular values will be replaced by one. It means that we can narrow the distribution of singular values. In the following experiments, we set the scalar δ to three different values and take `TF10` and `TF11` as the underdetermined examples. We test the distributions of the coefficient matrix of corresponding normal equations, the iterative process of the BAGMRES method and the PCGLS method. The results of TF10 and TF11 with varying scalar δ are presented in Figure13-15 and Figure 16-18 respectively.

As for the overdetermined cases, we take `abb313` as the first numerical examples. The singular values distribution and iterative processes of this example are illustrated by Figure 19-21.

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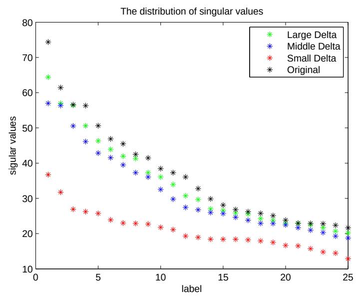

Figure 19: The distribution of singular values in `abb313`

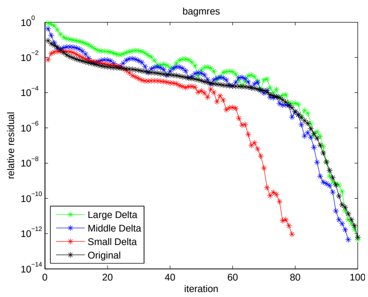

Figure 20: The iterative process of BAGMRE in `abb313`

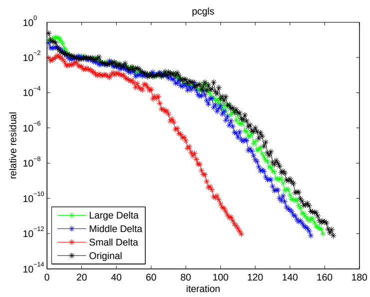

Figure 21: The iterative process of PCGLS in `abb313`

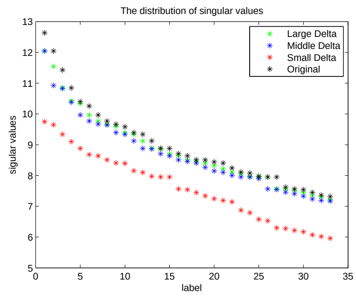

Figure 22: The distribution of singular values in `cat_ears_4_1`

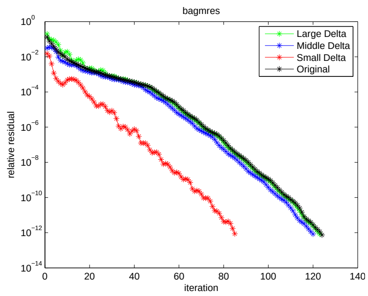

Figure 23: The iterative process of BAGMRE in `cat_ears_4_1`

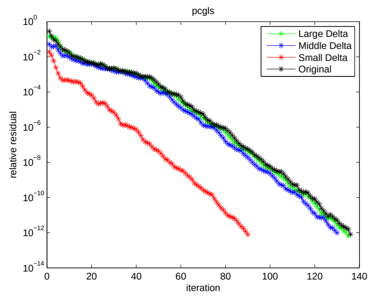

Figure 24: The iterative process of PCGLS in `cat_ears_4_1`

Similarly, the singular value distribution and iterative process regarding to different δ of the example `cat_ears_4_1` are presented in Figure 22-24.

In the above twelve figures, we classify the δ into three classes: the large delta, the middle delta and the small delta. The different δ stand for different preconditioners, upon which we denote the corresponding singular value distribution and iterative process by colorful points and lines. Theoretically, the small delta is able to reset most largest singular values while the large delta reset least largest singular values. Furthermore, required data of the experiments is presented in Table 3 and Table 4, in which k stands for the step of the Lanczos bidiagonalization process, $iter_{BAGMRES}$ and $iter_{PCGLS}$ represent the number of iterations of the BAGMRES method and the PCGLS method, respectively.

From Figure 13, Figure 16, Figure 19 and Figure 22, we can observe that the preconditioner with smaller δ indeed narrows the singular value distribution better than the ones led by larger δ. However, we fail to replace the largest singular values by one, although the improvement has brought us better convergence that is shown in Figure 14-15, Figure 17-18, Figure 20-21 and Figure 23-24. Through Table 3 and Table 4, we can also find that the number of iterations decreases

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Example	TF10		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		99	333
$\delta = 0.8$	2	99	340
$\delta = 0.3$	6	97	314
$\delta = 0.05$	22	83	250
Example	TF11		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		216	1000
$\delta = 0.8$	2	216	995
$\delta = 0.3$	5	216	972
$\delta = 0.05$	24	200	872

Table 3: The information along with the change of scalar δ in underdetermined cases `TF10` and `TF11`

Example	abb313		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		101	165
$\delta = 0.8$	2	101	159
$\delta = 0.3$	5	98	152
$\delta = 0.05$	24	80	112
Example	cat_ears_4_1		
	k	$iter_{BAGMRES}$	$iter_{PCGLS}$
Nonprec		125	136
$\delta = 0.8$	2	124	135
$\delta = 0.3$	5	121	130
$\delta = 0.05$	40	86	90

Table 4: The information along with the change of scalar δ in underdetermined cases `abb313` and `cat_ears_4_1`

obviously while the δ decreasing. In small-scale problem, the Lanczos-based preconditioner can reset the largest singular values closer to one than in large-scale problems, which is easy to testify by a simple numerical deduction. We suppose that the reason why the preconditioner fails to reset the largest singular values to one, just decreasing them instead, is the accumulation of calculation errors and the assumption

$$\bar{U}_m = \hat{U}$$
$$\bar{V} = \hat{V}.$$

From another experiment, the matrix B constructed in Lanczos bidiagonalization

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process has approximately equal singular values with coefficient matrix A. Merely focusing on the numerical value, the gap between the singular values of B and A may be underestimated and even ignored. Nevertheless, the gap will be enlarged when we assume the above equalities without considering the calculation errors. In the above experiments, we can also notice that the different δ influence the iterative process distinctly in different method so the perturbation analysis of the Lanczos-based preconditioner may give us a theoretical explanation of the difference between the theory and the numerical experiment. This supposition is remained to be testified in the future work.

5 Conclusions

To the overdetermined and the underdetermined least squares problems, we choose the BA-GMRES method and the PCGLS method to solve them respectively. Variants of the Lanczos bidiagonalization process are defined in the situation that coefficient matrices are not square, and the algorithm of modified Lanczos bidiagonalization is illustrated as conclusion. When we suffer from the ill-conditioned system matrices, the preconditioners based on modified Lanczos bidiagonalization, P structured for the overdetermined cases and the underdetermined cases respectively, are imposed on iterative Krylov subspace methods to accelerate convergence. Finally we prove our statements with numerical experiments and conclude that the preconditioner defined in this paper is effective to solve least squares problems in overdetermined and underdetermined cases.

Acknowledgements. *This research is supported by NSFC (61370147, 61170309), 973 Program (2013CB329404).*

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Classical Model of Prandtl's Boundary Layer Theory for Radial Viscous Flow: Application of $(G'/G)-$ Expansion Method

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Abstract

In this paper, the exact closed-form solutions of the Prandtl's boundary layer equation for radial flow models with uniform or vanishing mainstream velocity are derived by using the $(G'/G)-$expansion method. Many new exact solutions are found for the boundary layer equation, which are expressed by the hyperbolic, trigonometric and rational functions. The solutions are valid for all values of the parameter β. It is shown that the $(G'/G)-$expansion method is effective and can be used for many other nonlinear differential equations of mathematical physics.

Keywords: $(G'/G)-$Expansion method; Prandtl's boundary layer equation; Exact solutions

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1 Introduction

Many real world problems in nonlinear science associated with mechanical, structural, aeronautical, ocean, electrical, and control systems can be summarized as solving nonlinear differential equations which arise from mathematically modelling such problems. Therefore, the study of nonlinear differential equations has been an active area of research for the past few years. Investigating integrability and finding exact solutions to such nonlinear differential equations have extensive applications in many scientific fields such as hydrodynamics, fluid dynamics, general relativity, condensed matter physics, solid-state physics, nonlinear optics, neurodynamics, fibre-optic communication and so on. These exact solutions, if reported are helpful for the numerical analyst to verify the complex numerical codes and are also useful in stability analysis for solving special nonlinear models.

In recent years, much attention has been devoted to the development of several powerful and useful methods for finding exact and approximate solutions of nonlinear differential equations. These research methods for solving nonlinear differential equations include the bilinear method and multilinear method [1], classical Lie symmetry method [2], nonclassical Lie group approach [3], Clarkson-Kruskal's direct method [4], deformation mapping method [5], homogenous balance method [6], Weierstrass elliptic function expansion method [7], F-expansion method [8], transformed rational function method [9], auxiliary equation method [10], sine–cosine method [11], tanh-function method [12], Backlund transformation method [13], simplest equation method [14, 15], exponential function rational expansion method [16] and so forth.

Prandtl [17] initiated the concept of a boundary layer in large Reynolds number flows in 1904 and he also showed how the Navier-Stokes equation could be simplified to yield approximate solutions. Prandtl introduced boundary layer theory to understand the flow behavior of a viscous Newtonian fluid near a solid boundary. Prandtl's boundary layer equations arise in various physical models of fluid mechanics. The equations of the boundary layer theory have been the subject of considerable interest, since they represent an important simplification of the original Navier-Stokes equations. These equations arise in the study of steady flows produced by wall jets, free jets and liquid jets, the flow past a stretching plate/surface, flow

induced due to a shrinking sheet and so on. These boundary layer equations are usually solved subject to specific boundary conditions depending upon the physical model investigation. Blasius [18] solved the Prandtl's boundary layer equations for a flat moving plate problem and found a power series solution of the model. Falkner and Skan [19] generalized the Blasius problem by considering the boundary layer flow over an wedge inclined at certain angle. Sakiadis [20] studied the boundary layer flow over a continuously moving rigid surface with a constant speed. Crane [21] was the first one who investigated the boundary layer flow due to a stretching surface and developed the exact solutions of boundary layer equations. Gupta and Gupta [22] extended the Crane's work and for the first time introduced the concept of heat transfer with the stretching sheet boundary layer flow. Schlichting [23] was the first to apply the boundary layer theory to the steady flow produced by a free two-dimensional jet emerging into a fluid at rest and solved the resulting ordinary differential equation numerically. Later, Bickley [24] solved the differential equation analytically. The concept of the boundary layer to laminar jets is discussed fully in standard texts on boundary layer theory such as by Schlichting [25] and Rosenhead [26]. More recently, the similarity solution of axisymmetric non-Newtonian wall jet with swirl effects was obtained by Kolar [27]. Naz et al. [28] and Mason [29] studied the general boundary layer equations for two-dimensional and radial flows by using the classical Lie group approach and recently Naz et al. [30] provided the similarity solutions of the Prandtl's boundary layer equations by implementing the non-classical symmetry method.

The $(G'/G)-$expansion method is a powerful mathematical tool for finding exact solutions of certain nonlinear ordinary differential equations. The $(G'/G)-$expansion method was introduced by Wang in [31] for constructing the exact solutions of some nonlinear evolution equations. To express the applicability and effectiveness of the $(G'/G)-$expansion method, further research has been accomplished by a diverse group of researchers (see, for example, papers $[32-34]$). The importance of our present work is to find some new class of exact closed-form solutions of Prandtl's boundary layer equation for radial flow models with constant or uniform main stream velocity by employing the $(G'/G)-$expansion method.

2 Mathematical model

The Prandtl's boundary layer equation, for the stream function $\phi(r,\theta)$, for radial flow with uniform or vanishing mainstream velocity is [26]

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2-\frac{1}{r}\frac{\partial\phi}{\partial r}\frac{\partial^2\phi}{\partial\theta^2}-\nu\frac{\partial^3\phi}{\partial\theta^3}=0, \tag{1}$$

where (r,θ) denote the cylindrical polar coordinates and ν is the kinematic viscosity. The velocity components $u(r,\theta)$ and $v(r,\theta)$, in the r and θ directions, are related to stream function $\phi(r,\theta)$ as

$$u(r,\theta)=\frac{1}{r}\frac{\partial\phi}{\partial\theta},\quad v(r,\theta)=-\frac{1}{r}\frac{\partial\phi}{\partial r}. \tag{2}$$

By the use of Lie group theoretic method of infinitesimal transformations [2], the general form of similarity solution for equation (1) is

$$\phi(r,\theta)=r^{2-\beta}H(\xi),\quad \xi=\frac{\theta}{r^\beta}, \tag{3}$$

where β is the constant determined from further conditions and $\xi=\theta/r^\beta$ is the similarity variable. By the substitution of Eq. (3) into Eq. (1), we obtain the third-order nonlinear ordinary differential equation in $H(\xi)$, viz.,

$$\nu\frac{d^3H}{d\xi^3}+(2-\beta)H\frac{d^2H}{d\xi^2}+(2\beta-1)\left(\frac{dH}{d\xi}\right)^2=0. \tag{4}$$

Equation (4) is the general form of Prandtl's boundary layer equation for radial flow of a viscous incompressible fluid. The boundary layer equation is usually solved subject to certain boundary conditions depending upon the particular physical model under investigation. Here, we find the exact closed-form solutions of Eq. (4) using the $(G'/G)-$expansion method. The paper is organised as follows. In Section 3, we provide a brief summary of the $(G'/G)-$expansion method. In Sections 4, we apply this method to solve nonlinear Prandtl's boundary layer equation for radial flow. Finally, some concluding remarks are presented in Section 5.

3 A description of the $(G'/G)-$expansion method

In this section, we present a brief summary of the $(G'/G)-$expansion method for solving nonlinear ordinary differential equations. The essence of the $(G'/G)-$expansion method is given in the following steps:

Step 1: We consider a general form of a nonlinear ordinary differential equation

$$P\left[U(z), \frac{dU}{dz}, \frac{d^2U}{dz^2}, \frac{d^3U}{dz^3}, ...\right] = 0, \tag{5}$$

where U is an unknown function of z and P is a polynomial in U and its various derivatives.

Step 2: According to the $(G'/G)-$expansion method, one assumes that the solution of ODE (5) can be written as a polynomial in (G'/G) as follows:

$$U(z) = \sum_{i=0}^{M} \beta_i \left(\frac{G'}{G}\right)^i, \tag{6}$$

where $G = G(z)$ satisfies the second-order linear ODE with constant coefficients, namely

$$\frac{d^2G}{dz^2} + \lambda\frac{dG}{dz} + \mu G = 0, \tag{7}$$

with β_i $(i = 0, 1, 2, ..., M)$, λ and μ being constants to be determined. The integer M is found by considering the homogenous balance between the highest order derivatives and nonlinear terms appearing in ODE (5).

Step 3: The positive integer M can be accomplished by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (5) as follows:

If we define the degree of $U(z)$ as $D[U(z)] = M$, then the degree of other expressions is defined by

$$\begin{aligned} D\left[\frac{d^qU(z)}{dz^q}\right] &= M + q, \\ D\left[U^r\left(\frac{d^qU(z)}{dz^q}\right)^s\right] &= Mr + s(q + M). \end{aligned} \tag{8}$$

Therefore, we can get the value of M in Eq. (6).

Step 4: We substitute Eq. (6) into Eq. (5) and then use ODE (7) to collect all terms with same order of (G'/G) together. The left-hand side of (5) is then converted into polynomial in (G'/G). Now by equating each coefficient of this polynomial to zero, we obtain a system of algebraic equations for β_i, λ and μ.

Step 5: Since the three types of general solutions of Eq. (7) are well known, we substitute the values of β_i and the general solutions of Eq. (7) into Eq. (6) and obtain three types of solutions of the ODE (5).

4 Application of the $(G'/G)-$expansion method

In this section, we employ the $(G'/G)-$expansion method to obtain solutions of Prandtl's boundary layer Eq. (4).

We assume that the solutions of Eq. (4) are of the form

$$H(\xi) = \sum_{i=0}^{M} A_i \left(\frac{G'(\xi)}{G(\xi)} \right)^i, \tag{9}$$

where $G(\xi)$ satisfies the second-order linear ODE with constant coefficients, viz.,

$$\frac{d^2G}{d\xi^2} + \lambda \frac{dG}{d\xi} + \mu G = 0 \tag{10}$$

with λ and μ being constants.

The balancing procedure yields $M = 1$, so the solution of the ODE (4) is of the form

$$H(\xi) = A_0 + A_1 \left(\frac{G'(\xi)}{G(\xi)} \right). \tag{11}$$

Now substituting Eq. (11) into Eq. (4), making use of the ODE (10), collecting all terms with same powers of (G'/G) and equating each coefficient to zero, yields the

following system of algebraic equations:

$$\begin{aligned}
2\beta A_1^2\mu^2 - \beta A_0A_1\lambda\mu - A_1\lambda^2\mu\nu + 2A_0A_1\lambda\mu - 2A_1\mu^2\nu - A_1^2\mu^2 &= 0, \\
3\beta A_1^2\lambda\mu - \beta A_0A_1\lambda^2 - 2\beta A_0A_1\mu - A_1\lambda^3\nu + 2A_0A_1\lambda^2 - 8A_1\lambda\mu\nu + 4A_0A_1\mu &= 0, \\
\beta A_1^2\lambda^2 - 3\beta A_0A_1\lambda + 2\beta A_1^2\mu - 7A_1\lambda^2\nu + A_1^2\lambda^2 + 6A_0A_1\lambda - 8A_1\mu\nu + 2A_1^2\mu &= 0, \\
\beta A_1^2\lambda - 2\beta A_0A_1 - 12A_1\lambda\nu + 4A_1^2\lambda + 4A_0A_1 &= 0, \\
3A_1^2 - 6A_1\nu &= 0.
\end{aligned}$$

Solving this system of algebraic equations, with the aid of Mathematica, we obtain

$$\lambda = 2\sqrt{\mu}, \quad A_0 = \lambda\nu, \quad A_1 = 2\nu. \tag{12}$$

Substituting these values of A_0, A_1 and the corresponding solution of ODE (4) into Eq. (11), we obtain the following three types of solutions of Eq. (1):

Case 1: When $\lambda^2 - 4\mu > 0$

For this case we obtain the hyperbolic function solution given by

$$H(\xi) = \lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \delta\frac{C_1\sinh(\delta\xi) + C_2\cosh(\delta\xi)}{C_1\cosh(\delta\xi) + C_2\sinh(\delta\xi)}\right), \tag{13}$$

where $\delta = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

Reverting back to the original variables (r, θ), the corresponding stream function is given by

$$\phi(r,\theta) = r^{2-\beta}\left[\lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \delta\frac{C_1\sinh\left(\delta\frac{\theta}{r^\beta}\right) + C_2\cosh\left(\delta\frac{\theta}{r^\beta}\right)}{C_1\cosh\left(\delta\frac{\theta}{r^\beta}\right) + C_2\sinh\left(\delta\frac{\theta}{r^\beta}\right)}\right)\right]. \tag{14}$$

Case 2: When $\lambda^2 - 4\mu < 0$

Here we obtain the trigonometric function solution

$$H(\xi) = \lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \epsilon\frac{-C_1\sin(\epsilon\xi) + C_2\cos(\delta\xi)}{C_1\cos(\epsilon\xi) + C_2\sin(\epsilon\xi)}\right), \tag{15}$$

where $\epsilon = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants. The corresponding stream function is given as

$$\phi(r,\theta) = r^{2-\beta}\left[\lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \epsilon\frac{-C_1\sin\left(\epsilon\frac{\theta}{r^\beta}\right) + C_2\cos\left(\epsilon\frac{\theta}{r^\beta}\right)}{C_1\cos\left(\epsilon\frac{\theta}{r^\beta}\right) + C_2\sin\left(\epsilon\frac{\theta}{r^\beta}\right)}\right)\right]. \tag{16}$$

Case 3: When $\lambda^2 - 4\mu = 0$

For this case we obtain the rational function solution

$$H(\xi) = \lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right). \tag{17}$$

In the form of stream function, the solution is expressed as

$$\phi(r,\theta) = r^{2-\beta}\left[\lambda\nu + 2\nu\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\frac{\theta}{r^\beta}}\right)\right], \tag{18}$$

where C_1 and C_2 are arbitrary constants.

5 Concluding remarks

We have employed the (G'/G)-expansion method for obtaining exact closed-form solutions of the well-known Prandtl's boundary layer equation for radial flow models with uniform main stream velocity. The advantage of this method is that in this method, there is no need to apply the initial and boundary conditions at the outset. This method yields a general solution with free parameters which can be identified by the specific conditions. Also the general solutions obtained by (G'/G)-expansion method are not approximate solutions. Prandtl's boundary layer equations arise in various physical models of fluid dynamics and thus the exact solutions obtained maybe very useful and significant for the explanation of some practical physical models dealing with Prandtl's boundary layer theory.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.1, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

On properties of meromorphic solutions for a certain q-difference Painlevé equation

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Abstract

The main purpose of this paper is to investigate some properties on transcendental meromorphic solutions of a certain q-difference Painevé equation

$$f(qz)+f(z)+f(\frac{z}{q})=\frac{az+b}{f(z)}+c,$$

where a,b and c are complex constants such that $|a|+|b|\neq 0$. We obtain some results on the value distribution of $f(z)$ and $\Delta_q f(z):=f(qz)-f(z)$, and the non-existence of rational solutions, which extend some earlier results by Qi and Yang, Chen et al.

Key words: q-difference equation; solution; zero order.
Mathematical Subject Classification (2010): 39A 50, 30D 35.

1 Introduction and Main Results

In this paper, we shall assume that readers are familiar with the basic theorems and the standard notations of the Nevanlinna value distribution theory of meromorphic functions such as $m(r,f)$, $N(r,f)$, $T(r,f)$, $\cdots$, (see Hayman [12], Yang [19] and Yi and Yang [20]). We also use $S(r,f)$ to denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r on a set $F\subset[1,+\infty$ of logarithmic density 1, where the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty}\frac{1}{\log r}\int_{[1,r]\cap F}\frac{1}{t}dt.$$

Throughout this paper, the set F of logarithmic density 1 can be not necessarily the same at each occurrence.

A century ago, Painlevé and his colleagues [15] classified all equations of Painlevé type of the form

$$w''(z)=F(z;w;w'),$$

where F is rational in w and w' and (locally) analytic in z. They singled out a list of 50 equations, six of which could not be integrated in terms of known functions. These equations are now known as the differential Painlevé equations. The first two of these equations are P_I and P_{II}:

$$w''=6w^2+z,\quad w''=2w^2+zw+\alpha,$$

where α is a complex constant.

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Differential Painlevé equations have been an important research subject in the field of the Mathematics and the Physics since the beginning of last century. They occur in many physical situations—-plasma physics, statistical mechanics, nonlinear waves, and so on. Therefore, Painlevé equations have attracted much interest as the reduction of solution equations which are solvable by inverse scattering transformations, and so on.

In the past 22 years, the discrete Painlevé equations have become important research problems (see [7]). For example, the discrete P_I equation can be expressed by

$$y_{n+1} + y_{n-1} = \frac{an+b}{y_n} + c,$$

and the discrete P_{II} equation can be expressed by

$$y_{n+1} + y_{n-1} = \frac{(an+b)y_n + c}{1 - y_n^2},$$

where a, b, c are real constants, $n \in \mathbb{N}$.

In 2006-2007, Halburd and Korhonen used the analogues of Nevanlinna value distribution theory to single out the difference Painlevé I and II equations from the following form

$$w(z+1) + w(z-1) = R(z, w), \tag{1}$$

where $R(z, w)$ is rational in w and meromorphic in z (see [9, 10, 11]). They obtained that if (1) has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation, or (1) can be transformed by a linear change in w to some difference equations, which include the difference Painlevé I equation

$$w(z+1) + w(z-1) = \frac{az+b}{w(z)} + c, \tag{2}$$

and the difference Painlevé II equation

$$w(z+1) + w(z-1) = \frac{(az+b)w(z) + c}{1 - w(z)^2}, \tag{3}$$

where a, b, c are complex constants.

Chen et al [4, 5, 16] studied some properties of finite order transcendental meromorphic solutions of (2)-(3), and obtained a lot of interesting results.

Recently, there were lots of results about q-difference operators, q-difference equations, and so on (see [2, 6, 8, 18, 21, 22]), by applying the analogue of Logarithmic Derivative Lemma on q-difference operators, which was firstly established by Barnett, Halburd, Korhonen and Morgan [1] in 2007. By comparing these results of differences and q-differences, we find that the usual shift $f(z+c)$ of a meromorphic function are replaced by the q-difference $f(qz)$, and the difference $\Delta_c f = f(z+c) - f(z)$ are replaced by $\Delta_q f(z) = f(qz) - f(z)$, $q \in \mathbb{C} \backslash \{0, 1\}$.

In 2015, Qi and Yang [17] investigated the following equations

$$f(qz) + f(\frac{z}{q}) = \frac{az+b}{f(z)} + c, \tag{4}$$

$$f(qz) + f(\frac{z}{q}) = \frac{(az+b)f(z) + c}{1 - f(z)^2}, \tag{5}$$

which can be seen as q-difference analogues of (2) and (3), and obtained some theorems as follows.

Theorem 1.1 *[17, Theorem 1.1]. Let $f(z)$ be a transcendental meromorphic solution with zero order of equation (4), and a, b, c be three constants such that a, b cannot vanish simultaneously. Then,*

(i) $f(z)$ has infinitely many poles.

(ii) If $a \neq 0$, then $f(z)$ has infinitely many finite values.

(iii) If $a = 0$ and $f(z)$ takes a finite value A finitely often, then A is a solution of $2z^2 - cz - b = 0$.

Theorem 1.2 *[17, Theorem 1.2]. Let a, b, c and $|q| \neq 1$ be four constants, (i) if $a \neq 0$, then equation (4) has no rational solution;*

(ii) if $a = 0$, then the rational solutions of the equation (4) must satisfy $f(z) = B + \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are relatively prime polynomials and satisfy $\deg P < \deg Q$ and $2z^2 - cz - b = 0$.

Theorem 1.3 *[17, Theorem 1.3]. Let a, b, c be constants with $ac \neq 0$, and $f(z)$ be a transcendental meromorphic solution with zero order of equation (5). Then $f(z)$ has infinitely many poles and infinitely many finite values.*

Inspired by the above results, we further investigate some properties of transcendental meromorphic solutions of the q-difference Painlevé equation

$$f(qz) + f(z) + f(\frac{z}{q}) = \frac{az + b}{f(z)} + c, \tag{6}$$

which is different from (4) and (5) to some extent, and obtain the following theorems.

Theorem 1.4 *Let a, b, c be complex constants such that $|a| + |b| \neq 0$, and $f(z)$ be a zero-order transcendental meromorphic solution of the q-difference Painlevé equation (6).*

(i) If $a \neq 0$, $p(z)$ is a polynomial of degree $k(\geq 0)$ and $|q| \neq 1$, then $f(z) - p(z)$ has infinitely many zeros; if $a = 0$, then the Borel exceptional values of $f(z)$ can only come from the set $E = \{z|\ 3z^2 - cz - b = 0\}$;

(ii) $f(z)$ and $\Delta_q f(z)$ have infinitely many poles, where $\Delta_q f(z) = f(qz) - f(z)$.

Theorem 1.5 *Let a, b, c be complex constants such that $|a| + |b| \neq 0$.*

(i) If $a \neq 0$, then (6) has no rational solution.

(ii) If $a = 0$, then (6) has a nonzero constant solution $f(z) = B$, where B satisfies $3B^2 - cB - b = 0$. Furthermore, if $c^2 + 12b = 0$, then (6) has no nonconstant rational solution.

2 Some Lemmas

To prove our results, we require some lemmas as follows.

Lemma 2.1 *[14, Theorem 2.5] Let $f(z)$ be a transcendental meromorphic solution of order zero of a q-difference equation of the form*

$$U_q(z, f)P_q(z, f) = Q_q(z, f),$$

where $U_q(z, f), P_q(z, f)$ and $Q_q(z, f)$ are q-difference polynomials such that the total degree $\deg U_q(z, f) = n$ in $f(z)$ and its q-shifts, whereas $\deg Q_q(z, f) \leq n$. Moreover, we assume that $U_q(z, f)$ contains just one term of maximal total degree in $f(z)$ and its q-shifts. Then

$$m(r, P_q(z, f)) = o(T(r, f)),$$

on a set of logarithmic density 1.

Remark 2.1 *The above lemma can be called see as a type of a q-difference analogue of Clunie lemma, recently proved by Barnett et al.; see [1, Theorem 2.1].*

Remark 2.2 *Here, a q-difference polynomial of $f(z)$ for $q \in \mathbb{C}\backslash\{0,1\}$ is a polynomial in $f(z)$ and finitely many of its q-shifts $f(qz), \ldots, f(q^n z)$ with meromorphic coefficients in the sense that their Nevanlinna characteristic functions are $o(T(r,f))$ on a set of logarithmic density 1.*

Lemma 2.2 *[1, Theorem 2.5] Let $f(z)$ be a nonconstant zero-order meromorphic solution of $P_q(z,f) = 0$, where $P_q(z,f)$ is a q-difference polynomial in $f(z)$. If $P_q(z,a) \not\equiv 0$ for slowly moving target $a(z)$, then*

$$m(r, \frac{1}{f-a}) = o(T(r,f))$$

on a set of logarithmic density 1.

Lemma 2.3 *[21, Theorem 1.1 and 1.3] Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$T(r, f(qz)) = (1+o(1))T(r,f), \qquad N(r, f(qz)) = (1+o(1))N(r,f),$$

on a set of lower logarithmic density 1.

Lemma 2.4 *(Valiron-Mohon'ko) ([13]). Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in $f(z)$,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z) f(z)^i}{\sum_{j=0}^{n} b_j(z) f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies that

$$T(r, R(z, f(z))) = dT(r,f) + O(\Psi(r)),$$

where $d = \max\{m,n\}$ and $\Psi(r) = \max_{i,j}\{T(r,a_i), T(r,b_j)\}$.

3 Proof of Theorem 1.4

Suppose that $f(z)$ is a zero-order transcendental meromorphic solution of (6).

(i) If$a \neq 0$, and $p(z)$ is a polynomial of degree $k(\geq 0)$. Let $p(z) = a_k z^k + \cdots + a_1 z + a_0$. Let $g(z) = f(z) - p(z)$. Substituting $f(z) = g(z) + p(z)$ into equation (6), we have

$$g(qz) + p(qz) + g(z) + p(z) + g(\frac{z}{q}) + p(\frac{z}{q}) = \frac{az+b}{g(z)+p(z)} + c.$$

It follows that

$$\begin{aligned} P_q(z,g) := &\left[g(qz) + p(qz) + g(z) + p(z) + g(\frac{z}{q}) + p(\frac{z}{q})\right][g(z)+p(z)] \\ &- (az+b) - c\,[g(z)+p(z)] = 0. \end{aligned} \tag{7}$$

From (7), we have

$$P_q(z,0) = \left[p(qz) + p(z) + p(\frac{z}{q})\right] p(z) - (az+b) - cp(z). \tag{8}$$

If $p(z) \equiv 0$, then $P_q(z,0) = -(az+b) \not\equiv 0$. If $k=0$ and $p(z) = a_0 \equiv \alpha \in \mathbb{C} \setminus \{0\}$, then $P_q(z,0) = 3\alpha^2 - (az+b) - c\alpha \not\equiv 0$. If $k \geq 1$ and a_k is a nonzero constant, then, we have from (8) that

$$P_q(z,0) = \left[p(qz) + p(z) + p(\frac{z}{q})\right] p(z) - (az+b) - cp(z) = (q^k + 1 + \frac{1}{q^k})a_k^2 z^{2k} + \cdots. \quad (9)$$

Since $|q| \neq 1$, we have $q^k + 1 + \frac{1}{q^k} \neq 0$, then $P_q(z,0) \not\equiv 0$. Thus, we have by Lemma 2.2 that

$$m(r, \frac{1}{g}) = S(r,g).$$

Then, we get

$$N\left(r, \frac{1}{f-p}\right) = N\left(r, \frac{1}{g}\right) = T(r,g) + S(r,g) = T(r,f) + S(r,f). \quad (10)$$

Since $f(z)$ is transcendental, $f(z) - p(z)$ has infinitely many zeros.

If $a = 0$ and $p(z) = \beta \notin E$, then we have

$$P_q(z,0) = 3\beta^2 - c\beta - b \not\equiv 0.$$

Set $g(z) = f(z) - \beta$, by using the same argument as above, we can obtain $N(r, \frac{1}{f-\beta}) = T(r,f) + S(r,f)$.. Therefore, we can obtain that the Borel exceptional values of $f(z)$ can only come from the set $E = \{z | 3z^2 - cz - b = 0\}$.

(ii) From (6), we have

$$f(z)\left[f(qz) + f(z) + f(\frac{z}{q})\right] = az + b + cf(z). \quad (11)$$

It follows from Lemma 2.1 and (11) that

$$m\left(r, f(qz) + f(z) + f(\frac{z}{q})\right) = S(r,f). \quad (12)$$

By applying Lemma 2.4 for (6), we have

$$T\left(r, f(qz) + f(z) + f(\frac{z}{q})\right) = T(r,f) + S(r,f). \quad (13)$$

And by Lemma 2.3 we get

$$\begin{aligned} N\left(r, f(qz) + f(z) + f(\frac{z}{q})\right) &\leq N(r, f(qz)) + N(r, f(z)) + N\left(r, f(\frac{z}{q})\right) \\ &= 3(1+o(1))N(r,f) \end{aligned} \quad (14)$$

on a set of lower logarithmic density 1. Thus, by combining (12)-(14), we have

$$T(r,f) \leq 3(1+o(1))N(r,f) + S(r,f). \quad (15)$$

Since $f(z)$ is transcendental, $f(z)$ has infinitely many poles.

Next, we prove that $\Delta_q f(z)$ has infinitely many poles. Set $z = qw$, then we can rewrite (6) as the form

$$f(q^2 w) + f(qw) + f(w) = \frac{aqw + b}{f(qw)} + c. \quad (16)$$

Then it follows from (16) that

$$f(qw)\left[f(q^2w)+f(qw)+f(w)\right]=aqw+b+cf(qw). \tag{17}$$

Since $\Delta_q f(w)=f(qw)-f(w)$, we have $f(qw)=\Delta_q f(w)+f(w)$ and $f(q^2w)=\Delta_q f(qw)+\Delta_q f(w)+f(w)$. Substituting them into (17), we get

$$[\Delta_q f(w)+f(w)]\,[\Delta_q f(qw)+2\Delta_q f(w)+3f(w)]=(aqw+b)+c\,[\Delta_q f(w)+f(w)]\,,$$

i.e.,

$$\begin{aligned}-3f(w)^2=&[\Delta_q f(qw)+5\Delta_q f(w)-c]\,f(w)-(aqw+b)\\&+[\Delta_q f(qw)+2\Delta_q f(w)-c]\,\Delta_q f(w).\end{aligned} \tag{18}$$

Since $f(z)$ is a zero-order transcendental meromorphic function and $z=qw$, by Lemma 2.3, we get that $f(w)$ is of zero order. Thus, by Lemma 2.3 again, we have that $f(w),\Delta_q f(w),\Delta_q f(qw)$ are of zero-order. Then by Lemma 2.3 again, we have

$$N(r,\Delta_q f(qw))\leq N(r,\Delta_q f(w))+S(r,f). \tag{19}$$

Thus, from (18) and (19) we have

$$\begin{aligned}2N(r,f(w))=&N\left(r,[\Delta_q f(qw)+3\Delta_q f(w)-c]\,f(w)-(aqw+b)\right.\\&\left.+[\Delta_q f(qw)+\Delta_q f(w)-c]\,\Delta_q f(w)\right)\\\leq&N(r,f(w))+5N(r,\Delta_q f(w))+O(\log r)+S(r,f).\end{aligned}$$

That is,

$$N(r,f(w))\leq 5N(r,\Delta_q f(w))+S(r,f). \tag{20}$$

Then, it follows from (15) and (20) that

$$T(r,f(w))\leq 15N(r,\Delta_q f(w))+S(r,f). \tag{21}$$

Since $f(z)$ is transcendental, that is, $f(w)$ is transcendental, we have from (21) that $\Delta_q f(w)$ has infinitely many poles, that is, $\Delta_q f(z)$ has infinitely many poles.

Therefore, we complete the proof of Theorem 1.4.

4 Proof of Theorem 1.5

Sppose that $f(z)$ is a nonzero rational solution of (6), and has poles $z_1,z_2,\ldots,z_k$. Then, we let

$$\frac{\alpha_{is_i}}{(z-z_i)^{s_i}}+\cdots+\frac{\alpha_{is_1}}{(z-z_i)},\quad i=1,2,\ldots,k$$

be the principal parts of $f(z)$ at z_i respectively, where $\alpha_{is_i}\neq 0,\ldots,\alpha_{is_1}$ are constants, Thus, we can write $f(z)$ as the following form

$$f(z)=\sum_{i=1}^{k}\left(\frac{\alpha_{is_i}}{(z-z_i)^{s_i}}+\cdots+\frac{\alpha_{is_1}}{(z-z_i)}\right)+\beta_0+\beta_1 z+\cdots+\beta_m z^m, \tag{22}$$

where $\beta_0,\beta_1,\ldots,\beta_m$ are constants.

Next, we affirm that $\beta_m = \cdots = \beta_1 = 0$. Suppose that $\beta_m \neq 0 (m \geq 1)$. For sufficiently large z, by (22), we have

$$f(z) = \beta_m z^m (1 + o(1)), \tag{23}$$

$$f(qz) = \beta_m q^m z^m (1 + o(1)), \tag{24}$$

$$f(\frac{z}{q}) = \beta_m q^{-m} z^m (1 + o(1)). \tag{25}$$

By (6), we have

$$\left[f(qz) + f(z) + f(\frac{z}{q}) \right] f(z) = az + b + cf(z). \tag{26}$$

Substituting (23)-(25) into (26), we have

$$(1 + q^m + q^{-m}) \beta_m^2 z^{2m} (1 + o(1)) = az + b + c\beta_m z^m (1 + o(1)).$$

Since $|q| \neq 1$, we have $1 + q^m + q^{-m} \neq 0$. And since $\beta_m \neq 0$, we can see the above equation is a contradiction for sufficiently large z. Hence we have $\beta_1 = \cdots = \beta_m = 0$.

(i) Suppose that $a \neq 0$. If $\beta_0 \neq 0$, then for sufficiently large z, by (23)-(25), we have

$$f(qz) = f(z) = f(\frac{z}{q}) = \beta_0 + o(1). \tag{27}$$

Substituting (27) into (26), we conclude that

$$(3\beta_0 + o(1))(\beta_0 + o(1)) = az + b + c(\beta_0 + o(1)),$$

which is a contradiction to the assumption that $a \neq 0$. Thus, $\beta_0 = 0$. Then we have $\beta_0 = \beta_1 = \cdots = \beta_m = 0$. Thus, $f(z)$ can be rewritten by (22) as

$$f(z) = \frac{P(z)}{R(z)}, \tag{28}$$

where

$$P(z) = pz^k + p_{k-1} z^{k-1} + \cdots + p_0, \quad R(z) = rz^t + r_{t-1} z^{t-1} + \cdots + r_0, \tag{29}$$

where $p, p_{k-1}, \ldots, p_0$ and $r, r_{t-1}, \ldots, r_0$ are constants such that $pr \neq 0$ and $k < t$. Then substituting (28) into (6), we have

$$\begin{aligned} &P(qz)P(z)R(z)R(\frac{z}{q}) + P(z)^2 R(qz)R(\frac{z}{q}) + P(\frac{z}{q})P(z)R(qz)R(z) \\ &= (az + b)R(qz)R(z)^2 R(\frac{z}{q}) + cP(z)R(qz)R(z)R(\frac{z}{q}). \end{aligned} \tag{30}$$

Then since $k < t$, we can see that the degree of the left side of (30) does not exceed $2k + 2t$, and the degree of the right side of (30) is equal to $1 + 4t$ by $a \neq 0$. Thus, we can get a contradiction. Therefore, we have that (6) has no nonzero rational solution when $a \neq 0$.

(ii) Suppose that $a = 0$. If $f(z) = B$ is a nonzero constant solution of (6), we can easily get from (6) that B satisfies $3B^2 - cB - b = 0$. Now, we prove that (6) has no rational solution if $a = 0$ and $c^2 + 12b = 0$. Suppose that $f(z)$ is a nonconstant rational solution of (6). Since $\beta_m = 0 (m \geq 1)$, $f(z)$ can be rewritten as the form (28), where

$P(z)$ and $R(z)$ satisfy (29) with $k \le t$. Suppose that $k < t$. Substituting (28) into (6), we have

$$\begin{aligned}P(qz)P(z)R(z)R(\frac{z}{q}) + P(z)^2R(qz)R(\frac{z}{q}) + P(\frac{z}{q})P(z)R(qz)R(z)\\ = bR(qz)R(z)^2R(\frac{z}{q}) + cP(z)R(qz)R(z)R(\frac{z}{q}).\end{aligned} \tag{31}$$

If $k < t$, then it follows from (31) that there exists only one term $bR(qz)R(z)^2R(\frac{z}{q})$ with maximal degree, which is a contradiction. Thus, we have $k = t$. Then, it follows by (29) and (30) that

$$\begin{aligned}&\frac{pq^kz^k + p_{k-1}q^{k-1}z^{k-1} + \cdots + p_0}{rq^tz^t + r_{t-1}q^{t-1}z^{t-1} + \cdots + r_0} + \frac{pz^z + p_{k-1}z^{k-1} + \cdots + p_0}{rz^t + r_{t-1}z^{t-1} + \cdots + r_0}\\ &+ \frac{pq^{-k}z^k + p_{k-1}q^{-(k-1)}z^{k-1} + \cdots + p_0}{rq^{-t}z^t + r_{t-1}q^{-(t-1)}z^{t-1} + \cdots + r_0}\\ &= \frac{b(rz^t + r_{t-1}z^{t-1} + \cdots + r_0)}{pz^k + p_{k-1}z^{k-1} + \cdots + p_0} + c.\end{aligned} \tag{32}$$

Then it follows from (32) that

$$3B^2 - cB - b = 0,$$

as $z \to \infty$, where $B = \frac{p}{r} \neq 0$. Therefore, $f(z)$ can be rewritten as

$$f(z) = B + \frac{G(z)}{H(z)}, \tag{33}$$

where $G(z)$ and $H(z)$ are relatively prime polynomials and satisfy $\deg G(z) = \mu < \deg H(z) = \nu$, B is a constant satisfying $3B^2 - cB - b = 0$. Denote

$$G(z) = \xi z^\mu + \xi_{\mu-1}z^{\mu-1} + \cdots + \xi_0, \quad H(z) = \eta z^\nu + \eta_{\nu-1}z^{\nu-1} + \cdots + \eta_0, \tag{34}$$

where $\xi, \xi_{\mu-1}, \ldots, p_0$ and $\eta, \eta_{\nu-1}, \ldots, \eta_0$ are constants such that $\xi\eta \neq 0$. Substituting (34) into (6) and noting $3B^2 - cB - b = 0$, we have

$$\begin{aligned}(4B - c)G(z)H(qz)H(z)H(\frac{z}{q}) + BG(qz)H(z)^2H(\frac{z}{q}) + BG(\frac{z}{q})H(z)^2H(qz)\\ = -G(qz)G(z)H(z)H(\frac{z}{q}) - G(z)^2H(qz)H(\frac{z}{q}) - G(\frac{z}{q})G(z)H(z)H(qz).\end{aligned} \tag{35}$$

By observing the coefficients and degrees of all terms of the above equation, and combining with $\nu > \mu$, we have that the term with maximal degree of (35) is

$$\left[(4B - c) + Bq^{\mu-\nu} + Bq^{\nu-\mu}\right]\xi\eta^3z^{\mu+3\nu}.$$

Since $3B^2 - cB - b = 0$ and $c^2 + 12b = 0$, we have $B = \frac{c}{6}$. And by $|q| \neq 1$, we can get that $(4B - c) + Bq^{\mu-\nu} + Bq^{\nu-\mu} \neq 0$. In fact, if $(4B - c) + Bq^{\mu-\nu} + Bq^{\nu-\mu} = 0$, *i.e.*

$$B = \frac{c}{4 + q^{\mu-\nu} + q^{\nu-\mu}}.$$

Then, we have

$$\frac{c}{4 + q^{\mu-\nu} + q^{\nu-\mu}} = \frac{c}{6}.$$

By solving the above equation, we get $|q| = 1$, a contradiction. Thus, (35) is a contradiction for sufficiently large z. Therefore, if $a = 0$ and $c^2 + 12b = 0$, then (6) has no nonconstant rational solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors drafted the manuscript, read and approved the final manuscript.

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Acknowledgements

This work was supported by the National Natural Science Foundation of China (11301233, 11561033), the Natural Science Foundation of Jiangxi Province in China (20151BAB201008, 20151BAB201004), and the Youth Science Foundation of Education Bureau of Jiangxi Province in China (GJJ14644, GJJ14271).

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New approximation of fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces

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Abstract

We prove necessary and sufficient conditions for the strong convergence of the modified two-step iteration process to the fixed point of asymptotically demicontractive mappings in real Banach spaces.

2010 Mathematics Subject Classification: 47J25, 65J15.

Key words and phrases: Iteration process, asymptotically demicontractive mapping, Banach space

1 Introduction

Let K be a nonempty subset of a real Banach space X and X^* be its dual space. We denote by J the normalized duality mapping from X into 2^{X^*} defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If X is strictly convex, then J is single-valued. In the sequel, we shall denote the single-valued duality mapping by j.

Let $T : K \to K$ be a mapping.

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Definition 1.1. T is called a *k-strictly asymptotically pseudo-contractive mapping* with sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ if for all $x, y \in K$ there exists $j(x-y) \in J(x-y)$ and a constant $k \in [0,1)$ such that

$$\begin{aligned}&\langle (I-T^n)x-(I-T^n)y, j(x-y)\rangle \\ &\geq \frac{1}{2}(1-k)\left\|(I-T^n)x-(I-T^n)y\right\|^2-\frac{1}{2}(k_n^2-1)\left\|x-y\right\|^2\end{aligned} \tag{1.1}$$

for all $n \in \mathbb{N}$.

Definition 1.2. T is called an *asymptotically demicontractive mapping* with sequence $\{k_n\} \subset [0,\infty)$, $\lim_{n\to\infty} k_n = 1$ if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and for all $x \in K$ and $x^* \in F(T)$, there exists $k \in [0,1)$ and $j(x-x^*) \in J(x-x^*)$ such that

$$\langle x-T^n x, j(x-x^*)\rangle \geq \frac{1}{2}(1-k)\left\|x-T^n x\right\|^2-\frac{1}{2}(k_n^2-1)\left\|x-x^*\right\|^2 \tag{1.2}$$

for all $n \in \mathbb{N}$.

Definition 1.3. $T : K \to K$ is called *uniformly L-Lipschitizian* if there exists a constant $L > 0$ such that

$$\left\|T^n x-T^n y\right\| \leq L\left\|x-y\right\|, \tag{1.3}$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

The classes of k-strictly asymptotically pseudo-contractive and asymptotically demicontractive mappings are introduced by Liu [3]. It is easy to see that a k-strictly asymptotically pseudo-contrative mapping with a non-empty fixed point set $F(T)$ is asymptotically demicontractive.

In Hilbert spaces, it is shown in [3] that (1.1) and (1.2) are equivalent to the following inequalities:

$$\left\|T^n x-T^n y\right\| \leq k_n^2\left\|x-y\right\|^2+k\left\|(I-T^n)x-(I-T^n)y\right\|^2$$

and

$$\left\|T^n x-T^n y\right\|^2 \leq k_n^2\left\|x-y\right\|^2+\left\|x-T^n x\right\|^2,$$

respectively.

By using the modified Mann iteration method [4] introduced by Schu [7], Liu [3] proved a convergence theorem for the iterative approximation of fixed points of k-strictly asymptotically pseudo-contractive mappings and asymptotically demicontractive mappings in Hilbert spaces.

Osilike [6] extended the results of Liu [3] about the iterative approximation of fixed points of k-strictly asymptotically demicontractive mappings from Hilbert spaces to much more general real q-uniformly smooth Banach spaces, $1 < q < \infty$ and specifically proved the following results.

Theorem 1.4. *Let $q > 1$ and X be a real q-uniformly smooth Banach space. Let K be a closed convex and bounded subset of X and $T : K \to K$ a completely continuous uniformly L-Lipschitizian asymptotically demicontractive mapping with a sequence $k_n \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the conditions*

(i) $0 \leq \alpha_n, \beta_n \leq 1$, $n \geq 1$;

(ii) $0 < \epsilon \leq c_q \alpha_n^{q-1}(1 + L\beta_n)^q \leq \frac{1}{2}q(1-k)(1+L)^{-(q-2)} - \epsilon$ *for all* $n \geq 1$ *and for some* $\epsilon > 0$; *and*

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$.

Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \quad n \geq 1 \end{cases}$$

converges strongly to a fixed point of T.

Remark 1.5. For Hilbert spaces, in Theorem 1.4, if we put $q = 2$, $c_q = 1$ and $\beta_n = 0$, then Theorems 1 and 2 of Liu [3] follow.

Recently Chidume and Măruşter [1] made a comprehensive and very useful survey on the main convergence properties of the modified Mann iteration method for the demicontractive mappings.

The purpose of this work is to prove necessary and sufficient conditions for the strong convergence of the modified two-step iteration process to the fixed point of asymptotically demicontractive mappings in real Banach spaces. Our results extend and improve the results of Igbokwe [2], Liu [3], Moore and Nnoli [5].

2 Main results

The following results are useful:

Lemma 2.1. ([8]) *For all $\varrho, \varsigma \in X$ and $j(\varrho + \varsigma) \in J(\varrho + \varsigma)$,*

$$\|\varrho + \varsigma\|^2 \leq \|\varrho\|^2 + 2\mathrm{Re}\,\langle \varsigma, j(\varrho + \varsigma)\rangle.$$

Lemma 2.2. ([2]) *Let X be a normed space and K be a nonempty convex subset of X. Let $T : K \to K$ be uniformly L-Lipschitzian mapping and let $\{t_n\}$ and $\{\beta_n\}$ be the sequences in $[0, 1]$. For arbitrary $\varrho_1 \in K$, generate the sequence $\{\varrho_n\}$ by*

$$\begin{cases} \varrho_{n+1} = (1-t_n)\varrho_n + t_n T^n \varsigma_n, \\ \varsigma_n = (1-\beta_n)\varrho_n + \beta_n T^n \varrho_n, \quad n \geq 1. \end{cases}$$

Then

$$\|\varrho_n - T\varrho_n\| \leq \|\varrho_n - T^n \varrho_n\| + L(1+L)^2 \left\|\varrho_{n-1} - T^{n-1}\varrho_{n-1}\right\|. \tag{2.1}$$

We now prove our main results.

Lemma 2.3. *Let X be a real Banach space and K be a nonempty convex subset of X. Let $T : K \to K$ be an uniformly $L-$Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $\varrho_1 \in K$, generate the sequence $\{\varrho_n\}$ by*

$$\begin{cases} \varrho_{n+1} = (1-t_n)\varrho_n + t_n T^n \varsigma_n, \\ \varsigma_n = (1-\beta_n)\varrho_n + \beta_n T^n \varrho_n, \quad n \geq 1, \end{cases} \tag{2.2}$$

where $\{t_n\}$ and $\{\beta_n\}$ are the sequences in $[0,1]$ satisfying
(i) $\sum_{n=1}^{\infty} t_n = \infty$,
(ii) $\lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} \beta_n$.
Then (a) the sequence $\{\varrho_n\}$ is bounded,
(b) $\liminf_{n\to\infty} \|\varrho_{n+1} - T^n \varrho_{n+1}\| = 0$,
(c) $\liminf_{n\to\infty} \|\varrho_n - T^n \varrho_n\| = 0$,
(d) $\liminf_{n\to\infty} \|\varrho_n - T\varrho_n\| = 0$.

Proof. Since T is asymptotically demicontractive, then

$$\langle \varrho - T^n\varrho, j(\varrho - \varrho^*)\rangle \geq \frac{1}{2}(1-k)\|\varrho - T^n\varrho\|^2 - \frac{1}{2}(k_n^2 - 1)\|\varrho - \varrho^*\|^2$$

and hence

$$\|\varrho - T^n\varrho\| \leq \sqrt{\frac{(2\|\varrho - T^n\varrho\| + (k_n^2-1)\|\varrho - \varrho^*\|)\|\varrho - \varrho^*\|}{1-k}}.$$

Therefore, by the triangle inequality,

$$\|\varrho - \varrho^*\| \leqslant \|T^n\varrho - \varrho^*\| + \sqrt{\frac{(2\|\varrho - T^n\varrho\| + (k_n^2-1)\|\varrho - \varrho^*\|)\|\varrho - \varrho^*\|}{1-k}}. \tag{2.3}$$

Now we shall prove that

$$\liminf_{n\to\infty} \|\varrho_{n+1} - T^n\varrho_{n+1}\| = 0.$$

If $\varrho_n = T\varrho_n$ for all $n \geqslant m$ for some $m \in \mathbb{N}$, then (2.3) trivially holds, as we have

$$\begin{aligned} \|\varrho_{n+1} - T^n\varrho_{n+1}\| &= \|\varrho_{n+1} - T^nT\varrho_{n+1}\| = \left\|\varrho_{n+1} - T^{n+1}\varrho_{n+1}\right\| \\ &= 0 \end{aligned}$$

for all $n \geq m$.

Suppose now that there exists the smallest positive integer n_0 such that $\varrho_{n_0} \neq T\varrho_{n_0}$. Put

$$\begin{aligned} a_0 := &\|T^{n_0}\varrho_{n_0} - \varrho^*\| \\ &+ \sqrt{\frac{\left(2\|\varrho_{n_0} - T^{n_0}\varrho_{n_0}\| + (k_{n_0}^2 - 1)\|\varrho_{n_0} - \varrho^*\|\right)\|\varrho_{n_0} - \varrho^*\|}{1-k}} + 1. \end{aligned}$$

Then clearly

$$\|\varrho_{n_0} - \varrho^*\| \leq a_0. \tag{2.4}$$

To prove that $\liminf_{n\to\infty} \|\varrho_{n+1} - T^n\varrho_{n+1}\| = 0$, we shall assume, to the contrary, that $\liminf_{n\to\infty} \|\varrho_{n+1} - T^n\varrho_{n+1}\| = 2\delta > 0$. Then there exists $n_0' \in \mathbb{N}$ such that $\|\varrho_{n+1} - T^n\varrho_{n+1}\| \geqslant \delta$ for all $n \geq n_0'$.

Also, by $\lim_{n\to\infty} k_n = 1$ and (ii), we may suppose that

$$\begin{aligned} t_n &\leq \min\left\{\frac{1}{1+2L}, \frac{(1-k)\delta^2}{24(1+L)(1+2L)a_0^2}\right\}, \\ \beta_n &\leq \min\left\{\frac{1}{1+L}, \frac{(1-k)\delta^2}{24L(1+L)a_0^2}\right\}, \\ k_n^2 - 1 &\leq \frac{(1-k)\delta^2}{24a_0^2} \end{aligned} \tag{2.5}$$

for all $n \geq n_0'$.

We now show that the sequence $\{\varrho_n\}$ is bounded. By induction we shall show that

$$\|\varrho_n - \varrho^*\| \leqslant a_0 \tag{2.6}$$

for all $n \geq n_0'$.

It is clear that (2.6) holds for $n = n_0$. Assume it is true for some $n > N := \max\{n_0, n_0'\}$, that is, $\|\varrho_n - \varrho^*\| \leq a_0$ for some $n \geq N$. Then

$$\begin{aligned} \|\varrho_n - T^n\varrho_n\| &\leq \|\varrho_n - \varrho^*\| + \|T^n\varrho_n - \varrho^*\| \\ &\leq (1+L)\|\varrho_n - \varrho^*\| \\ &\leq (1+L)a_0, \end{aligned}$$

$$\begin{aligned} \|\varsigma_n - \varrho^*\| &= \|(1-\beta_n)\varrho_n + \beta_n T^n\varrho_n - \varrho^*\| \\ &= \|\varrho_n - \varrho^* - \beta_n(\varrho_n - T^n\varrho_n)\| \\ &\leq \|\varrho_n - \varrho^*\| + \beta_n\|\varrho_n - T^n\varrho_n\| \\ &\leq a_0 + (1+L)a_0\beta_n \\ &\leq 2a_0, \end{aligned}$$

$$\begin{aligned} \|\varrho_n - T^n\varsigma_n\| &\leq \|\varrho_n - \varrho^*\| + \|T^n\varsigma_n - \varrho^*\| \\ &\leq \|\varrho_n - \varrho^*\| + L\|\varsigma_n - \varrho^*\| \\ &\leq (1+2L)a_0, \end{aligned}$$

and

$$\begin{aligned} \|\varrho_{n+1} - \varrho^*\| &= \|(1-t_n)\varrho_n + t_n T^n\varsigma_n - \varrho^*\| \\ &= \|\varrho_n - \varrho^* - t_n(\varrho_n - T^n\varsigma_n)\| \\ &\leq \|\varrho_n - \varrho^*\| + t_n\|\varrho_n - T^n\varsigma_n\| \\ &\leq a_0 + (1+2L)a_0 t_n \\ &\leq 2a_0. \end{aligned} \tag{2.7}$$

On the other hand, by Lemma 2.1,

$$\begin{aligned}\|\varrho_{n+1}-\varrho^*\|^2 &= \|(1-t_n)\varrho_n + t_n T^n \varsigma_n - \varrho^*\|^2\\ &= \|\varrho_n - \varrho^* - t_n(\varrho_n - T^n\varsigma_n)\|^2\\ &\le \|\varrho_n-\varrho^*\|^2 - 2t_n\left\langle \varrho_n - T^n\varsigma_n, j(\varrho_{n+1}-\varrho^*)\right\rangle\\ &= \|\varrho_n-\varrho^*\|^2 - 2t_n\left\langle \varrho_{n+1} - T^n\varrho_{n+1}, j(\varrho_{n+1}-\varrho^*)\right\rangle\\ &\quad + 2t_n\left\langle T^n\varsigma_n - T^n\varrho_{n+1}, j(\varrho_{n+1}-\varrho^*)\right\rangle + 2t_n\left\langle \varrho_{n+1}-\varrho_n, j(\varrho_{n+1}-\varrho^*)\right\rangle.\end{aligned}$$

Since T is asymptotically demicontractive mapping, we obtain

$$\begin{aligned}\|\varrho_{n+1}-\varrho^*\|^2 \le \|\varrho_n-\varrho^*\|^2 &- (1-k)t_n\|\varrho_{n+1}-T^n\varrho_{n+1}\|^2\\ &+ \left(k_n^2-1\right)t_n\|\varrho_{n+1}-\varrho^*\|^2\\ &+ 2(1+L)t_n\|\varrho_{n+1}-\varrho_n\|\|\varrho_{n+1}-\varrho^*\|\\ &+ 2Lt_n\|\varsigma_n-\varrho_n\|\|\varrho_{n+1}-\varrho^*\|.\end{aligned} \tag{2.8}$$

Consider the following estimates,

$$\begin{aligned}\|\varsigma_n-\varrho_n\| &= \|(1-\beta_n)\varrho_n+\beta_n T^n\varrho_n-\varrho_n\|\\ &= \beta_n\|\varrho_n-T^n\varrho_n\|\\ &\le (1+L)a_0 t_n,\end{aligned}$$

and

$$\begin{aligned}\|\varrho_{n+1}-\varrho_n\| &= \|(1-t_n)\varrho_n+t_nT^n\varsigma_n-\varrho_n\|\\ &= t_n\|\varrho_n-T^n\varsigma_n\|\\ &\le (1+2L)a_0t_n,\end{aligned}$$

so that (2.8), takes the form

$$\begin{aligned}\|\varrho_{n+1}-\varrho^*\|^2 \le \|\varrho_n-\varrho^*\|^2 &- (1-k)t_n\|\varrho_{n+1}-T^n\varrho_{n+1}\|^2\\ &+ \left(k_n^2-1\right)t_n\|\varrho_{n+1}-\varrho^*\|^2\\ &+ 2(1+L)(1+2L)a_0t_n^2\|\varrho_{n+1}-\varrho^*\|\\ &+ 2L(1+L)a_0t_n\beta_n\|\varrho_{n+1}-\varrho^*\|.\end{aligned}$$

Then, by (2.5),

$$\begin{aligned}\|\varrho_{n+1}-\varrho^*\|^2 &\le \|\varrho_n-\varrho^*\|^2 - (1-k)\delta^2 t_n\\ &\quad + 4a_0^2\left[\left(k_n^2-1\right)+(1+L)(1+2L)t_n+L(1+L)\beta_n\right]t_n\\ &\le \|\varrho_n-\varrho^*\|^2-(1-k)\delta^2t_n+\frac{1}{2}(1-k)\delta^2t_n\end{aligned}$$

and hence

$$\|\varrho_{n+1}-\varrho^*\|^2 \leqslant \|\varrho_n-\varrho^*\|^2-\frac{1}{2}(1-k)\delta^2t_n. \tag{2.9}$$

Thus $\|\varrho_{n+1}-\varrho^*\| \le \|\varrho_n-\varrho^*\| \le a_0$ and so we proved (2.6). Therefore, we proved (a).

From (2.9) we have that for every $r > N$,

$$\begin{aligned}\frac{1}{2}(1-k)\delta^2 \sum_{n=N}^{r} t_n &\le \sum_{n=N}^{r} (\|\varrho_n - \varrho^*\|^2 - \|\varrho_{n+1} - \varrho^*\|^2) \\ &\le \|\varrho_N - \varrho^*\|^2 .\end{aligned}$$

Hence we have $\sum_{n=1}^{\infty} t_n < \infty$, a contradiction with the condition (i). Therefore, our assumption $\delta > 0$ was wrong. Thus

$$\liminf_{n\to\infty} \|\varrho_{n+1} - T^n \varrho_{n+1}\| = 0. \tag{2.10}$$

Therefore, we proved (b).

Now according to Lemma 2.1, substituting $\varrho = u + v$ and $\varsigma = -v$, we obtain

$$\|u+v\|^2 \ge \|u\|^2 + 2\langle v, j(u)\rangle ,$$

which is mainly due to Igbokwe [2].

By (2.2) we have

$$\begin{aligned}\|\varrho_{n+1} - T^n \varrho_{n+1}\|^2 &= \|(1-t_n)\varrho_n + t_n T^n \varsigma_n - T^n \varrho_{n+1}\|^2 \\ &= \|\varrho_n - T^n \varrho_n - t_n(\varrho_n - T^n \varsigma_n) - (T^n \varrho_{n+1} - T^n \varrho_n)\|^2 .\end{aligned} \tag{2.11}$$

Then by (2.11) we get

$$\begin{aligned}\|\varrho_{n+1} - T^n \varrho_{n+1}\|^2 \ge{}& \|\varrho_n - T^n \varrho_n\|^2 \\ &- 2\langle t_n(\varrho_n - T^n \varsigma_n) + (T^n \varrho_{n+1} - T^n \varrho_n), j(\varrho_n - T^n \varrho_n)\rangle .\end{aligned}$$

Thus

$$\begin{aligned}\|\varrho_n - T^n \varrho_n\|^2 \le{}& \|\varrho_{n+1} - T^n \varrho_{n+1}\|^2 \\ &+ 2\langle t_n(\varrho_n - T^n \varsigma_n) + (T^n \varrho_{n+1} - T^n \varrho_n), j(\varrho_n - T^n \varrho_n)\rangle \\ \le{}& \|\varrho_{n+1} - T^n \varrho_{n+1}\|^2 \\ &+ 2\|t_n(\varrho_n - T^n \varsigma_n) + (T^n \varrho_{n+1} - T^n \varrho_n)\| \|\varrho_n - T^n \varrho_n\| .\end{aligned} \tag{2.12}$$

Further,

$$\begin{aligned}\|t_n(\varrho_n - T^n \varsigma_n) + (T^n \varrho_{n+1} - T^n \varrho_n)\| &\le t_n \|\varrho_n - T^n \varsigma_n\| + \|T^n \varrho_{n+1} - T^n \varrho_n\| \\ &\le (1+2L)a_0 t_n + L\|\varrho_{n+1} - \varrho_n\| \\ &\le (1+2L)a_0 t_n + L(1+2L)a_0 t_n \\ &= (1+L)(1+2L)a_0 t_n.\end{aligned}$$

Therefore, from (2.12), we get

$$\|\varrho_n - T^n \varrho_n\|^2 \le \|\varrho_{n+1} - T^n \varrho_{n+1}\|^2 + 2(1+L)^2(1+2L)a_0^2 t_n. \tag{2.13}$$

From (2.13), (ii) and (b),

$$\liminf_{n\to\infty} \|\varrho_n - T^n \varrho_n\| = 0. \tag{2.14}$$

Thus we proved (c).

At last, from (2.14) and Lemma 2.2, we obtain (d). This completes the proof. □

Theorem 2.4. *Let X be a real Banach space and K be a nonempty convex subset of X. Let $T : K \to K$ be an uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. For arbitrary $\varrho_1 \in K$, generate the sequence $\{\varrho_n\}$ by*

$$\begin{cases} \varrho_{n+1} = (1 - t_n)\varrho_n + t_n T^n \varsigma_n, \\ \varsigma_n = (1 - \beta_n)\varrho_n + \beta_n T^n \varrho_n, \quad n \geq 1, \end{cases}$$

where $\{t_n\}$ and $\{\beta_n\}$ are the sequences in $[0, 1]$ satisfying

(i) $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $\lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} \beta_n$.

If T is completely continuos, then $\{\varrho_n\}$ converges strongly to some fixed point of T in K.

Proof. From Lemma 2.3, $\liminf_{n\to\infty} \|\varrho_n - T\varrho_n\| = 0$. Therefore, there exists a subsequence $\{\varrho_{n_j}\}$ of $\{\varrho_n\}$ such that $\lim_{j\to\infty} \left\|\varrho_{n_j} - T\varrho_{n_j}\right\| = 0$. Since $\{\varrho_{n_j}\}$ is bounded and T is completely continuous, then $\{T\varrho_{n_j}\}$ has a subsequence $\{T\varrho_{n_{j_k}}\}$, which converges strongly. Hence $\{\varrho_{n_{j_k}}\}$ converges strongly. Let $\lim_{k\to\infty} \varrho_{n_{j_k}} = p$. Then $\lim_{k\to\infty} T\varrho_{n_{j_k}} = Tp$. Thus we have $\lim_{k\to\infty} \left\|\varrho_{n_{j_k}} - T\varrho_{n_{j_k}}\right\| = \|p - Tp\| = 0$. Hence $p \in F(T)$. From (2.9) and Lemma 2.3 it follows that $\lim_{n\to\infty} \|\varrho_n - p\| = 0$. This completes the proof. $\square$

Remark 2.5. 1. We generalize the results of Liu [3] from Hilbert spaces to more general Banach spaces. Moreover the boundedness assumption on the subset K is removed.

2. One can see that, with $\sum_{n=1}^{\infty} t_n = \infty$, the condition $\sum_{n=1}^{\infty} t_n^2 < \infty$ is not always true. Let us take $t_n = \frac{1}{\sqrt{n}}$. Then obviously $\sum_{n=1}^{\infty} t_n = \infty$, but $\sum_{n=1}^{\infty} t_n^2 = \infty$. Hence the results of Igbokwe [2] are need to be improve.

3. We improve the results of Moore and Nnoli [5] by removing the conditions like $\liminf_{n\to\infty} d(\varrho_n, F(T)) = 0$.

Acknowledgment

This study was supported by research funds from Dong-A University.

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VISCOSITY APPROXIMATION OF SOLUTIONS OF FIXED POINT AND VARIATIONAL INCLUSION PROBLEMS

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Abstract. In this paper, fixed point and variational inclusion problems are investigated based on a viscosity approximation method. Strong convergence theorems are established without the aid of metric projections in the framework of Hilbert spaces.

Keywords: maximal monotone operator; fixed point; proximal point algorithm; zero point.

2010 AMS Subject Classification: 47H05, 65D15, 90C33.

1. Introduction

A very common problem in diverse areas of mathematics and physical sciences consist of finding a solution which satisfies certain constraints. This problem is referred to as the convex feasibility problem. It can be described as follows: Suppose $C_1, C_2, \cdots, C_r$, where r is some positive integer, are finitely many nonempty convex closed subset of a Hilbert space H with $C = \cap_{i=1}^r \neq \emptyset$. The convex feasibility problem is to find a point in C. In the real world, many important problems have reformulations which require finding fixed points of some nonlinear operators, for instance, evolution equations, complementarity problems, mini-max problems, variational inequalities and zero point problems; see [1-13] and the references therein.

In this paper, we are concerned with the problem of finding a common solution of fixed point and inclusion problems. Many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as this problem. One of the most popular methods for solving inclusion problems goes back to the work of Browder [14]. The basic ideas is to reduce inclusion problems to fixed point problems of nonlinear operators. In this paper, we study a regularization method for two monotone and a nonexpansive mappings. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a viscosity approximation method is introduced. A strong convergence theorem of common solutions is established. In Section 4, applications of the main results are discussed.

2. Preliminaries

In what follows, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty, convex and closed subset of H. Let $S : C \to C$ be a mapping. $Fix(S)$ stands for the fixed point set of S; that is, $Fix(S) := \{x \in C : x = Sx\}$. Recall that S is said to be *κ-contractive* iff there exists a constant $\kappa \in (0,1)$ such that

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$\|Sx - Sy\| \leq \kappa\|x - y\|$, $\forall x, y \in C$. It is well known that every contractive mapping has a unique fixed point in metric spaces. The Picard iterative algorithm $x_{n+1} = Sx_n$ converge to the fixed point of S. S is said to be *nonexpansive* iff $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in C$. If C is a bounded, closed, and convex subset of H, then $F(S)$ is not empty; see [15] and the references therein. Since the nonexpansivity of S, the Picard iterative algorithm may not converge to fixed points of S. The Mann iterative algorithm is powerful and efficient to study fixed points of nonexpansive mappings. However, in infinite dimensional spaces, the Mann iterative algorithm is only weak convergence. To obtain strong convergence of the Mann iterative algorithm, different regularization methods have been investigated recently; see [16]-[29] and the references therein.

Let $A : C \to H$ be a mapping. Recall that A is said to be *monotone* iff $\langle Ax-Ay, x-y\rangle \geq 0$, $\forall x, y \in C$. Recall that A is said to be *inverse-strongly monotone* iff there exists a constant $\alpha > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$, $\forall x, y \in C$. For such a case, A is also said to be *α-inverse-strongly monotone*. It is not hard to see that every inverse-strongly monotone mapping is monotone and continuous. Recall that a set-valued mapping $B : H \rightrightarrows H$ is said to be *monotone* iff, for all $x, y \in H$, $f \in Bx$ and $g \in By$ imply $\langle x - y, f - g\rangle \geq 0$. In this paper, we use $B^{-1}(0)$ to stand for the zero point of B. A monotone mapping $B : H \rightrightarrows H$ is *maximal* iff the graph $Graph(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$, for all $(y, g) \in Graph(B)$ implies $f \in Bx$. For a maximal monotone operator B on H, and $r > 0$, we may define the single-valued resolvent $J_r : H \to Dom(B)$, where $Dom(B)$ denote the domain of B. It is known that J_r is firmly nonexpansive, and $B^{-1}(0) = F(J_r)$.

In this paper, we study fixed points of nonexpansive mappings and zero points of two monotone mappings based on a viscosity approximation method. Strong convergence theorems are established in the framework of Hilbert spaces. The results obtained in this paper mainly improve the corresponding results in [23]-[29]. In order to prove our main results, we also need the following lemmas.

Lemma 2.1 [30] *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition $a_{n+1} \leq (1-t_n)a_n + t_nb_n + c_n$, $\forall n \geq 0$, where $\{t_n\}$ is a number sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$, $\{b_n\}$ is a number sequence such that $\limsup_{n\to\infty} b_n \leq 0$, and $\{c_n\}$ is a positive number sequence such that $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.2. [31] *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal monotone operator on H. Then $(A + B)^{-1}(0) = F(J_r(I - rA))$.*

Lemma 2.3. [32] *Let H be a Hilbert space, and A an maximal monotone operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have $J_\lambda x = J_\mu\left(\left(1 - \frac{\mu}{\lambda}\right)J_\lambda x + \frac{\mu}{\lambda}x\right)$, where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.*

Lemma 2.4. [14] *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T be a nonexpansive mapping on C. Then $I - T$ is demiclosed at origin.*

3. Main results

Theorem 3.1. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal*

monotone operator on H. Let S be a fixed κ-contraction and let T be a nonexpansive mapping on C. Assume $Dom(B) \subset C$ and $(A+B)^{-1}(0) \cap Fix(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$ and let $\{r_n\}$ be a positive real number sequence in $(0, 2\alpha)$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$, $y_n = \alpha_n Sx_n + (1-\alpha_n)Tx_n$, $x_{n+1} \approx (I+r_nB)^{-1}(y_n - r_nAy_n)$, $\forall n \geq 0$. Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - (I+r_nB)^{-1}(y_n - r_nAy_n)\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 2\alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in (A+B)^{-1}(0) \cap Fix(T)$, where $\bar{x} = Proj_{(A+B)^{-1}(0)\cap Fix(T)} S\bar{x}$.

Proof. First, we show that $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Using the restrictions imposed on $\{r_n\}$, one see that $I - r_nA$ is nonexpansive. Indeed, we have

$$\begin{aligned}
&\|(I - r_nA)x - (I - r_nA)y\|^2 \\
&\leq \|x - y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

That is, $\|(I - r_nA)x - (I - r_nA)y\| \leq \|x - y\|$. Fixing $p \in (A+B)^{-1}(0) \cap Fix(T)$, we find that

$$\begin{aligned}
\|y_n - p\| &\leq \alpha_n\|Sx_n - p\| + (1-\alpha_n)\|Tx_n - p\| \\
&\leq \alpha_n\|Sx_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - p\| + \alpha_n\|Sp - p\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|e_n\| + \|(I + r_nB)^{-1}(y_n - r_nAy_n) - p\| \\
&\leq e_n + \|(y_n - r_nAy_n) - (I - r_nA)p\| \\
&\leq e_n + \big(1 - \alpha_n(1-\kappa)\big)\|x_n - p\| + \alpha_n(1-\kappa)\frac{\|Sp - p\|}{1-\kappa} \\
&\leq \max\{\|x_n - p\|, \frac{\|Sp - p\|}{1-\kappa}\} + e_n \\
&\vdots \\
&\leq \max\{\|x_0 - p\|, \frac{\|Sp - p\|}{1-\kappa}\} + \sum_{i=0}^{\infty} e_i < \infty.
\end{aligned}$$

This proves that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$. Notice that

$$\|y_n - y_{n-1}\| \leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|.$$

Setting $z_n = y_n - r_nAy_n$, one further has

$$\begin{aligned}
\|z_n - z_{n-1}\| &\leq \|y_n - y_{n-1}\| + \|r_n - r_{n-1}\|\|Ay_{n-1}\| \\
&\leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + |r_n - r_{n-1}|\|Ay_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|.
\end{aligned} \tag{3.1}$$

Putting $J_{r_n} = (I + r_n B)^{-1}$, it follows from Lemma 2.3 that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq e_n + e_{n-1} + \|J_{r_{n-1}} z_{n-1} - J_{r_{n-1}}\big(\frac{r_{n-1}}{r_n} z_n + (1 - \frac{r_{n-1}}{r_n}) J_{r_n} z_n\big)\| \\
&\leq e_n + e_{n-1} + \|(1 - \frac{r_{n-1}}{r_n})(J_{r_n} z_n - z_{n-1}) + \frac{r_{n-1}}{r_n}(z_n - z_{n-1})\| \\
&\leq e_n + e_{n-1} + \frac{|r_n - r_{n-1}|}{r_n}\|z_n - J_{r_n} z_n\| + \|z_n - z_{n-1}\|,
\end{aligned}$$

which implies from (3.1) that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq e_n + e_{n-1} + \frac{|r_n - r_{n-1}|}{r_n}\|z_n - J_{r_n} z_n\| + \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| \\
&\quad + |r_n - r_{n-1}|\|Ay_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\| \\
&\leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + e_n + e_{n-1} \\
&\quad + |r_n - r_{n-1}|\Big(\|Ay_{n-1}\| + \frac{\|J_{r_n} z_n - z_n\|}{r_n}\Big) + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|.
\end{aligned}$$

From the restrictions imposed on the control sequences, we have

$$\sum_{n=1}^{\infty}\Big(e_n + e_{n-1} + |r_n - r_{n-1}|\Big(\|Ay_{n-1}\| + \frac{\|J_{r_n} z_n - z_n\|}{r_n}\Big) + |\alpha_n - \alpha_{n-1}|\|Sx_{n-1} - x_{n-1}\|\Big) < \infty.$$

Using Lemma 2.1, we find $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $\|\cdot\|^2$ is convex, we have $\|y_n - p\|^2 \leq \alpha_n\|Sx_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2$, from which it follows that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \|(y_n - r_n Ay_n) - (p - r_n Ap)\|^2 + 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2 \\
&\leq \|y_n - p\|^2 - r_n(2\alpha - r_n)\|Ay_n - Ap\|^2 + 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2 \\
&\leq \alpha_n\|Sx_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ay_n - Ap\|^2 \\
&\quad + 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
r_n(2\alpha - r_n)\|Ay_n - Ap\|^2 &\leq \alpha_n\|Sx_n - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2e_n\|(I + r_n B)^{-1}(y_n - r_n Ay_n) - p\| + e_n^2.
\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\|Ay_n - Ap\| = 0. \tag{3.2}$$

Put $\lambda_n = (I + r_n B)^{-1}(y_n - r_n Ay_n)$. Since $(I + r_n B)^{-1}$ is firmly nonexpansive, one has

$$\begin{aligned}
\|\lambda_n - p\|^2 &\leq \langle (y_n - r_n Ax_n) - (p - r_n Ap), \lambda_n - p\rangle \\
&\leq \frac{1}{2}\big(\|y_n - p\|^2 + \|\lambda_n - p\|^2 - \|y_n - \lambda_n - r_n(Ay_n - Ap)\|^2\big) \\
&\leq \frac{1}{2}\big(\|y_n - p\|^2 + \|\lambda_n - p\|^2 - \|y_n - \lambda_n\|^2 + 2r_n\|\lambda_n - y_n\|\|Ay_n - Ap\|\big).
\end{aligned}$$

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It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 \le e_n^2 + \alpha_n\|Sx_n-p\|^2 + \|x_n-p\|^2 - \|y_n-\lambda_n\|^2 \\ + 2r_n\|\lambda_n-y_n\|\|Ay_n-Ap\| + 2e_n\|\lambda_n-p\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|y_n-\lambda_n\|^2 \le e_n^2 + \alpha_n\|Sx_n-p\|^2 + \|x_n-p\|^2 - \|x_{n+1}-p\|^2 \\ + 2r_n\|\lambda_n-y_n\|\|Ay_n-Ap\| + 2e_n\|\lambda_n-p\|.\end{aligned}$$

Using the restrictions imposed on the control sequences and (3.2), we arrive at

$$\lim_{n\to\infty}\|y_n-\lambda_n\| = 0. \tag{3.3}$$

Note that $\|x_n-Tx_n\| \le \|x_n-x_{n+1}\| + \|\lambda_n-y_n\| + \|y_n-Tx_n\| + e_n$. This finds from (3.3) $\lim_{n\to\infty}\|x_n-Tx_n\| = 0$.

Next, we show that

$$\limsup_{n\to\infty}\langle S\bar{x}-\bar{x}, y_n-\bar{x}\rangle \le 0, \tag{3.4}$$

where $\bar{x}$ is the unique fixed point of the mapping $Proj_{(A+B)^{-1}(0)\cap Fix(T)}S$. To show this inequality, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $\limsup_{n\to\infty}\langle S\bar{x}-\bar{x}, y_n-\bar{x}\rangle = \lim_{i\to\infty}\langle S\bar{x}-\bar{x}, y_{n_i}-\bar{x}\rangle \le 0$, Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to $\hat{x}$. Without loss of generality, we assume that $y_{n_i} \rightharpoonup \hat{x}$. Since $\|x_n-y_n\| \le \|x_n-x_{n+1}\| + \|\lambda_n-y_n\| + e_n$, one has $x_{n_i} \rightharpoonup \hat{x}$. Using Lemma 2.4, one has $\hat{x} \in Fix(T)$. Since $y_n - r_nAy_n \in \lambda_n + r_nB\lambda_n$, that is, $\frac{y_n-\lambda_n-r_nAy_n}{r_n} \in B\lambda_n$. Let $\mu \in B\nu$. Since B is monotone, we find that $\langle \frac{y_n-\lambda_n}{r_n} - \mu - Ay_n, \lambda_n-\nu\rangle \ge 0$. Hence, one has $0 \le \langle -A\hat{x}-\mu, \hat{x}-\nu\rangle$. This implies that $-A\hat{x} \in B\hat{x}$, that is, $\hat{x} \in (A+B)^{-1}(0)$. This shows (3.4) holds. Notice that

$$\begin{aligned}\|y_n-\bar{x}\|^2 &\le \alpha_n\langle Sx_n-S\bar{x}, y_n-\bar{x}\rangle + \alpha_n\langle S\bar{x}-\bar{x}, y_n-\bar{x}\rangle + (1-\alpha_n)\|Tx_n-p\|\|y_n-\bar{x}\| \\ &\le \big(1-\alpha_n(1-\kappa)\big)\|x_n-\bar{x}\|\|y_n-\bar{x}\| + \alpha_n\langle S\bar{x}-\bar{x}, y_n-\bar{x}\rangle.\end{aligned}$$

It follows that $\|y_n-\bar{x}\|^2 \le (1-\alpha_n(1-\kappa))\|x_n-\bar{x}\|^2 + 2\alpha_n\langle S\bar{x}-\bar{x}, y_n-\bar{x}\rangle$.
Hence, we have

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &\le \|(y_n-r_nAy_n)-(I-r_nA)\bar{x}\|^2 + 2e_n\|\lambda_n-\bar{x}\| + e_n^2 \\ &\le (1-\alpha_n(1-\kappa))\|x_n-\bar{x}\|^2 + 2\alpha_n\langle S\bar{x}-\bar{x}, y_n-\bar{x}\rangle + 2e_n\|\lambda_n-\bar{x}\| + e_n^2.\end{aligned}$$

An application of Lemma 2.1 to the above inequality yields that $\lim_{n\to\infty}\|x_n-\bar{x}\| = 0$. This completes the proof.

4. Applications

Let C be a nonempty closed and convex subset of a Hilbert space H. Let i_C be the indicator function of C, that is, $i_C(x) = \infty, x \notin C$, $i_C(x) = 0, x \in C$. Since i_C is a proper lower and semicontinuous convex function on H, the subdifferential ∂i_C of i_C is maximal monotone. So, we can define the resolvent J_r of ∂i_C for $r > 0$, i.e., $J_r := (I + r\partial i_C)^{-1}$. Letting $x = J_ry$, we find that

$$y \in x + r\partial i_C x \iff y \in x + rN_Cx \iff x = Proj_Cy,$$

where $Proj_C$ is the metric projection from H onto C and $N_Cx := \{e \in H : \langle e, v-x\rangle, \forall v \in C\}$.

Theorem 4.1. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $T : C \to C$ be a nonexpansive mapping. Assume that $VI(C,A) \cap Fix(T)$ is not empty. Let $S : C \to C$ be a fixed κ-contraction. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$, $y_n = \alpha_n Sx_n + (1-\alpha_n)Tx_n$, $x_{n+1} \approx Proj_C(y_n - r_n Ay_n)$, $\forall n \geq 0$. Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - Proj_C(y_n - r_n Ay_n)\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 2\alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in VI(C,A) \cap Fix(T)$, where $\bar{x} = Proj_{VI(C,A)\cap Fix(T)} S\bar{x}$.*

Proof. Putting $B = \partial i_C$ in Theorem 3.1, we find that $J_{r_n} = Proj_C$. This finds from Theorem 3.1 the desired conclusion immediately.

Next, we consider the problem of finding a solution of a Ky Fan inequality [7], which is known as an equilibrium problem in the terminology of Blum and Oettli; see [33] and the references therein.

Let B be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } B(x,y) \geq 0, \quad \forall y \in C. \tag{4.1}$$

To study equilibrium problem (4.1), we may assume that B satisfies the following restrictions:

(R-a) $B(y,x) + B(x,y) \leq 0$, $\forall x, y \in C$;
(R-b) $B(x,x) = 0$, $\forall x \in C$;
(R-c) $B(x,y) \geq \limsup_{t\downarrow 0} B(tz + (1-t)x, y)$, $\forall x, y, z \in C$,
(R-d) $y \mapsto B(x,y)$, $\forall x \in C$, is lower semi-continuous and convex.

The following lemmas can be found in [22] and [33].

Lemma 4.2. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $B : C \times C \to \mathbb{R}$ be a bifunction with (R-a), (R-b), (R-c) and (R-d). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that $rB(z,y) + \langle y - z, z - x\rangle \geq 0$, $\forall y \in C$. Further, define*

$$T_r x = \Big\{ z \in C : rB(z,y) + \langle y - z, z - x\rangle \geq 0, \ \forall y \in C \Big\} \tag{4.2}$$

for all $r > 0$ and $x \in H$. Then T_r is single-valued and firmly nonexpansive and $F(T_r) = EP(F)$ is closed convex.

Lemma 4.3. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let B be a bifunction from $C \times C$ to $\mathbb{R}$ with (R-a), (R-b), (R-c) and (R-d). Let A_B be a multivalued mapping of H into itself defined by*

$$A_B x = \begin{cases} \{z \in H : \langle y - x, z\rangle \leq B(x,y), \ \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{4.3}$$

Then A_B is a maximal monotone operator with domain $D(A_B) \subset C$, $EP(B) = A_B^{-1}(0)$, where $FP(B)$ stands for the solution set of (4.1), and $T_r x = (I + rA_B)^{-1}x$, $\forall x \in H$, $r > 0$, where T_r is defined as in (4.2).

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Theorem 4.4. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $B : C \times C \to \mathbb{R}$ be a bifunction with (R-a), (R-b), (R-c) and (R-d). Let $T : C \to C$ be a nonexpansive mapping. Assume that $EP(B) \cap Fix(T)$ is not empty. Let $S : C \to C$ be a fixed κ-contraction and let $T_{r_n} = (I+r_nA_B)^{-1}$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$ and $x_{n+1} \approx T_{r_n}(\alpha_n Sx_n + (1-\alpha_n)Tx_n)$, $\forall n \geq 0$, Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - T_{r_n}(\alpha_n Sx_n + (1-\alpha_n)Tx_n)\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 2\alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in EP(B) \cap Fix(T)$, where $\bar{x} = Proj_{EP(B)\cap Fix(T)} S\bar{x}$.*

Proof. Putting $A = 0$ in Theorem 3.1, we find that $J_{r_n} = T_{r_n}$. From Theorem 3.1, we draw the desired conclusion immediately.

Recall that a mapping $T : C \to T$ is said to be α-strictly pseudocontractive iff there exits a constant $\alpha \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \alpha\|(I - T)x - (I - T)y\|^2 + \|x - y\|^2, \quad \forall x, y \in C.$$

The class of strictly pseudocontractive mappings was first introduced by Browder and Petryshyn [28]. It is known if T is α-strictly pseudocontractive, then $I - T$ is $\frac{1-\alpha}{2}$-inverse strongly monotone.

Finally, we consider the problem of common fixed point problems of nonlinear mappings.

Theorem 4.5. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T_1 be a nonexpansive mapping and let T_2 be a α-strictly pseudocontractive mapping on C. Let S be a fixed κ-contraction on C. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$, $y_n = \alpha_n Sx_n + (1-\alpha_n)T_1x_n$, $x_{n+1} \approx (1-r_n)y_n + r_nT_2y_n$, $\forall n \geq 0$, Let the criterion for the approximate computation of x_{n+1} be $\|x_{n+1} - (1 - r_n)y_n - r_nT_2y_n\| \leq e_n$, where $\sum_{n=1}^{\infty} e_n < \infty$. Assume that the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, and $0 < r \leq r_n \leq r' < 1 - \alpha$, where r and r' are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(T_1) \cap Fix(T_2)$, where $\bar{x} = Proj_{Fix(T_1)\cap Fix(T_2)} S\bar{x}$.*

Proof. Putting $A = I - T_2$, we find A is $\frac{1-\alpha}{2}$-inverse strongly monotone. We also have $VI(C, A) = Fix(T_2)$ and $r_nT_2y_n + (1 - r_n)y_n = Proj_C(y_n - r_nAy_n)$. In view of Theorem 3.1, we obtain the desired result immediately.

Acknowledgements

This article was supported by the National Natural Science Foundation of China under grant No.11401152.

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ON THE STABILITY OF ADDITIVE ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

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ABSTRACT. In this paper, we solve the following additive ρ-functional inequalities

$$N\left(f(x+y)-f(x)-f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right),t\right)\geq\frac{t}{t+\varphi(x,y)} \tag{0.1}$$

and

$$N\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)-f(x)-f(y)\right),t\right)\geq\frac{t}{t+\varphi(x,y)} \tag{0.2}$$

in fuzzy normed spaces, where ρ is a fixed real number with $\rho\neq 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [10] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [6, 12, 27]. In particular, Bag and Samanta [2], following Cheng and Mordeson [5], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [11]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 16, 17] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 16, 17, 18] Let X be a real vector space. A function $N: X\times\mathbb{R}\to[0,1]$ is called a *fuzzy norm* on X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

(N_1) $N(x,t)=0$ for $t\leq 0$;
(N_2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
(N_3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;
(N_4) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$.
(N_6) for $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [15, 16].

Definition 1.2. [2, 16, 17, 18] Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x\in X$ such that $\lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty}x_n=x$.

2010 *Mathematics Subject Classification.* Primary 46S40, 39B52, 39B62, 26E50, 47S40.

Key words and phrases. fuzzy Banach space; additive ρ-functional inequality; Hyers-Ulam stability.

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Definition 1.3. [2, 16, 17, 18] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [24] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 9, 13, 14, 19, 22, 23, 25]).

Park [20, 21] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we solve the additive ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in fuzzy Banach spaces by using the direct method.

In Section 3, we solve the additive ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in fuzzy Banach spaces by using the direct method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

2. Additive ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $\rho \neq 1$. We need the following lemma to prove the main results.

Lemma 2.1. *Let $f : X \to Y$ be a mapping satisfying*

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \tag{2.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is additive.

Proof. Letting $x = y = 0$ in (2.1), we get $-f(0) = 0$ and so $f(0) = 0$.

Replacing y by x in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) = \rho(f(x+y) - f(x) - f(y))$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. □

Theorem 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$N\left(f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \varphi(x,y)} \tag{2.3}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t + \frac{1}{2}\Phi(x,x)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.3), we get

$$N\left(f(2x) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x,x)} \tag{2.5}$$

and so

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)}$$

for all $x \in X$. Hence

$$\begin{aligned}
&N\left(2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right) \qquad (2.6)\\
&\geq \min\left\{N\left(2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, N\left(2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right)\right\}\\
&= \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right), \frac{t}{2^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right), \frac{t}{2^{m-1}}\right)\right\}\\
&\geq \min\left\{\frac{\frac{t}{2^l}}{\frac{t}{2^l} + \varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{\frac{t}{2^{m-1}}}{\frac{t}{2^{m-1}} + \varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\}\\
&= \min\left\{\frac{t}{t + 2^l \varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{t}{t + 2^{m-1}\varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\}\\
&\geq \frac{t}{t + \frac{1}{2}\sum_{j=l+1}^{m} 2^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}\right)}
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (2.2) and (2.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

By (2.3),

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right.$$
$$\left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),2^nt\right)\geq\frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. So

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right.$$
$$\left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),t\right)\geq\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}=\frac{t}{t+2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\frac{t}{t+2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}=1$ for all $x,y\in X$ and all $t>0$,

$$A(x+y)-A(x)-A(y)=\rho\left(2A\left(\frac{x+y}{2}\right)-A(x)-A(y)\right)$$

for all $x,y\in X$. By Lemma 2.1, the mapping $A:X\to Y$ is Cauchy additive, as desired. □

Corollary 2.3. *Let $\theta\geq 0$ and let p be a real number with $p>1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f:X\to Y$ be a mapping satisfying*

$$N\left(f(x+y)-f(x)-f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right),t\right)$$
$$\geq\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\tag{2.7}$$

for all $x,y\in X$ and all $t>0$. Then $A(x):=N\text{-}\lim_{n\to\infty}2^nf(\frac{x}{2^n})$ exists for each $x\in X$ and defines an additive mapping $A:X\to Y$ such that

$$N\left(f(x)-A(x),t\right)\geq\frac{(2^p-2)t}{(2^p-2)t+2\theta\|x\|^p}$$

for all $x\in X$ and all $t>0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y):=\theta(\|x\|^p+\|y\|^p)$ for all $x,y\in X$, as desired. □

Theorem 2.4. *Let $\varphi:X^2\to[0,\infty)$ be a function such that*

$$\Phi(x,y):=\sum_{j=0}^{\infty}\frac{1}{2^j}\varphi\left(2^jx,2^jy\right)<\infty$$

for all $x,y\in X$. Let $f:X\to Y$ be a mapping satisfying (2.3). Then $A(x):=N\text{-}\lim_{n\to\infty}\frac{1}{2^n}f\left(2^nx\right)$ exists for each $x\in X$ and defines an additive mapping $A:X\to Y$ such that

$$N\left(f(x)-A(x),t\right)\geq\frac{1}{t+\frac{1}{2}\Phi(x,x)}$$

for all $x\in X$ and all $t>0$.

Proof. It follows from (2.5) that

$$N\left(f(x)-\frac{1}{2}f(2x),\frac{1}{2}t\right)\geq\frac{t}{t+\varphi(x,x)}$$

and so

$$N\left(f(x)-\frac{1}{2}f(2x),t\right) \geq \frac{2t}{2t+\varphi(x,x)} = \frac{t}{t+\frac{1}{2}\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying (2.7). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x)-A(x),t\right) \geq \frac{(2-2^p)t}{(2-2^p)t+2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

3. Additive ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a fuzzy number with $\rho \neq 1$.

Lemma 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho\left(f(x+y) - f(x) - f(y)\right) \tag{3.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is additive.

Proof. Letting $y = 0$ in (3.1), we get $2f\left(\frac{x}{2}\right) - f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f(x+y) - f(x) - f(y) = 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho(f(x+y) - f(x) - f(y))$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. □

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right), t\right) \geq \frac{t}{t+\varphi(x,y)} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t+\Phi(x,0)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = 0$ in (3.3), we get

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x, 0)} \tag{3.5}$$

for all $x \in X$. Hence

$$\begin{aligned} & N\left(2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right) \qquad (3.6) \\ & \geq \min\left\{N\left(2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, N\left(2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right), t\right)\right\} \\ & = \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right), \frac{t}{2^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right), \frac{t}{2^{m-1}}\right)\right\} \\ & \geq \min\left\{\frac{\frac{t}{2^l}}{\frac{t}{2^l} + \varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{\frac{t}{2^{m-1}}}{\frac{t}{2^{m-1}} + \varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\} \\ & = \min\left\{\frac{t}{t + 2^l\varphi\left(\frac{x}{2^l}, 0\right)}, \cdots, \frac{t}{t + 2^{m-1}\varphi\left(\frac{x}{2^{m-1}}, 0\right)}\right\} \\ & \geq \frac{t}{t + \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^j}, 0\right)} \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (3.2) and (3.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

By (3.3),

$$\begin{aligned} & N\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right. \\ & \quad \left. - \rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 2^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} & N\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right. \\ & \left. - \rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), t\right) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = \frac{t}{t + 2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{t}{t+2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$2A\left(\frac{x+y}{2}\right) - A(x) - A(y) = \rho\left(A(x+y) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 3.1, the mapping $A : X \to Y$ is Cauchy additive, as desired. $\square$

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \quad (3.7)$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. $\square$

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y\right) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{t}{t + \Phi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (3.5) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{t}{2}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{2}f(2x), t\right) \geq \frac{2t}{2t + \varphi(2x, 0)} = \frac{t}{t + \frac{1}{2}\varphi(2x, 0)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.2. $\square$

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. $\square$

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On the Difference equation $x_{n+1} = Ax_n + \dfrac{B\sum\limits_{i=0}^{k} x_{n-i}}{C+D\prod\limits_{i=0}^{k} x_{n-i}}$

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Abstract

The main objective of this paper is to study the global stability of the positive solutions and the periodic character of the difference equation

$$x_{n+1} = \frac{A\sum\limits_{i=0}^{k} x_{n-i}}{B + C\prod\limits_{i=0}^{k} x_{n-i}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $A,\ B$ and C are positive real numbers and the initial conditions $x_{-k},\ x_{-k+1}, ...,\ x_{-},\ x_0$ are nonnegative real numbers.

Keywords: difference equations, stability, global stability, periodic solutions.
Mathematics Subject Classification: 39A10

1 Introduction

Difference equations have always played an important role in the construction and analysis of mathematical models of biology, ecology, probability theory, genetics, number theory, physics, economic process, and so forth.

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

Ahmed [1] investigated the global asymptotic stability and the periodic character for the rational difference equation,

$$x_{n+1} = \frac{\alpha x_{n-l}}{\beta + \gamma \prod_{i=l}^{k} x_{n-2i}^{p_i}}, \qquad n = 0, 1, ...,$$

where the parameters $\alpha,\ \beta,\ \gamma,\ p_1,\ p_2,\ ...,\ p_k$ are nonnegative real numbers, and $l,\ k$ are nonnegative integers such that $l \leq k$ and the initial conditions x_{-2k}, x_{-2k+1}, ..., x_{-1}, x_0 are arbitrary nonnegative real numbers.

Wang et al. [2] studied the asymptotic behavior of the solutions of the nonlinear difference equation

$$x_{n+1} = \frac{\sum_{i=0}^{l} A_{s_i}\ x_{n-s_i}}{B + C \sum_{j=0}^{k} x_{n-t_j}}, \qquad n = 0, 1, ...,$$

where the initial conditions $x_{-m},\ x_{-m+1},\ ...,\ x_{-1},\ x_0$ are positive real numbers, $m = max\left\{s_1,\ ...,\ s_l,\ t_1,\ ...,\ t_k\right\}$, $s_1,\ ...,\ s_l,\ t_1,\ ...,\ t_k$ are nonnegative integers, and A_{s_i}, B, C are arbitrary positive real numbers.

Zayed et al. [3] investigated the boundedness character, the periodic character, the convergence and the global stability of positive solutions of the difference equation

$$x_{n+1} = \frac{A + \sum_{i=0}^{k} \alpha_i\ x_{n-i}}{\sum_{i=0}^{k} \beta_i\ x_{n-i}}, \qquad n = 0, 1, ...,$$

where the coefficients $A,\ \alpha_i,\ ,\beta_i$ and the initial conditions $x_{-k},\ x_{-k+1},\ ...,\ x_{-1},\ x_0$ are positive real numbers, while k is a positive integer number.

In [4] Ibrahim et al. studied the global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-m}}{\beta + \gamma \prod_{j=0}^{k} x_{n-i_j}}, \qquad n = 0, 1, ...,$$

where the parameters $\alpha,\ \beta,\ \gamma$ and initial conditions are non-negative real numbers, $\{i_0 < i_1 < ... < i_k\}$ is a set of nonnegative even integers and m is an odd positive integer

Hamza et al. [5] studied the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{A \prod_{i=l}^{k} x_{n-2i-1}}{B + C \prod_{j=0}^{k-1} x_{n-2i}}, \qquad n = 0, 1, ...,$$

where $A,\ B,\ C$ are nonnegative parameters and $l,\ k$ are nonnegative integers for $l < k$. They discussed the existence of unbounded solutions under certain conditions for $l = 0$.

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In [6] El-Metwally investigated the global stability character and the oscillatory of the solutions of the following difference equation

$$y_{n+1} = \frac{\alpha y_n \prod\limits_{i=l}^{k} x_{n-2i-1}}{\beta+\gamma \sum\limits_{i=0}^{k} y^p_{n-2i-1} \prod\limits_{i=0}^{k} y_{n-2i-1}}, \qquad n = 0, 1, ...,$$

where $\alpha,\ \beta,\ \gamma,\ p \in (0,\ \infty)$ with the initial conditions $y_0,\ y_{-1},\ ...,\ y_{-2k},\ y_{-2k-1} \in (0,\ \infty)$. For more results in the direction of this study, see, for example, [1–27] and the papers therein.

The aim of this paper to study some qualitative behavior of the positive solutions of a higher order difference equation

$$x_{n+1} = Ax_n + \frac{B \sum\limits_{i=0}^{k} x_{n-i}}{C+D \prod\limits_{i=0}^{k} x_{n-i}}, \qquad n = 0,\ 1,\ ..., \tag{1}$$

where the parameters $A,\ B,\ C$ and D are positive real numbers and the initial conditions $x_{-k},\ x_{-k+1}, ...,\ x_{-},\ x_0$ are nonnegative real numbers.

2 Preliminaries

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 *(Equilibrium Point)*
A point $\overline{x} \in I$ is called an equilibrium point of the difference equation (2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of the difference equation (2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 2 *(Stability)*
Let $\overline{x} \in (0,\ \infty)$ be an equilibrium point of the difference equation (2). Then, we have

(i) The equilibrium point $\overline{x}$ of the difference equation (2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of the difference equation (2) is called locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of the difference equation (2) is called global attractor if for all $x_{-k}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of the difference equation (2) is called globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of the difference equation (2).
(v) The equilibrium point $\overline{x}$ of the difference equation (2) is called unstable if $\overline{x}$ is not locally stable.

Definition 3 *(Periodicity)*
A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Definition 4 *The linearized equation of the difference equation (2) about the equilibrium $\overline{x}$ is the linear difference equation*

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0, \tag{4}$$

where

$$p_i = \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}.$$

Theorem 1 *[1]: Assume that $p_i \in R,\ i = 1, 2, ..., k$ and k is non-negative integer. Then*

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ... \ .$$

3 Change of variables

By using the change of variables $x_n = \left(\frac{C}{D}\right)^{\frac{1}{k+1}} y_n$, the equation (1) reduces to the following difference equation

$$y_{n+1} = Ay_n + \frac{r \sum_{i=0}^{k} y_{n-i}}{1 + \prod_{i=0}^{k} y_{n-i}}, \qquad n = 0,\ 1,\ ..., \tag{5}$$

where $r = \frac{B}{C}$ and the initial conditions $y_n,\ y_{n-1}, ...,\ y_{n-k+1},\ y_{n-k}$ are positive real numbers.

4 Local Stability of the Equilibrium Point

In this section, we study the local stability character of the equilibrium point of Eq.(5).

Eq.(5) has equilibrium point and is given by

$$\overline{y} = A\overline{y} + \frac{r \sum_{i=0}^{k} \overline{y}_{n-i}}{1 + \prod_{i=0}^{k} \overline{y}_{n-i}},$$

or

$$\overline{y}\,(1 - A)\left(1 + \overline{y}^{k+1}\right) = r(k+1)\overline{y}.$$

Thus $\overline{y}_1 = 0$ is always an equilibrium point of Eq. (5). If $A < 1$ and $\frac{r(k+1)}{1-A} > 1$; then the only positive equilibrium point $\overline{y}_2$ of Eq. (5) is given by

$$\overline{y}_2 = \left(\frac{r(k+1)}{1-A} - 1\right)^{\frac{1}{k+1}}.$$

Theorem 2 *The equilibrium $\overline{y}_1$ of Eq. (5) is locally asymptotically stable if*

$$A + r(k+1) < 1.$$

Proof: Let $f:(0,\ \infty)^{k+1}\longrightarrow(0,\ \infty)$ be a continuous function defined by

$$f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})=Au_n+\frac{r\sum\limits_{i=0}^{k}u_{n-i}}{1+\prod\limits_{i=0}^{k}u_{n-i}}. \tag{7}$$

Therefore, it follows that

$$\begin{aligned}
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_n} &= A+\frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-r\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=1}^{k}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2},\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-1}} &= \frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-ru_n\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=2}^{k}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2},\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-2}} &= \frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-ru_nu_{n-1}\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=3}^{k}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2},\\
&\ \ \vdots\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-k}} &= \frac{r\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)-r\left(\sum\limits_{i=0}^{k}u_{n-i}\right)\left(\prod\limits_{i=0}^{k-1}u_{n-i}\right)}{\left(1+\prod\limits_{i=0}^{k}u_{n-i}\right)^2}.
\end{aligned}$$

At $\overline{y}_1=0$, we have

$$\begin{aligned}
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_n} &= A+r\\
\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-1}} &= ...=\frac{\partial f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})}{\partial u_{n-k}}=r,
\end{aligned}$$

and the linearized equation of Eq. (5) about $\overline{y}_1=0$, is the equation

$$z_{n+1}-(A+r)\,z_n-rz_{n-l}-...-ry_{n-k}=0,$$

It follows by Theorem 1 that, Eq. (5) is asymptotically stable if and only if

$$|A+r|+|r|+...+|r|<1,$$

and so

$$A+r\,(k+1)<1.$$

The proof is complete.

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Theorem 3 *The equilibrium $\overline{y}_1$ of Eq. (5) is unstable if $A + r(k+1) > 1$.*

Theorem 4 *The equilibrium $\overline{y}_2$ of Eq. (5) is stable if*

$$Ar + (1-A)(1 - rk - A) < r.$$

Proof: At $\overline{y}_2 = \left(\frac{r(k+1)}{1-A} - 1\right)^{\frac{1}{k+1}}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial u_n} &= A + \frac{r\left(1+\frac{r(k+1)}{1-A}-1\right) - r(k+1)\left(\frac{r(k+1)}{1-A}-1\right)}{\left(1+\frac{r(k+1)}{1-A}-1\right)^2} \\
&= A + \frac{r\left(\frac{r(k+1)}{1-A}\right) - r(k+1)\left(\frac{r(k+1)-1+A}{1-A}\right)}{\left(\frac{r(k+1)}{1-A}\right)^2} = A + \frac{\left(\frac{r(k+1)}{1-A}\right)(r - r(k+1) + 1 - A)}{\left(\frac{r(k+1)}{1-A}\right)^2} \\
&= A + \frac{(r - rk - r + 1 - A)}{\left(\frac{r(k+1)}{1-A}\right)} = A + \frac{(1-A)(1-rk-A)}{r(k+1)} \\
\frac{\partial f}{\partial u_{n-1}} &= \ldots = \frac{\partial f}{\partial u_{n-k}} = \frac{(1-A)(1-rk-A)}{r(k+1)},
\end{aligned}$$

and the linearized equation of Eq. (5) about $\overline{y}_2 = \left(\frac{r(k+1)}{1-A} - 1\right)^{\frac{1}{k+1}}$, is the equation

$$z_{n+1} - \left(A + \frac{(1-A)(1-rk-A)}{r(k+1)}\right) z_n - \frac{(1-A)(1-rk-A)}{r(k+1)} z_{n-l} - \ldots - \frac{(1-A)(1-rk-A)}{r(k+1)} y_{n-k} = 0,$$

It follows by Theorem A that, Eq.(5) is stable if and only if

$$\left|A + \frac{(1-A)(1-rk-A)}{r(k+1)}\right| + \left|\frac{(1-A)(1-rk-A)}{r(k+1)}\right| + \ldots + \left|\frac{(1-A)(1-rk-A)}{r(k+1)}\right| < 1,$$

for $rk + A < 1$ we get

$$A + \frac{(1-A)(1-rk-A)}{r} < 1.$$

The proof is complete.

5 Existence of Boundedness Solutions

Here we look at the boundedness nature of solutions of Eq.(5).

Theorem 5 *Every solution of Eq.(5) is bounded if $A + r(k+1) < 1$.*

Proof: Let $\{y_n\}_{n=0}^{\infty}$ be a solution of Eq.(5). It follows from Eq.(5) that

$$0 \le y_{n+1} = Ay_n + \frac{r\sum\limits_{i=0}^{k} y_{n-i}}{1+\prod\limits_{i=0}^{k} y_{n-i}} < Ay_n + r\sum_{i=0}^{k} y_{n-i} < (A + r(k+1))\,\overline{y}.$$

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this equation is locally asymptotically stable if $A+r\left(k+1\right)<1,$ and converges to the equilibrium point $\overline{y}$. Therefore

$$\limsup_{n\to\infty} y_n \leq \left(A+r\left(k+1\right)\right)\overline{y}.$$

Hence, the solution is bounded.

Theorem 6 *Every solution of Eq.(5) is unbounded if* $A>1$.

Proof: Let $\{y_n\}_{n=0}^{\infty}$ be a solution of Eq.(5). Then from Eq.(5) we see that

$$y_{n+1}=Ay_n+\frac{r\sum\limits_{i=0}^{k}y_{n-i}}{1+\prod\limits_{i=0}^{k}y_{n-i}}>Ay_n+r\sum_{i=0}^{k}y_{n-i}>A\overline{y}.$$

This equation is unbounded because $A>1$, and $\lim\limits_{n\to\infty}y_n=\infty$. Then by using ratio test $\{y_n\}_{n=0}^{\infty}$ is unbounded from above.

6 Global Stability of the Equilibrium Point

In this section we study the global stability of the positive solutions of Equation (1).

Theorem 7 *The following statements are true*

(a) If $A+r(k+1)<1$ *then the equilibrium point* $\overline{y}_1=0$ *is a global attractor of equation (1).*

(b) If $rk+A<1$ *then the equilibrium point* $\overline{y}_2=\left(\frac{r(k+1)}{1-A}-1\right)^{\frac{1}{k+1}}$ *is a global attractor of equation (1).*

Proof. **(a)** From Eq. (7) we can see that the function is increasing of all arguments. Now, we can see that the function $F(y_n,\ y_{n-1},\ ...,\ y_{n-k})$ increasing in $y_n,\ y_{n-1},\ ...,\ y_{n-k+1}$ and x_{n-k}. Then

$$\begin{aligned}&\left[Ay+\frac{r(k+1)y}{1+y^{k+1}}-y\right](y-\overline{y}_1)\\ \leq\ &\left[Ay+r(k+1)y-y\right](y-0)\\ \leq\ &-\left(1-A-r(k+1)\right)y^2<0\end{aligned}$$

If $A+r(k+1)<1$, then $F(y,\ y,\ ...,\ y)$ satisfies the inequality

$$\left[F(y,\ y,\ ...,\ y)-y\right](y-\overline{y}_1)<0,\quad \text{for } \overline{y}_1=0.$$

According to Theorem 1.10 page 15 in [1], then $\overline{x}_1$ is a global attractor of Eq. (1). This completes the proof.

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(b) If $rk+A<1$, then we can see that the function $f(u_n,\ u_{n-1},\ u_{n-2},\ ...,\ u_{n-k})$ defined by Eq. (7) increasing of all arguments. Suppose that (m,M) is a solution of the system

$$M=f(M,\ M,\ ...,\ M) \quad \text{and} \quad m=f(m,\ m,\ ...,\ m).$$

Then from Equation (1), we see that

$$M=AM+\frac{r(k+1)M}{1+M^{k+1}}, \text{ and } m=Am+\frac{r(k+1)m}{1+m^{k+1}},$$

then

$$\begin{aligned}(1-A)+(1-A)M^{k+1} &= r(k+1),\\ (1-A)+(1-A)m^{k+1} &= r(k+1),\end{aligned}$$

Subtracting this two equations, we obtain

$$(1-A)\left(M^{k+1}-m^{k+1}\right)=0$$

under the condition $A\neq 1$,we see that $M=m$. According to Theorem 1.15 page 18 in [1], we see that $\overline{y}_2$ is a global attractor of Equation (1).

7 Existence of Periodic Solutions

In this section we investigate the existence of periodic solutions of Eq.(5).

Theorem 8 *If k is even, then equation (5) has not prime period two solution.*

Proof: Equation (5) can be expressed that

$$y_{n+1}=Ay_n+\frac{r\left(y_n+y_{n-1}+y_{n-2}+...+y_{n-k}\right)}{1+y_ny_{n-1}y_{n-2}...y_{n-k}},$$

For $k=2m$ is even, then $y_n,\ y_{n-2},\ y_{n-4},\ ...,\ y_{n-k-2},\ y_{n-k}$ are even and $y_{n-1},\ y_{n-3},\ y_{n-5},\ ...,\ y_{n-k-3},\ y_{n-k-1}$ are odd. Suppose that exists there distinct positive solutions

$$...p,\ q,\ p,\ q,\ ...,$$

of Equation (5). Then

$$p=Aq+\frac{r\left(\left(m+1\right)q+mp\right)}{1+q^{m+1}p^m} \text{ and } q=Ap+\frac{r\left(\left(m+1\right)p+mq\right)}{1+p^{m+1}q^m}.$$

Therefore,

$$\begin{aligned}p-Aq+q^{m+1}p^{m+1}-Aq^{m+1}p^{m+1} &= r\left(m+1\right)q+rmp, &(7)\\ q-Ap+p^{m+1}q^{m+1}-Ap^{m+1}q^{m+1} &= r\left(m+1\right)p+rmq, &(8)\end{aligned}$$

By subtracting (8) from (7), we have

$$(1 + A + r)(p - q) = 0$$

Since $r + A + 1 \neq 0$, then $p = q$. This is a contradiction. Thus, the proof is completed.

Theorem 9 *If k is odd, then equation (5) has not prime period two solution.*

Proof: When $k = 2m + 1$ is odd, then $y_n,\ y_{n-2},\ y_{n-4},\ \ldots,\ y_{n-k-3},\ y_{n-k-1}$ are even and $y_{n-1},\ y_{n-3},\ y_{n-5},\ \ldots,\ y_{n-k-2},\ y_{n-k}$ are odd.

First suppose that there exists distinct positive solutions

$$\ldots p,\ q,\ p,\ q,\ \ldots,$$

of Equation (5). Then

$$p = Aq + \frac{r\left(\left(m+1\right)q + \left(m+1\right)p\right)}{1 + q^{m+1}p^{m+1}},$$

and

$$q = Ap + \frac{r\left(\left(m+1\right)p + \left(m+1\right)q\right)}{1 + p^{m+1}q^{m+1}}.$$

Therefore,

$$p - Aq + q^{m+1}p^{m+2} - Aq^{m+2}p^{m+1} = r\left(m+1\right)q + r\left(m+1\right)p, \tag{9}$$

$$q - Ap + p^{m+1}q^{m+2} - Ap^{m+2}q^{m+1} = r\left(m+1\right)p + r\left(m+1\right)q, \tag{10}$$

Subtracting (10) from (9), we get

$$(p - q)\left((A+1)p^{m+1}q^{m+1} + 1 + A\right) = 0$$

Since $A+1 \neq 0$, then $p = q$. This is a contradiction. Thus, the proof is completed.

8 Numerical Examples

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (5).

Example 1. The zero solution of the difference equation (5) is local stability if $k = 3$, $A = 0.2$, $r = 0.1$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$ (See Fig. 1).

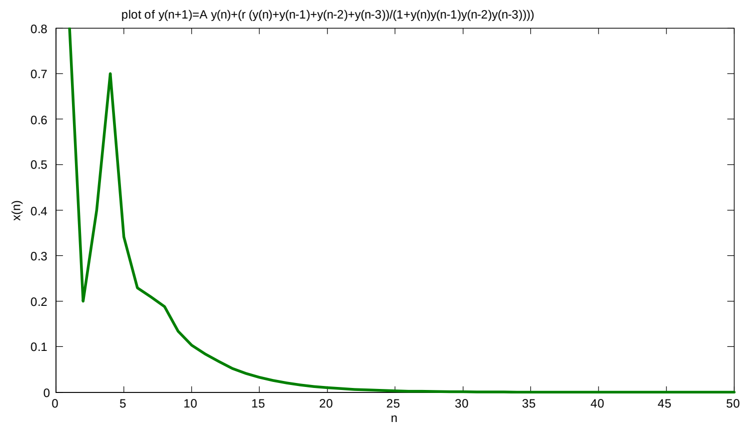

Figure 1. Plot the behavior of the zero solution of equation (5).

Example 2. The positive solution of the difference equation (5) is local stability if $k = 3$, $A = 0.6$, $r = 0.2$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$ (See Fig. 2).

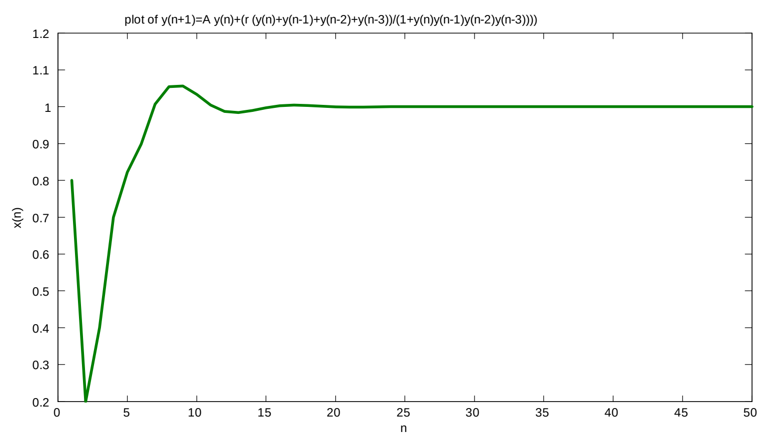

Figure 2. Plot the behavior of the positive solution of equation (5).

Example 3. The solution of the difference equation (5) is global stability if $k = 3$, $A = 0.02$, $r = 0.33$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$ (See Fig. 3).

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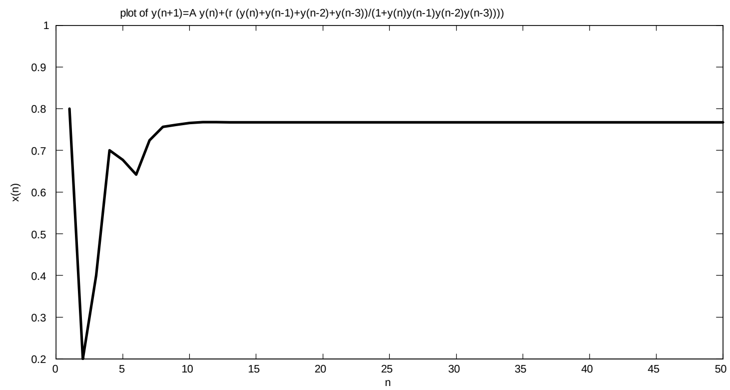

Figure 3. Plot the behavior of the positive solution of equation (5).

Example 4. Figure (4) shows the equation (5) is unbounded when $k = 3$, $A = 1.1$, $r = 0.1$ and the initial conditions $x_{-3} = 0.8$, $x_{-2} = 0.2$, $x_{-1} = 0.4$ and $x_0 = 0.7$.

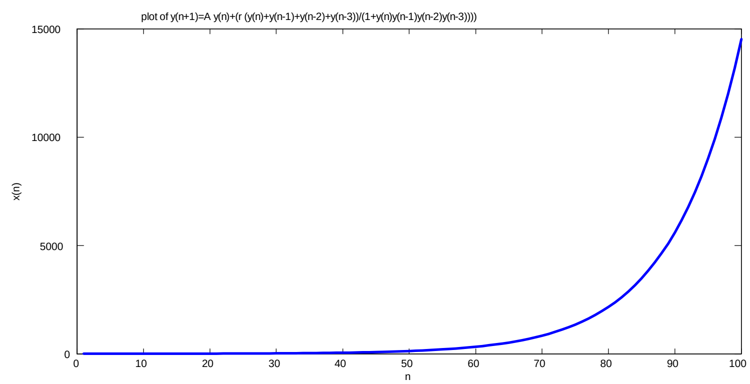

Figure 4. Plot the behavior of the solution of equation (5).

Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

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A Kind of Generalized Fuzzy Integro-differential Equations of Mixed Type and Strong Fuzzy Henstock Integrals*

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December 13, 2015

Abstract

In this paper, we prove the existence theorem of solutions for a kind of discontinuous fuzzy integro-differential equation of mixed type by using the definition of the $\omega - ACG^*$ for a fuzzy-number-valued function and a generalized controlled convergence theorem of strong fuzzy Henstock integrals.

Keywords: Fuzzy number; $\omega - ACG^*$; Discontinuous fuzzy Integro-differential equation; Controlled convergence theorem; Strong fuzzy Henstock integrals.

1 INTRODUCTION

The Cauchy problems for fuzzy differential equations have been studied by several authors [11, 9, 12, 16, 17, 18] on the metric space (E^n, D) of normal fuzzy convex set with the distance D given by the maximum of the Hausdorff distance between the corresponding level sets. In [16], the author has been proved the Cauchy problem has a uniqueness result if f was continuous and bounded. In [11, 12], the authors presented a uniqueness result when f satisfies a Lipschitz condition. For a general reference to fuzzy differential equations, see a recent

*The authors would like to thank National Natural Science Foundation of China (No.11161041, 61472056, 61262022) and the PhD Research Startup Foundation of Chongqing University of Posts and Telecommunications (No. A2014-90) and Basic and Advanced Research Projects in Chongqing (No.cstc2015jcyjA00015).

book by Lakshmikantham and Mohapatra [13] and references therein. In 2002, Xue and Fu [26] established solutions to fuzzy differential equations with right-hand side functions satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets.

However, there are discontinuous systems in which the right-hand side functions $\tilde{f} : [a, b] \times E^n \to E^n$ are not integrable in the sense of Kaleva [11] on certain intervals and their solutions are not absolute continuous functions. Recently, Wu and Gong [24, 25] have combined the fuzzy set theory [27] and nonabsolute integration theory [10], and discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva[11] integration. In order to complete the theory of fuzzy calculus and to meet the solving need of transferring a fuzzy differential equation into a fuzzy integral equation, Gong and Shao [7, 8] have defined the strong fuzzy Henstock integrals and discussed some of their properties and the controlled convergence theorem. So, in [19, 20, 21, 22, 23], the authors used the strong fuzzy Henstock integrals [8], and deal with the Cauchy problem of discontinuous fuzzy systems. In this paper, according to the idea of [4] and using the concept of generalized differentiability [2], the operator j which is the isometric embedding from (E^n, D) onto its range in the Banach space X and the generalized controlled convergence theorems for the strong fuzzy Henstock integrals, we will deal with the Cauchy problem of discontinuous fuzzy integro-differential equations of mixed type as following:

$$\begin{cases} x'(t) = \tilde{f}(t, x(t), \int_0^t k_1(t, s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(t, s)\tilde{h}(s, x(s))\mathrm{d}s), \\ x(0) = x_0, \quad x_0 \in E^n, t \in I_a = [0, a], a \in R^+ \end{cases} \tag{1}$$

where $\tilde{f}, \tilde{g}, \tilde{h}, x$ will be assumed strong fuzzy Henstock integrable and k_1, k_2 are real-valued functions.

To make our analysis possible, in section 2, we will first recall some basic results of fuzzy numbers. In section 3, we give some definitions of $\omega - ACG^*$ of fuzzy-number-valued function. In addition, we present the concept of strong fuzzy Henstock integral and a generalized controlled convergence theorem for the strong fuzzy Henstock integrals. In section 4, we deal with the Cauchy problem of discontinuous fuzzy integro-differential equation of mixed type. And in section 5, we present some concluding remarks.

2 PRELIMINARIES

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two nonempty bounded subset of R^n. The distance between A and B is defined by the Hausdorff metric [6]:

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \parallel a - b \parallel, \sup_{b\in B} \inf_{a\in A} \parallel b - a \parallel\}.$$

Denote $E^n = \{u : R^n \to [0,1] | u$ satisfies (1)-(4) below$\}$ is a fuzzy number space. where

(1)u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2)u is fuzzy convex, i.e. $u(\lambda x+(1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

(3)u is upper semi-continuous,

(4)$[u]^0 = cl\{x \in R^n | u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n | u(x) \geq \alpha\}$. Then from above (1)-(4), it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha < 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows [6]:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $u, v \in E^n$ and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0, \infty)$

$$D(u, v) = \sup\{d_H([u]^\alpha, [v]^\alpha) : \alpha \in [0,1]\},$$

where d is the Hausdorff metric defined in $P_k(R^n)$. Then it is easy see that D is a metric in E^n. Using the results [5], we know that

(1) (E^n, D) is a complete metric space,

(2) $D(u+w, v+w) = D(u, v)$ for all $u, v, w \in E^n$,

(3)$D(\lambda u, \lambda v) = |\lambda| D(u, v)$ for all $u, v, w \in E^n$ and $\lambda \in R$.

The metric space (E^n, D) has a linear structure, it can be imbedded isomorphically as a cone in a Banach space of function $u^* : I \times S^{n-1} \longrightarrow R$, where S^{n-1} is the unit sphere in R^n, with an imbedding function $u^* = j(u)$ defined by

$$u^*(r, x) = \sup_{\alpha \in [u]^\alpha} < \alpha, x >$$

for all $< r, x > \in I \times S^{n-1}$. (see [5])

Theorem 1 *There exist a real Banach space X such that E^n can be imbedding as a convex cone C with vertex 0 into X. Furthermore the following conclusions hold:*

(1) the imbedding j is isometric,

(2) addition in X induces addition in E^n,

(3) multiplication by nonnegative real number in X induces the corresponding operation in E^n,

(4) $C - C$ is dense in X,

(5) C is closed.

A fuzzy-number-valued function $f : [a, b] \to E^n$ is said to satisfy the condition (H) on $[a, b]$, if for any $x_1 < x_2 \in [a, b]$ there exists $u \in E^n$ such that $f(x_2) = f(x_1) + u$. We call u is the H-difference of $f(x_2)$ and $f(x_1)$, denoted $f(x_2) -_H f(x_1)$ ([11]).

For brevity, we always assume that it satisfies the condition (H) when dealing with the operation of subtraction of fuzzy numbers throughout this paper.

It is well-known that the H-derivative for fuzzy-number-functions was initially introduced by Puri and Ralescu [17] and it is based in the condition (H) of sets. We note that this definition is fairly strong, because the family of fuzzy-number-valued functions H-differentiable is very restrictive. For example, the fuzzy-number-valued function $f : [a, b] \to E^n$ defined by $f(x) = C \cdot g(x)$, where C is a fuzzy number, $\cdot$ is the scalar multiplication (in the fuzzy context) and $g : [a, b] \to R^+$, with $g'(t_0) < 0$, is not H-differentiable in t_0 (see [2]). To avoid the above difficulty, in this paper we consider a more general definition of a derivative for fuzzy-number-valued functions enlarging the class of differentiable fuzzy-number-valued functions, which has been introduced in [2] and [3].

Definition 1 ([2]) *Let $\tilde{f} : (a, b) \to E^n$ and $x_0 \in (a, b)$. We say that $\tilde{f}$ is differentiable at x_0, if there exists an element $\tilde{f}'(t_0) \in E^n$, such that*

(1) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0+h) -_H \tilde{f}(x_0), \tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)}{h} = \tilde{f}'(x_0)$$

or

(2) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0 + h), \tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

or

(3) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0 + h) -_H \tilde{f}(x_0), \tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

or

(4) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0+h), \tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)}{h} = \tilde{f}'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}\cdot$ and $-\frac{1}{h}\cdot$, respectively).

4

3 THE STRONG FUZZY HENSTOCK INTEGRAL AND ITS CONTROLLED CONVERGENCE THEOREM

In this section we shall give the definition of the strong Henstock integral for fuzzy-number-valued functions [7, 8] on a finite interval, which is an extension of the usual fuzzy Kaleva integral in [11]. In addition, we define the properties of $\omega - AC^*$ and $\omega - ACG^*$ for fuzzy-number-valued functions. In particular, we shall prove a controlled convergence theorems for the strong fuzzy Henstock integrals.

Definition 2 ([10, 14]) *Let $\delta(x)$ be a positive function defined on the interval $[a, b]$. A division $P = \{[x_{i-1}, x_i] : \xi_i\}$ is said to be $\delta-$fine if the following conditions are satisfied:*
(1) $a - x_0 < x_1 < \cdots < x_n = b$;
(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$.
For brevity, we write $P = \{[u, v]; \xi\}$

Definition 3 ([7, 8]) *A fuzzy-number-valued function $\tilde{f}$ is said to be strong Henstock integrable on $[a, b]$ if there exists a additive fuzzy-number-valued function $\tilde{F}$ on $[a, b]$ such that for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ and for any δ-fine division $P = \{([u, v], \xi)\}$ of $[a, b]$, we have*

$$\begin{aligned}
&\sum_{i\in K_n} D(\tilde{f}(\xi_i)(v_i - u_i), \tilde{F}([u_i, v_i])) \\
&+ \sum_{j\in I_n} D(\tilde{f}(\xi_j)(v_j - u_j), (-1)\cdot \tilde{F}([u_j, v_{j-1}])) \\
< \ & \varepsilon.
\end{aligned}$$

where $K_n = \{i \in \{1, 2, \cdot\cdot, n\}$ such that $\tilde{F}([x_{i-1}, x_i])$ is a fuzzy number and $I_n = \{j \in \{1, 2, \cdot\cdot, n\}$ such that $\tilde{F}([x_j, x_{j-1}])$ is a fuzzy number. We write $\tilde{f} \in SFH[a, b]$.

Definition 4 ([10, 14]) *A real-valued function F is strong absolute continuous ($F \in AC^*$) on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a $\eta > 0$ such that for every finite or infinite sequence of non-overlapping interval $\{[a_i, b_i]\}$, satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \mathcal{O}(F; [a_i, b_i]) < \varepsilon$, where where $\mathcal{O}$ denotes the oscillation of f over $[a_i, b_i]$, i.e.,*

$$\mathcal{O}(f, [a_i, b_i]) = \sup\{|F(x) - F(y)|; x, y \in [a_i, b_i]\}.$$

A real-valued function F is said to be ACG^* on X if X is the union of a sequence of sets $\{X_i\}$ such that on each X_i the function F is $AC^*(X_i)$.

Definition 5 *A fuzzy-number-valued function f defined on $X \subset [a, b]$ is said to be weak generalized absolute continuous ($\tilde{f} \in \omega - ACG^*(X)$) if for every $\lambda \in [0, 1]$, the real-valued function $f^-_\lambda(x)$ and $f^+_\lambda(x)$ are ACG^*.*

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Theorem 2 *If $\tilde{f}$ is strong fuzzy Henstock integrable on $[a,b]$, then its primitive F is $\omega - ACG^*$ on $[a,b]$.*

Proof. For every $\varepsilon > 0$, there is a function $\delta(\xi) > 0$ such that for any δ-fine partial division $P = \{[u,v],\xi\}$ in $[a,b]$, we have

$$\sum D(F([u,v]), f(\xi)(v-u)) < \varepsilon.$$

We assume that $\delta(\xi) \leq 1$. Let

$$X_{n,i} = \{x \in [a,b] : D(f(x),\tilde{0}) \leq n, \frac{1}{n} < \delta(x) \leq \frac{1}{n-1}, x \in [a+\frac{i-1}{n}, a+\frac{i}{n})\}$$

for $n = 2,3,\cdots, i = 1,2,\cdots$. Fixed $X_{n,i}$ and let $\{[a_k,b_k]\}$ be any finite sequence of non-overlapping intervals with $a_k, b_k \in X_{n,i}$ for all k. Then $\{([a_k,b_k],a_k)\}$ is a δ-fine partial division of $[a,b]$. Furthermore, if $a_k \leq u_k \leq v_k \leq b_k$, then $\{([a_k,u_k],a_k)\},\{([a_k,v_k],a_k)\}$ are δ-fine partial division of $[a,b]$. Thus

$$\begin{aligned}
\sum D(F(u_k),F(v_k)) &\leq \sum D(F(a_k),F(u_k)) + \sum D(F(b_k),F(v_k)) \\
&+ \sum D(F(a_k),F(b_k)) \\
&\leq 3\varepsilon + \sum D(f(a_k)(u_k-a_k),\tilde{0}) + \sum D(f(b_k)(b_k-v_k),\tilde{0}) \\
&+ \sum D(f(a_k)(b_k-a_k),\tilde{0}) \leq 3\varepsilon + 3n\sum(b_k-a_k).
\end{aligned}$$

Choose $\eta \leq \frac{\varepsilon}{3n}$ and $\sum(b_k-a_k) < \eta$. Then

$$\sum \mathcal{O}(F,[a_k,b_k]) \leq 3\varepsilon + \varepsilon.$$

Therefore, F is $\omega - AC^*(X_{n,i})$. Consequently, F is $\omega - ACG^*$ on $[a,b]$.

Theorem 3 *If there exists a fuzzy-number-valued function F is continuous and $\omega - ACG^*$ on $[a,b]$ such that $F'(x) = f(x)$ a.e. in $[a,b]$, then f is strong fuzzy Henstock integrable on $[a,b]$ with primitive F.*

Proof. Let F be the primitive of f and $F'(x) = f(x)$ for $x \in [a,b]\setminus S$ where S is of measure zero. For $\xi \in [a,b]\setminus S$, given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that whenever $\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ we have

$$D(F([u,v]), f(\xi)(v-u)) \leq \varepsilon|v-u|.$$

Since F is continuous and $\omega - ACG^*$ on $[a,b]$, there is a sequence of closed sets $\{X_i\}$ such that $\cup_i X_i = [a,b]$ and F is $\omega - AC^*(X_i)$ for each i. Let $Y_1 = X_1, Y_i = X_i \setminus (X_1 \cup X_2 \cdots \cup X_{i-1})$ for $i = 1,2,\cdots$ and S_{ij} denote the set of points $x \in S \cap Y_i$ such that $j-1 \leq D(f,\tilde{0}) < j$. Obviously, S_{ij} are pairwise disjointed and their union is the set S. Since F is also $\omega - AC^*(S_{ij})$, there is a $\eta_{ij} < \varepsilon 2^{-i-j} j^{-1}$ such that for any sequence of non-overlapping intervals $\{I_k\}$ with at least one endpoint of I_k belonging to S_{ij} and satisfying $\sum_k |I_k| < \eta_{ij}$

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we have $\sum_k D(F(I_k), \tilde{0}) < \varepsilon 2^{-i-j}$. Again, $F(I)$ denotes $F(v) -_H F(u)$ where $I = [u, v]$. Choose G_{ij} to be the union of a sequence of open intervals such that $|G_{ij}| < \eta_{ij}$ and $G_{ij} \supset S_{ij}$ where $|G_{ij}|$ denotes the total length of G_{ij}. Now for $\xi \in S_{ij}$, put $(\xi - \delta(\xi), \xi + \delta(\xi)) \subset G_{ij}$. Hence we have defined a positive function $\delta(\xi)$.

Take any $\delta-$fine division $P = \{[u, v]; \xi\}$. Split the $\sum$ over P into partial sums $\sum_1$ and $\sum_2$ in which $\xi \bar{\in} S$ and $\xi \in S$ respectively and we obtain

$$\begin{aligned} D(f(\xi)(v-u), F([a,b])) &\leq \sum_1 D(f(\xi)(v-u), F([a,b])) \\ &+ \sum_2 D(F([a,b]), \tilde{0}) + \sum_2 D(f(\xi)(v-u), \tilde{0}) \\ &< \varepsilon(b-a) + \sum_{i,j} \varepsilon 2^{-i-j} + \sum_2 j\eta_{ij} \\ &< \varepsilon(b-a) + 2\varepsilon. \end{aligned}$$

That is to say, f is strong fuzzy Henstock integrable to F on $[a, b]$.

Definition 6 *A sequence of fuzzy-number-valued functions $\{G_n(x)\}$ is said to be weak uniformly $ACG^*(U\omega - ACG^*)$ if for every $\lambda \in [0, 1]$, the real-valued functions $\{G_n(x)\}^-_\lambda$ and $\{G_n(x)\}^+_\lambda$ are $UACG^*$.*

Theorem 4 (Controlled Convergence theorem) *If a sequence of strong fuzzy Henstock integrable $\{f_n\}$ satisfies the following conditions:*

(1) $f_n(x) \to f(x)$ almost everywhere in $[a, b]$ as $n \to \infty$;

(2) the primitives $F_n(x) = (SFH)\int_a^x f_n(s)\mathrm{d}x$ of f_n are $\omega - ACG^*$ uniformly in n;

(3) the primitives $F_n(x)$ are equicontinuous on $[a, b]$,
then $f(x)$ is strong fuzzy Henstock integrable on $[a, b]$ and we have

$$\lim_{n\to\infty} (SFH) \int_a^b f_n(x)\mathrm{d}x = (SFH) \int_a^b f(x)\mathrm{d}x.$$

If condition (1) and (2) are replaced by condition (4):

(4) $g(x) \leq f(x) \leq h(x)$ almost everywhere on $[a, b]$, where $g(x)$ and $h(x)$ are steong fuzzy Henstock integrable.

Proof. In view of condition (3), $F(x)$ exist as the limit of $F_n(x)$ and is continuous. In fact, for $\forall\lambda \in [0, 1]$, $(F_n(x))^-_\lambda$ and $(F_n(x))^+_\lambda$ is uniformly ACG^* on $[a, b]$. By the Controlled Convergence theorem of real valued strong Henstock integral([14] Theorem 7.6), $F(x)$ is continuous. Because $F^-_\lambda(x)$ and $F^+_\lambda(x)$ is Henstock integrable on $[a, b]$, it follows condition (2) that F is $\omega - ACG^*$. From theorem 3.2, it remains to show that $F'(x) = f(x)$ almost everywhere. Hence we obtain $f(x)$ is strong fuzzy Henstock integrable on $[a, b]$.

Next, we put $G(x) = (SFH)\int_a^x F(t)\mathrm{d}t$, in view of condition (3), for $\forall\lambda \in [0, 1]$, we have

$$\lim_{n\to\infty} (F_n(x))^-_\lambda = G^-_\lambda(x) = F^-_\lambda(x)$$

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and

$$\lim_{n\to\infty}(F_n(x))^+_\lambda = G^+_\lambda(x) = F^+_\lambda(x).$$

So, let $x = b$, we have

$$\lim_{n\to\infty}(SFH)\int_a^b f_n(x)\mathrm{d}x = (SFH)\int_a^b f(x)\mathrm{d}x.$$

This completes the proof.

4 AN EXISTENCE RESULT OF GENERALIZED FUZZY INTEGRO-DIFFERENTIAL EQUATIONS

By using the Controlled Convergence theorem of strong fuzzy Henstock integral, in this section, we prove a theorem for the existence of solution to the Cauchy problem (1). For any bounded subset A of the Banach space X we denote $\alpha(A)$ the Kuratowski measure of non-compactness of A, i.e the infimum of all $\varepsilon > 0$ such that there exist a finite covering of A by sets of diameter less than ε. For the properties of α we refer to [1] for example.

Lemma 1 ([1]) *Let $H \subset C(I_\gamma, X)$ be a family of strong equicontinuous functions. Then*

$$\alpha(H) = \sup_{t\in I_\gamma}\alpha(H(t)) = \alpha(H(I_\gamma))$$

where $\alpha(H)$ denote the Kuratowski measure of non-compactness in $C(I_\gamma, X)$ and the function $t \to \alpha(H(t))$ is continuous.

Theorem 5 ([1]) *Let D be a closed convex subset of X, and let F be a continuous function from D into itself. If for $x \in D$ the implication*

$$\bar{V} = c\bar{o}n(\{x\} \cup F(V)) \Rightarrow V$$

is relatively compact, then F has a fixed point.

Theorem 6 *If the fuzzy-number-valued function $\tilde{f} : I_a \longrightarrow E^n$ is (SFH) integrable, then*

$$\int_I \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv}\tilde{f}(I),$$

where $\overline{conv}\tilde{f}(I)$ is the closure of the convex of $\tilde{f}(I)$, I is an arbitrary subinterval of I_a, and $|I|$ is the length of I..

Proof. Because of $j \circ \tilde{f}$ is abstract (SH) integrable in a Banach Space, by using the mean valued theorem of (SH) integrals, we have

$$(SH)\int_I j \circ \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv} j \circ \tilde{f}(I) = |I| \cdot j \circ \overline{conv}\tilde{f}(t).$$

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In additional, there exists $(SH)\int_I j\circ\tilde{f}(t)\mathrm{d}t = j\circ\int_I \tilde{f}(t)\mathrm{d}t$.

So, we have $j\circ\int_I \tilde{f}(t)\mathrm{d}t \in |I|\cdot\overline{conv}j\circ\tilde{f}(I)$. And the set $\{|I|\cdot\overline{conv}\tilde{f}(I)\}$ is a closed convex set, we have

$$\int_I \tilde{f}(t)\mathrm{d}t \in |I|\cdot\overline{conv}\tilde{f}(I).$$

Definition 7 *A fuzzy-number-valued function $\tilde{f}: I_a\times E^n \longrightarrow E^n$ is L^1-Carathéodory if the following conditions hold:*

(1) the fuzzy mapping $(x,y)\in E^n\times E^n$ is measurable for all $t\longrightarrow \tilde{f}(t,x,y)$;

(2) the fuzzy mapping $t\in I_a$ is continuous for all $(x,y)\longrightarrow \tilde{f}(t,x,y)$.

We observer that the problem (1) is equivalent to the integral eqution:

$$x(t)=x_0+\int_0^t \tilde{f}(z,x(z),\int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z$$

or

$$x(t)=x_0+(-1)\cdot\int_0^t \tilde{f}(z,x(z),\int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z. \tag{2}$$

Now, we define a notion of a solution.

Definition 8 *A $\omega-ACG^*$ function $x: I_a\to E^n$ is said to be the generalized solutions of the problem (1) if it satisfies the following conditions:*

(1) $x(0)=x_0$;

(2)

$$x'(t)=\tilde{f}(t,x(t),\int_0^t k_1(t,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(t,s)\tilde{h}(s,x(s))\mathrm{d}s).$$

for a. e. $t\in I_a$.

Definition 9 *A continuous function $x: I_a\to E^n$ is said to be the solutions of problem (2) if*

$$x(t)=x_0+\int_0^t \tilde{f}(z,x(z),\int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z$$

or

$$x(t)=x_0+(-1)\cdot\int_0^t \tilde{f}(z,x(z),\int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z.$$

for every $t\in I_a$

For every fuzzy number $x\in C(I_a,E^n)$, we define the norm of x by:

$$H(x,\tilde{0})=\sup_{t\in I_a} D(x,\tilde{0}).$$

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Let

$$B(p) = \{x \in C(I_a, E^n) | H(x, \tilde{0}) \leq H(x, \tilde{0}) + p, p > 0\}.$$

Obviously, $B(p)$ is closed and convex in E^n. Define the operator $F : C(I_a, E^n) \to C(I_a, E^n)$ by:

$$F(x)(t) = x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z$$

where integrals are in the sense of strong fuzzy Henstock integral.

Let

$$\Gamma(p) = \{F(x) \in C(I_a, E^n) | x \in B(p)\}$$

for each $p > 0$. Let $r(K)$ be the spectral radius of the integral operator K defined by

$$K(u)(t) = \int_0^c k(t,s)u(s)\mathrm{d}s,$$

where the kernel $k \in C(I_a \times I_a, R), u \in C(I_a, E^n)$ and c denotes any fixed valued in I_a.

Next, we give the main result in this section.

Theorem 7 *Suppose that for each $\omega - ACG^*$ function $x : I_a \to E^n$, the functions $\tilde{g}(\cdot, x(\cdot)), \tilde{f}(\cdot, x(\cdot)), \int_0^{(\cdot)} k_1(\cdot, s)\tilde{g}(s, x(s))\mathrm{d}s$, and $\int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s$ are (SFH) integrable, $\tilde{g}, \tilde{f}$, and $\tilde{h}$ are fuzzy L^1-Caratheodory functions. Let $k_1, k_2 : I_a \times I_a \to R^+$ be measurable functions such that $k_1(t, \cdot), k_2(t, \cdot)$ are continuous.*

Assume that there exists $p_0 > 0$ and positive constants L, L_1 and d_1, such that

$$\alpha(j \circ \tilde{g}(I, X)) \leq L\alpha(j \circ X), \quad I \subset I_a, X \subset B(p_0),$$
$$\alpha(j \circ \tilde{h}(I, X)) \leq L_1\alpha(j \circ X), \quad I \subset I_a, X \subset B(p_0),$$
$$\alpha(j \circ \tilde{f}(t, A, C, D)) \leq d_1 \cdot \max\{\alpha(j \circ A), \alpha(j \circ C), \alpha(j \circ D)\} \quad A, C, D \subset B(p_0),$$

where $\tilde{g}(I, X) = \{\tilde{g}(t, x(t)) | t \in I, x \in X\}, \tilde{h}(I, X) = \{\tilde{h}(t, x(t)) | t \in I, x \in X\}$ and

$$\tilde{f}(t, A, C, D) = \{\tilde{f}(t, x_1, x_2, x_3) | (x_1, x_2, x_3) \in A \times C \times D\}$$

where α denotes the Kuratowski measure of non-compactness.

Moreover, let $\Gamma(p_0)$ be equicontinuous, equibounded, and uniformly $\omega-ACG^*$ on I_a. Then, there exists at least on solution of problem (1) on I_c, for some $0 < c \leq a$, such that $d_1 \cdot c < 1$ and $d_1 \cdot c \cdot L \cdot r(K)$.

Proof. By equicontinuity and equiboundedness of $\Gamma(p_0)$ there exists a number $c, 0 < c \leq a$ such that

$$\begin{aligned}
&H(\int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
=&\sup_{t \in I_c} D(\int_0^t \tilde{f}(z, x(z), \int_0^z k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s, \int_0^a k_2(z,s)\tilde{h}(s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
\leq& p_0,
\end{aligned}$$

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where $p_0 > 0, x \in B(p_0)$. By the definition of F, we have

$$
\begin{aligned}
&H(F(x)(t),\tilde{0})\\
=&H(x_0+\int_0^t \tilde{f}(z,x(z),\int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z,\tilde{0})\\
\leq&H(x_0,\tilde{0})+H(\int_0^t \tilde{f}(z,x(z),\int_0^z k_1(z,s)\tilde{g}(s,x(s))\mathrm{d}s,\int_0^a k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z,\tilde{0})\\
\leq&H(x_0,\tilde{0})+p_0, \quad t\in I_c, x_0\in E^n.
\end{aligned}
$$

Using Theorem 4, we deduce that the fuzzy-number-valued function F is continuous.

Obviously, there exists $V \subset B$ such that $\overline{V} = \overline{conv}(\{x\} \cup F(V))$ for every $x \in B(p_0)$. Next, we will prove that V is relatively compact.

In fact, let $V(t) = \{v(t) \in E^n | v \in V\}$ for $t \in I_c$. Since $V \subset B(p_0)$ and $F(V) \subset \Gamma(p_0)$, then $V \subset \overline{V}$ is equicontinuous. By Lemma 1, we get that $t \to v(t) = \alpha(j \circ V(t))$ is continuous on I_c. For fixed $t \in I_c$, we divide the interval $[0, t]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = it/m, i = 0, 1, 2 \cdots, m$. Let $V([t_i, t_{i+1}]) = \{u(s) : u \in V, t_i \leq s \leq t_{i+1}, i = 1, 2, \cdots, m-1\}$ By Lemma 1 and the continuity of v, there exists $s_i \in I_i = [t_i, t_{i+1}]$ such that

$$\alpha(j \circ V([t_i, t_{i+1}])) = \sup_{t\in I_c}\{\alpha(j \circ V(s))|t_i \leq s \leq t_{i+1}\} := v(s_i).$$

For fixed $z \in [0, t]$, we divide the interval $[0, z]$ into m parts: $0 = z_0 < z_1 < \cdots < z_m = z$, where $z_j = jz/m, j = 0, 1, 2 \cdots, m$. Let $V([z_j, z_{j+1}]) = \{u(s)|u \in V, z_j \leq s \leq z_{j+1}\}, j = 0, 1, 2, \cdots, m-1$. By Lemma 1 and the continuity of v, there exists $s_j \in I_j = [z_j, z_{j+1}]$ such that

$$\alpha(j \circ V([z_j, z_{j+1}])) = \sup_{t\in I_c}\{\alpha(j \circ V(s))|z_j \leq s \leq z_{j+1}\} := v(s_j).$$

Furthermore, we divide the interval $[0, c]$ into m parts: $0 = r_0 < r_1 < \cdots < r_m = c$, where $r_k = kc/m, k = 0, 1, 2 \cdots, m$. Let $V([r_k, r_{k+1}]) = \{u(s)|u \in V, r_k \leq s \leq r_{k+1}\}, j = 0, 1, 2, \cdots, m-1$. By Lemma 1 and the continuity of v, there exists $s_k \in I_k = [r_k, r_{k+1}]$ such that

$$\alpha(j \circ V([r_k, r_{k+1}])) = \sup_{t\in I_c}\{\alpha(j \circ V(s))|r_k \leq s \leq r_{k+1}\} := v(s_k).$$

By Theorem 3 and Theorem 4, we have

$$\begin{aligned}
F(x)(t) = x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \tilde{f}(z, x(z), \sum_{j=0}^{m-1} \int_{z_j}^{z_{j+1}} k_1(z,s)\tilde{g}(s, x(s))\mathrm{d}s,\\
\sum_{k=0}^{m-1} \int_{r_k}^{r_{k+1}} k_2(z,s)\tilde{h}(s,x(s))\mathrm{d}s)\mathrm{d}z \in x_0\\
+ \sum_{i=0}^{m-1}(t_{i+1}-t_i)\overline{conv}\tilde{f}(I_i, V(I_i), \sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j))),\\
\sum_{k=0}^{m-1}(r_{j+1}-r_j)\overline{conv}(k_2(I_i,I_j)\tilde{h}(I_k,V(I_k))),
\end{aligned}$$

where $k(I,J) = \{k(t,s)|t \in I, s \in J\}$ and $\tilde{g}(I, V(I)) = \{\tilde{g}(t, x(t))|t \in I, x \in V\}$.

Using the condition in assumption and the properties of noncompactness α ([1]), we have

$$\begin{aligned}
&\alpha(j \circ F(V)(t))\\
&\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\overline{conv}\alpha(j \circ \tilde{f}(I_i, V(I_i), \sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j))),\\
&\sum_{k=0}^{m-1}(r_{j+1}-r_j)\overline{conv}(k_2(I_i,I_j)\tilde{h}(I_k,V(I_k))))\\
&\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)d_1 \max\{(\alpha(j \circ V(I_i)), \alpha j \circ (\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j)))),\\
&\alpha j \circ (\sum_{k=0}^{m-1}(r_{j+1}-r_j)\overline{conv}(k_2(I_i,I_j)\tilde{h}(I_k,V(I_k)))).
\end{aligned}$$

We observe that if

$$\begin{aligned}
\alpha(j \circ V(I_i)) = \max\{(\alpha(j \circ V(I_i)), \alpha j \circ (\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j)))),\\
\alpha(j \circ (\sum_{k=0}^{m-1}(r_{j+1}-r_j)\overline{conv}(k_2(I_i,I_j)\tilde{h}(I_k,V(I_k))))),
\end{aligned}$$

then

$$\alpha(j \circ V(t)) = \alpha j \circ (\overline{conv}(\{x(t)\} \cup F(V(t))))\alpha(j \circ F(V(t))) \leq d_1 \cdot c \cdot \alpha(j \circ V(t))$$

for every $t \in I_c$. Because $d_1 \cdot c < 1$, we have $\alpha(j \circ V) < \alpha(j \circ V)$. This is a contradiction.

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If

$$\alpha(j \circ (\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j))))))$$
$$= \max\{\alpha(j \circ V(I_i)), \alpha j \circ (\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j)))),$$
$$\alpha(j \circ (\sum_{k=0}^{m-1}(r_{j+1}-r_j)\overline{conv}(k_2(I_i,I_j)\tilde{h}(I_k,V(I_k)))))\},$$

we have

$$\begin{aligned}
&\alpha(j \circ F(V)(t)) \\
&\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1 \cdot \sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_i,I_j)\alpha(j \circ \tilde{g}(I_j,V(I_j))) \\
&\leq \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot d_1 \cdot L \cdot \sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_i,I_j)\alpha(j \circ V(I_j)) \\
&\leq \frac{c}{m}\cdot \sum_{j=0}^{m-1}(z_{j+1}-z_j)\alpha(j \circ V(I_j))\sum_{i=0}^{m-1}k_1(I_i,I_j).
\end{aligned}$$

For $j = 0, 1, 2, \ldots, m-1$, there exists $q_j = 0, 1, 2, \ldots, m-1$ such that $k_1(I_i, I_j) \leq k_1(I_{q_j}, I_j)$. So,

$$\alpha(j \circ F(V)(t)) \leq d_1 \cdot c \cdot L \cdot \sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)v(s_j), \quad s_j \in I_j.$$

Hence

$$\begin{aligned}
&\alpha(j \circ F(V)(t)) \\
&\leq d_1 \cdot c \cdot L \cdot \sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)(v(s_j)-v(p_j)) \\
&+ d_1 \cdot c \cdot L \cdot \sum_{j=0}^{m-1}(z_{j+1}-z_j)k_1(I_{q_j},I_j)v(p_j).
\end{aligned}$$

By the continuity of v, we have $j \circ v(s_j) - j \circ v(p_j) < \varepsilon$. Therefore, we have

$$\alpha(j \circ F(V)(t)) \leq d_1 \cdot c \cdot L \cdot \int_0^c k_1(t,s)v(s)\mathrm{d}s$$

for $t \in I_c$. Since $V = \overline{conv}(\{x\} \cup F(V))$, we have $\alpha(j \circ V(t)) \leq \alpha(j \circ F(V)(t))$, so, $v(t) \leq d_1 \cdot c \cdot L \cdot \int_0^c k_1(t,s)v(s)\mathrm{d}s$. By Gronwalls inequality, we have $\alpha(j \circ V(t)) = 0$ for $t \in I_c$. By Arzelá−Ascoli's theorem, we have V is relatively. Consequently,

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by Theorem 5, F has a fixed point. That is to say that problem (1) have at least solutions.

Similary, if

$$\alpha(j\circ(\sum_{k=0}^{m-1}(r_{j+1}-r_j)\overline{conv}(k_2(I_i,I_j)\tilde{h}(I_k,V(I_k)))))$$
$$=\max\{\alpha(j\circ V(I_i)),\alpha j\circ(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{conv}(k_1(I_i,I_j)\tilde{g}(I_j,V(I_j)))),$$
$$\alpha(j\circ(\sum_{k=0}^{m-1}(r_{j+1}-r_j)\overline{conv}(k_2(I_i,I_j)\tilde{h}(I_k,V(I_k)))))\},$$

then we have $\alpha(j\circ V(t))\leq\alpha(j\circ F(V)(t))$. By Arzel$\acute{a}-$Ascoli's theorem, the set V is relatively. By Theorem 5, F has a fixed point which is a solution of the problem (1).

5 CONCLUSIONS

In this paper, we give the definition of the $\omega-ACG^*$ for a fuzzy-number-valued function and a generalized controlled convergence theorem. In addition, we deal with the Cauchy problem of discontinuous fuzzy integro-differential equations of mixed type involving the strong fuzzy Henstock integral in fuzzy number space. The function governing the equations is supposed to be discontinuous with respect to some variables and satisfy nonabsolute fuzzy integrablility. Our result improves the result given in Ref. [11, 2] and [26] (where uniform continuity was required), as well as those referred therein.

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On the Generalized Stieltjes Transform of Fox's Kernel Function and its Properties in the Space of Generalized Functions

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Abstract

In this paper, a Stieltjes transform enfolding some Fox's H-function has been investigated on certain class of generalized functions named as Boehmians. By developing two spaces of Boehmians, the extended transform has been inspected and some general properties are also obtained. An inverse problem is also discussed in some detail.

Keywords: Fox's H-function; Stieltjes transform; Laplace transform; Boehmian space; Distribution space.

1 Introduction

The Fox's H-function is a generalization of the Meijer G-function introduced by Charles Fox [15]. It is defined by the compact notation adopted for

$$H_{p,q}^{m,n}(\omega) = H_{p,q}^{m,n}\left[\omega \left| \begin{array}{l} (a_j, \alpha_j)_{j=1,2,\ldots,p} \\ (b_j, \beta_j)_{j=1,2,\ldots,q} \end{array} \right.\right]$$

and has an exemplification in terms of the Barnes-type integral [2]

$$H_{p,q}^{m,n}(\omega) = \frac{1}{2\pi i}\int_{\mathcal{L}} \jmath_{p,q}^{m,n}(\varsigma)\,\omega^{\varsigma} d\varsigma,$$

where $\mathcal{L}$ is a path in the complex plane, $\omega^{\varsigma} = \exp\{\varsigma(\log|\omega| + i\arg\omega)\}$, and

$$\jmath_{p,q}^{m,n}(\varsigma) = \frac{\boldsymbol{a}(\varsigma)\,\boldsymbol{b}(\varsigma)}{\boldsymbol{c}(\varsigma)\,\boldsymbol{d}(\varsigma)},$$

where

$$\begin{aligned} \boldsymbol{a}(\varsigma) &:= \prod_{1}^{m}\Gamma\left(b_j - \beta_j\varsigma\right), \boldsymbol{b}(\varsigma) := \prod_{1}^{n}\Gamma(1 - a_j + \alpha_j\varsigma) \\ \boldsymbol{c}(\varsigma) &:= \prod_{m+1}^{q}\Gamma\left(1 - b_j - \beta_j\varsigma\right) \text{ and } \boldsymbol{d}(\varsigma) := \prod_{n+1}^{p}\Gamma(a_j + \alpha_j\varsigma), \end{aligned}$$

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with $m,p,q \in \mathbb{N}$, $a_j, b_j \in \mathbb{C}, \alpha_j, \beta_j \in \mathbb{R}^+, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ satisfying $0 < n < p$ and $0 < m < q$, and $\mathbb{C}, \mathbb{R}^+$ and $\mathbb{N}$ denote, respectively, the sets of complex numbers, positive real numbers and positive integers.

We refer to the survey article by Braaksma [2] and the book of Charles Fox [15] for asymptotic behaviour of Fox's H-functions.

Fox's H-functions being an extreme generalization of the generalized hypergeometric functions ${}_pF_q$ are utilized for applications in a large variety of problems connected with statistical distribution theory, structures of random variables, generalized distributions, Mathai's pathway models, versatile integrals, reaction, diffusion, reaction diffusion, engineering, communication, fractional differential and integral equations and many areas of theoretical physics and statistical distribution theory as well.

Recently, utility and importance of H-functions are realized due to their occurrence as kernels of certain integral transforms.

The generalized Stieltjes transform of a function $\varphi(t)$ of one variable with kernel involving Fox's H-function is defined by [5, (1.3)]

$$\chi_g^s(\varphi)(\omega) = \int_0^\infty \omega^{\square 1} H_{2,2}^{1,2}\left[\left(\frac{\xi}{\omega}\right)^\lambda \middle| \begin{array}{l} (a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2) \end{array}\right]\varphi(\xi)\,\mathtt{d}\xi, \tag{1}$$

where $H_{p,q}^{m,n}[\omega]$ is the usual notation of the Fox H-function.

An interesting fact that we find it worthwhile to be mentioned here is that the transform under consideration is a modulation of the Laplace transform

$$\chi_\ell(\varphi)(\omega) = \int_0^\infty H_{2,2}^{1,2}\left[(\xi\omega)^\lambda \middle| \begin{array}{l} (a_1,\alpha_1) \\ (e_1,\gamma_1),(e_2,\gamma_2) \end{array}\right]\varphi(\xi)\,\mathtt{d}\xi \tag{2}$$

that rectified after some iterations and an appropriate choice on its parameter.

Denote by $\mathcal{J}_{c,d}$ the Fréchet space of smooth functions φ defined for all ξ $(0 < \xi < \infty)$ by the set $\{\boldsymbol{\delta}_{c,d,k}\}$ of seminorms where

$$\boldsymbol{\delta}_{c,d,k}(\varphi) = \sup_{0<\xi<\infty}\left|\varrho_{c,d}(\log\xi)(\xi D_\xi)^k\sqrt{\xi}\varphi(\xi)\right| < \infty \tag{3}$$

for every choice of $k\,(k \in \mathbb{N}_0)$,

$$\varrho_{c,d}(\log\xi) = \begin{cases} \xi^c, & 1 \le \xi < \infty \\ \xi^d, & 0 < \xi < 1 \end{cases},$$

c and d are being real numbers.

The strong dual of continuous linear forms on $\mathcal{J}_{c,d}$ is denoted by $\mathcal{J}_{c,d}^{'}$.

Let p_1 and q_1 be real numbers defined by $p_1 = \min\left(\operatorname{Re}\frac{b_j}{\beta_j}\right)$ $(j = 1,2,...,m)$, $q_1 = \max\left(\operatorname{Re}\frac{a_j \square 1}{\alpha_j}\right)$ $(j = 1,2,...,n)$ and related by the pair of inequalities $c + \dfrac{1}{2} + \lambda q_1 < 0$ and $d + \dfrac{1}{2} + \lambda p_1 > 0$.

Then, the extended transform of a distribution $f \in \mathcal{J}_{c,d}^{'}$ is defined as the application of $f(t) \in \mathcal{J}_{c,d}^{'}$ to its kernel (see [5, Theorem 3.1])

$$\omega^{\square 1} H_{2,2}^{1,2}\left[\left(\frac{\xi}{\omega}\right)^\lambda \middle| \begin{array}{l} (a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2) \end{array}\right]$$

giving, by kernel method,

$$\chi_g^s(f)(\omega) = \left\langle f(\xi), \omega^{\square 1} H_{2,2}^{1,2}\left[\left(\frac{\xi}{\omega}\right)^\lambda \middle| \begin{array}{l} (a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2) \end{array}\right]\right\rangle, \tag{4}$$

where ω is a complex number not lying on the negative real axis.

For our consecutive investigation, we denote by $\mathcal{I}_{c,d}$ the subset of those integrable functions of $\mathcal{J}_{c,d}$ assigned by the set $\{\boldsymbol{\delta}_{c,d,k}\}$ and its strong dual $\mathcal{I}^{'}_{c,d}$ of distributions. Then, indeed, $\mathcal{I}_{c,d} \subseteq \mathcal{J}_{c,d}$ and, hence, $\mathcal{J}^{'}_{c,d} \subseteq \mathcal{I}^{'}_{c,d}$. Denote by $\mathcal{D}$ the standard notation of the space of smooth functions whose supports are compact subset of $(0, \infty)$. Then, it is easy to check that $\mathcal{D} \subset \mathcal{I}_{c,d}$ and that topology of $\mathcal{D}$ is stronger than the topology induced on it by $\mathcal{I}_{c,d}$. Hence, the restriction to any $f \in \mathcal{I}^{'}_{c,d}$ to $\mathcal{D}$ is in $\mathcal{D}^{'}$, where $\mathcal{D}^{'}$ is the space of disributions.

We need to establish the following theorem.

THEOREM 1 Given $\varphi \in \mathcal{I}_{c,d}$. Then, $\chi^{s}_{g}(\varphi) \in \mathcal{I}_{c,d}$.
PROOF Let $\varphi \in \mathcal{I}_{c,d}$ be given. For the convenience of the reader, we write

$$H^{1,2}_{2,2}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right] = H^{1,2}_{2,2}\left[\left(\frac{y}{\xi}\right)^{\lambda} \middle| \begin{array}{l} (a_1, \alpha_1), (1 - b_1 - \beta_1, \lambda\beta_1) \\ (e_1, \gamma_1), (e_2, \gamma_2) \end{array}\right].$$

By aid of (3) and (1) and simple computation we write

$$\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_{\xi})^{k}\sqrt{\xi}\left(\chi^{s}_{g}\right)(\varphi)(\xi)\right| \leq \int_{0}^{\infty}\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_{\xi})^{k}\sqrt{\xi}\xi^{\square 1}H^{1,2}_{2,2}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| \times |\varphi(y)|\,\mathtt{d}y.$$

This can also be revised to give

$$\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_{\xi})^{k}\sqrt{\xi}\left(\chi^{s}_{g}\right)(\varphi)(\xi)\right| \leq \int_{0}^{\infty}\left|\varrho_{c,d}(\log\xi)(\xi\mathcal{D}_{\xi})^{k}\left(\xi^{\square 1}\right)^{\frac{1}{2}}H^{1,2}_{2,2}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| \times |\varphi(y)|\,\mathtt{d}y.$$

By utilizing the Property 2.8

$$\mathcal{D}^{k}_{z}\left\{z^{w}H^{m,n}_{p,q}\left[cz^{\sigma}\middle| \begin{array}{l} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array}\right]\right\} = z^{w \square k}H^{m,n+1}_{p+1,q+1}\left[cz^{\sigma}\middle| \begin{array}{l} (-w, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (k - w, \sigma) \end{array}\right]$$

of Kilbas and Saigo [1, p.33] we get

$$\left|\varrho_{c,d}(\log\xi)(\xi D_{\xi})^{k}\sqrt{\xi}\left(\chi^{s}_{g}\right)(\varphi)(\xi)\right| \leq \int_{0}^{\infty}\left|\varrho_{c,d}(\log\xi)\,\xi^{\square\frac{1}{2}}\,\widehat{H^{1,3}_{3,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right||\varphi(y)|\,\mathtt{d}y,$$

where

$$\widehat{H^{1,3}_{3,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right] = H^{1,3}_{3,3}\left[\left(\frac{y}{\xi}\right)^{\lambda}\middle| \begin{array}{l} \left(\frac{1}{2}, \lambda\right), (a_1, \alpha_1), (1 - b_1 - \beta_1, \lambda\beta_1) \\ (e_1, \gamma_1), (e_2, \gamma_2), \left(k - \frac{1}{2}, \lambda\right) \end{array}\right].$$

Therefore, the asymptotic properties of H-functions, for large ξ, imply

$$\sup_{0<\xi<\infty}\left|\varrho_{c,d}(\log\xi)\,\xi^{\square\frac{1}{2}}\,\widehat{H^{1,3}_{3,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| = \sup_{0<\xi<\infty}\left|\xi^{c}\xi^{\square\frac{1}{2}}\,\widehat{H^{1,3}_{3,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| < M_1,$$

where M_1 is some positive constant. Similarly, for small ξ, it implies

$$\sup_{0<\xi<\infty}\left|\varrho_{c,d}(\log\xi)\,\xi^{\square\frac{1}{2}}\,\widehat{H^{1,3}_{3,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| = \sup_{0<\xi<\infty}\left|\xi^{d}\xi^{\square\frac{1}{2}}\,\widehat{H^{1,3}_{3,3}}\left[\left(\frac{y}{\xi}\right)^{\lambda}\right]\right| < M_2,$$

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where M_2 is a postive constant.
Let $M = \max\{M_1, M_2\}$. Then, by the preceding two formulas, we have

$$\sup_{0<\xi<\infty} \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} \left(\chi_g^s\right) (\varphi) (\xi) \right| \leq M \int_0^\infty |\varphi(y)| \, \mathtt{d}y < \infty,$$

since φ is integrable.

The proof of this theorem is finished.

Definition 2 Let $f \in \mathcal{I}'_{c,d}$. Then, the Stieltjes transform χ_g^s of $f \in \mathcal{I}'_{c,d}$ is defined by the inner product

$$\left\langle \chi_g^s (f) (\omega), \varphi(\omega) \right\rangle = \left\langle f(\omega), \chi_g^s (\varphi)(\omega) \right\rangle, \tag{6}$$

where $\varphi \in \mathcal{I}_{c,d}$ is aritrary.
The inner product on the left hand side of (6) is well-defined by Theorem 1. Hence, it may be noted from Equation 6 that the Stieltjes transform of $f \in \mathcal{I}'_{c,d}$ is a distribution in $\mathcal{I}'_{c,d}$.

2 Generalized Distributions; Boehmian Spaces

We always assume that readers are aquainted with the concept of Boehmian spaces, if it were otherwise, we would refer to [4], [6 − 14] and [16, 17].

Let us now prove the following Theorems that legitimate the existence of our Boehmian spaces.

The following definition is important for our next investigation.

Definition 3 Given $\varphi, \psi \in \mathcal{I}_{c,d}$, then, for φ and ψ, the product $\otimes$ is defined by

$$(\varphi \otimes \psi)(\omega) = \int_0^\infty \varphi\left(\xi^{\square 1}\omega\right) \frac{\psi(\xi)}{\xi} \mathtt{d}\xi, \tag{7}$$

provided the integral exists.
Theorem 4 Given $\varphi \in \mathcal{I}_{c,d}$, then $\varphi \otimes \psi \in \mathcal{I}_{c,d}$, for every $\psi \in \mathcal{D}$.
Proof On account of (3), we write

$$\begin{aligned} \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} (\varphi \otimes \psi)(\xi) \right| &\leq \int_0^\infty \left| y^{\square 1} \psi(y) \right| \\ &\quad \times \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} \varphi\left(y^{\square 1}\xi\right) \right| \mathtt{d}y \\ &\leq A^* \int_0^\infty \left| y^{\square 1} \psi(y) \right| \mathtt{d}y. \end{aligned}$$

Let $[a_1, a_2]$ be a closed interval containing the support of ψ. Since $\varphi \in \mathcal{I}_{c,d}$, it by considering supremum over all $\xi\,(0 < \xi < \infty)$ follows that

$$\boldsymbol{\delta}_{c,d,k} (\varphi \otimes \psi) \leq A^* \int_{a_1}^{a_2} \left| y^{\square 1} \psi(y) \right| \mathtt{d}y < \infty,$$

for some constant A^*.

Hence, the proof of this theorem is finished.

Let $\curlyvee$ be the product of Mellin type given by

$$(\varphi \curlyvee \psi)(y) = \int_0^\infty \xi^{\square 1} \varphi\left(\xi^{\square 1} y\right) \psi(\xi) \, \mathtt{d}\xi. \tag{8}$$

We generate the space $\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\curlyvee\right)\right)$ where Δ is the subset of $\mathcal{D}$ of sequences $\left(\delta_n\right)$ such that

$$\left.\begin{array}{l}\text{(i)}\ \displaystyle\int_0^\infty \delta_n\left(\xi\right)\mathtt{d}\xi=1;\\ \text{(ii)}\ \left|\delta_n\left(\xi\right)\right|<A,\ A\in\mathbb{R},\ A>0;\\ \text{(iii)}\ \operatorname{supp}\delta_n\left(\xi\right)\subseteq\left(a_n,b_n\right),\ a_n,b_n\rightarrow0 \text{ as } n\rightarrow\infty,\end{array}\right\}\qquad(9)$$

$n\in\mathbb{N},\ \xi\in(0,\infty)$.

In what follows we shall make a free use of the properties of the product $\curlyvee$ that we briefly describe them as follows :

(i) $\varphi_1\curlyvee\varphi_2=\varphi_2\curlyvee\varphi_1$;
(ii) $\left(\varphi_1\curlyvee\varphi_2\right)\curlyvee\varphi_3=\varphi_1\curlyvee\left(\varphi_2\curlyvee\varphi_3\right)$;
(iii) $\left(\alpha\varphi_1\right)\curlyvee\varphi_2=\alpha\left(\varphi_1\curlyvee\varphi_2\right)$;
(iv) $\varphi_1\curlyvee\left(\varphi_2+\varphi_3\right)=\varphi_1\curlyvee\varphi_2+\varphi_1\curlyvee\varphi_3$.

Following theorem follows from elementary rules of integral calculus. Hence, its proof is deleted.

Theorem 5 Given $\varphi_n,\varphi,\varphi_1,\varphi_2\in\mathcal{I}_{c,d}$, $\alpha\in\mathbb{C}$, and $\psi,\psi_1,\psi_2\in\mathcal{D}$ such that $\varphi_n\rightarrow\varphi$ as $n\rightarrow\infty$, then the following are true :

(i) $\varphi_n\otimes\psi\rightarrow\varphi\otimes\psi$ as $n\rightarrow\infty$.
(ii) $\varphi_1\otimes\left(\psi_1+\psi_2\right)=\varphi_1\otimes\psi_1+\varphi_2\otimes\psi_2$.
(iii) $\alpha\left(\varphi\otimes\psi\right)=\alpha\varphi\otimes\psi=\varphi\otimes\left(\alpha\psi\right)$.

Theorem 6 Given $\varphi\in\mathcal{I}_{c,d}$ and $\psi_1,\psi_2\in\mathcal{D}$, then $\varphi\otimes\left(\psi_1\curlyvee\psi_2\right)=\left(\varphi\otimes\psi_1\right)\otimes\psi_2$.

Proof Let $\varphi\in\mathcal{I}_{c,d}$ and $\psi_1,\psi_2\in\mathcal{D}$. Then, by aid of the integrals (7) and (8), we write

$$\begin{aligned}\left(\varphi\otimes\left(\psi_1\curlyvee\psi_2\right)\right)(\omega)&=\int_0^\infty\varphi\left(\xi^{\square1}\omega\right)\frac{\left(\psi_1\curlyvee\psi_2\right)(\xi)}{\xi}\mathtt{d}\xi\\&=\int_0^\infty\psi_2\left(y\right)y^{\square1}\int_0^\infty\varphi\left(\xi^{\square1}\omega\right)\frac{\psi_1\left(\xi y^{\square1}\right)}{\xi}\mathtt{d}y\mathtt{d}\xi\\&=\int_0^\infty\psi_2\left(y\right)\frac{\int_0^\infty\varphi\left(y^{\square1}z^{\square1}\omega\right)z^{\square1}\psi\left(z\right)\mathtt{d}z}{y}\mathtt{d}y\\&=\int_0^\infty\psi_2\left(y\right)\frac{\left(\varphi\otimes\psi_1\right)\left(y^{\square1}\omega\right)}{y}\mathtt{d}y.\end{aligned}$$

The proof of this theorem is finished.

Theorem 7 Given $\left(\delta_n\right)\in\Delta$ and $\varphi\in\mathcal{I}_{c,d}$, then $\varphi\otimes\delta_n\in\mathcal{I}_{c,d}$.

Proof Let $\varphi\in\mathcal{I}_{c,d}$ and $\left(\delta_n\right)\in\Delta$ be given. Then, by (3) and the Identity (i) of (9) we have

$$\begin{aligned}\left|\varrho_{c,d}\left(\log\xi\right)\left(\xi\mathcal{D}_\xi\right)^k\sqrt{\xi}\left(\varphi\otimes\delta_n-\varphi\right)(\xi)\right|&=\int_0^\infty\left|\varrho_{c,d}\left(\log\xi\right)\left(\xi\mathcal{D}_\xi\right)^k\sqrt{\xi}\varphi_y\left(\xi\right)\right|\\&\quad\times\left|\delta_n\left(y\right)\right|\mathtt{d}y,\end{aligned}\qquad(10)$$

where $\varphi_y\left(\xi\right)=\varphi\left(\xi y^{\square1}\right)y^{\square1}-\varphi\left(\xi\right)$. Since $\varphi_y\left(\xi\right)\in\mathcal{I}_{c,d}$, we from (10), get that

$$\left|\varrho_{c,d}\left(\log\xi\right)\left(\xi\mathcal{D}_\xi\right)^k\sqrt{\xi}\left(\varphi\otimes\delta_n-\varphi\right)(\xi)\right|\leq A\int_0^\infty\left|\delta_n\left(y\right)\right|\mathtt{d}y,\qquad(11)$$

where A is some positive constant.

Hence, by the identities (ii) and (iii) of (9), Equation (11) can be expressed as

$$\left|\varrho_{c,d}\left(\log\xi\right)\left(\xi\mathcal{D}_\xi\right)^k\sqrt{\xi}\left(\varphi\otimes\delta_n-\varphi\right)(\xi)\right|\leq AA_1\left(b_n-a_n\right)\rightarrow0\text{ as }n\rightarrow\infty.\qquad(12)$$

Hence, the proof of this theorem is finished.

THEOREM 8 Given $\varphi \in \mathcal{I}_{c,d}$, then, for every $(\delta_n) \in \Delta$, we have $\varphi \otimes \delta_n \to \varphi$ in $\mathcal{I}_{c,d}$ as $n \to \infty$.
PROOF Let F_n be a compact subset of $(0, \infty)$ containing $\operatorname{supp} \delta_n$, for all n. Then, on account of (i) of (9), we get

$$\begin{aligned} \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} (\varphi \otimes \delta_n - \varphi)(\xi) \right| \leq & \int_{F_n} \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} \varphi \left(x^{\square 1} \xi \right) \right| \\ & \times \frac{|\delta_n(x)|}{x} \mathrm{d}x \\ & + \int_{F_n} \left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} \varphi(\xi) \right| \\ & \times |\delta_n(x)| \, \mathrm{d}x. \end{aligned} \quad (13)$$

Therefore, (13) gives

$$\left| \varrho_{c,d} (\log \xi) (\xi \mathcal{D}_\xi)^k \sqrt{\xi} (\varphi \otimes \delta_n - \varphi)(\xi) \right| \leq A_1 \int_{F_n} \frac{|\delta_n(x)|}{x} \mathrm{d}x + A_2 \int_{F_n} |\delta_n(x)| \, \mathrm{d}x.$$

Considering the supremum over all ξ, $0 < \xi < \infty$, implies

$$\boldsymbol{\delta}_{c,d,k} (\varphi \otimes \delta_n - \varphi) < \infty,$$

for any choice of the real numbers c, d and $k \in \mathbb{N}_0$. Thus, we find that

$$\varphi \otimes \delta_n \to \varphi \text{ in } \mathcal{I}_{c,d} \text{ as } n \to \infty.$$

The proof has been completed .

COROLLARY 9 Given $(\delta_n) \in \Delta$ and $\varphi_1 \otimes \delta_n = \varphi_2 \otimes \delta_n$, then $\varphi_1 = \varphi_2$ for all $\varphi_1, \varphi_2 \in \mathcal{I}_{c,d}$.

The space $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ is constructed.

Addition and multiplication by a scalar in $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ are defined by

$$\left[\frac{\varphi_n}{\delta_n} \right] + \left[\frac{\psi_n}{\varepsilon_n} \right] =: \left[\frac{\varphi_n \otimes \delta_n + \psi_n \otimes \delta_n}{\delta_n \curlyvee \varepsilon_n} \right] \text{ and } \mu \left[\frac{\varphi_n}{\delta_n} \right] =: \left[\frac{\mu \varphi_n}{\delta_n} \right] \quad (\mu \in \mathbb{C}).$$

An extension of $\otimes$ and differentiation to $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ is given as follows

$$\left[\frac{\varphi_n}{\delta_n} \right] \otimes \left[\frac{\psi_n}{\varepsilon_n} \right] =: \left[\frac{\varphi_n \otimes \psi_n}{\delta_n \curlyvee \varepsilon_n} \right] \text{ and } \mathcal{D}^\alpha \left[\frac{\varphi_n}{\delta_n} \right] =: \left[\frac{\mathcal{D}^\alpha \varphi_n}{\delta_n} \right] \quad (\alpha \in \mathbb{R}).$$

Given $\left[\frac{\varphi_n}{\delta_n} \right] \in \mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ and $\varpi \in \mathcal{I}_{c,d}$. Then, $\otimes$ can be extended to $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee)) \times \mathcal{I}_{c,d}$ by

$$\left[\frac{\varphi_n}{\delta_n} \right] \otimes \varpi =: \left[\frac{\varphi_n \otimes \varpi}{\delta_n} \right].$$

$\beta_n \xrightarrow{\delta} \beta$ in $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ if there can be (δ_n) in Δ satisfying $(\beta_n \otimes \delta_k), (\beta \otimes \delta_k) \in \mathcal{I}_{c,d}$ $(k, n \in \mathbb{N})$ and $(\beta_n \otimes \delta_k) \to (\beta \otimes \delta_k)$ in $\mathcal{I}_{c,d}$ as $n \to \infty$ $(k \in \mathbb{N})$. This can be expressed to mean :

$\beta_n \xrightarrow{\delta} \beta$ $(n \to \infty)$ in $\mathcal{B}((\mathcal{I}_{c,d}, \otimes); (\mathcal{D}, \curlyvee))$ if there are $\varphi_{n,k}$ and $\varphi_k \in \mathcal{I}_{c,d}$, and $(\delta_k) \in \Delta$ where $\beta_n = \left[\frac{\varphi_{n,k}}{\delta_k} \right], \beta = \left[\frac{\varphi_k}{\delta_k} \right]$ and for each $k \in \mathbb{N}$ we have $f_{n,k} \to f_k$ as $n \to \infty$ in $\mathcal{I}_{c,d}$.

$\beta_n \xrightarrow{\Delta} \beta$ in $\mathcal{B}\left(\left(\mathcal{I}_{c,d}, \otimes\right); \left(\mathcal{D}, \curlyvee\right)\right)$, in a sense of Δ, if there can be $(\delta_n) \in \Delta$ where $(\beta_n - \beta) \otimes \delta_n \in \mathcal{I}_{c,d}$ $(\forall n \in \mathbb{N})$ and that $(\beta_n - \beta) \otimes \delta_n \to 0$ as $n \to \infty$ in $\mathcal{I}_{c,d}$.

By techniques similar to above, the space $\mathcal{B}\left(\left(\mathcal{I}_{c,d}, \curlyvee\right); \left(\mathcal{D}, \curlyvee\right)\right)$ can similarly be generated. In $\mathcal{B}\left(\left(\mathcal{I}_{c,d}, \curlyvee\right); \left(\mathcal{D}, \curlyvee\right)\right)$, addition and multiplication by a scalar has the following meanings

$$\left[\frac{\varphi_n}{\delta_n}\right] + \left[\frac{\psi_n}{\varepsilon_n}\right] =: \left[\frac{\varphi_n \curlyvee \delta_n + \psi_n \curlyvee \delta_n}{\delta_n \curlyvee \varepsilon_n}\right] \text{ and } \rho \left[\frac{\varphi_n}{\delta_n}\right] =: \left[\frac{\alpha \varphi_n}{\delta_n}\right] \quad (\rho \in \mathbb{C}).$$

We extend $\curlyvee$ and the differentiation to $\mathcal{B}\left(\left(\mathcal{I}_{c,d}, \curlyvee\right); \left(\mathcal{D}, \curlyvee\right)\right)$ as

$$\left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \left[\frac{\psi_n}{\varepsilon_n}\right] = \left[\frac{\varphi_n \curlyvee \psi_n}{\delta_n \curlyvee \varepsilon_n}\right], \ \mathcal{D}^{\alpha} \left[\frac{\varphi_n}{\delta_n}\right] = \left[\frac{\mathcal{D}^{\alpha} \varphi_n}{\delta_n}\right],$$

α being real number.

Given $\left[\frac{\varphi_n}{\delta_n}\right] \in \mathcal{B}\left(\left(\mathcal{I}_{c,d}, \curlyvee\right); \left(\mathcal{D}, \curlyvee\right)\right)$ and $\varpi \in \mathcal{I}_{c,d}$. We define $\curlyvee$ for $\mathcal{B}\left(\left(\mathcal{I}_{c,d}, \curlyvee\right); \left(\mathcal{D}, \curlyvee\right)\right) \times \mathcal{I}_{c,d}$ as

$$\left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \varpi =: \left[\frac{\varphi_n \curlyvee \varpi}{\delta_n}\right].$$

Convergence in $\mathcal{B}\left(\left(\mathcal{I}_{c,d}, \curlyvee\right); \left(\mathcal{D}, \curlyvee\right)\right)$ is as follows :

$\beta_n \xrightarrow{\delta} \beta$ $(n \to \infty)$ in $\mathcal{B}\left(\left(\mathcal{I}_{c,d}, \curlyvee\right); \left(\mathcal{D}, \curlyvee\right)\right)$ if and only if there can be (δ_n) in Δ such that $(\beta_n \curlyvee \delta_k), (\beta \curlyvee \delta_k) \in \mathcal{I}_{c,d}$ $(\forall k, n \in \mathbb{N})$ and $(\beta_n \curlyvee \delta_k) \to (\beta \curlyvee \delta_k)$ in $\mathcal{I}_{c,d}$ as $n \to \infty$ $(\forall k \in \mathbb{N})$. Or, if there can be found $\varphi_{n,k}, \varphi_k \in \mathcal{I}_{c,d}$, $(\delta_k) \in \Delta$, $\beta_n = \left[\frac{\varphi_{n,k}}{\delta_k}\right]$, $\beta = \left[\frac{\varphi_k}{\delta_k}\right]$ and $f_{n,k} \to f_k$ as $n \to \infty$ in $\mathcal{I}_{c,d}$ $(k \in \mathbb{N})$.

$\beta_n \xrightarrow{\Delta} \beta$ $(n \to \infty)$, in $\mathcal{B}\left(\left(\mathcal{I}_{c,d}, \curlyvee\right); \left(\mathcal{D}, \curlyvee\right)\right)$, if there can be $(\delta_n) \in \Delta$ satisfying $(\beta_n - \beta) \curlyvee \delta_n \in \mathcal{I}_{c,d}$ and $(\beta_n - \beta) \curlyvee \delta_n \to 0$ as $n \to \infty$ in $\mathcal{I}_{c,d}$.

3 The Generalized χ_g^s Transform of $\mathcal{B}\left(\left(\mathcal{I}_{c,d}, \curlyvee\right); \left(\mathcal{D}, \curlyvee\right)\right)$

We devote this section to the definition of the generalized Stieltjes transform and to derive some desired properties. The following theorem specifies the relation between $\curlyvee$ and $\otimes$.
THEOREM 10 Given $\varphi \in \mathcal{I}_{c,d}$, then $\chi_g^s (\varphi \curlyvee \psi)(\omega) = \left(\chi_g^s (\varphi) \psi\right)(\omega)$ for every $\psi \in \mathcal{D}$.
PROOF Let $\varphi \in \mathcal{I}_{c,d}$ and $\psi \in \mathcal{D}$ be given. Then, by (1), we have

$$\begin{aligned} \chi_g^s (\varphi \curlyvee \psi)(\omega) &= \int_0^{\infty} \omega^{\square 1} H_{2,2}^{1,2} \left[\left(\frac{\xi}{\omega}\right)^{\lambda} \middle| \begin{array}{l} (a_1, \alpha_1), (1 - b_1 - \beta_1, \lambda \beta_1) \\ (e_1, \gamma_1), (e_2, \gamma_2) \end{array} \right] \\ &\quad \times (\varphi \curlyvee \psi)(\omega) \, \mathrm{d}\xi, \end{aligned}$$

which can be expressed after setting the variables and using Fubini's theorem as

$$\begin{aligned} \chi_g^s (\varphi \curlyvee \psi)(\omega) &= \int_0^{\infty} \psi(y) \int_0^{\infty} \omega^{\square 1} \\ &\quad \times H_{2,2}^{1,2} \left[\left(\frac{z}{y^{\square 1} \omega}\right)^{\lambda} \middle| \begin{array}{l} (a_1, \alpha_1), (1 - b_1 - \beta_1, \lambda \beta_1) \\ (e_1, \gamma_1), (e_2, \gamma_2) \end{array} \right] \varphi(z) \, \mathrm{d}z \mathrm{d}y. \end{aligned} \tag{14}$$

Simple motivation on (14) yields

$$\begin{aligned}\chi_g^s(\varphi \curlyvee \psi)(\omega) &= \int_0^\infty \psi(y) \int_0^\infty \left(y\omega^{\square 1}\right) \\ &\times H_{2,2}^{1,2}\left[\left(\frac{z}{y^{\square 1}\omega}\right)^\lambda \middle| \begin{matrix}(a_1,\alpha_1),(1-b_1-\beta_1,\lambda\beta_1) \\ (e_1,\gamma_1),(e_2,\gamma_2)\end{matrix}\right]\varphi(z)\,\mathrm{d}z\mathrm{d}y.\end{aligned}$$

Hence, the above equation is interpreted to mean

$$\chi_g^s(\varphi \curlyvee \psi)(\omega) = \int_0^\infty \psi(y)\, y^{\square 1}\left(\chi_g^s\right)(\varphi)\left(y\omega^{\square 1}\right)\mathrm{d}y.$$

Hence, the proof of this theorem is finished.

In view of the preceeding result we give the definition of χ_g^s transform of $\left[\frac{\varphi_n}{\delta_n}\right]$ in the space $\mathcal{B}\left((\mathcal{I}_{c,d},\curlyvee),(\mathcal{D},\curlyvee)\right)$ as

$$\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right]\right) =: \left[\frac{\chi_g^s\varphi_n}{\delta_n}\right] \tag{15}$$

which belongs to $\mathcal{B}\left((\mathcal{I}_{c,d},\otimes),(\mathcal{D},\curlyvee)\right)$ by means of Theorem 10.

Theorem 11 The operator $\widehat{\chi_g^s}$ is well - defined and linear, mapping from $\mathcal{B}\left((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee)\right)$ into $\mathcal{B}\left((\mathcal{I}_{c,d},\otimes);(\mathcal{D},\curlyvee)\right)$.

Proof Let $\left[\frac{\varphi_n}{\delta_n}\right] = \left[\frac{\psi_n}{\varepsilon_n}\right]$ in the sense of $\mathcal{B}\left((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee)\right)$. Then, by the concept of equivalent classes, $\frac{\varphi_n}{\delta_n}$ and $\frac{\psi_n}{\varepsilon_n}$ are equivalent in $\mathcal{B}\left((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee)\right)$. Thus, it has been obtained $\varphi_n \curlyvee \varepsilon_m = \psi_n \curlyvee \delta_m$.

Applying χ_g^s to the sides of the above equation and employing Theorem 10 imply

$$\chi_g^s\varphi_n \otimes \varepsilon_m = \chi_g^s\psi_n \otimes \delta_m \ \ (\forall n,m \in \mathbb{N}).$$

That is,

$$\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right] = \left[\frac{\chi_g^s\psi_n}{\varepsilon_n}\right].$$

To show that the $\widehat{\chi_g^s} : \mathcal{B}\left((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee)\right) \to \mathcal{B}\left((\mathcal{I}_{c,d},\otimes);(\mathcal{D},\curlyvee)\right)$ is linear, let $\rho_1 = \left[\frac{\varphi_n}{\delta_n}\right], \rho_2 = \left[\frac{\psi_n}{\varepsilon_n}\right] \in \mathcal{B}\left((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee)\right)$. Then, addition of Boehmians of $\mathcal{B}\left((\mathcal{I}_{c,d},\curlyvee);(\mathcal{D},\curlyvee)\right)$ and Equation 15, suggest to write

$$\widehat{\chi_g^s}(\rho_1+\rho_2) = \left[\frac{\chi_g^s(\varphi_n \curlyvee \varepsilon_n) + \chi_g^s(\psi_n \curlyvee \delta_n)}{\delta_n \curlyvee \varepsilon_n}\right].$$

By aid of Theorem 10, we obtain

$$\widehat{\chi_g^s}(\rho_1+\rho_2) = \left[\frac{\chi_g^s\varphi_n \otimes \varepsilon_n + \chi_g^s\psi_n \otimes \delta_n}{r_n \curlyvee \varepsilon_n}\right].$$

Employing the product $\otimes$ that assigned to the $\mathcal{B}\left((\mathcal{I}_{c,d},\otimes);(\mathcal{D},\curlyvee)\right)$ gives

$$\widehat{\chi_g^s}(\rho_1+\rho_2) = \left[\frac{\chi_g^s\varphi_n}{\delta_n}\right] + \left[\frac{\chi_g^s\psi_n}{\varepsilon_n}\right].$$

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Hence, we have obtained that

$$\widehat{\chi_g^s}\left(\rho_1+\rho_2\right)=\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right]\right)+\widehat{\chi_g^s}\left(\left[\frac{\psi_n}{\varepsilon_n}\right]\right).$$

Also, it is easy for readers to check that

$$\lambda\widehat{\chi_g^s}\left(\rho_1\right)=\widehat{\chi_g^s}\left(\lambda\rho_1\right)\ \ (\lambda\in\mathbb{C}).$$

Hence, the proof of this theorem is finished.

Theorem 12 The mapping $\widehat{\chi_g^s}:\mathcal{B}\left(\left(\mathcal{I}_{c,d},\Upsilon\right);\left(\mathcal{D},\Upsilon\right)\right)\to\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\Upsilon\right)\right)$ is an isomorphism.

Proof Given $\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]=\left[\frac{\chi_g^s\psi_n}{\varepsilon_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\Upsilon\right)\right)$. Then, by virtue of Theorem 10, we get

$$\chi_g^s\varphi_n\otimes\varepsilon_m=\chi_g^s\psi_m\otimes\delta_n\ \ (m,n\in\mathbb{N}).$$

Once again, Theorem 10 implies

$$\chi_g^s\left(\varphi_n\otimes\varepsilon_m\right)=\chi_g^s\left(\psi_m\otimes\delta_n\right).$$

Hence $\varphi_n\otimes\varepsilon_m=\psi_m\otimes\delta_n$. Therefore,

$$\left[\frac{\varphi_n}{\delta_n}\right]=\left[\frac{\psi_n}{\varepsilon_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\Upsilon\right);\left(\mathcal{D},\Upsilon\right)\right).$$

This proves that the above mapping is an injection. surjectivity of $\widehat{\chi_g^s}$ is obvious.
The proof is finished.

Definition 13 Let $\rho^*\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\Upsilon\right)\right),\ \rho^*=\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]$. Then, we the inverse mapping $\widehat{\chi_g^s}$ is defined as

$$\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\right)=\left[\frac{\left(\chi_g^s\right)^{\square 1}\left(\chi_g^s\varphi_n\right)}{\delta_n}\right]=\left[\frac{\varphi_n}{\delta_n}\right],$$

for each $(\delta_n)\in\Delta$.

Theorem 14 Let $\rho^*=\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\Upsilon\right)\right)$ for some $\left[\frac{\varphi_n}{\delta_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\Upsilon\right)\right)$ and $\phi,\psi\in\mathcal{D}$. Then we have

(i) $\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\otimes\phi\right)=\left[\frac{\varphi_n}{\delta_n}\right]\Upsilon\,\phi,$

(ii) $\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right]\Upsilon\,\psi\right)=\rho^*\otimes\psi.$

Proof Assume $\rho^*=\left[\frac{\chi_g^s\varphi_n}{\delta_n}\right]\in\mathcal{B}\left(\left(\mathcal{I}_{c,d},\otimes\right);\left(\mathcal{D},\Upsilon\right)\right)$ be given. Then, by Theorem 10, we write

$$\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\otimes\phi\right)=\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\left[\frac{\chi_g^s\varphi_n\otimes\phi}{\delta_n}\right]\right)=\left[\frac{\left(\chi_g^s\right)^{\square 1}\left(\chi_g^s\varphi_n\otimes\phi\right)}{\delta_n}\right].$$

Hence,

$$\left(\widehat{\chi_g^s}\right)^{\square 1}\left(\rho^*\otimes\phi\right)=\left[\frac{\varphi_n\,\Upsilon\,\phi}{\delta_n}\right].$$

Therefore,

$$\left(\widehat{\chi_g^s}\right)^{\square 1}(\rho^* \otimes \phi) = \left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \phi.$$

To prove the second identity, we apply Theorem 10 to get

$$\widehat{\chi_g^s}\left(\left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \psi\right) = \widehat{\chi_g^s}\left(\left[\frac{\varphi_n \curlyvee \psi}{\delta_n}\right]\right) = \rho^* \otimes \psi.$$

This finishes the proof of the theorem.

Conclusion : This paper provides some integral products which were implemented to extend a new type of Stieltjes transforms enfolding Fox's H-functions as kernels to generalized functions. The generalized Sitieltjes transform was formed to satisfy the desired properties of the classical transform. It may be concluded here that the employed Stieltjes transform method is a very efficient technique in extending integral transforms to generalized functions and could lead to a promising approach for many integrals of special functions kernels .

AMS Classification : Primary 54C40, 14E20; Secondary 46E25, 20C20.

Competing Interests : The authors declare that they have no competing interests.

Authors Contributions : Both of the authors contributed equally to the manuscript and read and approved the final draft of the paper.

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Decision making based on interval-valued intuitionistic fuzzy soft sets and its algorithm *

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December 14, 2015

Abstract: This paper investigates an approach to interval-valued intuitionistic fuzzy soft sets in decision making by means of grey relational analysis and D-S theory of evidence. An algorithm based on this approach in decision making is presented.

Keywords: Interval-valued intuitionistic fuzzy soft set; Decision making; Grey relational analysis; D-S theory of evidence.

1 Introduction

In 1999, Molodtsov [18] initiated soft sets as a mathematical tool for dealing with vagueness and uncertainties. Compared with some traditional tools for dealing with uncertainties, such as probability theory, fuzzy set theory [32], rough set theory [23], soft set theory has the advantage of freeing from the inadequacy of the parametrization tools of those theories.

Recently, many efforts have been devoted to further generalizations and extensions of Molodtsov's soft sets. Maji et al. [19, 20] defined fuzzy soft sets and intuitionistic fuzzy soft sets by combining soft sets with fuzzy sets and and intuitionistic fuzzy sets, Yang et al. [31] defined the interval-valued fuzzy soft sets. Jiang et al. [7] proposed a more general soft set model called interval-valued intuitionistic fuzzy soft set, which is a substantial and important combination of the soft set and the interval-valued intuitionistic fuzzy set. The intuitionistic fuzzy soft set theory makes descriptions of the objective world more realistic, practical and accurate in some cases, making it very promising.

With the rapid development of soft set theory, there has been some progress on the practical applications, especially the use of soft sets in decision making. Roy et al. [25] discussed score value as the evaluation basis to find an optimal choice object in fuzzy soft sets. But Kong et al. [10] argued that the Roy's method was incorrect by using a counter example to discuss two evaluation bases

*This work is supported by the National Social Science Foundation of China (No. 12BJL087).

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of choice value and score value, and they proposed a revised algorithm. Later Feng et al. [5] applied level soft sets to discuss fuzzy soft sets based decision making and subsequently extended the approach to interval-valued fuzzy soft set based decision making [6]. Jiang et al. [8] generalize the approach to solve intuitionistic fuzzy soft sets. Based on Feng' works, Basu et al. [2] further investigated the previous methods to fuzzy soft sets in decision making and introduced the mean potentiality approach, which was showed more efficient and more accurate than the previous methods. Zhang [36] proposed a rough set approach to intuitionistic fuzzy soft set based decision making. Li et al. [15] investigated decision making based on intuitionistic fuzzy soft sets. Li et al. [16] considered fuzzy soft set based decision making for applications in medical diagnosis. Ma et al. [22] presented the algorithm to solve decision making problems based on interval-valued intuitionistic fuzzy soft sets. Qin et al. [24] present an adjustable approach to interval-valued intuitionistic fuzzy soft set based decision making by using reduct intuitionistic fuzzy soft sets and level soft sets of intuitionistic fuzzy soft sets.

All of the above methods for soft sets in decision making are mainly based on the level soft set to obtain useful information such as choice values and score values. However, the existing methods have their limitations. For example, it is very difficult for decision maker to select a suitable level soft set to reduce subjectivity and uncertainty (see [36]). Moreover, there has been rather little work completed for interval-valued intuitionistic fuzzy soft set based decision making. Then it is necessary to pay attention to this issue.

Grey relational analysis, initiated by Deng [4], is an important method to reflect uncertainty in grey system theory, which is utilized for generalizing estimates under small samples and uncertain conditions. It has been successfully applied in solving decision-making problems [9, 27, 28, 35]. D-S theory of evidence, proposed by Dempster [3] and Shafer [26], is a powerful method for combining accumulative evidence of changing prior opinions in the light of new evidences [26]. Compared to probability theory, this theory captures more information to support decision making by identifying the uncertain and unknown evidence. It provides a mechanism to derive solutions from various vague evidences without knowing much prior information. Therefore, combining both theories enables the decision makers to take advantage of both methods' merits and make evaluation experts to deal with uncertainty and risk confidently. The hybrid model is effective and practical under uncertainty [27, 29]. It is very meaningful to extend the hybrid model to interval-valued fuzzy soft set based decision making Thus, this not only allows us to avoid selecting a suitable level soft set, but also helps reducing humanistic and subjective in nature to raise the choices decision level.

The purpose of this paper is to investigate decision making based on the interval-valued intuitionistic fuzzy soft sets.

2 Preliminaries

Throughout this paper, U denotes an initial universe, E denotes the set of all possible parameters, 2^U denotes the family of all subsets of U. We only consider the case where U and E are both nonempty finite sets. $\mathrm{Int}[0,1]$ denotes a set of all closed subintervals of $[0,1]$.

2.1 Interval-valued intuitionistic fuzzy soft sets

Definition 2.1 ([1]). *An interval-valued intuitionistic fuzzy set $\widetilde{X}$ over U is an object having the form $\widetilde{X} = \{(x, \mu_{\widetilde{X}}(x), \nu_{\widetilde{X}}(x)) \mid x \in U\}$ $(e \in A)$, where $\mu_{\widetilde{X}} : U \to Int[0,1]$ and $\nu_{\widetilde{X}} : U \to Int\ [0,1]$ satisfy $0 \leqslant sup\ \mu_{\widetilde{X}}(x) + sup\ \nu_{\widetilde{X}}(x) \leqslant 1$ for all $x \in U$.*

$\mu_{\widetilde{X}}(x)$ and $\nu_{\widetilde{X}}(x)$ are called the membership degree and non-membership degree of the element $x \in U$ to $\widetilde{X}$.

The set of all interval-valued intuitionistic fuzzy subsets of U is denoted by $IVIF(U)$.

Definition 2.2 ([18]). *Let $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to 2^U$.*

Definition 2.3 ([7]). *Let $A \subseteq E$. A pair (F, A) is called an interval-valued intuitionistic fuzzy soft set over U, where F is a mapping given by $F : A \to IVIF(U)$.*

In other words, an interval-valued intuitionistic fuzzy soft set over U is a parameterized family of interval-valued intuitionistic fuzzy subsets of U. For any $e \in A$, $F(e)$ is referred as the set of e-approximate elements of (F, A) and can be written as:

$$F(e) = \{(x, \mu_{F(e)}(x), \nu_{F(e)}(x)) \mid x \in U\}\ (e \in A),$$

where $\mu_{F(e)} : U \to Int[0,1]$ and $\nu_{F(e)} : U \to Int[0,1]$ satisfy $0 \leqslant sup\ \mu_{F(e)}(x) + sup\ \nu_{F(e)}(x) \leqslant 1$. $\mu_{F(e)}(x)$ and $\nu_{F(e)}(x)$ are called the membership degree and non-membership degree that x holds e, respectively. $\pi_{F(e)}(x) = 1 - \mu_{F(e)}(x) - \nu_{F(e)}(x)$ is called the hesitating degree of x holds e.

The set of all interval-valued intuitionistic fuzzy soft subsets of U is denoted by $IVIFS(U)$.

Example 2.4. *Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a set of houses and let $A = \{e_1, e_2, e_3, e_4,\} \subseteq E$ be a set of status of houses where e_j $(j = 1, 2, 3, 4)$ stand for the parameters "beautiful", "modern", "cheap" and "in the green surroundings", respectively.*

Now, we consider an interval-valued intuitionistic fuzzy soft set (F, A) over U, which describes "the attractiveness of the houses" to this decision maker and its tabular representation is shown in Table 1.

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Obviously, we can see that the precise evaluation for each object on each parameter is unknown while the lower and upper limits of such an evaluation are given. For example, we cannot present the precise membership degree and non-membership degree of how beautiful house h_1 is, however, house h_1 is at least beautiful on the membership degree of 0.6 *and it is at most beautiful on the membership degree of* 0.8*; house h_1 is not at least beautiful on the non-membership degree of 0.1 and it is not at most beautiful on the non-membership degree of 0.2.*

Table 1: Tabular representation of the interval-valued intuitionistic soft set (F, A)

	e_1	e_2	e_3	e_4
h_1	[0.6, 0.8],[0.1,0.2]	[0.7, 0.8],[0.15,0.2]	[0.75, 0.85],[0.1,0.15]	[0.8, 0.9],[0.01,0.1]
h_2	[0.8, 0.9],[0.05,0.1]	[0.6, 0.7],[0.15,0.21]	[0.5, 0.6],[0.2,0.35]	[0.65, 0.75],[0.2,0.25]
h_3	[0.6, 0.7],[0.2,0.25]	[0.5, 0.7],[0.2,0.3]	[0.6, 0.8],[0.1,0.18]	[0.66, 0.77],[0.2,0.22]
h_4	[0.65, 0.78],[0.15,0.21]	[0.7, 0.75],[0.15,0.25]	[0.68, 0.75],[0.1,0.2]	[0.69, 0.78],[0.1,0.2]

2.2 Basic concepts of D-S theory of evidence

D-S theory of evidence is a new important reasoning method under uncertainty. It has an advantage to deal with subjective judgments and to synthesize the uncertainty knowledge [34].

A frame of discernment, denoted Θ, is a finite nonempty set of mutually exclusive and exhaustive hypotheses, denoted $\{A_1, A_2, \cdots, A_n\}$ and $A_i \cap A_j = \emptyset$. 2^Θ denotes the set of all subsets of Θ.

Definition 2.5 ([26])**.** *Let Θ be a frame of discernment. A basic probability assignment function (or Mass function) on Θ is defined a mapping $m : 2^\Theta \to [0,1]$, m satisfies*

$$m(\emptyset) = 0\ ,\ \sum_{A \subseteq \Theta} m(A) = 1 \ for\ A \in 2^\Theta.$$

For any $A \subseteq \Theta$, A is called as focal elements if $m(A) > 0$, $m(A)$ represents the belief measuser that one is willing to commit exactly to A, given a certain piece of evidence.

Definition 2.6 ([26])**.** *Let Θ be the frame of discernment and $m : 2^\Theta \to [0,1]$ be a Mass function. Then a belief function on Θ is defined a mapping Bel : $2^\Theta \to [0,1]$, Bel satisfies*

$$Bel(\emptyset) = 0,\ Bel(\Theta) = 1,\ Bel(A) = \sum_{B \subseteq A} m(B)\ for\ A \subseteq \Theta.$$

$Bel(A)$ can be interpreted as a global belief measure that the hypothesis A is true, and represents the imprecision and uncertainty in the decision-making process. In the case of single hypothesis, $Bel(A) = m(A)$.

Definition 2.7 ([26]). *Let Θ be the frame of discernment. Suppose there are two Mass functions are m_1 and m_2 over Θ, induced by two independent items of evidences $A_1, A_2, \cdots, A_s$ and $B_1, B_2, \cdots, B_t$, respectively. D-S rule of evidence combination is defined and denoted as follows:*

$$m(A) = m_1 \oplus m_2(A) = \begin{cases} \frac{1}{1-K} \sum\limits_{A_i \cap B_j = A} m_1(A_i)m_2(B_j), \ \forall\ A \subseteq \Theta, A \neq \emptyset, \\ 0, \ A = \emptyset, \end{cases}$$

where $K = \sum\limits_{A_i \cap B_j = \emptyset} m_1(A_i)m_2(B_j) < 1.$

K is called the conflict probability and reflects the extent of the conflict between the evidences. Coefficient $\frac{1}{1-K}$ is called normalized factor, its role is to avoid the probability of assigning non-0 to empty set $\emptyset$ in the combination.

D-S rule of evidence combination can be generalized to multiple Mass functions, the belief measure resulting from the combination of multiply evidences A_i is as follows:

$$m_1 \oplus m_2 \cdots \oplus m_n(A) = \frac{1}{1-K} \sum_{\bigcap_{i=1}^n A_i = A, A_i \subset \Theta} m_1(A_1)m_2(A_2)\cdots m_n(A_n),$$

where $K = \sum\limits_{\bigcap_{i=1}^n A_i = \emptyset, A_i \subset \Theta} m_1(A_1)m_2(A_2)\cdots m_n(A_n) < 1.$

D-S rule of evidence combination can increase belief measure of hypotheses and reduce the uncertain degree to improve reliability.

Example 2.8. *Let $\Theta = \{A_1, A_2\}$ be the frame of discernment. Suppose there are two Mass functions m_1 and m_2 over Θ, induced by the independent items of evidences A_1, A_2, given by*

$m_1(A_1) = 0.3, \quad m_1(A_2) = 0.4, \quad m_1(\Theta) = 0.3,$

$m_2(A_1) = 0.4, \quad m_2(A_2) = 0.3, \quad m_2(\Theta) = 0.3.$

Combining the two evidences by D-S rule of evidence combination leads to:

$m(A_1) = m_1 \oplus m_2(A_1) = \frac{m_1(A_1)m_2(A_1)+m_1(A_1)m_2(\Theta)+m_1(\Theta)m_2(A_1)}{1-K} = 0.44,$

$m(A_2) = m_1 \oplus m_2(A_2) = \frac{m_1(A_2)m_2(A_2)+m_1(A_2)m_2(\Theta)+m_1(\Theta)m_2(A_2)}{1-K} = 0.44,$

$m(\Theta) = m_1 \oplus m_2(\Theta) = \frac{m_1(\Theta)m_2(\Theta)}{1-K} = 0.12,$

where $K = m_1(A_1)m_2(A_2) + m_1(A_2)m_2(A_1) = 0.25.$

3 An approach to interval-valued intuitionistic fuzzy soft sets in decision making

Recently, research on soft sets based decision making has attracted more and more attention. The works of Roy et al. [10, 25, 5, 2, 11] are fundamental and significant. Later other authors like Qin et al. further studied and proposed an adjustable approach to interval-valued intuitionistic fuzzy soft set based decision making using the level soft sets and reductions . Generally, there does not exist

any unique or uniform criterion for the evaluation of decision alternatives under uncertain condition. However, it is very difficult for decision makers to select suitable level soft sets and discuss reduct intuitionistic fuzzy soft sets.

Now we investigate interval-valued intuitionistic fuzzy soft sets based decision making by means of grey relational analysis and D-S theory of evidence. It is divided three phases: First, grey relational analysis is applied to calculate the grey mean relational degree and the uncertain degree of each parameter is obtained. Second, the corresponding Mass function with respect to each parameter is constructed by the uncertain degree of each parameter. Third, we apply D-S rule of evidence combination to aggregate individual alternatives into a collective alternative, by which the candidate alternatives are ranked and the best alternative is obtained.

In the following, we consider the decision making problem with m mutually exclusive alternatives x_i and n evaluation parameters (or indexes) e_j. d_{ij} denotes the degree that the alternative x_i satisfies the parameter e_j. Put

$$\Theta = \{x_1, x_2, \cdots, x_m\} \text{ and } A = \{e_1, e_2, \cdots, e_n\}.$$

Define $F : A \to IVIF(\Theta)$ by $F(e_j) = \{(x_i, \mu_{F(e_j)}(x_i), \nu_{F(e_j)}(x_i)) \mid x_i \in \Theta\}$ $(e_j \in A)$ where $\mu_{F(e_j)} : U \to Int[0,1]$ and $\nu_{F(e_j)} : U \to Int[0,1]$ satisfy $0 \leqslant sup\ \mu_{F(e_j)}(x_i) + sup\ \nu_{F(e_j)}(x_i) \leqslant 1$. Then (F, A) is an interval-valued intuitionistic fuzzy soft set over Θ. Denote $\mu_{F(e_j)}(x_i) = [\mu_{ij}^-, \mu_{ij}^+]$, $\nu_{F(e_j)}(x_i) = [\nu_{ij}^-, \nu_{ij}^+]$, $a_{ij} = (\mu_{F(e_j)}(x_i), \nu_{F(e_j)}(x_i))$. $D = (a_{ij})_{m\times n}$ is called an interval-valued intuitionistic fuzzy soft decision matrix induced by (F, A). Here, we see the set of parameters as a item of evidences information.

The key to solve decision problems by using D-S theory of evidence is how to obtain the uncertain degree of evidences (or parameters).

First, inspired by Xu [12], we define the score function of as follows.

Definition 3.1. *Suppose that (F, A) is an interval-valued intuitionistic fuzzy soft over Θ. Suppose that $D = (a_{ij})_{m\times n}$ is an interval-valued intuitionistic fuzzy soft decision matrix induced by (F, A). Denote $\mu_{F(e_j)}(x_i) = [\mu_{ij}^-, \mu_{ij}^+]$, $\nu_{F(e_j)}(x_i) = [\nu_{ij}^-, \nu_{ij}^+]$, $a_{ij} = (\mu_{F(e_j)}(x_i), \nu_{F(e_j)}(x_i))$. Then score function of d_{ij} is defined and denoted as*

$$s(a_{ij}) = (\mu_{ij}^- + \mu_{ij}^+ - \nu_{ij}^- - \nu_{ij}^+)/2 + \alpha(\mu_{ij}^+ + \nu_{ij}^+ - \mu_{ij}^- - \nu_{ij}^-)/2.$$

By Definition 4.1, we can convert d_{ij} into real numbers. $s(a_{ij})$ presents the global degree that the alternative x_i holds the parameter e_j. Obviously, $0 \leqslant s(a_{ij}) \leqslant 1$. α is called a risk factor. For $\alpha = 0, > 0, < 0$, they imply the attitude of decision makers for risk is neutral, positive, oppose, respectively. Decision makers can select a α value according to their risk preference. In this paper, we pick $\alpha = 0$.

To obtain Mass functions of each alternative with respect to each parameter, we consider score function values may be negative, so we should normalize the

score function values by the following formula:

$$d_{ij} = \frac{s(a_{ij}) - min_{1\leqslant i\leqslant m}s(a_{ij})}{max_{1\leqslant i\leqslant m}s(a_{ij}) - min_{1\leqslant i\leqslant m}S(a_{ij})}, \quad 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n.$$

Hence, we can get normalized matrix of score function values $D = (d_{ij})_{m\times n}$.

Next, inspired by the paper [12], we define the grey mean relational degree and the uncertain degree of the parameter as follows.

Definition 3.2. *Let $\Theta = \{x_1, x_2, \cdots, x_m\}$, $A = \{e_1, e_2, \cdots, e_n\}$ and let (F, A) be an intuitionistic fuzzy soft set on Θ. Suppose that $D = (d_{ij})_{m\times n}$ is normalized matrix of score function values. For any i, j, denote*

$$\widetilde{d_i} = \frac{1}{n}\sum_{j=1}^{n} d_{ij}, \quad \triangle d_{ij} = |d_{ij} - \widetilde{d_i}|,$$

$$r_{ij} = \frac{min_{1\leqslant j\leqslant n}min_{1\leqslant i\leqslant m} \ \triangle d_{ij} + \rho \ max_{1\leqslant j\leqslant n}max_{1\leqslant i\leqslant m} \ \ \triangle d_{ij}}{\triangle d_{ij} + \rho \ max_{1\leqslant j\leqslant n}max_{1\leqslant i\leqslant m} \ \triangle d_{ij}}, \ \rho \in (0, 1),$$

$$DOI(e_j) = \frac{1}{m}(\sum_{i=1}^{m}(r_{ij})^q)^{\frac{1}{q}} \ (j = 1, 2, \cdots, n).$$

r_{ij} is called the grey mean relational degree between d_{ij} and $\widetilde{d_i}$. $DOI(e_j)$ is called q order uncertain degree of the parameter e_j.

ρ aims to expand or compress the range of the grey relational coefficient. Decision makers can select q, ρ values according to different circumstance. To obtain strong distinguishing effectiveness, we pick $q = 2, \ \rho = 0.5$ in this paper. We call $DOI(e_j)$ the uncertain degree of e_j for short.

It is worthy to notice that the method to obtain the uncertain degree varies from different situation in Definition 4.2. General speaking, since a index (or parameter) is specially more matching the mean of the index set than other indexes, it contains more satisfying information for decision making and the uncertain degree of the index information is lower. Then, in this paper we just consider grey mean relational degree between d_{ij} and $\widetilde{d_i}$.

Definition 3.3 ([36])**.** *Let $X = (x_1, x_2, \cdots, x_m)$ be a finite difference information sequence, where there exists $x_{i_k} \neq 0$ for $k = 1, 2, \cdots, m$ and $1 \leqslant i_k \leqslant m$. Then the information structure image sequence $Y = (y_1, y_2, \cdots, y_m)$ is given by $y_i = \frac{x_i}{\sum\limits_{i=1}^{m} x_i}$.*

In the normalized matrix of score function values $D = (d_{ij})_{m\times n}$, the information structure image sequence with respect to a parameter e_j is denoted by $d_j = \{\widetilde{d_{1j}}, \widetilde{d_{2j}}, \widetilde{d_{3j}}, \cdots, \widetilde{d_{mj}}\}$, where $\widetilde{d_{ij}} = \frac{d_{ij}}{\sum\limits_{i=1}^{m} d_{ij}}$. Then we obtain an information structure image matric $\widetilde{D} = (\widetilde{d_{ij}})_{m\times n}$ induced by d_j $(j = 1, 2, \cdots, n)$.

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D-S theory of evidence is a powerful method for combining accumulative evidence of changing prior opinions in the light of new evidences [26]. The primary procedure of combining the known evidences or information with other evidences is to construct suitable Mass functions of evidences.

Now, by the uncertain degree of each parameter, we can obtain Mass function of each alternative with respect to each parameter.

Theorem 3.4. *Let* $\Theta = \{x_1, x_2, \cdots, x_m\}$, $A = \{e_1, e_2, \cdots, e_n\}$ *and let* (F, A) *be an intuitionistic fuzzy soft set on* Θ. *Suppose that* $D = (d_{ij})_{m\times n}$ *is the normalized matrix of score function values and* $DOI(e_j)$ *is the uncertain degree of* e_j. *Denote* $\widetilde{d_{ij}} = \frac{d_{ij}}{\sum\limits_{i=1}^{m} d_{ij}}$. *For any* i, j, *we define functions* $m_{e_j}(j = 1, 2, \cdots, n)$ *with respect to the parameter* e_j, *it satisfies:*

$$m_{e_j}(x_i) = \widetilde{d_{ij}}\ (1 - DOI(e_j)), \quad m_{e_j}(\Theta) = 1 - \sum_{i=1}^{m} m_j(i).$$

Then $m_{e_j}(j = 1, 2, \cdots, n)$ *are Mass functions.*

In a normalized matrix of score function values $D = (d_{ij})_{m\times n}$, denote $m_{e_j}(x_i)$, $m_{e_j}(\Theta)$ by $m_j(i)$ and $m_j(m+1)$, respectively. $m_j(i)$ implies the belief measure that holds the alternative x_i with the parameter e_j and $m_j(m+1)$ implies the belief measure of the whole uncertainty with parameter e_j.

Next, using D-S rule of evidence combination to compose m_j $(j = 1, 2, \cdots, n)$, we get the belief measure of each alternative with all the parameters, by which the candidate alternatives are ranked and thus the best alternative is obtained.

4 Algorithm

4.1 Algorithm

Based on the above analysis, the detailed step-wise procedure as an algorithm is given as follows:

Input: An interval-value intuitionistic fuzzy soft set (F, A).

Output: The optimal decision-making results.

Step 1. Input an interval-value intuitionistic fuzzy soft set (F, A) and construct an interval-value intuitionistic fuzzy soft decision matrix induced by (F, A).

Step 2. Compute the normalized matrix of score function values $(D = (d_{ij})_{m\times n})$.

Step 3. Compute the mean of all the score function values $(\widetilde{d_i})$ with respect to each alternative.

Step 4. Compute the difference information between d_{ij} and $\widetilde{d_i}$.

Step 5. Compute the gray mean relational degree between d_{ij} and $\widetilde{d_i}$.

Step 6. Compute the uncertain degree $DOI(e_j)$ of each parameter e_j.

Step 7. Compute the information structure image sequence $\widetilde{d_{ij}}$ with respect to each parameter e_j by Definition 3.3.

Step 8. Compute Mass function values of the alternative x_i and Θ with respect to the parameter e_j by Theorem 3.4.

Step 9. Compute belief measure of each alternative x_i by combining these Mass functions $m_{e_j}(j = 1, 2, \cdots, n)$ respectively by Definition 2.8.

Step 10. The optimal decision is to select x_k if $c_k = max_i \{Bel(x_i)\}$. k has more than one value then any one of x_k may be optimal choices .

4.2 An illustrative example

Suppose that a fund manager in a wealth management wants to invest a company. Suppose that the set of four potential investment companies $U = \{x_1, x_2, x_3, x_4\}$ which are characterized by a set of parameters $A = \{e_1, e_2, e_3, e_4\}$. For $i = 1, 2, 3, 4$, the parameters e_i stand for "risk", "growth ,"socio-political issues" ,and "environmental impacts", respectively. The fund manager provide his/her assessment of each investment company on each parameter as an interval-valued intuitionistic fuzzy soft set (F, A). Its tabular representation is shown in Table 2.

Table 2: Tabular representation of the interval-valued intuitionistic soft set (F, A)

	e_1	e_2	e_3	e_4
x_1	[0.4, 0.5],[0.3,0.4]	[0.4, 0.6],[0.2,0.4]	[0.1, 0.3],[0.5,0.6]	[0.5, 0.7],[0.2,0.3]
x_2	[0.4, 0.5],[0.4,0.5]	[0.5, 0.8],[0.1,0.2]	[0.3, 0.6],[0.3,0.4]	[0.6, 0.7],[0.1,0.3]
x_3	[0.3, 0.5],[0.4,0.5]	[0.1, 0.3],[0.2,0.4]	[0.7, 0.8],[0.1,0.2]	[0.5, 0.7],[0.1,0.2]
x_4	[0.2, 0.4],[0.4,0.5]	[0.6, 0.7],[0.2,0.3]	[0.5, 0.6],[0.2,0.3]	[0.7, 0.8],[0.1,0.2]

Now, we suppose that the four mutually exclusive and exhaustive investment companies consist a frame of discernment, denoted $\Theta = \{x_1, x_2, x_3, x_4\}$. And we consider the set of parameters $A = \{e_1, e_2, e_3, e_4\}$ as a set of evidences.

Step 1. Construct an interval-valued intuitionistic fuzzy soft decision matrix induced by (F, A) as follows:

$$\begin{pmatrix} ([0.4, 0.5], [0.3, 0.4]) & ([0.4, 0.6], [0.2, 0.4]) & ([0.1, 0.3], [0.5, 0.6]) & ([0.5, 0.7], [0.2, 0.3]) \\ ([0.4, 0.5], [0.4, 0.5]) & ([0.5, 0.8], [0.1, 0.2]) & ([0.3, 0.6], [0.3, 0.4]) & ([0.6, 0.7], [0.1, 0.3]) \\ ([0.3, 0.5], [0.4, 0.5]) & ([0.1, 0.3], [0.2, 0.4]) & ([0.7, 0.8], [0.1, 0.2]) & ([0.5, 0.7], [0.1, 0.2]) \\ ([0.2, 0.4], [0.4, 0.5]) & ([0.6, 0.7], [0.2, 0.3]) & ([0.5, 0.6], [0.2, 0.3]) & ([0.7, 0.8], [0.1, 0.2]) \end{pmatrix}$$

Step 2. Compute the normalized matrix of score function values as follows:

$$D = (d_{ij})_{4\times4} = \begin{pmatrix} 1.0000 & 0.5000 & 0 & 0 \\ 0.6000 & 1.0000 & 0.4737 & 0.4000 \\ 0.4000 & 0 & 1.0000 & 0.4000 \\ 0 & 0.8333 & 0.6842 & 1.0000 \end{pmatrix}$$

Step 3. Compute the mean of all parameters with respect to each investment company x_i as follows:

$$\widetilde{d_1} = 0.3750,\ \widetilde{d_2} = 0.6184,\ \widetilde{d_3} = 0.4500, \widetilde{d_4} = 0.6294$$

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Step 4. Compute the difference information between d_{ij} and $\widetilde{d_i}$, and construct the difference matrix as follows:

$$\triangle D = \begin{pmatrix} 0.6250 & 0.1250 & 0.3750 & 0.3750 \\ 0.0184 & 0.3816 & 0.1447 & 0.2184 \\ 0.0500 & 0.4500 & 0.5500 & 0.0500 \\ 0.6294 & 0.2039 & 0.0548 & 0.3706 \end{pmatrix}$$

Step 5. Compute the gray mean relational degree between d_{ij} and $\widetilde{d_i}$ based on $\triangle D$ as follows:

$$(r_{ij})_{4\times 4} = \begin{pmatrix} 0.3545 & 0.7576 & 0.4830 & 0.4830 \\ 1.0000 & 0.4784 & 0.7251 & 0.6248 \\ 0.9134 & 0.4356 & 0.3852 & 0.9134 \\ 0.3528 & 0.6423 & 0.9015 & 0.4861 \end{pmatrix}$$

Step 6. Compute the uncertain degree of each parameter e_j by Definition 3.2 as follows:

$$DOI(e_1) = 0.3609,\ \ DOI(e_2) = 0.2963,\ \ DOI(e_3) = 0.3279,\ \ DOI(e_4) = 0.3254.$$

Step 7. Compute the information structure image sequence with respect to each parameter and construct the matrix as follows:

$$\widetilde{D} = (\widetilde{d_{ij}})_{4\times 4} = \begin{pmatrix} 0.5000 & 0.2143 & 0 & 0 \\ 0.3000 & 0.4286 & 0.2195 & 0.2222 \\ 0.2000 & 0 & 0.4634 & 0.2222 \\ 0 & 0.3571 & 0.3171 & 0.5556 \end{pmatrix}$$

Step 8. Let $2^{\Theta} = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \Theta\}$. Compute Mass function values of x_i and Θ with respect to the parameter e_j by Theorem 3.4:

$$(m_j(i))_{4\times 4} = \begin{pmatrix} 0.3195 & 0.1508 & 0 & 0 \\ 0.1917 & 0.3016 & 0.1475 & 0.1499 \\ 0.1278 & 0 & 0.3115 & 0.1499 \\ 0 & 0.2513 & 0.2131 & 0.3748 \end{pmatrix}$$

and

$$m_1(5) = 0.3609,\quad m_2(5) = 0.2963,\quad m_3(5) = 0.3279,\quad m_4(5) = 0.3254,$$

$$\frac{1}{4}\sum_{j=1}^{4} m_j(5) = 0.3276.$$

Step 9. We combine these Mass functions and compute each belief measure of each candidate x_i respectively as follows:

$$Bel(\{x_1\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_1\}) = 0.1098,$$
$$Bel(\{x_2\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_2\}) = 0.3298,$$
$$Bel(\{x_3\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_3\}) = 0.1700,$$
$$Bel(\{x_4\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\{x_4\}) = 0.3309,$$
$$Bel(\{x_5\}) = m_1 \oplus m_2 \oplus m_3 \oplus m_4(\Theta) = 0.0595.$$

Then the final rang order is $x_4 \succ x_2 \succ x_3 \succ x_1$.

Step 10. x_4 is the optimal investment company for $max_i \{Bel(x_i)\} = 0.3309$.

From the above results, the belief measure of the uncertainty with respect to the whole candidates Θ is declined from 0.3276 to 0.0595, after applying grey relational analysis to construct the corresponding Mass functions for different evidences and then using the rule of evidence combination to compose these information. This implies the above algorithm is effective and practical under uncertainties. It not only allows us to avoid selecting the suitable level soft set, but also helps reducing humanistic and subjective in nature to raise the choices decision level. Moreover, it broadens the application field of the grey system theory and D-S theory of evidence.

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PRODUCT-TYPE OPERATORS FROM WEIGHTED ZYGMUND SPACES TO BLOCH-ORLICZ SPACES

YONG YANG AND ZHI-JIE JIANG

ABSTRACT. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The boundedness and compactness of the product-type operators $D^nM_uC_\varphi$, $D^nC_\varphi M_u$, $M_uD^nC_\varphi$, $C_\varphi D^nM_u$, $M_uC_\varphi D^n$ and $C_\varphi M_uD^n$ from weighted Zygmund spaces to Bloch-Orlicz spaces are characterized by constructing some test functions in weighted Zygmund spaces.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. For $\alpha > 0$, the weighted Zygmund space $\mathcal{Z}^\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$b_{\mathcal{Z}^\alpha}(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha|f''(z)| < \infty.$$

It is a Banach space with the norm

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + b_{\mathcal{Z}^\alpha}(f).$$

If $\alpha = 1$, then it becomes the famous Zygmund space, usually denoted by $\mathcal{Z}$. For some results of weighted Zygmund spaces and some concrete operators on them, see, for example, [9, 22, 24, 43, 56] and the references therein.

Next we introduce the Bloch-Orlicz space which was defined by Ramos Fernández in [32]. Let Ψ be a Young's function, i.e., Ψ is a strictly increasing convex function on $[0, +\infty)$ such that $\Psi(0) = 0$ and $\lim_{t\to+\infty} \Psi(t) = +\infty$. The Bloch-Orlicz space $\mathcal{B}^\Psi$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)\Psi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. The Minkowski's functional

$$\|f\|_\Psi = \inf\left\{k > 0 : S_\Psi\left(\frac{f'}{k}\right) \le 1\right\}$$

defines a seminorm for $\mathcal{B}^\Psi$, where

$$S_\Psi(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)\Psi(|f(z)|).$$

2000 *Mathematics Subject Classification.* Primary 47B38; Secondary 47B33, 47B37.

Key words and phrases. Weighted Zygmund space, Bloch-Orlicz space, Product-type operator, Test function, Boundedness, Compactness.

$\mathcal{B}^\Psi$ becomes a Banach space with the norm $\|f\|_{\mathcal{B}^\Psi} = |f(0)| + \|f\|_\Psi$. Ramos Fernández in [32] proved that it is isometrically equal to a special μ_Ψ-Bloch space, where

$$\mu_\Psi(z) = \frac{1}{\Psi^{-1}(\frac{1}{1-|z|^2})}, \ \ z \in \mathbb{D}.$$

Consequently, a equivalent norm on $\mathcal{B}^\Psi$ is given by $\|f\|_{\mathcal{B}^\Psi} = |f(0)| + b_{\mathcal{B}^\Psi}(f)$, where

$$b_{\mathcal{B}^\Psi}(f) = \sup_{z\in\mathbb{D}} \mu_\Psi(z)|f'(z)|.$$

Clearly, the quantity $b_{\mathcal{B}^\Psi}(f)$ is a seminorm on the space $\mathcal{B}^\Psi$ and a norm on the quotient space $\mathcal{B}^\Psi/\mathbb{P}_0$, where $\mathbb{P}_0$ is the set of all constant functions. The Bloch-Orlicz space generalizes some spaces. For example, if $\Psi(t) = t^p$ with $p > 0$, then $\mathcal{B}^\Psi$ coincides with the weighted Bloch space $\mathcal{B}^\alpha$, where $\alpha = 1/p$; if $\Psi(t) = t\log(1 + t)$, then $\mathcal{B}^\Psi$ coincides with the Log-Bloch space (see [2]).

Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator $W_{\varphi,u}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,u}f(z) = u(z)f(\varphi(z)), \ z \in \mathbb{D}.$$

If $u \equiv 1$, it becomes the composition operator, usually denoted by C_φ. If $\varphi(z) = z$, it becomes the multiplication operator, usually denoted by M_u. Since $W_{\varphi,u} = M_uC_\varphi$, it is a product-type operator. For some studies on weighted composition operators, see, for example, [1, 4, 7, 10, 19, 22, 29, 42, 49, 50] and the references therein.

Let $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The nth differentiation operator D^n on $H(\mathbb{D})$ is defined by

$$D^nf(z) = f^{(n)}(z), \ \ z \in \mathbb{D},$$

where $f^{(0)} = f$. If $n = 1$, it is the well-known differentiation operator D. Zhu in [57] introduced the following, so-called, generalized weighted composition operator:

$$D^n_{\varphi,u}f(z) = u(z)f^{(n)}(\varphi(z)), \ z \in \mathbb{D}.$$

If $n = 0$, it becomes the weighted composition operator. Since $D^n_{\varphi,u} = M_uC_\varphi D^n$, it is also a product-type operator. For generalized weighted composition operators, see, for example, [3, 28, 47, 53, 54, 59, 60] and the references therein. Before the operator $D^n_{\varphi,u}$ some other product-type operators were introduced and studied. For example, the next product-type operators

$$M_uC_\varphi D, \ C_\varphi M_uD, \ M_uDC_\varphi, \ C_\varphi DM_u, \ DC_\varphi M_u, \ DM_uC_\varphi$$

were studied by Sharma in [34]. They were also studied on weighted Bergman spaces by Stević et al. in [51] and [52]. However, a normally systematic study of product-type operators started by Stević et al. since the publication of papers [21] and [25]. Before that there were a few papers in the topic, e.g., [8]. The publication of paper [21] first attracted some attention in product-type operators DC_φ and $C_\varphi D$ (see, e.g., [23, 30, 39, 41] and the references therein). The publication of paper [25] attracted some attention in product-type operators involving integral-type ones (see, e.g., [16, 26, 37, 43, 48] and the references therein). Recently there is a great interest in various product-type operators between two given spaces of holomorphic functions (see, e.g., [11, 12, 17, 31, 33, 36, 38, 40, 45, 57] and the references therein).

Before this paper some product-type operators from Zygmund spaces or weighted Zygmund spaces to some other spaces were studied, for example, in [3, 13, 14, 18, 27]. In this paper we consider the following product-type operators:

$$D^nM_uC_\varphi, \ D^nC_\varphi M_u, \ M_uD^nC_\varphi, \ C_\varphi D^nM_u, \ M_uC_\varphi D^n, \ C_\varphi M_uD^n. \tag{1}$$

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The boundedness and compactness of operators in (1) from Zygmund spaces to Bloch-Orlicz spaces were characterized in [14]. As a continuation and completeness of our work, we consider the same problems for operators in (1) from weighted Zygmund spaces with $\alpha \neq 1$ to Bloch-Orlicz spaces. Because these operators are more complicated than those above mentioned, we need seek some other test functions in weighted Zygmund spaces to achieve our objective.

Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is bounded if there exists a positive constant K such that $\|Lf\|_Y \leq K\|f\|_X$ for all $f \in X$. The operator $L : X \to Y$ is compact if it maps bounded sets into relatively compact sets. The norm of the operator $L : X \to Y$ is defined by

$$\|L\|_{X\to Y} = \sup_{\|f\|_X \leq 1} \|Lf\|_Y.$$

In this paper, the letter C denotes a positive constant which may differ from one occurrence to the other. The notation $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$. When $a \lesssim b$ and $b \lesssim a$, we write $a \asymp b$.

2. Preliminaries and test functions

We first state the following result which was essentially proved in [35] and [46].

Lemma 2.1. *For $\alpha > 0$ and $f \in \mathcal{Z}^\alpha$. Then*

(a) For $0 < \alpha < 1$, $|f(z)| \leq \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}}$ and $|f'(z)| \leq \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}}$.
(b) For $\alpha = 1$, $|f(z)| \leq \|f\|_{\mathcal{Z}}$ and $|f'(z)| \leq \|f\|_{\mathcal{Z}} \log \frac{e}{1-|z|^2}$.
(c) For $1 < \alpha < 2$, $|f(z)| \leq \frac{1}{(\alpha-1)(2-\alpha)}\|f\|_{\mathcal{Z}^\alpha}$ and $|f'(z)| \leq \frac{2}{\alpha-1}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-1}}$.
(d) For $\alpha = 2$, $|f(z)| \leq 2\|f\|_{\mathcal{Z}^2} \log \frac{e}{1-|z|^2}$ and $|f'(z)| \leq \frac{e}{1-|z|^2}\|f\|_{\mathcal{Z}^2}$.
(e) For $\alpha > 2$, $|f(z)| \leq \frac{1}{(\alpha-1)(\alpha-2)}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-2}}$ and $|f'(z)| \leq \frac{2}{\alpha-1}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha-1}}$.

The following result directly follows from the corresponding result for the Bloch type spaces when a function f is replaced by f' (see, e.g., [55]).

Lemma 2.2. *For each $k \in \mathbb{N}$ and $k \geq 2$, there exists a positive constant C_k independent of $f \in \mathcal{Z}^\alpha$ and $z \in \mathbb{D}$ such that*

$$|f^{(k)}(z)| \leq \frac{C_k\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha+k-2}}.$$

Let $w \in \mathbb{D}$ and $i \in \mathbb{N}_0$. It is easily shown that the next function is in the space $\mathcal{Z}^\alpha$

$$r_{w,i}(z) = \frac{(1-|w|^2)^{2+i}}{(1-\overline{w}z)^{\alpha+i}},\ z \in \mathbb{D}.$$

The following result provides the needed test functions for the cases $0 < \alpha < 1$, $1 < \alpha < 2$, $\alpha = 2$ and $\alpha > 2$.

Lemma 2.3. *(a) If $0 < \alpha < 1$, then for each fixed $k \in \{2, 3, \ldots, n+1\}$, there exist constants $a_{0,k}$, $a_{1,k}$, $\ldots$, $a_{n+1,k}$ such that the function*

$$f_{w,k}(z) = \sum_{i=0}^{n+1} a_{i,k} r_{w,i}(z)$$

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satisfies

$$f_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\alpha+k-2}} \quad \text{and} \quad f_{w,k}^{(j)}(w) = 0 \tag{2}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$.

(b) If $1 < \alpha \leq 2$, *then for each fixed* $k \in \{1, 2, \dots, n+1\}$, *there exist constants* $b_{0,k}$, $b_{1,k}$, $\dots$, $b_{n+1,k}$ *such that the function*

$$g_{w,k}(z) = \sum_{i=0}^{n+1} b_{i,k} r_{w,i}(z)$$

satisfies

$$g_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\alpha+k-2}} \quad \text{and} \quad g_{w,k}^{(j)}(w) = 0 \tag{3}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$.

(c) If $\alpha > 2$, *then for each fixed* $k \in \{0, 1, \dots, n+1\}$, *there exist constants* $c_{0,k}$, $c_{1,k}$, $\dots$, $c_{n+1,k}$ *such that the function*

$$h_{w,k}(z) = \sum_{i=0}^{n+1} c_{i,k} r_{w,i}(z)$$

satisfies

$$h_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\alpha+k-2}} \quad \text{and} \quad h_{w,k}^{(j)}(w) = 0 \tag{4}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$.

Proof. (a). From a calculation, it follows that (2) is equivalent to the following system

$$\begin{cases} \sum\limits_{i=0}^{n+1} (\alpha+i) a_{i,k} = 0 \\ \sum\limits_{i=0}^{n+1} (\alpha+i)(\alpha+i+1) a_{i,k} = 0 \\ \quad \cdots\cdots\cdots \\ \sum\limits_{i=0}^{n+1} \prod\limits_{j=0}^{k-1} (\alpha+i+j) a_{i,k} = 1 \\ \quad \cdots\cdots\cdots \\ \sum\limits_{i=0}^{n+1} \prod\limits_{j=0}^{n} (\alpha+i+j) a_{i,k} = 0. \end{cases} \tag{5}$$

Hence, we only need to prove that there exist constants $a_{0,k}$, $a_{1,k}$, $\dots$, $a_{n+1,k}$ such that the system (5) holds. By Lemma 3 in [47], the determinant of the system (5) equals to $\prod_{j=1}^{n+1} j!$, which is different from zero. So there exist constants $a_{0,k}$, $a_{1,k}$, $\dots$, $a_{n+1,k}$ such that the system (5) holds. Results (b) and (c) can be proved similarly, so we omit. ☐

Let $w \in \mathbb{D}$ and

$$q_w(z) = \Big(1 + \log^2 \frac{e}{1-\overline{w}z}\Big) \log^{-1} \frac{e}{1-|w|^2}.$$

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Lemma 2.4. *For the function q_w, it follows that*

$$q_w^{(k)}(w) = c_k \frac{\overline{w}^k}{(1-|w|^2)^k} + d_k \frac{\overline{w}^k}{(1-|w|^2)^k} \log^{-1} \frac{e}{1-|w|^2}, \tag{6}$$

where $c_k > 0$ for each $k \geq 1$, $d_1 = 0$ and $d_k > 0$ for each $k \geq 2$.

Proof. By a direct computation, we have

$$q'_w(z) = 2\frac{\overline{w}}{1-\overline{w}z} \log \frac{e}{1-\overline{w}z} \log^{-1} \frac{e}{1-|w|^2}, \tag{7}$$

and

$$q''_w(z) = 2\frac{\overline{w}^2}{(1-\overline{w}z)^2} \log \frac{e}{1-\overline{w}z} \log^{-1} \frac{e}{1-|w|^2} + 2\frac{\overline{w}^2}{(1-\overline{w}z)^2} \log^{-1} \frac{e}{1-|w|^2}. \tag{8}$$

Also, from a direct computation, we see that for $k \geq 2$

$$\begin{aligned} q_w^{(k)}(z) = {} & 2(k-1)! \frac{\overline{w}^k}{(1-\overline{w}z)^k} \log \frac{e}{1-\overline{w}z} \log^{-1} \frac{e}{1-|w|^2} \\ & + \big[k-1+2(k-1)!\big] \frac{\overline{w}^k}{(1-\overline{w}z)^k} \log^{-1} \frac{e}{1-|w|^2}. \end{aligned} \tag{9}$$

Set $c_k = 2(k-1)!$, $d_1 = 0$ and $d_k = k-1+2(k-1)!$ for $k \geq 2$. Then (6) follows from (7)-(9). □

Remark 2.1. *Let X_w be the functions in Lemmas 2.3 and 2.4. Then*

$$\sup_{w\in\mathbb{D}} \|X_w\|_{\mathcal{Z}^\alpha} \lesssim 1, \tag{10}$$

and $X_w \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$. In fact, if X_w are the functions in Lemma 2.3, then this remark follows from the facts that $\sup_{w\in\mathbb{D}} \|r_{w,i}\|_{\mathcal{Z}^\alpha} \lesssim 1$ and $r_{w,i} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$; if X_w is the function in Lemma 2.4, then it follows from [44].

Stević in [47] used Faà di Bruno's formula of the following version

$$(f\circ\varphi)^{(n)}(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z)) B_{n,k}(\varphi'(z), \dots, \varphi^{(n-k+1)}(z)), \tag{11}$$

where $B_{n,k}(x_1, ..., x_{n-k+1})$ is the Bell polynomial. See [15] for the Faà di Bruno's formula. For $n \in \mathbb{N}$ the sum can go from $k = 1$ since $B_{n,0}(\varphi'(z), ..., \varphi^{(n-k+1)}(z)) = 0$, however we will keep the summation since for $n = 0$ the only existing term $B_{0,0}$ is equal to 1. From (11) and the Leibniz formula the next lemma follows.

Lemma 2.5. *Let f, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then*

$$\big(u(z)f(\varphi(z))\big)^{(n+1)} = \sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big).$$

3. Boundedness the product-type operators

We first characterize the boundedness of the operator $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$.

Theorem 3.1. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, C_{n+1}^j the binomial coefficient and $0 < \alpha < 1$. Then the following statements are equivalent.*

(a) The operator $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.
(b) The functions u and φ satisfy the following conditions:

$$I_0 := \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,0}\big(\varphi'(z),\ldots,\varphi^{(j+1)}(z)\big)\Big| < \infty,$$

$$I_1 := \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=1}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,1}\big(\varphi'(z),\ldots,\varphi^{(j)}(z)\big)\Big| < \infty,$$

and

$$I_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\ldots,\varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \infty$$

for each $k \in \{2, 3, \ldots, n+1\}$.

Moreover, if the operator $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^nM_uC_\varphi\big\|_{\mathcal{Z}^\alpha\to\mathcal{B}^\Psi/\mathbb{P}_0} \asymp \sum_{k=0}^{n+1} I_k.$$

Proof. $(a) \Rightarrow (b)$. Let $h_k(z) = z^k \in \mathcal{Z}$, $k = 0, 1, \ldots, n+1$. Then applying the operator $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ to the function h_0, we have

$$I_0 = \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,0}(\varphi'(z),\ldots,\varphi^{(j+1)}(z))\Big| \le C\big\|D^nM_uC_\varphi\big\|. \quad (12)$$

By the fact $\|\varphi\|_\infty \le 1$, the boundedness of $D^nM_uC_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$, the triangle inequality and (12), we have

$$I_1 \le I_0 + C\|D^nM_uC_\varphi\|. \quad (13)$$

Assume now that we have proved the following inequalities

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=l}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,l}(\varphi'(z),\ldots,\varphi^{(j-l+1)}(z))\Big| \le C\|D^nM_uC_\varphi\| \quad (14)$$

for each $l \in \{0, 1, ..., k-1\}$ and a $k \le n+1$. Applying Lemma 2.5 to the function h_k, and noticing that $h_k^{(s)}(z) \equiv 0$ for $s > k$, we get

$$\begin{aligned}\big(D^nM_uC_\varphi h_k\big)'(z) &= \sum_{j=0}^{k} h_k^{(j)}(\varphi(z)) \sum_{i=j}^{n+1} C_{n+1}^i u^{(n+1-i)}(z)B_{i,j}(\varphi'(z),\ldots,\varphi^{(i-j+1)}(z))\\ &= \sum_{j=0}^{k} k\cdots(k-j+1)(\varphi(z))^{k-j} \sum_{i=j}^{n+1} C_{n+1}^i u^{(n+1-i)}(z)B_{i,j}(\varphi'(z),\ldots,\varphi^{(i-j+1)}(z)). \quad (15)\end{aligned}$$

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From (15), the boundedness of function φ and the triangle inequality, by noticing that the coefficient at

$$\sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))$$

is independent of z and finally using hypothesis (14) we easily obtain

$$L_k := \sup_{z\in\mathbb{D}} \mu_\Psi(z) \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)) \Big| \leq C\|D^n M_u C_\varphi\|. \tag{16}$$

By induction we see that (16) holds for each $k \in \{0, 1, \dots, n+1\}$.

For a fixed $w \in \mathbb{D}$ and a fixed $k \in \{2, 3, \dots, n+1\}$, by Lemma 2.3 (a) there exists a function

$$f_{w,k}(z) = \sum_{i=0}^{n+1} a_{i,k} r_{\varphi(w),i}(z)$$

such that

$$f_{w,k}^{(k)}(\varphi(w)) = \frac{\overline{\varphi(w)}^k}{(1-|\varphi(w)|^2)^{\alpha+k-2}} \quad \text{and} \quad f_{w,k}^{(j)}(\varphi(w)) = 0 \tag{17}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$, and

$$\sup_{w\in\mathbb{D}} \|f_{w,k}\|_{\mathcal{Z}^\alpha} \leq C. \tag{18}$$

Then by (17), (18) and the boundedness of $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} I_k(w) &:= \frac{\mu_\Psi(w)|\varphi(w)|^k \Big| \sum\limits_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(w) B_{j,k}\big(\varphi'(w), \dots, \varphi^{(j-k+1)}(w)\big) \Big|}{(1-|\varphi(w)|^2)^{\alpha+k-2}} \\ &\leq \|D^n M_u C_\varphi f_{w,k}\|_{\mathcal{B}^\Psi} \leq C\|D^n M_u C_\varphi\|. \end{aligned} \tag{19}$$

From (19) we see that

$$\sup_{z\in\mathbb{D}} I_k(z) \leq C\big\|D^n M_u C_\varphi\big\|,$$

which leads to

$$\sup_{|\varphi(z)|>1/2} \frac{\mu_\Psi(z) \Big| \sum\limits_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)) \Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} \leq C\|D^n M_u C_\varphi\|. \tag{20}$$

On the other hand, by (16) we have

$$\sup_{|\varphi(z)|\leq 1/2} \frac{\mu_\Psi(z) \Big| \sum\limits_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)) \Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} \leq C\|D^n M_u C_\varphi\|. \tag{21}$$

Hence from (20) and (21) we obtain

$$I_k \leq C\|D^n M_u C_\varphi\| < \infty. \tag{22}$$

$(b) \Rightarrow (a)$. By Lemmas 2.1, 2.2 and 2.5, for all $f \in \mathcal{Z}^\alpha$ we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z)\sum_{k=0}^{n+1} \big|f^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \Big(\frac{1}{1-\alpha}(I_0+I_1) + \sum_{k=2}^{n+1} C_k I_k\Big)\|f\|_{\mathcal{Z}^\alpha}. \qquad (23)
\end{aligned}$$

It is clear that

$$|(D^n M_u C_\varphi f)(0)| \le C\|f\|_{\mathcal{Z}^\alpha}. \qquad (24)$$

Hence, from (23) and (24) it follows that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

Clearly, if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0$ is also bounded. By the definition of the norm in the quotient spaces, and using the same functions in the proofs of (12), (13) and (22), we obtain

$$I_k \le C\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha\to\mathcal{B}^\Psi/\mathbb{P}_0},$$

for each $k \in \{0,1,2,\dots,n+1\}$, and then

$$\sum_{k=0}^{n+1} I_k \le C\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha\to\mathcal{B}^\Psi/\mathbb{P}_0}. \qquad (25)$$

By (23) we have

$$\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha\to\mathcal{B}^\Psi/\mathbb{P}_0} \le C\sum_{k=0}^{n+1} I_k. \qquad (26)$$

The asymptotic expression of $\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha\to\mathcal{B}^\Psi/\mathbb{P}_0}$ follows from (25) and (26). □

Remark 3.1. *In fact, from the fact $z^k \in \mathcal{Z}^\alpha$, in the proof of Theorem 3.1 we have seen that if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then $L_k < \infty$ for all $\alpha > 0$.*

Theorem 3.2. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, C_{n+1}^j the binomial coefficient and $1 < \alpha < 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.
(b) The functions u and φ are such that $I_0 < \infty$ and for each $k \in \{1,2,\dots,n+1\}$

$$M_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \infty.$$

Moreover, if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^n M_u C_\varphi\big\|_{\mathcal{Z}^\alpha\to\mathcal{B}^\Psi/\mathbb{P}_0} \asymp I_0 + \sum_{k=1}^{n+1} M_k.$$

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Proof. $(a) \Rightarrow (b)$. Let $h_0(z) \equiv 1 \in \mathcal{Z}^\alpha$. Then $I_0 < \infty$. For a fixed $w \in \mathbb{D}$ and each fixed $k \in \{1, 2, \dots, n+1\}$, by Lemma 2.3 (b) there exists a function

$$g_{w,k}(z) = \sum_{i=0}^{n+1} b_{i,k} r_{\varphi(w),i}(z)$$

such that

$$g_{w,k}^{(k)}(\varphi(w)) = \frac{\overline{\varphi(w)}^k}{(1-|\varphi(w)|^2)^{\alpha+k-2}} \quad \text{and} \quad g_{w,k}^{(j)}(\varphi(w)) = 0 \tag{27}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$. Moreover,

$$\sup_{w\in\mathbb{D}} \|g_{w,k}\|_{\mathcal{Z}^\alpha} \le C. \tag{28}$$

Then from (27), (28) and the boundedness of $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} M_k(w) :=& \frac{\mu_\Psi(w)|\varphi(w)|^k \Big| \sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(w) B_{j,k}\big(\varphi'(w), \dots, \varphi^{(j-k+1)}(w)\big)\Big|}{(1-|\varphi(w)|^2)^{\alpha+k-2}} \\ \le& \|D^n M_u C_\varphi g_{\varphi(w),k}\|_{\mathcal{B}^\Psi} \le C\|D^n M_u C_\varphi\|. \end{aligned} \tag{29}$$

From (29) we see

$$\sup_{z\in\mathbb{D}} M_k(z) \le C\big\|D^n M_u C_\varphi\big\|, \tag{30}$$

and then

$$\sup_{|\varphi(z)|>1/2} \frac{\mu_\Psi(z)\Big| \sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} \le C\|D^n M_u C_\varphi\|. \tag{31}$$

On the other hand, by using the fact $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$, we get

$$\sup_{|\varphi(z)|\le1/2} \frac{\mu_\Psi(z)\Big| \sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} \le C\|D^n M_u C_\varphi\|. \tag{32}$$

Hence from (31) and (32) we see that $M_k < \infty$ for each $k \in \{1, 2, ..., n+1\}$.

$(b) \Rightarrow (a)$. By Lemmas 2.1, 2.2 and 2.5, for all $f \in \mathcal{Z}^\alpha$ we have

$$\begin{aligned} &\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f)'(z)\big| \\ &= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big| \sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big| \\ &\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f^{(k)}(\varphi(z))\big| \Big| \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big| \\ &\le \Big(\frac{I_0}{(\alpha-1)(2-\alpha)} + \frac{2M_1}{\alpha-1} + \sum_{k=2}^{n+1} C_k M_k\Big)\|f\|_{\mathcal{Z}^\alpha}. \end{aligned} \tag{33}$$

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It is clear that

$$|(D^n M_u C_\varphi f)(0)| \le C\|f\|_{\mathcal{Z}^\alpha}. \tag{34}$$

Hence from (33) and (34) it follows that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. Similarly is obtained the asymptotic formula of $\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0}$, hence we omit. □

Theorem 3.3. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, C_{n+1}^j the binomial coefficient and $\alpha = 2$. Then the following statements are equivalent.*

(a) *The operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded.*
(b) *The functions u and φ satisfy the following conditions:*

$$R_0 := \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z), \dots, \varphi^{(j+1)}(z)\big)\Big| \log\frac{e}{1-|\varphi(z)|^2} < \infty.$$

and for each $k \in \{1, 2, \dots, n+1\}$

$$R_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^k} < \infty.$$

Moreover, if the operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^n M_u C_\varphi\big\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0} \asymp \sum_{k=0}^{n+1} R_k.$$

Proof. $(a) \Rightarrow (b)$. By using Lemma 2.3 (b), we can prove that $R_k < \infty$ for each $k \in \{1, 2, \dots, n+1\}$, so we do not give the proof again. For a fixed $w \in \mathbb{D}$, by Lemma 2.4 there exists a function

$$s_{\varphi(w)}(z) = p_{\varphi(w)}(z) + \sum_{i=0}^{n+1} d_i r_{\varphi(w),i}(z)$$

such that

$$s_{\varphi(w)}(\varphi(w)) = \log\frac{e}{1-|\varphi(w)|^2} \quad \text{and} \quad s_{\varphi(w)}^{(j)}(\varphi(w)) = 0 \tag{35}$$

for each $j \in \{1, 2, \dots, n+2\}$, moreover, $\sup_{w\in\mathbb{D}} \|s_{\varphi(w)}\|_{\mathcal{Z}^2} \le C$. Then from these and the boundedness of $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} R_0(w) &:= \mu_\Psi(w)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(w) B_{j,0}\big(\varphi'(w), \dots, \varphi^{(j+1)}(w)\big)\Big| \log\frac{e}{1-|\varphi(w)|^2} \\ &\le \|D^n M_u C_\varphi s_{\varphi(w)}\|_{\mathcal{B}^\Psi} \le C\big\|D^n M_u C_\varphi\big\|. \end{aligned} \tag{36}$$

Then from (36) it follows that $R_0 < \infty$.

$(b) \Rightarrow (a)$. From Lemmas 2.1, 2.2 and 2.5, for all $f \in \mathcal{Z}^2$ we have

$$\begin{aligned} &\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f)'(z)\big| \\ &= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big| \end{aligned}$$

$$\leq \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f^{(k)}(\varphi(z))\big| \Big| \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|$$
$$\leq \Big(2R_0 + eR_1 + \sum_{k=2}^{n+1} C_k R_k\Big)\|f\|_{\mathcal{Z}^2}. \tag{37}$$

It is clear that

$$|(D^n M_u C_\varphi f)(0)| \leq C\|f\|_{\mathcal{Z}^2}. \tag{38}$$

Hence from (37) and (38) it follows that the operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded. The asymptotic expression of $\|D^n M_u C_\varphi\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0}$ can be similarly obtained. □

Theorem 3.4. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $\alpha > 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

(b) The functions u and φ satisfy

$$S_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \infty, \ k = 0, \dots, n+1.$$

Moreover, if the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded, then

$$\big\|D^n M_u C_\varphi\big\|_{\mathcal{Z}^\alpha \to \mathcal{B}^\Psi/\mathbb{P}_0} \asymp \sum_{k=0}^{n+1} S_k.$$

Proof. Similarly to the proofs of Theorems 3.1-3.3, this result can be proved. □

Remark 3.2. *By using the similar methods and techniques, the boundedness of the operators $D^n C_\varphi M_u$, $C_\varphi D^n M_u$, $M_u D^n C_\varphi$, $M_u C_\varphi D^n$ and $C_\varphi M_u D^n$ from weighted Zygmund spaces to Bloch-Orlicz spaces can be characterized, so we omit.*

4. Compactness of the product-type operators

The first result is an alternative to Proposition 3.11 in [5], which characterizes the compactness in terms of sequential convergence. So the proof is omitted.

Lemma 4.1. *Let $T \in \{D^n M_u C_\varphi, D^n C_\varphi M_u, M_u D^n C_\varphi, C_\varphi D^n M_u, M_u C_\varphi D^n, C_\varphi M_u D^n\}$. Then the bounded operator $T : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact if and only if for every bounded sequence $\{f_j\}$ in $\mathcal{Z}^\alpha$ such that $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, it follows that $\lim_{j\to\infty} \|Tf_j\|_{\mathcal{B}^\Psi} = 0$.*

The following lemma was proved in [46].

Lemma 4.2. *(a) If $0 < \alpha < 2$ and $\{f_j\}$ is a bounded sequence in $\mathcal{Z}^\alpha$ which uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j \to \infty$, then*

$$\lim_{j\to\infty} \sup_{z\in\mathbb{D}} |f_j(z)| = 0.$$

(b) If $0 < \alpha < 1$ and $\{f_j\}$ is a bounded sequence in $\mathcal{Z}^\alpha$ which uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j \to \infty$, then

$$\lim_{j\to\infty} \sup_{z\in\mathbb{D}} |f_j'(z)| = 0.$$

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Now we characterize the compactness of the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$.

Theorem 4.1. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $0 < \alpha < 1$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact.

(b) The functions u and φ satisfy $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$, and for each $k \in \{2, 3, \dots, n+1\}$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} = 0.$$

Proof. $(a) \Rightarrow (b)$. Suppose that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact. Clearly the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. By Remark 2.1, $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$. Consider a sequence $\{\varphi(z_i)\}$ in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1$ as $i \to \infty$. If such a sequence does not exist, then the last condition in (b) obviously holds. Without loss of generality, we may suppose that $|\varphi(z_i)| > 1/2$ for all $i \in \mathbb{N}$. For each fixed $k \in \{2, 3, \dots, n+1\}$, using this sequence we define the function sequence $f_{i,k}(z) = f_{\varphi(z_i),k}(z)$, $i \in \mathbb{N}$. Then by Lemma 2.3 (a) we have that $\sup_{i\in\mathbb{N}} \|f_{i,k}\|_{\mathcal{Z}^\alpha} \le C$ and $f_{i,k} \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$, moreover

$$f_{i,k}^{(k)}(\varphi(z_i)) = \frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^{\alpha+k-2}} \quad \text{and} \quad f_{i,k}^{(j)}(\varphi(z_i)) = 0 \tag{39}$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$. By Lemma 4.1 and (39), we have

$$\lim_{i\to\infty} \frac{\mu_\Psi(z_i)\big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,k}(\varphi'(z_i), \dots, \varphi^{(j-k+1)}(z_i))\big|}{(1-|\varphi(z_i)|^2)^{\alpha+k-2}} = 0. \tag{40}$$

$(b) \Rightarrow (a)$. We first check that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. We observe that the last condition in (b) implies that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$ such that for all $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$ and for each $k \in \{2, 3, \dots, n+1\}$

$$\frac{\mu_\Psi(z)\big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))\big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \varepsilon. \tag{41}$$

From the fact $L_k < \infty$ for each $k \in \{2, 3, \dots, n+1\}$, and (41), we have

$$I_k \le \varepsilon + \frac{L_k}{(1-\eta^2)^{\alpha+k-2}}. \tag{42}$$

From (42) and the fact $L_k < \infty$, it follows that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

To prove that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact, by Lemma 4.1 we just need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{Z}^\alpha$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{Z}^\alpha} \le M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

For such chosen ε and η, by using (39), Lemma 2.1 and Lemma 2.2, we have

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$$
\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big| \Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z),\dots,\varphi^{(j+1)}(z)\big)\Big|\big|f_i(\varphi(z))\big| \\
&\quad + \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=1}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,1}\big(\varphi'(z),\dots,\varphi^{(j)}(z)\big)\Big|\big|f_i'(\varphi(z))\big| \\
&\quad + \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z) \sum_{k=2}^{n+1} \big|f_i^{(k)}(\varphi(z))\big| \Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le L_0 \sup_{z\in\mathbb{D}} \big|f_i(\varphi(z))\big| + L_1 \sup_{z\in\mathbb{D}} \big|f_i'(\varphi(z))\big| + \sum_{k=2}^{n+1} L_k \sup_{|z|\le\eta} \big|f_i^{(k)}(z)\big| + C\varepsilon. \qquad (43)
\end{aligned}
$$

From (43), Lemma 4.2 and the fact $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ implies that for each $k \in \mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$, we finally get

$$\lim_{i\to\infty} \sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| = 0. \qquad (44)$$

It is clear that

$$\lim_{i\to\infty} \big|(D^n M_u C_\varphi f_i)(0)\big| = 0. \qquad (45)$$

From (44) and (45) we obtain

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

This shows that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact. □

Theorem 4.2. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $1 < \alpha < 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact.

(b) The functions u and φ are such that $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$, and for each $k \in \{1, 2, \dots, n+1\}$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)\big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} = 0.$$

Proof. $(a) \Rightarrow (b)$. Suppose that the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact. Obviously the operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. Then $L_k < \infty$ for each $k \in \{0, 1, \dots, n+1\}$. Consider a sequence $\{\varphi(z_i)\}_{i\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1$ as $i \to \infty$. If such a sequence does not exist, then the last condition in (b) obviously holds. Without loss of generality,

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we may suppose that $|\varphi(z_i)| > 1/2$ for all $i \in \mathbb{N}$. For each fixed $k \in \{1, 2, \ldots, n+1\}$, by using this sequence we define the function sequence $g_{i,k}(z) = g_{\varphi(z_i),k}(z),\ i \in \mathbb{N}$. Then from Lemma 2.3 (b) we see that $\sup_{i\in\mathbb{N}} \|g_{i,k}\|_{\mathcal{Z}^\alpha} \le C$ and $g_{i,k} \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$, moreover

$$g_{i,k}^{(k)}(\varphi(z_i)) = \frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^{\alpha+k-2}} \quad \text{and} \quad g_{i,k}^{(j)}(\varphi(z_i)) = 0 \tag{46}$$

for each $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$. From Lemma 4.1 and (46), for each fixed $k \in \{1, 2, \ldots, n+1\}$ we have

$$\lim_{i\to\infty} \frac{\mu_\Psi(z_i)\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i) B_{j,k}(\varphi'(z_i), \ldots, \varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^{\alpha+k-2}} = 0. \tag{47}$$

$(b) \Rightarrow (a)$. We first check that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded. We observe that the last condition in (b) implies that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$ such that for all $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$ and for each $k \in \{1, 2, \ldots, n+1\}$

$$\frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} < \varepsilon. \tag{48}$$

From the fact $L_k < \infty$ for each $k \in \{0, 1, \ldots, n+1\}$, and (48), we have

$$M_k \le \varepsilon + \frac{L_k}{(1-\eta^2)^{\alpha+k-2}}. \tag{49}$$

From (49) and the fact $I_0 = L_0 < \infty$, it follows that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is bounded.

In order to prove that $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact, by Lemma 4.1 we just need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{Z}^\alpha$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{Z}^\alpha} \le M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then $\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0$. For such chosen ε and η, by using (46), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z), \ldots, \varphi^{(j+1)}(z)\big)\Big|\big|f_i(\varphi(z))\big| \\
&\quad + \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z) \sum_{k=1}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le L_0 \sup_{z\in\mathbb{D}} \big|f_i(\varphi(z))\big| + \sum_{k=1}^{n+1} L_k \sup_{|z|\le\eta} \big|f_i^{(k)}(z)\big| + C\varepsilon.
\end{aligned} \tag{50}$$

From (50), Lemma 4.2 and the fact $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ implies that for each $k \in \mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$, we

get

$$\lim_{i\to\infty}\sup_{z\in\mathbb{D}}\mu_\Psi(z)\big|(D^nM_uC_\varphi f_i)'(z)\big|=0. \tag{51}$$

It is clear that

$$\lim_{i\to\infty}\big|(D^nM_uC_\varphi f_i)(0)\big|=0. \tag{52}$$

From (51) and (52) we obtain

$$\lim_{i\to\infty}\|D^nM_uC_\varphi f_i\|_{\mathcal{B}^\Psi}=0.$$

This shows that the operator $D^nM_uC_\varphi:\mathcal{Z}^\alpha\to\mathcal{B}^\Psi$ is compact. □

Theorem 4.3. *Let φ be an analytic self-map of $\mathbb{D}$, $u\in H(\mathbb{D})$ and $\alpha=2$. Then the following statements are equivalent.*

(a) The operator $D^nM_uC_\varphi:\mathcal{Z}^2\to\mathcal{B}^\Psi$ is compact.
(b) The functions u and φ are such that $L_k<\infty$ for each $k\in\{0,1,\dots,n+1\}$,

$$\lim_{|\varphi(z)|\to1}\mu_\Psi(z)\Big|\sum_{j=0}^{n+1}C_{n+1}^{j}u^{(n+1-j)}(z)B_{j,0}\big(\varphi'(z),\dots,\varphi^{(j+1)}(z)\big)\Big|\log\frac{e}{1-|\varphi(z)|^2}=0,$$

and for each $k\in\{1,2,\dots,n+1\}$

$$\lim_{|\varphi(z)|\to1}\frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1}C_{n+1}^{j}u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^k}=0.$$

Proof. $(a)\Rightarrow(b)$. Suppose that the operator $D^nM_uC_\varphi:\mathcal{Z}^2\to\mathcal{B}^\Psi$ is compact. Clearly the operator $D^nM_uC_\varphi:\mathcal{Z}^2\to\mathcal{B}^\Psi$ is bounded. Then $L_k<\infty$ for each $k\in\{0,1,\dots,n+1\}$. Consider a sequence $\{\varphi(z_i)\}_{i\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_i)|\to1$ as $i\to\infty$. If such a sequence does not exist, then the last two conditions in (b) obviously hold. Without loss of generality, we may suppose that $|\varphi(z_i)|>1/2$ for all $i\in\mathbb{N}$. For each fixed $k\in\{1,2,\dots,n+1\}$, by using this sequence we define the function sequence $g_{i,k}(z)=g_{\varphi(z_i),k}(z)$, $i\in\mathbb{N}$. Then from Lemma 2.3 (b) we see that $\sup_{i\in\mathbb{N}}\|g_{i,k}\|_{\mathcal{Z}^2}\le C$ and $g_{i,k}\to0$ uniformly on every compact subset of $\mathbb{D}$ as $i\to\infty$, moreover

$$g_{i,k}^{(k)}(\varphi(z_i))=\frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^k}\quad\text{and}\quad g_{i,k}^{(j)}(\varphi(z_i))=0 \tag{53}$$

for each $j\in\{0,1,\dots,n+1\}\setminus\{k\}$. From Lemma 4.1 and (53), for each fixed $k\in\{1,2,\dots,n+1\}$ we have

$$\lim_{i\to\infty}\frac{\mu_\Psi(z_i)\Big|\sum\limits_{j=k}^{n+1}C_{n+1}^{j}u^{(n+1-j)}(z_i)B_{j,k}(\varphi'(z_i),\dots,\varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^k}=0. \tag{54}$$

Now consider another function sequence $q_i(z)=q_{\varphi(z_i)}(z)$. Then by Lemma 2.4 we have

$$q_i^{(k)}(\varphi(z_i))=c_k\frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^k}+d_k\frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^k}\log^{-1}\frac{e}{1-|\varphi(z_i)|^2}, \tag{55}$$

where $c_k > 0$ for each $k \geq 1$, $d_1 = 0$ and $d_k > 0$ for each $k \geq 2$. Moreover, $\sup_{i\in\mathbb{N}} \|q_i\|_{\mathcal{Z}^2} \leq C$, and $q_i \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$. From Lemma 4.1, we get

$$\lim_{i\to\infty} \|D^n M_u C_\varphi q_i\|_{\mathcal{B}^\Psi} = 0. \tag{56}$$

By (55) and the triangle inequality, we have

$$\begin{aligned}
&\mu_\Psi(z_i)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i)B_{j,0}\big(\varphi'(z_i),\dots,\varphi^{(j+1)}(z_i)\big)\Big|\Big(\log\frac{e}{1-|\varphi(z_i)|^2}+\log^{-1}\frac{e}{1-|\varphi(z_i)|^2}\Big)\\
&\leq \|D^n M_u C_\varphi q_i\|_{\mathcal{B}^\Psi} + \sum_{k=1}^{n+1}\frac{c_k\mu_\Psi(z_i)|\varphi(z_i)|^k\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i)B_{j,k}(\varphi'(z_i),\dots,\varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^k}\\
&+\sum_{k=1}^{n+1}\frac{d_k\mu_\Psi(z_i)|\varphi(z_i)|^k\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i)B_{j,k}(\varphi'(z_i),\dots,\varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^k}\log^{-1}\frac{e}{1-|\varphi(z_i)|^2}.
\end{aligned}\tag{57}$$

Therefore, taking the limit in (57) as $i \to \infty$, from (54), (56) and the fact

$$\log^{-1}\frac{e}{1-|\varphi(z_i)|^2} \to 0 \ \text{ as } \ i\to\infty,$$

we get

$$\lim_{i\to\infty}\mu_\Psi(z_i)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z_i)B_{j,0}\big(\varphi'(z),\dots,\varphi^{(j+1)}(z_i)\big)\Big|\log\frac{e}{1-|\varphi(z_i)|^2} = 0.$$

$(b) \Rightarrow (a)$. We first check that $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded. We observe that the conditions in (b) imply that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$, such that for any $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$

$$\mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,0}\big(\varphi'(z),\dots,\varphi^{(j+1)}(z)\big)\Big|\log\frac{e}{1-|\varphi(z)|^2} < \varepsilon \tag{58}$$

and

$$\frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}(\varphi'(z),\dots,\varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^k} < \varepsilon \tag{59}$$

for each $k \in \{1,2,\dots,n+1\}$. From the fact $L_0 < \infty$ and (58), we see

$$R_0 \leq \varepsilon + L_0\log\frac{e}{1-\eta^2}. \tag{60}$$

From (59) and the fact $L_k < \infty$ for each $k \in \{1,2,\dots,n+1\}$, we see

$$R_k \leq \varepsilon + \frac{L_k}{(1-\eta^2)^k}. \tag{61}$$

Then from (60), (61) and Theorem 3.3, it follows that $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is bounded.

In order to prove that $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is compact, by Lemma 4.1 we just need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{Z}^2$ such that $\sup_{i\in\mathbb{N}}\|f_i\|_{\mathcal{Z}^2} \leq M$ and $f_i \to 0$ uniformly

on any compact subset of $\mathbb{D}$ as $i \to \infty$, then $\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0$. For such chosen ε and η, by using (58), (59), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}\big(\varphi'(z),\dots,\varphi^{(j+1)}(z)\big)\Big|\big|f_i(\varphi(z))\big| \\
&\quad + \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z) \sum_{k=1}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sum_{k=0}^{n+1} L_k \sup_{|z|\le\eta} \big|f_i^{(k)}(z)\big| + C\varepsilon. \qquad (62)
\end{aligned}$$

From (62) and the fact $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ implies that for each $k \in \mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$, we get

$$\lim_{i\to\infty} \sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| = 0. \qquad (63)$$

It is clear that

$$\lim_{i\to\infty} \big|(D^n M_u C_\varphi f_i)(0)\big| = 0. \qquad (64)$$

From (63) and (64) we obtain

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

Hence this shows that the operator $D^n M_u C_\varphi : \mathcal{Z}^2 \to \mathcal{B}^\Psi$ is compact. □

Theorem 4.4. *Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $\alpha > 2$. Then the following statements are equivalent.*

(a) The operator $D^n M_u C_\varphi : \mathcal{Z}^\alpha \to \mathcal{B}^\Psi$ is compact.

(b) The functions u and φ are such that $L_k < \infty$ and for each $k \in \{0, 1, \dots, n+1\}$

$$\lim_{|\varphi(z)|\to1} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{\alpha+k-2}} = 0.$$

Proof. Similarly to the proofs of Theorems 4.1-4.3, this result can be proved. □

Remark 4.1. *By using the similar methods and techniques, the compactness of the operators $D^n C_\varphi M_u$, $C_\varphi D^n M_u$, $M_u D^n C_\varphi$, $M_u C_\varphi D^n$ and $C_\varphi M_u D^n$ from weighted Zygmund spaces to Bloch-Orlicz spaces can be characterized, so we omit.*

Acknowledgments. This work was supported by the Key Fund Project of Sichuan Provincial Department of Education (no.15ZA0221) and the Cultivation Project of Sichuan University of Science and Engineering (no.2015PY04).

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Union soft p-ideals and union soft sub-implicative ideals in BCI-algebras

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Abstract. The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notion of union soft p-ideals(sub-implicative ideals) are introduced, and related properties are investigated. Conditions for a union soft ideal to be a union soft p-ideal(sub-implicative ideal) are established. Characterizations of a union soft p-ideal(sub-implicative ideal) are considered, and a new union soft p-ideal(sub-implicative ideal) from an old one is constructed.

1. Introduction

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [16]. In response to this situation Zadeh [17] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [18]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [14]. Maji et al. [13] and Molodtsov [14] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [14] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [13] described the application of soft set theory to a decision making problem. Maji et

[0]**2010 Mathematics Subject Classification**: 06F35; 03G25; 06D72.
[0]**Keywords**: Exclusive set, Union soft ideal, Union soft p-ideal, Union soft sub-implicative ideal.
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al. [12] also studied several operations on the theory of soft sets. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun [9] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [1, 3, 6, 8, 10] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we discuss applications of the union soft sets in p-ideals of BCI-algebras. We introduce the notion of union soft p-ideals, and investigated related properties. We provide conditions for a union soft ideal to be a union soft p-ideal, and establish characterizations of a union soft p-ideal. We construct a new union soft p-ideal from an old one.

Secondly, we define the notion of union soft sub-implicative ideals, and investigated related properties. We provide conditions for a union soft ideal to be a union soft sub-implicative ideal, and study characterizations of a union soft sub-implicative ideal. We find a new union soft sub-implicative ideal from an old one.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(a3) $(\forall x \in X)\ (x * x = 0)$,
(a4) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(a5) $(\forall x \in X)(0 * x = 0)$,

then X is called a BCK-algebra. A BCI-algebra X is said to be *p-semisimple* if $0 * (0 * x) = x$ for all $x \in X$. A BCI-algebra X is said to be *implicative* if $(x * (x * y)) * (y * x) = y * (y * x)$ for all $x, y \in X$.

In any BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra X has the following properties:

(b1) $(\forall x \in X)\ (x * 0 = x)$,
(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(b3) $(\forall x, y \in X)\ (0 * (x * y) = (0 * x) * (0 * y))$,
(b4) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$.
(b5) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,
(b6) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$,
(b7) $(\forall x, y, z \in X)\ (0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))$,

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(b8) $(\forall x, y \in X)$ $(0 * (0 * (x * y)) = (0 * y) * (0 * x))$.

A non-empty subset S of a BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset A of a BCI-algebra X is called an *ideal* of X if it satisfies:

(c1) $0 \in A$,
(c2) $(\forall x \in X)$ $(\forall y \in A)$ $(x * y \in A \Rightarrow x \in A)$.

Note that every ideal A of a BCI-algebra X satisfies:

$$(\forall x \in X)\,(\forall y \in A)\,(x \leq y \Rightarrow x \in A).$$

A non-empty subset A of a BCI-algebra X is called a *p-ideal* ([15]) of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)((x * z) * (y * z) \in A$ and $y \in A \Rightarrow x \in A)$.

Note that any p-ideal is an ideal, but the converse is not true in general.

Theorem 2.1. ([15]) *An ideal I of a BCI-algebra X is a p-ideal if and only if $0 * (0 * x) \in I$ implies $x \in I$ for any $x \in X$.*

For any elements x and y of a BCI-algebra X, $x^n * y$ denotes $x * (x * \cdots * (x * (x * y \cdots)$ in which x occurs n times. A non-empty subset A of a BCI-algebra X is called a *sub-implicative ideal* ([11]) of X if it satisfies (c1) and

(c4) $(\forall x, y, z \in X)((x^2 * y) * (y * x)) * z \in A$ and $z \in A \Rightarrow y^2 * x \in A)$.

Note that any sub-implicative ideal is an ideal, but the converse is not true in general.

Theorem 2.2. ([11]) *An ideal I of a BCI-algebra X is a sub-implicative ideal if and only if $(x^2 * y) * (y * x) \in I$ implies $y^2 * x \in I$ for any $x, y \in X$.*

We refer the reader to the book [7] for further information regarding BCI-algebras. A soft set theory is introduced by Molodtsov [14].

In what follows, let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe.* Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.3. ([14]) A soft set $\mathscr{F}_A$ over U is defined to be the set of ordered pairs

$$\mathscr{F}_A := \{(x, f_A(x)) : x \in E,\ f_A(x) \in \mathscr{P}(U)\},$$

where $f_A : E \to \mathscr{P}(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

The function f_A is called the approximate function of the soft set $\mathscr{F}_A$. The subscript A in the notation f_A indicates that f_A is the approximate function of $\mathscr{F}_A$.

In what follows, denote by $S(U)$ the set of all soft sets over U.

Definition 2.4. ([12]) For two soft sets $\mathscr{F}_A$ and $\mathscr{G}_B$ over a common universe U, we say that $\mathscr{F}_A$ is a *soft subset* of $\mathscr{G}_B$, denoted by $\mathscr{F}_A \tilde{\subset} \mathscr{G}_B$, if it satisfies:

(i) $A \subset B$,
(ii) For every $\epsilon \in A$, $\mathscr{F}(\epsilon)$ and $\mathscr{G}(\epsilon)$ are identical approximations.

Let $\mathscr{F}_A \in S(U)$ and let $\tau \subseteq U$. Then the *τ-exclusive set* of $\mathscr{F}_A$ is defined to be the set

$$e(\mathscr{F}_A;\tau) := \{x \in A \mid f_A(x) \subseteq \tau\}.$$

Obviously, we have the following properties:

(1) $e(\mathscr{F}_A;U) = A$,
(2) $f_A(x) = \cap\{\tau \subseteq U \mid x \in e(\mathscr{F}_A;\tau)\}$,
(3) $(\forall \tau_1, \tau_2 \subseteq U)\,(\tau_1 \subseteq \tau_2 \;\Rightarrow\; e(\mathscr{F}_A;\tau_1) \subseteq e(\mathscr{F}_A;\tau_2))$.

3. Union soft p-ideals

Definition 3.1. ([9]) Let $(U,E) = (U,X)$ where X is a BCI-algebra. Given a subalgebra A of E, we let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft deal* over U (briefly, *U-soft ideal*) if the approximate function f_A of $\mathscr{F}_A$ satisfies:

$$(\forall x \in A)\,(f_A(0) \subseteq f_A(x)), \tag{3.1}$$

$$(\forall x, y \in A)\,(f_A(x) \subseteq f_A(x*y) \cup f_A(y)). \tag{3.2}$$

Definition 3.2. Let $(U,E) = (U,X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft p-ideal* over U (briefly, *U-soft p-ideal*) if the approximate function f_A of $\mathscr{F}_A$ satisfies (3.1) and

$$(\forall x, y, z \in A)\,(f_A(x) \subseteq f_A((x*z)*(y*z)) \cup f_A(y)). \tag{3.3}$$

Example 3.3. Let $(U,E) = (U,X)$ where $X = \{0,1,a,b,c\}$ is a BCI-algebra ([10]) with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let τ_1, τ_2 and τ_3 be subsets of U such that $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0,\tau_1),(1,\tau_2),(a,\tau_3),(b,\tau_3),(c,\tau_2)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft p-ideal over U.

Theorem 3.4. *Let $(U,E) = (U,X)$ where X is a BCI-algebra. Then every U-soft p-ideal is a U-soft ideal.*

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Proof. Let $\mathscr{F}_A$ be a U-soft p-ideal over U where A is a subalgebra of E. Taking $z := 0$ in (3.3) and using (b1) we obtain

$$\begin{aligned} f_A(x) \subseteq & f_A((x*0)*(y*0)) \cup f_A(y) \\ = & f_A(x*y)) \cup f_A(y) \end{aligned}$$

for all $x, y \in A$. Therefore $\mathscr{F}_A$ is a U-soft ideal over U. □

The following example shows that the converse of Theorem 3.4 is not true.

Example 3.5. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3, 4\}$ is a BCI-algebra ([9]) with the following Cayley table:

$*$	0	1	2	a	b
0	0	0	0	a	b
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $\tau_1, \tau_2, \tau_3, \tau_4$ and τ_5 be subsets of U such that $\tau_1 \subsetneq \tau_3 \subsetneq \tau_4 \subsetneq \tau_5$ and $\tau_1 \subsetneq \tau_2 \subsetneq \tau_5$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (1, \tau_2), (2, \tau_3), (a, \tau_4), (b, \tau_5)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft ideal over U. But it is not a U-soft p-ideal over U, since

$$f_E(b) = \tau_5 \nsubseteq \tau_4 = \tau_1 \cup \tau_4 = f_E((b*b)*(a*b)) \cup f_E(a).$$

We provide some conditions for a U-soft ideal to be a U-soft p-ideal over U.

Theorem 3.6. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(1) *$\mathscr{F}_A$ is a U-soft p-ideal over U,*

(2) *$\mathscr{F}_A$ is a U-soft ideal over U and its approximate function f_A satisfies*

$$(\forall x, y, z \in A)\,(f_A(x*y) \subseteq f_A((x*z)*(y*z))). \tag{3.4}$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. By Theorem 3.4, $\mathscr{F}_A$ is a U-soft deal over U. Using (a1) and (b2), we have $0 = ((x*z)*(x*y))*(y*z) = ((x*z)*(y*z))*(x*y)$ for any $x, y, z \in A$. Hence $((x*y)*(x*y))*[((x*z)*(y*z))*(x*y)] = 0*0 = 0$. It follows from (3.3) and (3.1) that

$$\begin{aligned} f_A(x*y) \subseteq & f_A((x*y)*(x*y))*[((x*z)*(y*z))*(x*y)]) \cup f_A((x*z)*(y*z)) \\ = & f_A(0) \cup f_A((x*z)*(y*z)) \\ = & f_A((x*z)*(y*z)). \end{aligned}$$

Hence (3.4) holds.

Conversely, suppose that $\mathscr{F}_A$ is a U-soft ideal over U satisfying (3.4). Using (3.2) and (3.4), we have $f_A(x) \subseteq f_A(x*y) \cup f_A(y) \subseteq f_A((x*z)*(y*z)) \cup f_A(y)$ for any $x, y, z \in A$. Hence $\mathscr{F}_A$ is a U-soft p-ideal over U. This completes the proof. □

Lemma 3.7. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. If $\mathscr{F}_A$ is a U-soft ideal over U, then the approximate function f_A of $\mathscr{F}_A$ satisfies the following condition:*

$$(\forall x \in A)(f_A(0*(0*x)) \subseteq f_A(x)).$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft ideal over U. Note that $0 = (0*x)*(0*x) = (0*(0*x))*x$. Using (3.2) and (3.1), we have

$$\begin{aligned} f_A(0*(0*x)) \subseteq & f_A((0*(0*x)*x)) \cup f_A(x) \\ = & f_A(0) \cup f_A(x) \\ = & f_A(x) \end{aligned}$$

for any $x \in A$. This completes the proof. □

Theorem 3.8. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is a U-soft p-ideal over U,*
(ii) *$\mathscr{F}_A$ is a U-soft ideal over U and its approximate function f_A satisfies*

$$(\forall x \in A)\left(f_A(x) \subseteq f_A(0*(0*x))\right). \tag{3.5}$$

Proof. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. By Theorem 3.4, $\mathscr{F}_A$ is a U-soft deal over U. It follows from (3.3) and (3.1) that

$$\begin{aligned} f_A(x) \subseteq & f_A((x*x)*(0*x)) \cup f_A(0) \\ = & f_A(0*(0*x)) \end{aligned}$$

for any $x \in A$. Hence (3.5) holds.

Conversely, suppose that $\mathscr{F}_A$ is a U-soft ideal over U satisfying (3.5). By Lemma 3.7, we obtain $f_A(0*(0*((x*z)*(y*z)))) \subseteq f_A((x*z)*(y*z))$. It follows from (b7) and (b8) that $0*(0*(x*y)) = (0*y)*(0*x) = 0*(0*(x*z)*(y*z)))$. Using (3.5), we have $f_A(x*y) \subseteq f_A(0*(0*(x*y))) \subseteq f_A((x*z)*(y*z))$. By Theorem 3.6, $\mathscr{F}_A$ is a U-soft p-ideal over U. □

Lemma 3.9. ([9]) *Let $(U, E) = (U, X)$ where X is a BCI-algebra, Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is an U-soft ideal over U,*
(ii) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a ideal of A for any $\tau \subseteq U$.*

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Theorem 3.10. *Let $(U, E) = (U, X)$ where X is a BCI-algebra, Given a subalgebra A of E. let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is a U-soft p-ideal over U,*
(ii) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a p-ideal of A for any $\tau \subseteq U$.*

Proof. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. Then $\mathscr{F}_A$ is a U-soft ideal over U by Theorem 3.4. Hence $e(\mathscr{F}_A; \tau)$ is an ideal of A for all $\tau \subseteq U$ by Lemma 3.9. Let $\tau \subseteq U$ and let $x, y, z \in A$ be such that $(x * z) * (y * z) \in e(\mathscr{F}_A; \tau)$ and $y \in e(\mathscr{F}_A; \tau)$. Then $f_A((x * z) * (y * z)) \subseteq \tau$, $f_A(y) \subseteq \tau$, and so

$$f_A(x) \subseteq f_A((x * z) * (y * z)) \cup f_A(y) \subseteq \tau.$$

Hence $x \in e(\mathscr{F}_A; \tau)$. Thus $e(\mathscr{F}_A; \tau)$ is a p-ideal of A.

Conversely, suppose that the nonempty τ-exclusive set of $\mathscr{F}_A$ is a p-ideal of A for any $\tau \subseteq U$. Then $e(\mathscr{F}_A; \tau)$ is an ideal of A for all $\tau \subseteq U$. Hence $\mathscr{F}_A$ is a U-soft ideal over U by Lemma 3.9. Let $x \in A$ be such that $f_A(0 * (0 * x)) = \tau$. Then $0 * (0 * x) \in e(\mathscr{F}_A; \tau)$, and so $x \in e(\mathscr{F}_A; \tau)$ by Theorem 2.1. Hence $f_A(x) \subseteq f_A(0 * (0 * x))$. It follows from Theorem 3.8 that $\mathscr{F}_A$ is a U-soft p-ideal over U. □

The p-ideals $e(\mathscr{F}_A; \tau)$ in Theorem 3.10 are called the *exclusive p-ideals* of $\mathscr{F}_A$.

Theorem 3.11. *Let $(U, E) = (U, X)$ and $\mathscr{F}_A \in S(U)$ where X is a BCI-algebra and A is a subalgebra of E. For a subset τ of U, define a soft set $\mathscr{F}_A^*$ over U by*

$$f_A^* : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} f_A(x) & \text{if } x \in e(\mathscr{F}_A; \tau), \\ U & \text{otherwise.} \end{cases}$$

If $\mathscr{F}_A$ is a U-soft p-ideal over U, then so is $\mathscr{F}_A^*$.

Proof. If $\mathscr{F}_A$ is a U-soft p-ideal over U, then $e(\mathscr{F}_A; \tau)$ is a p-ideal of A for any $\tau \subseteq U$. Hence $0 \in e(\mathscr{F}_A; \tau)$, and so $f_A^*(0) = f_A(0) \subseteq f_A(x) \subseteq f_A^*(x)$ for all $x \in A$. Let $x, y, z \in A$. If $(x * z) * (y * z) \in e(\mathscr{F}_A; \tau)$ and $y \in e(\mathscr{F}_A; \tau)$, then $x \in e(\mathscr{F}_A; \tau)$ and so

$$\begin{aligned} f_A^*(x) &= f_A(x) \\ &\subseteq f_A((x * z) * (y * z)) \cup f_A(y) \\ &= f_A^*((x * z) * (y * z)) \cup f_A^*(y). \end{aligned}$$

If $(x * y) * (y * z) \notin e(\mathscr{F}_A; \tau)$ or $y \notin e(\mathscr{F}_A; \tau)$, then $f_A^*((x * z) * (y * z)) = U$ or $f_A^*(y) = U$. Hence

$$f_A^*(x) \subseteq U = f_A^*((x * z) * (y * z)) \cup f_A^*(y).$$

This shows that $\mathscr{F}_A^*$ is a U-soft p-ideal over U. □

Theorem 3.12. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then any p-ideal of E can be realized as an exclusive p-ideal of some U-soft p-ideal over U.*

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Proof. Let A be a p-ideal of E. For any subset $\tau \subsetneq U$, let $\mathscr{F}_A$ be a soft set over U defined by

$$f_A : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases}$$

Obviously, $f_A(0) \subseteq f_A(x)$ for all $x \in E$. For any $x, y, z \in E$, if $(x * z) * (y * z) \in A$ and $y \in A$ then $x \in A$. Hence

$$f_A((x * z) * (y * z)) \cup f_A(y) = \tau = f_A(x).$$

If $(x * z) * (y * z) \notin A$ or $y \notin A$ then $f_A((x * z) * (y * z)) = U$ or $f_A(y) = U$. It follows from (3.3) that

$$f_A(x) \subseteq U = f_A((x * z) * (y * z)) \cup f_A(y).$$

Therefore $\mathscr{F}_A$ is a U-soft p-ideal over U, and clearly $e(\mathscr{F}_A; \tau) = A$. This completes the proof. □

Example 3.13. Let $(U, E) = (U, X)$ where X is a BCI-algebra.
(1) $B(X) := \{x \in X | 0 * x = 0\}$. Then $B(X)$ is a p-ideal ([15]) of X. For any subset $\tau \subsetneq U$, let $\mathscr{F}_{B(X)}$ be a soft set over U defined by

$$f_{B(X)} : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in B(X), \\ U & \text{if } x \notin B(X). \end{cases}$$

Then $\mathscr{F}_{B(X)}$ is a U-soft p-ideal over U.
(2) $T_n(X) := \{x \in X | 0 * x^n = 0\}$, where $0 * x^n = (\cdots (0 * x) * \cdots) * x$ in which x appears n-times. Then $T_n(X)$ is a p-ideal ([15]) of X. For any subset $\tau \subsetneq U$, let $\mathscr{G}_{T_n(X)}$ be a soft set over U defined by

$$g_{T_n(X)} : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in T_n(X), \\ U & \text{if } x \notin T_n(X). \end{cases}$$

Then $\mathscr{G}_{T_n(X)}$ is a U-soft p-ideal over U.

Theorem 3.14. [Extension property] *Let $(U, E) = (U, X)$ where X is a p-semisimple BCI-algebra. Given subalgebras A and B of E, let $\mathscr{F}_A, \mathscr{F}_B \in S(U)$ such that*

(i) $\mathscr{F}_A \tilde{\subset} \mathscr{F}_B$,
(ii) $\mathscr{F}_B$ *a U-soft ideal over U.*

If $\mathscr{F}_A$ is a U-soft p-ideal over U, then so is $\mathscr{F}_B$.

Proof. Let $\tau \subseteq U$ be such that $e(\mathscr{F}_B; \tau) \neq \emptyset$. It follows from the condition (ii) and Lemma 3.9 that $e(\mathscr{F}_B; \tau)$ is an ideal. Assume that $\mathscr{F}_A$ is a U-soft p-ideal over U. Then $e(\mathscr{F}_A; \tau)$ is a p-ideal for every $\tau \subseteq U$ by Theorem 3.10. Let $x \in E$ and $\tau \subseteq U$ be such that $0 * (0 * x) \in e(\mathscr{F}_B; \tau)$. Since X is a p-semisimple BCI-algebra, $0 * (0 * x) = x$. Hence $x \in e(\mathscr{F}_B; \tau)$. Thus $e(\mathscr{F}_B; \tau)$ is a p-ideal by Theorem 2.1. By Theorem 3.10, $\mathscr{F}_B$ is a U-soft p-ideal over U. □

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4. Union soft sub-implicative ideals

Definition 4.1. Let $(U, E) = (U, X)$ where X is a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *union soft sub-implicative ideal* over U (briefly, *U-soft sub-implicative ideal*) if the approximate function f_A of $\mathscr{F}_A$ satisfies (3.1) and

$$(\forall x, y, z \in A)\left(f_A(y^2 * x) \subseteq f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z)\right). \tag{4.1}$$

Example 4.2. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2\}$ is a BCI-algebra ([11]) with the following Cayley table:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Let τ_1 and τ_2 be subsets of U such that $\tau_1 \subsetneq \tau_2$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (1, \tau_1), (2, \tau_2)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft sub-implicative ideal over U.

Theorem 4.3. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then every U-soft sub-implicative ideal is a U-soft ideal.*

Proof. Let $\mathscr{F}_A$ be a U-soft sub-implicative ideal over U where A is a subalgebra of E. Taking $y := x$ in (4.1) we obtain

$$\begin{aligned} f_A(x) &= f_A(x^2 * x) \\ &\subseteq f_A(((x^2 * x) * (x * x)) * z) \cup f_A(z) \\ &= f_A(x * z)) \cup f_A(z) \end{aligned}$$

for all $x, z \in A$. Therefore $\mathscr{F}_A$ is a U-soft ideal over U. □

The following example shows that the converse of Theorem 4.3 is not true.

Example 4.4. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3, 4\}$ is a BCI-algebra ([11]) with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	c
b	b	b	0	c
c	c	c	c	0

Let τ_1 and τ_2 be subsets of U such that $\tau_1 \subsetneq \tau_2$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (a, \tau_2), (b, \tau_2), (c, \tau_2)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft ideal over U. But it is not a U-soft sub-implicative ideal over U, since

$$f_E(a^2 * b) = f_E(a) = \tau_2 \nsubseteq \tau_1 = f_E(((b^2 * a) * (a * b)) * 0) \cup f_E(0).$$

Proposition 4.5. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. If $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, then the approximate function f_A of $\mathscr{F}_A$ satisfies the following condition:*

$$(\forall x, y \in A) \left(f_A(y^2 * x) \subseteq f_A((x^2 * y) * (y * x))\right). \tag{4.2}$$

.

Proof. Assume that $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U. For any $x, y \in A$, we have

$$\begin{aligned} f_A(y^2 * x) \subseteq& f_A(((x^2 * y) * (y * x)) * 0) \cup f_A(0) \\ =& f_A((x^2 * y) * (y * x)). \end{aligned}$$

This completes the proof. □

We provide conditions for a U-soft BCI-ideal to be a U-soft sub-implicative ideal over U.

Theorem 4.6. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. If $\mathscr{F}_A$ is a U-soft ideal over U satisfying the condition (4.2), then $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U.*

Proof. Assume that $\mathscr{F}_A$ is a U-soft ideal over U satisfying the condition (4.2). For any $x, y \in A$, we have

$$\begin{aligned} f_A(y^2 * x) \subseteq& f_A((x^2 * y) * (y * x)) \\ \subseteq& f_A((x^2 * y) * (y * x)) * z) \cup f_A(z) \end{aligned}$$

which proves the condition (4.1). This completes the proof. □

Corollary 4.7. *Let $(U, E) = (U, X)$ where X is an implicative BCI-algebra. Then every U-soft sub-implicative ideal is a U-soft ideal.*

Theorem 4.8. *Let $(U, E) = (U, X)$ where X is a p-semisimple BCI-algebra. For a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. The notions of a U-soft ideal over U and a U-soft sub-implicative ideal over U coincide.*

Proof. Note that $x^2 * y = y$ for all $x, y \in X$, since X is a p-semsimple BCI-algebra. Assume that $\mathscr{F}_A$ is a U-soft ideal over U. For any $x, y, z \in A$, we have

$$\begin{aligned} f_A(y^2 * x) =& f_A(x) \\ \subseteq& f_A(x * z) \cup f_A(z) \\ =& f_A((y^2 * x) * z)) \cup f_A(z) \\ =& f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z). \end{aligned}$$

Therefore $\mathscr{F}_A$ ia a U-soft sub-implicative ideal over U. □

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Theorem 4.9. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then every U-soft p-ideal is a U-soft sub-implicative ideal.*

Proof. Let $\mathscr{F}_A$ be a U-soft p-ideal over U, where A is a subalgebra of E. Then $\mathscr{F}_A$ is a U-soft ideal over U. Then $\mathscr{F}_A$ is a U-soft ideal over U by Theorem 3.4. Note that

$$\begin{aligned}(0^2 * (y^2 * x)) * ((x^2 * y) * (y * x)) &= 0 * ((x^2 * y) * (y * x)) * (0 * (y^2 * x)) \\ &= [(0 * (x^2 * y)) * (0 * (y * x))] * (0 * (y^2 * x)) \\ &= [((0 * x) * (0 * (x * y))) * (0 * (y * x))] * [(0 * y) * (0 * (y * x))] \\ &\leq ((0 * x) * (0 * (x * y))) * (0 * y) \\ &= ((0 * x) * (0 * y)) * (0 * (x * y)) \\ &= 0.\end{aligned}$$

For any $x, y \in A$, we have

$$\begin{aligned}f_A(y^2 * x) &\subseteq f_A(0^2 * (y^2 * x)) \\ &\subseteq f_A((0^2 * (y^2 * x)) * ((x^2 * y) * (y * x))) \cup f_A((x^2 * y) * (y * x)) \\ &\subseteq f_A(0) \cup ((x^2 * y) * (y * x)) \\ &= f_A((x^2 * y) * (y * x)).\end{aligned}$$

It follows from Theorem 4.6 that $\mathscr{F}_A$ is a U-soft sub-implicaticve ideal over U. □

The converse of Theorem 4.9 may not be true in general as seen in the following example.

Example 4.10. Let $(U, E) = (U, X)$ where $X = \{0, a, 1, 2, 3\}$ is a BCI-algebra with the following Cayley table:

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Let τ_1, τ_2 and τ_3 be subsets of U such that $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$. Define a soft set $\mathscr{F}_E$ over U as follows:

$$\mathscr{F}_E = \{(0, \tau_1), (a, \tau_2), (1, \tau_3), (2, \tau_3), (3, \tau_3)\}.$$

Routine calculations show that $\mathscr{F}_E$ is a U-soft sub-implicative ideal over U. But it is not a U-soft p-ideal over U, since

$$f_E(a) = \tau_2 \nsubseteq \tau_1 = f_E((a * 1) * (0 * 1)) \cup f_E(0).$$

Theorem 4.11. *Let $(U, E) = (U, X)$ where X is a BCI-algebra, Given a subalgebra A of E. let $\mathscr{F}_A \in S(U)$. Then the following are equivalent:*

(i) *$\mathscr{F}_A$ is a U-soft sub-implicative ideal over U,*
(ii) *The nonempty τ-exclusive set of $\mathscr{F}_A$ is a sub-implicative ideal of A for any $\tau \subseteq U$.*

Proof. Assume that $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U. Then $\mathscr{F}_A$ is a U-soft ideal over U by Theorem 4.3. Hence $e(\mathscr{F}_A;\tau)$ is an ideal of A for all $\tau \subseteq U$ by Lemma 3.9. Let $\tau \subseteq U$ and let $x, y, z \in A$ be such that $((x^2 * y) * (y * x)) * z \in e(\mathscr{F}_A;\tau)$ and $z \in e(\mathscr{F}_A;\tau)$. Then $f_A(((x^2 * y) * (y * x)) * z) \subseteq \tau$, $f_A(z) \subseteq \tau$, and so

$$f_A(y^2 * x) \subseteq f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z) \subseteq \tau.$$

Hence $y^2 * x \in e(\mathscr{F}_A;\tau)$. Thus $e(\mathscr{F}_A;\tau)$ is a sub-implicative ideal of A.

Conversely, suppose that the nonempty τ-exclusive set of $\mathscr{F}_A$ is a sub-implicative ideal of A for any $\tau \subseteq U$. Then $e(\mathscr{F}_A;\tau)$ is an ideal of A for all $\tau \subseteq U$. Hence $\mathscr{F}_A$ is a U-soft ideal over U by Lemma 3.9. Let $x, y \in A$ be such that $f_A((x^2 * y) * (y * x)) = \tau$. Then $(x^2 * y) * (y * x) \in e(\mathscr{F}_A;\tau)$, and so $y^2 * x \in e(\mathscr{F}_A;\tau)$ by Theorem 2.2. Hence $f_A(y^2 * x) \subseteq f_A((x^2 * y) * (y * x))$. It follows from Theorem 4.6 that $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U. $\square$

The sub-implicative ideals $e(\mathscr{F}_A;\tau)$ in Theorem 4.11 are called the *exclusive sub-implicative ideals* of $\mathscr{F}_A$.

Theorem 4.12. *Let $(U, E) = (U, X)$ and $\mathscr{F}_A \in S(U)$ where X is a BCI-algebra and A is a subalgebra of E. For a subset τ of U, define a soft set $\mathscr{F}^*_A$ over U by*

$$f^*_A : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} f_A(x) & \text{if } x \in e(\mathscr{F}_A;\tau), \\ U & \text{otherwise.} \end{cases}$$

If $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, then so is $\mathscr{F}^*_A$.

Proof. If $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, then $e(\mathscr{F}_A;\tau)$ is a sub-implicative ideal of A for any $\tau \subseteq U$. Hence $0 \in e(\mathscr{F}_A;\tau)$, and so $f^*_A(0) = f_A(0) \subseteq f_A(x) \subseteq f^*_A(x)$ for all $x \in A$. Let $x, y, z \in A$. If $((x^2 * y) * (y * x) * z) \in e(\mathscr{F}_A;\tau)$ and $z \in e(\mathscr{F}_A;\tau)$, then $y^2 * x \in e(\mathscr{F}_A;\tau)$ and so

$$\begin{aligned} f^*_A(y^2 * x) =& f_A(y^2 * x) \\ \subseteq& f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z) \\ =& f^*_A(((x^2 * y) * (y * x)) * z) \cup f^*_A(z). \end{aligned}$$

If $((x^2 * y) * (y * x)) * z \notin e(\mathscr{F}_A;\tau)$ or $z \notin e(\mathscr{F}_A;\tau)$, then $f^*_A((x^2 * y) * (y * x)) * z)) = U$ or $f^*_A(z) = U$. Hence

$$f^*_A(x) \subseteq U = f^*_A(((x^2 * y) * (y * x)) * z) \cup f^*_A(z).$$

This shows that $\mathscr{F}^*_A$ is a U-soft sub-implicative ideal over U. $\square$

Theorem 4.13. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. Then any sub-implicative ideal of E can be realized as an exclusive sub-implicative ideal of some U-soft sub-implicative ideal over U.*

Proof. Let A be a sub-implicative ideal of E. For any subset $\tau \subsetneq U$, let $\mathscr{F}_A$ be a soft set over U defined by

$$f_A : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases}$$

Obviously, $f_A(0) \subseteq f_A(x)$ for all $x \in E$. For any $x, y, z \in E$, if $((x^2 * y) * (y * x)) * z \in A$ and $z \in A$, then $y^2 * x \in A$. Hence $f_A(y^2 * x) = \tau = f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z)$. If $((x^2 * y) * (y * x)) * z \notin A$ or $z \notin A$, then $f_A(((x^2 * y) * (y * x)) * z) = U$ or $f_A(z) = U$. It follows from (4.1) that

$$f_A(y^2 * x) \subseteq U = f_A(((x^2 * y) * (y * x)) * z) \cup f_A(z).$$

Therefore $\mathscr{F}_A$ is a U-soft sub-implicative ideal over U, and clearly $e(\mathscr{F}_A; \tau) = A$. This completes the proof. □

Example 4.14. Let $(U, E) = (U, X)$ where X is a BCI-algebra and let $B(X) := \{x \in X | 0 * x = 0\}$. Then $B(X)$ is a sub-implicative ideal ([11]) of X. For any subset $\tau \subsetneq U$, let $\mathscr{F}_{B(X)}$ be a soft set over U defined by

$$f_{B(X)} : E \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau & \text{if } x \in B(X), \\ U & \text{if } x \notin B(X). \end{cases}$$

Then it is easy to see that $\mathscr{F}_{B(X)}$ is a U-soft sub-implicative ideal over U.

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On interval-valued fuzzy rough approximation operators *

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March 5, 2016

Abstract: Rough approximation operators based on approximation spaces are a key concept of rough set theory. This paper investigates rough approximation operators in interval-valued fuzzy (for short, IVF) environment by using constructive and axiomatic approaches. Moreover, IVF pseudo-closure operators are considered.

Keywords: IVF set; IVF relation; IVF approximate space; IVF rough set; IVF rough approximation operators.

1 Introduction

Rough set theory was proposed by Pawlak [16] as a mathematical tool for data reasoning. It may be seen as an extension of classical set theory, has been proved to be an effective approach to deal with intelligent systems characterized by insufficient and incomplete information, and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [17, 18, 19, 20]. The foundation of its object classification is an equivalence relation. The upper and lower approximation operations are two core notions of this theory. They can also be seen as the closure operator and the interior operator of the topology induced by an equivalence relation on the universe, respectively. In the real world, the equivalence

*This Work is partly supported by the Scientific Research Project from Guangxi Education Department(KY2015ZD039), the National Natural Science Foundation of China (11371003, 11461006, 11461005), the Natural Science Foundation of Guangxi (2011GXNSFA018154, 2012GXNSFGA060003), the Science and Technology Foundation of Guangxi (10169-1) and the Science Computing and Intelligent Information Processing of Guangxi Higher Education Key Laboratory(GXSCIIP201201).

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relation is, however, too restrictive for many practical applications. To address this issue, many interesting and meaningful extensions of Pawlak's rough sets have been presented in the literature. Equivalence relations can be replaced by tolerance relations [23], similarity relations [24], binary relations [7, 27].

Various fuzzy generalizations of rough approximations have been proposed in the literature [1, 2, 6, 10, 11, 15, 21, 26, 29]. The most common fuzzy rough set is obtained by replacing the crisp binary relations with fuzzy relations on the universe and the crisp subsets with the fuzzy sets.

There are mainly two approaches to the development of rough set theory. One is the constructive approach in which rough approximation operators are constructed by means of relations, partitions, coverings, neighborhood systems and so on. The constructive approach is suitable for practical applications of rough sets. The other one is the axiomatic approach. In this approach, a set of axioms is used to characterize rough approximation operators that guarantee the existence of certain types of relations which produce the same operators. This approach is appropriate for studying algebra structures of rough sets. Under this point of view, rough set theory may be interpreted as an extension of set theory with two additional unary operators.

As a generalization of Zadeh's fuzzy set, interval-valued fuzzy (IVF, for short) sets were introduced by Gorzalczany [4] and Turksen [25], and they were applied to the fields of approximate inference, signal transmission and controller, etc. Mondal et al. [14] defined topology of IVF sets and studied their properties.

By integrating Pawlak rough set theory with IVF set theory, Sun et al. [22] introduced IVF rough sets based on an IVF approximation space, defined IVF information systems and discussed their attribute reduction. Gong et al. [5] studied the knowledge discovery in IVF information systems. Zhang et al. [30] discussed $(\mathcal{I},\mathcal{T})$-IVF rough sets based on an IVF approximation space on two universes of discourse.

The purpose of this paper is to investigate IVF rough approximation operators by using constructive and axiomatic approaches.

2 Preliminaries

Throughout this paper, " interval-valued fuzzy " denote briefly by " IVF ". U denotes a nonempty finite set called the universe of discourse. I denotes $[0,1]$ and $[I]$ denotes $\{[a,b] : a,b \in I \ and \ a \leq b\}$. $\mathscr{P}(U)$ denotes the family of all subsets of U. $F^{(i)}(U)$ denotes the family of all IVF sets in U. $\bar{a}$ denotes $[a,a]$ for each $a \in [0,1]$.

2.1 IVF sets

For any $[a_j,b_j] \in [I](\ j=1,2)$, we define

$$[a_1,b_1]=[a_2,b_2] \iff a_1=a_2, b_1=b_2;$$

$$[a_1,b_1]\leq[a_2,b_2] \iff a_1\leq a_2, b_1\leq b_2;$$

$$[a_1, b_1] < [a_2, b_2] \iff [a_1, b_1] \leq [a_2, b_2] \text{ and } [a_1, b_1] \neq [a_2, b_2];$$

$$\bar{1} - [a_1, b_1] \text{ or } [a_1, b_1]^c = [1 - b_1, 1 - a_1].$$

Obviously, $([a, b]^c)^c = [a, b]$ for each $[a, b] \in [I]$.

Definition 2.1 ([4, 25])**.** *For each $\{[a_j, b_j] : j \in J\} \subseteq [I]$, we define*

$$\bigvee_{j\in J} [a_j, b_j] = [\bigvee_{j\in J} a_j, \bigvee_{j\in J} b_j] \text{ and } \bigwedge_{j\in J} [a_j, b_j] = [\bigwedge_{j\in J} a_j, \bigwedge_{j\in J} b_j],$$

where $\bigvee\limits_{j\in J} a_j = sup\ \{a_j : j \in J\}$ and $\bigwedge\limits_{j\in J} a_j = inf\ \{a_j : j \in J\}$.

Definition 2.2 ([4, 25])**.** *An IVF set A in U is defined by a mapping $A : U \to [I]$. Denote*

$$A(x) = [A^-(x), A^+(x)] \quad (x \in U).$$

Then $A^-(x)$ (resp. $A^+(x)$) is called the lower (resp. upper) degree to which x belongs to A. A^- (resp. A^+) is called the lower (resp. upper) IVF set of A.

The set of all IVF sets in U is denoted by $F^{(i)}(U)$.

Let $a, b \in I$. $\widetilde{[a, b]}$ represents the IVF set which satisfies $\widetilde{[a, b]}(x) = [a, b]$ for each $x \in U$. We denoted $\widetilde{[a, a]}$ by $\tilde{a}$.

We recall some basic operations on $F^{(i)}(U)$ as follows ([4, 25]): for any $A, B \in F^{(i)}(U)$ and $[a, b] \in [I]$,

(1) $A = B \iff A(x) = B(x)$ for each $x \in U$.
(2) $A \subseteq B \iff A(x) \leq B(x)$ for each $x \in U$.
(3) $A = B^c \iff A(x) = B(x)^c$ for each $x \in U$.
(4) $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in U$.
(5) $(A \cup B)(x) = A(x) \vee B(x)$ for each $x \in U$.
Moreover,

$$(\bigcup_{j\in J} A)(x) = \bigvee_{j\in J} A(x) \text{ and } (\bigcap_{j\in J} A)(x) = \bigwedge_{j\in J} A(x),$$

where $\{A_j : j \in J\} \subseteq F^{(i)}(U)$.

(6) $([a, b]A)(x) = [a, b] \wedge [A^-(x), A^+(x)]$ for each $x \in U$.
Obviously,

$$A = B \iff A^- = B^- \text{ and } A^+ = B^+ \ ; \ (\widetilde{[a, b]})^c = \widetilde{[a, b]^c} \quad ([a, b] \in [I]).$$

Definition 2.3 ([14])**.** *$A \in F^{(i)}(U)$ is called an IVF point in U, if there exist $[a, b] \in [I] - \{\bar{0}\}$ and $x \in U$ such that*

$$A(y) = \begin{cases} [a, b], & y = x, \\ \bar{0}, & y \neq x. \end{cases}$$

We denote A by $x_{[a,b]}$.

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If $[a,b] = \bar{1}$, then

$$x_{\bar{1}}(y) = \begin{cases} \bar{1}, & y = x, \\ \bar{0}, & y \neq x. \end{cases}$$

Remark 2.4. $A = \bigcup_{x \in U} (A(x)x_{\bar{1}})$.

2.2 Definition of IVF rough approximation operators

Recall that R is called an IVF relation on U if $R \in F^{(i)}(U \times U)$.

Definition 2.5 ([7, 22])**.** *Let R be an IVF relation on U. Then R is called*

(1) *reflexive, if $R(x,x) = \bar{1}$ for each $x \in U$.*

(2) *transitive, if $R(x,z) \geq R(x,y) \wedge R(y,z)$ for any $x,y,z \in U$.*

(3) *preorder, if R is reflexive and transitive.*

Definition 2.6 ([22])**.** *Let R be an IVF relation on U. The pair (U,R) is called an IVF approximation space. For each $A \in F^{(i)}(U)$, the IVF lower and the IVF upper approximation of A with respect to (U,R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two IVF sets and are respectively defined as follows:*

$$\underline{R}(A)(x) = \bigwedge_{y \in U} (A(y) \vee (\bar{1} - R(x,y))) \ (x \in U)$$

and

$$\overline{R}(A)(x) = \bigvee_{y \in U} (A(y) \wedge R(x,y)) \ (x \in U).$$

The the pair $(\underline{R}(A), \overline{R}(A))$ is called the IVF rough set of A with respect to (U,R).

$\underline{R} : F^{(i)}(U) \to F^{(i)}(U)$ and $\overline{R} : F^{(i)}(U) \to F^{(i)}(U)$ are called the IVF lower approximation operator and the IVF upper approximation operator, respectively. In general, we refer to $\underline{R}$ and $\overline{R}$ as the IVF rough approximation operators.

Remark 2.7. *Let (U,R) be an IVF approximation space. Then*

(1) *for each $x,y \in U$,*

$$\overline{R}(x_{\bar{1}})(y) = R(y,x) \quad \text{and} \quad \underline{R}((x_{\bar{1}})^c)(y) = \bar{1} - R(y,x).$$

(2) *for each $[a,b] \in [I]$, $\underline{R}(\widetilde{[a,b]}) \supseteq \widetilde{[a,b]} \supseteq \overline{R}(\widetilde{[a,b]})$.*

Proposition 2.8 ([22])**.** *Let (U,R) be an IVF approximation space. Then for each $A \in F^{(i)}(U)$,*

$$(\underline{R}(A))^- = \underline{R^+}(A^-), \ (\underline{R}(A))^+ = \underline{R^-}(A^+),$$

$$(\overline{R}(A))^- = \overline{R^-}(A^-) \quad \text{and} \quad (\overline{R}(A))^+ = \overline{R^+}(A^+).$$

3 IVF rough approximation operators

In this section, we deeply investigate IVF rough approximation operators.

3.1 Construction of IVF rough approximation operators

Theorem 3.1 ([28]). *Let (U,R) be an IVF approximation space. Then for any $A,B \in F^{(i)}(U)$, $\{A_j : j \in J\} \subseteq F^{(i)}(U)$ and $[a,b] \in [I]$,*

(1) $\underline{R}(\tilde{1}) = \tilde{1}$, $\overline{R}(\tilde{0}) = \tilde{0}$.

(2) $A \subseteq B \Longrightarrow \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$.

(3) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$.

(4) $\underline{R}(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \underline{R}(A_j)$, $\overline{R}(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} \overline{R}(A_j)$.

(5) $\underline{R}(\widetilde{[a,b]} \cup A) = \widetilde{[a,b]} \cup \underline{R}(A)$, $\overline{R}([a,b]A) = [a,b]\overline{R}(A)$.

Theorem 3.2 ([28]). *Let (U,R) be an IVF approximation space. Then*

$$\begin{array}{lcl}
(1)\ R\ is\ reflexive & \Longleftrightarrow & (ALR)\ \forall A \in F^{(i)}(U), \underline{R}(A) \subseteq A. \\
 & \Longleftrightarrow & (AUR)\ \forall A \in F^{(i)}(U), A \subseteq \overline{R}(A). \\
(2)\ R\ is\ transitive & \Longleftrightarrow & (ALT)\ \forall A \in F^{(i)}(U), \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)). \\
 & \Longleftrightarrow & (AUT)\ \forall A \in F^{(i)}(U), \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A).
\end{array}$$

Corollary 3.3 ([28]). *Let (U,R) be an IVF approximation space. If R is preorder, then*

$$\underline{R}(\underline{R}(A)) = \underline{R}(A) \quad and \quad \overline{R}(\overline{R}(A)) = \overline{R}(A) \quad (A \in F^{(i)}(U)).$$

Let $A \in F^{(i)}(U)$. Denote

$$\begin{array}{rcl}
A_\lambda & = & \{(x) \in U : A^-(x) \geq \lambda\} \quad (\lambda \in I), \\
A^\lambda & = & \{(x) \in U : A^+(x) \geq \lambda]\} \quad (\lambda \in I), \\
A_{\lambda^+} & = & \{(x) \in U : A^-(x) > \lambda\} \quad (\lambda \in [0,1)), \\
A^{\lambda^+} & = & \{(x) \in U : A^+(x) > \lambda]\} \quad (\lambda \in [0,1)).
\end{array}$$

Definition 3.4 ([4, 25]). *Let $A \in F^{(i)}(U)$ and $[\alpha,\beta] \in [I]$. Denote*

$$A_{[\alpha,\beta]} = \{x \in U : A^-(x) \geq \alpha, A^+(x) \geq \beta]\},$$

$$A_{[\alpha,\beta]^+} = \{x \in U : A(x) > [\alpha,\beta]\},$$

$$A_{(\alpha,\beta)} = \{x \in U : A^-(x) > \alpha, A^+(x) > \beta]\}.$$

Then $A_{[\alpha,\beta]}$ (resp.$A_{[\alpha,\beta]^+}, A_{(\alpha,\beta)}$) is called the $[\alpha,\beta]$-level (resp. strong $[\alpha,\beta]$-level, (α,β)-level) set of A.

Obviously, $A_{(\alpha,\beta)} \subseteq A_{[\alpha,\beta]^+} \subseteq A_{[\alpha,\beta]}$.

Proposition 3.5 ([4, 25]). *Let $A,B \in F^{(i)}(U)$ and $[\alpha,\beta] \in [I]$. Then*

(1) $A \subseteq B \Longrightarrow A_{[\alpha,\beta]^+} \subseteq B_{[\alpha,\beta]^+}$;

(2) $(A \cup B)_{[\alpha,\beta]^+} \supseteq A_{[\alpha,\beta]^+} \cup B_{[\alpha,\beta]^+}$;

(2) $(A \cap B)_{[\alpha,\beta]^+} = A_{[\alpha,\beta]^+} \cap B_{[\alpha,\beta]^+}$.

Let $R \in F^{(i)}(U \times U)$. Denote

$$\begin{aligned}
R_\lambda &= \{(x,y) \in U \times U : R^-(x,y) \geq \lambda\} \quad (\lambda \in I),\\
R^\lambda &= \{(x,y) \in U \times U : R^+(x,y) \geq \lambda]\} \quad (\lambda \in I),\\
R_{\lambda^+} &= \{(x,y) \in U \times U : R^-(x,y) > \lambda\} \quad (\lambda \in [0,1)),\\
R^{\lambda^+} &= \{(x,y) \in U \times U : R^+(x,y) > \lambda]\} \quad (\lambda \in [0,1)),\\
R_{[\alpha,\beta]} &= \{(x,y) \in U \times U : R(x,y) \geq [\alpha,\beta]\} \quad ([\alpha,\beta] \in [I]),\\
R_{[\alpha,\beta]^+} &= \{(x,y) \in U \times U : R(x,y) > [\alpha,\beta]\} \quad (\alpha < 1, [\alpha,\beta] \in [I]).
\end{aligned}$$

Proposition 3.6. *Let R be an IVF relation on U.*
(1) *If R is reflexive, then R_λ, R^λ, R_{λ^+}, R^{λ^+} and $R_{[\alpha,\beta]^+}$ are reflexive.*
(2) *If R is transitive, then R_λ, R^λ, R_{λ^+}, R^{λ^+} and $R_{[\alpha,\beta]^+}$ are transitive.*

Proof. (1) are obvious.
(2) For any $x,y,z \in U$, if $(x,y),(y,z) \in R_\lambda$, we have $R^-(x,y) \geq \lambda$ and $R^-(y,z) \geq \lambda$. Note that R is transitive. Then $R(x,z) \geq R(x,y) \wedge R(y,z)$ and so

$$R^-(x,z) \geq R^-(x,y) \wedge R^-(y,z) \geq \lambda.$$

Thus $(x,z) \in R_\lambda$. Hence R_λ is transitive.
Similarly, We can prove that R^λ, R_{λ^+} and R^{λ^+} are transitive.
For any $x,y,z \in U$, if $(x,y),(y,z) \in R_{[\alpha,\beta]^+}$, we have $R(x,y) > [\alpha,\beta]$ and $R(y,z) > [\alpha,\beta]$. Note that R is transitive. Then

$$R(x,z) \geq R(x,y) \wedge R(y,z) > [\alpha,\beta].$$

and so $(x,z) \in R_{[\alpha,\beta]^+}$. Hence $R_{[\alpha,\beta]^+}$ is transitive. □

Theorem 3.7. *Let (U,R) be an IVF approximation space. Then IVF rough approximation operator can be represented as follows: for each $A \in F^{(i)}(U)$,*

(1) $(\underline{R}(A))^- = \bigcup\limits_{\lambda\in I} \lambda \underline{R^{1-\lambda}}(A_\lambda) = \bigcup\limits_{\lambda\in I} \lambda \underline{R^{1-\lambda}}(A_{\lambda^+}),$
$= \bigcup\limits_{\lambda\in I} \lambda \underline{R^{(1-\lambda)^+}}(A_\lambda) = \bigcup\limits_{\lambda\in I} \lambda \underline{R^{(1-\lambda)^+}}(A_{\lambda^+});$

(2) $(\underline{R}(A))^+ = \bigcup\limits_{\lambda\in I} \lambda \underline{R_{1-\lambda}}(A^\lambda) = \bigcup\limits_{\lambda\in I)} \lambda \underline{R_{1-\lambda}}(A^{\lambda^+}),$
$= \bigcup\limits_{\lambda\in I} \lambda \underline{R_{(1-\lambda)^+}}(A^\lambda) = \bigcup\limits_{\lambda\in I} \lambda \underline{R_{(1-\lambda)^+}}(A^{\lambda^+});$

(3) $(\overline{R}(A))^- = \bigcup\limits_{\lambda\in I} \lambda \overline{R_\lambda}(A_\lambda) = \bigcup\limits_{\lambda\in I} \lambda \overline{R_{\lambda^+}}(A_\lambda),$
$= \bigcup\limits_{\lambda\in I} \lambda \overline{R_\lambda}(A_{\lambda^+}) = \bigcup\limits_{\lambda\in I} \lambda \overline{R_{\lambda^+}}(A_{\lambda^+});$

(4) $(\overline{R}(A))^+ = \bigcup\limits_{\lambda\in I} \lambda \overline{R^\lambda}(A^\lambda) = \bigcup\limits_{\lambda\in I} \lambda \overline{R^{\lambda^+}}(A^\lambda),$
$= \bigcup\limits_{\lambda\in I} \lambda \overline{R^\lambda}(A^{\lambda^+}) = \bigcup\limits_{\lambda\in I} \lambda \overline{R^{\lambda^+}}(A^{\lambda^+});$

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Proof. (1) For each $x \in U$, by Proposition 2.10,

$$\begin{aligned}
(\bigcup_{\lambda\in I} \lambda \underline{R^{1-\lambda}}(A_\lambda))(x) &= \bigvee\{\lambda \in I : x \in \underline{R^{1-\lambda}}(A_\lambda)\}\\
&= \bigvee\{\lambda \in I : (R^{1-\lambda})_s(x) \subseteq A_\lambda\}\\
&= \bigvee\{\lambda \in I : R^{+}(x,y) \geq 1-\lambda \ implies\ A^{-}(y) \geq \lambda\}\\
&= \bigvee\{\lambda \in I : 1-R^{+}(x,y) \leq \lambda \ implies\ A^{-}(y) \geq \lambda\}\\
&= \bigvee\{\lambda \in I : \bigwedge_{y\in U}(A^{-}(x) \vee (1-R^{+}(x,y))) \geq \lambda\}\\
&= \bigvee\{\lambda \in I : (\underline{R}(A))^{-}(x) \geq \lambda\}\\
&= (\underline{R}(A))^{-}(x).
\end{aligned}$$

Then $(\underline{R}(A))^{-} = \bigcup_{\lambda\in I} \lambda \underline{R^{1-\lambda}}(A_\lambda)$.

Similarly, we can prove that

$$(\underline{R}(A))^{-} = \bigcup_{\lambda\in[0,1)} \lambda \underline{R^{1-\lambda}}(A_{\lambda^+}) = \bigcup_{\lambda\in(0,1]} \lambda \underline{R^{(1-\lambda)^+}}(A_\lambda) = \bigcup_{\lambda\in(0,1)} \lambda \underline{R^{(1-\lambda)^+}}(A_{\lambda^+}).$$

(2) The proof is similar to (1).

(3) For each $x \in U$, by Proposition 2.10,

$$\begin{aligned}
(\bigcup_{\lambda\in I} \lambda(\overline{R_\lambda}(A_\lambda)))(x) &= \bigvee\{\lambda \in I : x \in \overline{R_\lambda}(A_\lambda)\}\\
&= \bigvee\{\lambda \in I : (R_\lambda)_s(x) \cap A_\lambda \neq \emptyset\}\\
&= \bigvee\{\lambda \in I : \exists y \in U, y \in A_\lambda \cap (R_\lambda)_s(x)\}\\
&= \bigvee\{\lambda \in I : \exists y \in U, A^{-}(y) \wedge R^{-}(x,y) \geq \lambda\}\\
&= \bigvee\{\lambda \in I : \bigvee_{y\in U}(A^{-}(y) \wedge R^{-}(x,y)) \geq \lambda\}\\
&= \bigvee\{\lambda \in I : (\overline{R}(A))^{-}(x) \geq \lambda\}\\
&= (\overline{R}(A))^{-}(x).
\end{aligned}$$

Then $\bigcup_{\lambda\in I} \lambda(\overline{R_\lambda}(A_\lambda)) = (\overline{R}(A))^{-}$.

Similarly, we can prove that

$$(\overline{R}(A))^{-} = \bigcup_{\lambda\in[0,1)} \lambda(\overline{R_{\lambda^+}}(A_\lambda)) = \bigcup_{\lambda\in[0,1)} \lambda(\overline{R_\lambda}(A_{\lambda^+})) = \bigcup_{\lambda\in[0,1)} \lambda(\overline{R_{\lambda^+}}(A_{\lambda^+})).$$

(4) The proof is similar to (3). □

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Theorem 3.8. *Let (U, R) be an IVF approximation space. Then IVF rough approximation operator can be represented as follows: for each $A \in F^{(i)}(U)$,*

$$\begin{aligned}\overline{R}(A) &= \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+})) = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]})) \\ &= \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]}}(A_{[\alpha,\beta]^+})).\end{aligned}$$

Proof. Denote $B = \bigcup\limits_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+}))$. By Proposition 2.10,

$$\begin{aligned}B^-(x) &= \bigvee_{\alpha\in I} (\alpha \wedge \overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+})(x)) \\ &= \bigvee\{\alpha \in I : x \in (\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+}))\} \\ &= \bigvee\{\alpha \in I : (R_{[\alpha,\beta]^+})_s(x) \cap A_{[\alpha,\beta]^+} \neq \emptyset\} \\ &= \bigvee\{\alpha \in I : \exists y \in U, R(x,y) > [\alpha,\beta] \text{ and } A(y) > [\alpha,\beta]\} \\ &= \bigvee\{\alpha \in I : \exists y \in U, A^-(y) \wedge R^-(x,y) > \alpha \text{ and } A^+(y) \wedge R^+(x,y) \\ &\quad \geq \beta \text{ or } A^-(y) \wedge R^-(x,y) \geq \alpha \text{ and } A^+(y) \wedge R^+(x,y) > \beta\} \\ &= \bigvee_{y\in U} (A^-(y) \wedge R^-(x,y)) = (\overline{R}(A))^-(x).\end{aligned}$$

Then $(\overline{R}(A))^- = B^-$. Similarly, we can prove that $(\overline{R}(A))^+ = B^+$. Hence

$$\overline{R}(A) = B = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]^+})).$$

Similarly, we can prove that

$$\overline{R}(A) = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]^+}}(A_{[\alpha,\beta]})) = \bigcup_{[\alpha,\beta]\in[I]} ([\alpha,\beta]\overline{R_{[\alpha,\beta]}}(A_{[\alpha,\beta]^+})).$$

□

3.2 Axiomatic characterizations of IVF rough approximation operators

In an axiomatic approach, rough sets are axiomatized by abstract operators. For the case of IVF rough sets, the primitive notion is the system $(F^{(i)}(U), \bigcap, \bigcup, c, L, H)$, where $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two IVF set operators. In this subsection, rough approximation operators in the IVF environment are characterized by some axioms.

Definition 3.9. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two IVF set operators. If*

$$(L(A))^c = H(A^c) \ (A \in F^{(i)}(U)),$$

then L, H are called two dual operators.

Remark 3.10. *$L, H : F^{(i)}(U) \to F^{(i)}(U)$ are two dual operators iff $(H(A))^c = L(A^c)$ for each $A \in F^{(i)}(U)$.*

Theorem 3.11. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists an IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axioms $(AL1)$ and $(AL2)$, or equivalently, H satisfies axioms $(AU1)$ and $(AU2)$:*

$$(AL1) \quad L(\widetilde{[a,b]} \cup A) = \widetilde{[a,b]} \cup L(A) \ (A \in F^{(i)}(U), [a,b] \in [I]),$$

$$(AL2) \qquad L(A \cap B) = \underline{R}(A) \cap L(B) \ (A, B \in F^{(i)}(U);$$

$$(AU1) \qquad H([a,b]A) = [a,b]H(A) \ (A \in F^{(i)}(U), [a,b] \in [I]),$$

$$(AU2) \qquad H(A \cup B) = H(A) \cup H(B) \ (A, B \in F^{(i)}(U)).$$

Proof. Note that $L, H : F^{(i)}(U) \to F^{(i)}(U)$ are two dual operators. Then $(AL1)$ and $(AL2)$ are equivalent to $(AU1)$ and $(AU2)$. We only need to prove that $L = \underline{R}$ and $H = \overline{R}$ iff H satisfies axioms $(AU1)$ and $(AU2)$.

Necessity. This is obvious.

Sufficiency. Assume that the operator H satisfies axioms $(AU1)$ and $(AU2)$. Define an IVF relation R on U by

$$R(x, y) = H(y_{\bar{1}})(x) \quad (x, y \in U).$$

Let $A \in F^{(i)}(U)$. Note that

$$\begin{aligned} H(A)(x) &= H(\bigcup_{y \in U}(A(y)y_{\bar{1}}))(x) = (\bigcup_{y \in U} H(A(y)y_{\bar{1}}))(x) = (\bigcup_{y \in U}(A(y)H(y_{\bar{1}})))(x) \\ &= \bigvee_{y \in U}(A(y) \wedge H(y_{\bar{1}})(x)) = \bigvee_{y \in U}(A(y) \wedge R(x, y)) = \overline{R}(A)(x) \end{aligned}$$

for each $x \in U$. Then $H(A) = \overline{R}(A)$. By Theorem 3.1(3),

$$L(A) = (H(A^c))^c = (\overline{R}(A^c))^c = \underline{R}(A).$$

Thus $L = \underline{R}$, $H = \overline{R}$. □

Theorem 3.12. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists a reflexive IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axiom $(AL1)$, $(AL2)$ and (ALR), or equivalently, H satisfies axiom $(AU1)$, $(AU2)$ and (AUR):*

$$(ALR) \quad L(A) \subseteq A \quad (A \in F^{(i)}(U));$$

$$(AUR) \quad A \subseteq H(A) \quad (A \in F^{(i)}(U)).$$

Proof. This holds by Theorems 3.2(1) and 3.11. □

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Theorem 3.13. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists a symmetric IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axiom $(AL1)$, $(AL2)$ and (ALS), or equivalently, H satisfies axiom $(AU1)$, $(AU2)$ and (AUS):*

$$(ALS) \quad L((x_{\bar{1}})^c)(y) = L((y_{\bar{1}})^c)(x) \ (x, y \in U);$$
$$(ALS) \quad H(x_{\bar{1}})(y) = H(y_{\bar{1}})(x) \ (x, y \in U).$$

Proof. This hold by Remark 2.9(1) and Theorem 3.11. □

Theorem 3.14. *Let $L, H : F^{(i)}(U) \to F^{(i)}(U)$ be two dual operators. Then there exists a transitive IVF relation R on U such that $L = \underline{R}$ and $H = \overline{R}$ iff L satisfies axiom $(AL1)$, $(AL2)$ and (ALT), or equivalently, H satisfies axiom $(AU1)$, $(AU2)$ and (AUT):*

$$(ALT) \quad L(A) \subseteq L(L(A)) \quad (A \in F^{(i)}(U));$$
$$(AUT) \quad H(H(A)) \subseteq H(A) \quad (A \in F^{(i)}(U)).$$

Proof. This holds by Theorems 3.2(2) and 3.11. □

4 IVF pseudo-closure operators in IVF approximation spaces

In this section, we investigate IVF pseudo-closure operators in IVF approximation spaces.

For each $[a, b] \in [I], X \in \mathscr{P}(U)$, we define

$$([a, b]X)(x) = \begin{cases} [a, b], & x \in X, \\ \bar{0}, & x \in U - X. \end{cases}$$

Denote

$$\mathscr{E}(U) = \{[a, b]X : [a, b] \in [I], X \in \mathscr{P}(U)\}.$$

Then $\mathscr{E}(U) \subseteq F^{(i)}(U)$.

Definition 4.1. *Let τ be an IVF topology on U. Define*

$$S_\tau(A) = \bigcup_{[\alpha,\beta] \in [I]} cl_\tau([\alpha, \beta]A_{[\alpha,\beta]}\} \quad (A \in F^{(i)}(U)).$$

Then $S_\tau : F^{(i)}(U) \to F^{(i)}(U)$ is called the IVF pseudo-closure operator induced by τ on U.

Theorem 4.2 ([25])**.** *Let $A \in F^{(i)}(U)$. Then*

$$A = \bigcup_{[\alpha,\beta] \in [I]} [\alpha, \beta]A_{[\alpha,\beta]} = \bigcup_{[\alpha,\beta] \in [I]} [\alpha, \beta]A_{(\alpha,\beta)}.$$

Theorems 4.3(5) and 4.4 below illustrate the meaning on IVF pseudo-closure operators.

Theorem 4.3. *Let τ be an IVF topology on U and let S_τ be the IVF pseudo-closure operator induced by τ on U. Then for any $A, B \in F^{(i)}(U)$,*
(1) $S_\tau(\tilde{0}) = \tilde{0}$.
(2) $A \subseteq S_\tau(A) \subseteq cl_\tau(A)$.
(3) $S_\tau(A \cup B) \supseteq S_\tau(A) \cup S_\tau(B)$. $S_\tau(A \cap B) \subseteq S_\tau(A) \cap S_\tau(B)$.
(4) $A \in \tau^c \Longrightarrow S_\tau(A) = A$.
(5) *S_τ coincides with cl_τ as operators from $\mathscr{E}(U)$ to $F^{(i)}(U)$.*

Proof. (1) For any $[\alpha,\beta] \in [I]$ and $x \in U$, since

$$([\alpha,\beta]\tilde{0}_{[\alpha,\beta]})(x) = [\alpha,\beta] \wedge \tilde{0}_{[\alpha,\beta]}(x) = \begin{cases} [0,0] \wedge \bar{1} = \bar{0}, & [\alpha,\beta] = \bar{0}, \\ [\alpha,\beta] \wedge \bar{0} = \bar{0}, & [\alpha,\beta] \in [I] - \{\bar{0}\}. \end{cases}$$

we have $[\alpha,\beta]\tilde{0}_{[\alpha,\beta]} = \tilde{0}$. Thus

$$S_\tau(\tilde{0}) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]\tilde{0}_{[\alpha,\beta]}) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\tilde{0}) = \tilde{0}.$$

(2) By Theorem 4.2,

$$A = \bigcup_{[\alpha,\beta]\in[I]} [\alpha,\beta]A_{[\alpha,\beta]} \subseteq \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) = S_\tau(A) \quad and$$

$$S_\tau(A) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) \subseteq cl_\tau(\bigcup_{[\alpha,\beta]\in[I]} [\alpha,\beta]A_{[\alpha,\beta]}) = cl_\tau(A).$$

(3) For any $A, B \in F^{(i)}(U)$, $[\alpha,\beta] \in [I]$ and $x \in U$ put

$$C(x) = \begin{cases} \bar{1}, & x \in A_{[\alpha,\beta]}, \\ \bar{0}, & x \in U - A_{[\alpha,\beta]} \end{cases}, \quad D(x) = \begin{cases} \bar{1}, & x \in B_{[\alpha,\beta]}, \\ \bar{0}, & x \in U - B_{[\alpha,\beta]}. \end{cases}$$

Obviously,

$$[\alpha,\beta]A_{[\alpha,\beta]} = \widetilde{[\alpha,\beta]} \cap C, \quad [\alpha,\beta]B_{[\alpha,\beta]} = \widetilde{[\alpha,\beta]} \cap D,$$

$$[\alpha,\beta](A_{[\alpha,\beta]} \cup B_{[\alpha,\beta]}) = \widetilde{[\alpha,\beta]} \cap (C \cup D)$$

and

$$[\alpha,\beta](A_{[\alpha,\beta]} \cap B_{[\alpha,\beta]}) = \widetilde{[\alpha,\beta]} \cap (C \cap D).$$

We can easily prove that

$$(A \cup B)_{[\alpha,\beta]} \supseteq A_{[\alpha,\beta]} \cup B_{[\alpha,\beta]} \quad and \quad (A \cap B)_{[\alpha,\beta]} = A_{[\alpha,\beta]} \cap B_{[\alpha,\beta]}.$$

By Proposition 2.6(5),

11

$$
\begin{aligned}
& S_\tau(A \cup B) \\
= & \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A\cup B)_{[\alpha,\beta]}) \supseteq \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A_{[\alpha,\beta]} \cup B_{[\alpha,\beta]})) \\
= & \bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap (C\cup D)) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau((\widetilde{[\alpha,\beta]} \cap C) \cup (\widetilde{[\alpha,\beta]} \cap D)) \\
= & \bigcup_{[\alpha,\beta]\in[I]} (cl_\tau(\widetilde{[\alpha,\beta]} \cap C) \cup cl_\tau(\widetilde{[\alpha,\beta]} \cap D)) \\
= & \Big(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap C) \Big) \cup \Big(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap D) \Big) \\
= & \Big(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) \Big) \cup \Big(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]B_{[\alpha,\beta]}) \Big) \\
= & S_\tau(A) \cup S_\tau(B).
\end{aligned}
$$

By Proposition 2.6(3),

$$
\begin{aligned}
& S_\tau(A \cap B) \\
= & \bigcap_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A\cap B)_{[\alpha,\beta]}) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta](A_{[\alpha,\beta]} \cap B_{[\alpha,\beta]})) \\
= & \bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap (C\cap D)) = \bigcup_{[\alpha,\beta]\in[I]} cl_\tau((\widetilde{[\alpha,\beta]} \cap C) \cap (\widetilde{[\alpha,\beta]} \cap D)) \\
\subseteq & \bigcup_{[\alpha,\beta]\in[I]} (cl_\tau(\widetilde{[\alpha,\beta]} \cap C) \cap cl_\tau(\widetilde{[\alpha,\beta]} \cap D)) \\
\subseteq & \Big(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap C) \Big) \cap \Big(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau(\widetilde{[\alpha,\beta]} \cap D) \Big) \\
= & \Big(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) \Big) \cap \Big(\bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]B_{[\alpha,\beta]}) \Big) \\
= & S_\tau(A) \cap S_\tau(B).
\end{aligned}
$$

(4) By (2) and Proposition 2.6(6),

$$cl_\tau(A) \subseteq S(cl_\tau(A)) \subseteq cl_\tau(cl_\tau(A)) = cl_\tau(A),$$

Note that $A \in \tau^c$. Then

$$S_\tau(A) = S_\tau(cl_\tau(A)) = cl_\tau(A) = A.$$

(5) Let $A \in \mathscr{E}(U)$. Then there exist $[a,b] \in [I]$ and $X \in \mathscr{P}(U)$ such that $A = [a,b]X$.

(i) If $[a,b] \neq \bar{0}$, then for each $x \in U$,

$$A_{[a,b]}(x) = ([a,b]X)_{[a,b]}(x) = \begin{cases} \bar{1}, & ([a,b]X)(x) \geq [a,b] \\ \bar{0}, & ([a,b]X)(x) \ngeq [a,b] \end{cases} = \begin{cases} \bar{1}, & x \in X, \\ \bar{0}, & x \in U - X. \end{cases}$$

Thus $A_{[a,b]} = X$. So

$$\begin{aligned} S_\tau(A) &= \bigcup_{[\alpha,\beta]\in[I]} cl_\tau([\alpha,\beta]A_{[\alpha,\beta]}) \\ &\supseteq cl_\tau([a,b]A_{[a,b]}) = cl_\tau([a,b]X) = cl_\tau(A). \end{aligned}$$

By (2), $S_\tau(A) \subseteq cl(A)$. Thus $S_\tau(A) = cl_\tau(A)$.
(ii) If $[a,b] = \bar{0}$, then $A = \tilde{0}$. By (1), $S_\tau(\tilde{0}) = \tilde{0}$. Thus $S_\tau(A) = cl_\tau(A)$.
By (i) and (ii),

$$S_\tau \text{ coincides with } cl_\tau \text{ as operators from } \mathscr{E}(U) \text{ to } F^{(i)}(U).$$

□

Theorem 4.4. *Let (U,R) be an IVF approximation space. If R is preorder, then*

$$\overline{R}(A) = S_{\tau_R}(A) \quad (A \in \mathscr{E}(U)).$$

Proof. For each $A \in \mathscr{E}(U)$, by Theorems 3.11(3) and 4.3(5),

$$\overline{R}(A) = cl_{\tau_R}(A) = S_{\tau_R}(A).$$

□

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.1, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

SOME WEIGHTED HERMITE-HADAMARD TYPE INEQUALITIES FOR GEOMETRICALLY–ARITHMETICALLY CONVEX FUNCTIONS ON THE CO-ORDINATES

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ABSTRACT. In this paper, the concept of GA-convex functions on the co-ordinates is introduced. By using a concept of GA-convex functions on the co-ordinates, Hermite-Hadamard type inequalities for this class of functions are settled.

1. INTRODUCTION

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ forenamed as convex in the classical touch [24], if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x,\ y \in I$ and $\lambda \in [0,1]$.

Indeed, a vast literature has been written on inequalities using classical convexity but one of the most celebrated is the Hermite-Hadamard inequelity. This double inequality is stated as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $a,\ b \in I$ with $a < b$. Then f is convex on $[a,b]$ iff

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This also reveals that (1.1) can be compulsary as a adequate and sufficient condition to function f to be convex on $[a,b]$.

Hermite-Hadmard inequality (1.1) has recieved considerable attention of many reserchers because of its various applications and usefulness in the field of mathematical inequalities itself as well as in other areas of mathematics. The inequality (1.1) has been extended to various forms by using various generalizations of the definition of classical convex functions and it has also been refined under different hypotheses, see for instance [6, 9, 10, 11, 15, 24, 32] and the references therein.

As stated above the classical convexity has been generalized to different forms and we mention below one of the generalizations of the classical convexity which is known as GA-convexity.

Definition 1. [18, 19] *A function $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be GA-convex function on I if*

$$f\left(x^{\lambda}y^{1-\lambda}\right) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x,\ y \in I$ and $\lambda \in [0,1]$, where $x^{\lambda}y^{1-\lambda}$ and $\lambda f(x)+(1-\lambda)f(y)$ are respectively the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

For results on Hermite-Hadamard type inequalities on GA-convex functions and their applications we refere to a recent articles of Latif [15] and Zhang et al. [32].

Date: March 4, 2016.

2000 *Mathematics Subject Classification.* Primary: 26A15, 26A51; Scondary 52A30.

The definition of classical convexity for functions of of one variables was extended to functions two variables as follows.

Definition 2. [5, 6] *Let $\Delta =: [a,b] \times [c,d] \subseteq \mathbb{R}^2$ with $a < b$ and $c < d$ be a bidimensional interval. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w)$$

holds for all $(x,y), (z,w) \in \Delta$ and $\lambda \in [0,1]$.

The Definition 2 of convex functions on Δ was modified as co-ordinated convex functions by Dragomir in [5].

Definition 3. [5] *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$, $y \in [c,d]$.*

Remark 1. [12] *It is clear that if a function $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then*

$$\begin{aligned} & f(tx + (1-t)z, sy + (1-s)w) \\ & \leq tsf(x,y) + t(1-s)f(x,w) + s(1-t)f(z,y) + (1-t)(1-s)f(z,w), \end{aligned}$$

holds for all $(t,s) \in [0,1] \times [0,1]$ and $x, z \in [a,b], y, w \in [c,d]$.

It is well-known that every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates but converse may not be true (see [5]).

The following inequalities of Hermite-Hadamard type for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ were established in [5, Theorem 1, page 778]:

Most recently, the notion of co-ordinated convexity has also been generalized in a diverse manner and as a result, the author [14] extended the defintion of GA-convex functions of one variable to GA-convex functions of two variables.

Definition 4. [14]*A function $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ is GA-convex on Δ if*

$$f\left(x^{\lambda} z^{1-\lambda}, y^{\lambda} w^{1-\lambda}\right) \leq \lambda f(x,y) + (1-\lambda) f(z,w)$$

holds for all $(x,y), (z,w) \in \Delta$ and $\lambda \in [0,1]$.

A modification in Definition 4 resulted in the notion of GA-convex functions on the co-ordinates on Δ.

Definition 5. [14] *A function $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ is said to be GA-convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \subseteq (0,\infty) \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \subseteq (0,\infty) \to \mathbb{R}$, $f_x(v) = f(x,v)$ are GA-convex where defined for all $x \in [a,b]$, $y \in [c,d]$.*

The following result holds as a consequence of the defintion of GA-convex fuctions on the co-ordinates on Δ.

Remark 2. *If a function $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ is GA-convex on the co-ordinates on Δ. Then*

$$\begin{aligned} & f\left(x^t z^{1-t}, y^s w^{1-s}\right) \\ & \leq tf\left(x, y^s w^{1-s}\right) + (1-t) f\left(z, y^s w^{1-s}\right) \\ & \leq t\left[sf(x,y) + (1-s) f(x,w)\right] + (1-t)\left[sf(z,y) + (1-s) f(z,w)\right] \\ & \leq tsf(x,y) + t(1-s)f(x,w) + s(1-t)f(z,y) + (1-t)(1-s)f(z,w) \end{aligned}$$

holds for all $(t,s) \in [0,1] \times [0,1]$ and $x, z \in [a,b], y, w \in [c,d]$.

In [13], some H-H type inequalities for GA-convex functions on the co-ordinates on Δ were also proved for GA-convex functions on the co-ordinates on Δ. For more results on H-H type inequalities for different generilazations of the defintion of of co-ordinated convex functions we refer the reader to [1], [2], [7]-[12], [16], [20]-[23], [27], [28] and closely related articles mentioned therein.

The main objective of the present paper is to establish some new weighted H-H type inequalities for the class of GA-convex functions on the co-ordinates on a rectangle from the plane in Section 2.

2. Weighted Inequalities for co-ordinated GA-convex functions

For the sake of convenience to the reader, we will use the following notations

$$L_1(t)=a^{\frac{1+t}{2}}b^{\frac{1-t}{2}},L_2(s)=c^{\frac{1+s}{2}}d^{\frac{1-s}{2}},U_1(t)=a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},U_2(s)=c^{\frac{1-s}{2}}d^{\frac{1+s}{2}}.$$

To obtain our main results, we first establish the following weighted identity.

Lemma 1. *Suppose that $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ has second order partial derivatives on Δ° and $[a,b]\times[c,d]\subseteq\Delta^\circ$ with $a<b$ and $c<d$. If $h:[a,b]\times[c,d]\to[0,\infty)$ is twice partially differentiable mapping and $f_{ts}\in L([a,b]\times[c,d])$, then we have*

$$\begin{aligned}
&\Phi(a,b,c,d;f,h)\\
&\quad=h(a,c)f(a,c)-h(a,d)f(a,d)-h(b,c)f(b,c)+h(b,d)f(b,d)\\
&\quad+\int_c^d h_y(a,y)f(a,y)\,dy-\int_c^d h_y(b,y)f(b,y)\,dy-\int_a^b h_x(x,d)f(x,d)\,dx\\
&\quad+\int_a^b h_x(x,c)f(x,c)\,dx+\int_a^b\int_c^d h_{xy}(x,y)f(x,y)\,dydx\\
&=\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\Big[\int_0^1\int_0^1 L_1(t)L_2(s)h(L_1(t),L_2(s))f_{ts}(L_1(t),L_2(s))\,dsdt\\
&\quad+\int_0^1\int_0^1 U_1(t)L_2(s)h(U_1(t),L_2(s))f_{ts}(U_1(t),L_2(s))\,dsdt\\
&\quad+\int_0^1\int_0^1 L_1(t)U_2(s)h(L_1(t),U_2(s))f_{ts}(L_1(t),U_2(s))\,dsdt\\
&\quad+\int_0^1\int_0^1 U_1(t)U_2(s)h(U_1(t),U_2(s))f_{ts}(U_1(t),U_2(s))\,dsdt\Big].
\end{aligned}\tag{2.1}$$

Proof. By letting $x=a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}$, $y=c^{\frac{1+s}{2}}d^{\frac{1-s}{2}}$ and by integration by parts with respect to y and then with respect to x, we have

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 L_1(t)L_2(s)h(L_1(t),L_2(s))f_{ts}(L_1(t),L_2(s))\,dsdt\\
&\quad=\int_a^{\sqrt{ab}}\int_c^{\sqrt{cd}} h(x,y)f_{xy}(x,y)\,dydx=h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right)\\
&-h\left(a,\sqrt{cd}\right)f\left(a,\sqrt{cd}\right)-h\left(\sqrt{ab},c\right)f\left(\sqrt{ab},c\right)+h(a,c)f(a,c)+\int_c^{\sqrt{cd}}h_y(a,y)f(a,y)\,dy\\
&\quad-\int_c^{\sqrt{cd}}h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy-\int_a^{\sqrt{ab}}h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx\\
&\quad+\int_a^{\sqrt{ab}}h_x(x,c)f(x,c)\,dx+\int_a^{\sqrt{ab}}\int_c^{\sqrt{cd}}h_{xy}(x,y)f(x,y)\,dydx.
\end{aligned}\tag{2.2}$$

Similarly, we obtain

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 U_1(t)\,L_2(s)\,h(U_1(t),L_2(s))\,f_{ts}(U_1(t),L_2(s))\,dsdt\\
&=h\left(b,\sqrt{cd}\right)f\left(b,\sqrt{cd}\right)-h(b,c)\,f(b,c)-h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right)\\
&\quad+h\left(\sqrt{ab},c\right)f\left(\sqrt{ab},c\right)-\int_c^{\sqrt{cd}}h_y(b,y)\,f(b,y)\,dy\\
&\quad+\int_c^{\sqrt{cd}}h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy-\int_{\sqrt{ab}}^b h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx\\
&\quad+\int_{\sqrt{ab}}^b h_x(x,c)\,f(x,c)\,dx+\int_{\sqrt{ab}}^b\int_c^{\sqrt{cd}}h_{xy}(x,y)\,f(x,y)\,dydx, \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 L_1(t)\,U_2(s)\,h(L_1(t),U_2(s))\,f_{ts}(L_1(t),U_2(s))\,dsdt\\
&=h\left(\sqrt{ab},d\right)f\left(\sqrt{ab},d\right)-h(a,d)\,f(a,d)-h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right)\\
&\quad+h\left(a,\sqrt{cd}\right)f\left(a,\sqrt{cd}\right)-\int_c^{\sqrt{cd}}h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy\\
&\quad+\int_c^{\sqrt{cd}}h_y(a,y)\,f(a,y)\,dy-\int_a^{\sqrt{ab}}h_x(x,d)\,f(x,d)\,dx\\
&\quad+\int_a^{\sqrt{ab}}h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx+\int_a^{\sqrt{ab}}\int_{\sqrt{cd}}^d h_{xy}(x,y)\,f(x,y)\,dydx \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(\ln b-\ln a)(\ln d-\ln c)}{4}\int_0^1\int_0^1 U_1(t)\,U_2(s)\,h(U_1(t),U_2(s))\,f_{ts}(U_1(t),U_2(s))\,dsdt\\
&=h(b,d)\,f(b,d)-h\left(b,\sqrt{cd}\right)f\left(b,\sqrt{cd}\right)-h\left(\sqrt{ab},d\right)f\left(\sqrt{ab},d\right)\\
&\quad+h\left(\sqrt{ab},\sqrt{cd}\right)f\left(\sqrt{ab},\sqrt{cd}\right)-\int_{\sqrt{cd}}^d h_y(b,y)\,f(b,y)\,dy\\
&\quad+\int_{\sqrt{cd}}^d h_y\left(\sqrt{ab},y\right)f\left(\sqrt{ab},y\right)dy-\int_{\sqrt{ab}}^b h_x(x,d)\,f(x,d)\,dx\\
&\quad+\int_{\sqrt{ab}}^b h_x\left(x,\sqrt{cd}\right)f\left(x,\sqrt{cd}\right)dx+\int_{\sqrt{ab}}^b\int_{\sqrt{cd}}^d h_{xy}(x,y)\,f(x,y)\,dydx. \qquad (2.5)
\end{aligned}$$

Adding (2.2)-(2.5), we get the desired identity. This completes the proof of the lemma. □

Lemma 2. *Let $u,\ v>0$, $\eta,k\in\mathbb{R}$ and $\eta\neq 0$. Then*

$$\begin{aligned}
\zeta(u,v;k,\eta)&=\int_0^1(1-kt)\,u^{\frac{1}{2}+\eta t}v^{\frac{1}{2}-\eta t}dt\\
&=\begin{cases}\frac{kv^{\frac{1}{2}-\eta}u^{\frac{1}{2}}\left[L(u^\eta,v^\eta)-u^\eta\right]}{\eta(\ln u-\ln v)}+v^{\frac{1}{2}-\eta}u^{\frac{1}{2}}L(u^\eta,v^\eta), & u\neq v,\\[2ex] \frac{u\left[1-(1-k)^2\right]}{2k}, & u=v,\end{cases}
\end{aligned}$$

where $L(u,v)$ *is the logarithmic mean*

$$L(u,v)=\begin{cases}\frac{v-u}{\ln v-\ln u}, & u\neq v,\\ u, & u=v.\end{cases}$$

Proof. The proof follows by integration by parts. □

Now we present some new weighted H-H type inequality for GA-convex functions on a rectangle from $\mathbb{R}^2$.

In what follows, we will use the following notation to make our presentation compact.

$$\begin{aligned}\sigma_1(u,v,z,w;q)=&\Big[\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(a,c)|^q\\&+\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\\&\times\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(b,d)|^q\Big]^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}\sigma_2(u,v,z,w;q)=&\Big[\zeta\left(u,v;1,-\frac{1}{2}\right)\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(a,c)|^q\\&+\zeta\left(u,v;1,-\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(u,v;-1,-\frac{1}{2}\right)\\&\times\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(u,v;-1,-\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(b,d)|^q\Big]^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}\sigma_3(u,v,z,w;q)=&\Big[\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;1,-\frac{1}{2}\right)|f_{ts}(a,c)|^q\\&+\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;-1,-\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\\&\times\zeta\left(z,w;1,-\frac{1}{2}\right)|f_{ts}(b,c)|^q+\zeta\left(u,v;1,\frac{1}{2}\right)\zeta\left(z,w;-1,-\frac{1}{2}\right)|f_{ts}(b,d)|^q\Big]^{\frac{1}{q}}\end{aligned}$$

and

$$\begin{aligned}\sigma_4(u,v,z,w;q)=&\Big[\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;-1,\frac{1}{2}\right)|f_{ts}(a,c)|^q\\&+\zeta\left(u,v;-1,\frac{1}{2}\right)\zeta\left(z,w;1,\frac{1}{2}\right)|f_{ts}(a,d)|^q+\zeta(u,v;-1)\\&\times\zeta(z,w;1)|f_{ts}(b,c)|^q+\zeta\left(u,v;-1,-\frac{1}{2}\right)\zeta\left(z,w;-1,-\frac{1}{2}\right)|f_{ts}(b,d)|^q\Big]^{\frac{1}{q}}.\end{aligned}$$

It is easy to observe that when $u=v=z=w=1$, then

$$\sigma_1(1,1,1,1;q)=\left[\frac{9}{4}|f_{ts}(a,c)|^q+\frac{3}{4}|f_{ts}(a,d)|^q+\frac{3}{4}|f_{ts}(b,c)|^q+\frac{1}{4}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},$$

$$\sigma_2(1,1,1,1;q)=\left[\frac{3}{4}|f_{ts}(a,c)|^q+\frac{1}{4}|f_{ts}(a,d)|^q\left|+\frac{9}{4}f_{ts}(b,c)\right|^q+\frac{3}{4}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},$$

$$\sigma_3(1,1,1,1;q)=\left[\frac{3}{4}|f_{ts}(a,c)|^q+\frac{9}{4}|f_{ts}(a,d)|^q+\frac{1}{4}|f_{ts}(b,c)|^q+\frac{3}{4}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}$$

and

$$\sigma_4(1,1,1,1;q)=\left[\frac{1}{4}\left|f_{ts}(a,c)\right|^q+\frac{3}{4}\left|f_{ts}(a,d)\right|^q+\frac{3}{4}\left|f_{ts}(b,c)\right|^q+\frac{9}{4}\left|f_{ts}(b,d)\right|^q\right]^{\frac{1}{q}}.$$

Theorem 1. *Let $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a twice partially differentiable mapping on Δ° and $[a,b]\times[c,d]\subseteq\Delta^\circ$ with $a<b$ and $c<d$. If $h:[a,b]\times[c,d]\to[0,\infty)$ is a twice partially differentiable mapping such that $f_{ts}\in L([a,b]\times[c,d])$ and $|f_{ts}|^q$ is GA-convex on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq 1$, then we get hands on:*

$$\begin{aligned}
|\Phi(a,b,c,d;f,h)| &\leq \left(\frac{1}{4}\right)^{\frac{1}{q}+1}(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty \\
&\times\left\{\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(a,b,c,d;q)\right. \\
&+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(a,b,c,d;q) \\
&+\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(a,b,c,d;q) \\
&\left.+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(a,b,c,d;q)\right\}, \quad (2.6)
\end{aligned}$$

where $\|h\|_\infty=\sup\limits_{(x,y)\in[a,b]\times[c,d]}h(x,y)$ and $\zeta(u,v;k,\eta)$ is defined in Lemma 2.

Proof. By virtue of Lemma 1, we have

$$\begin{aligned}
&|\Phi(a,b,c,d;f,h)| \\
&\leq\frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty}{4}\left[\int_0^1\int_0^1 L_1(t)L_2(s)\left|f_{ts}(L_1(t),L_2(s))\right|dsdt\right. \\
&+\int_0^1\int_0^1 U_1(t)L_2(s)\left|f_{ts}(U_1(t),L_2(s))\right|dsdt \\
&+\int_0^1\int_0^1 L_1(t)U_2(s)\left|f_{ts}(L_1(t),U_2(s))\right|dsdt \\
&\left.+\int_0^1\int_0^1 U_1(t)U_2(s)\left|f_{ts}(U_1(t),U_2(s))\right|dsdt\right]. \quad (2.7)
\end{aligned}$$

Now by using Hölder's inequality for double integrals and by the GA-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq 1$, we acquire

$$\begin{aligned}
&\int_0^1\int_0^1 L_1(t)L_2(s)\left|f_{ts}(L_1(t),L_2(s))\right|dsdt \\
&\leq\left(\int_0^1\int_0^1 L_1(t)L_2(s)dsdt\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1 L_1(t)L_2(s)\left|f_{ts}(L_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}} \\
&\leq\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;-1,\frac{1}{2}\right)\right. \\
&\times\left|f_{ts}(a,c)\right|^q+\zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)\left|f_{ts}(a,d)\right|^q+\zeta\left(a,b;1,\frac{1}{2}\right) \\
&\left.\times\zeta\left(c,d;-1,\frac{1}{2}\right)\left|f_{ts}(b,c)\right|^q+\zeta\left(a,b;1,\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)\left|f_{ts}(b,d)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Correspondingly

$$\begin{aligned}
&\int_0^1 \int_0^1 U_1(t) L_2(s) \left| f_{ts}(U_1(t), L_2(s)) \right| dsdt \\
&\leq \left(\frac{1}{4}\right)^{\frac{1}{q}} \left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \left[\zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;-1,\frac{1}{2}\right)\right. \\
&\quad \times \left|f_{ts}(a,c)\right|^q + \zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)\left|f_{ts}(a,d)\right|^q + \zeta\left(a,b;-1,-\frac{1}{2}\right) \\
&\quad \left. \times\zeta\left(c,d;-1,\frac{1}{2}\right)\left|f_{ts}(b,c)\right|^q + \zeta\left(a,b;-1,-\frac{1}{2}\right)\zeta\left(c,d;1,\frac{1}{2}\right)\left|f_{ts}(b,d)\right|^q\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \int_0^1 L_1(t) U_2(s) \left| f_{ts}(L_1(t), U_2(s)) \right| dsdt \\
&\leq \left(\frac{1}{4}\right)^{\frac{1}{q}} \left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \left[\zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;1,-\frac{1}{2}\right)\right. \\
&\quad \times \left|f_{ts}(a,c)\right|^q + \zeta\left(a,b;-1,\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)\left|f_{ts}(a,d)\right|^q + \zeta\left(a,b;1,\frac{1}{2}\right) \\
&\quad \left. \times\zeta\left(c,d;1,-\frac{1}{2}\right)\left|f_{ts}(b,c)\right|^q + \zeta\left(a,b;1,\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)\left|f_{ts}(b,d)\right|^q\right]^{\frac{1}{q}},
\end{aligned}$$

by similar argument

$$\begin{aligned}
&\int_0^1 \int_0^1 U_1(t) U_2(s) \left| f_{ts}(U_1(t), U_2(s)) \right| dsdt \\
&\leq \left(\frac{1}{4}\right)^{\frac{1}{q}} \left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \left[\zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;1,-\frac{1}{2}\right)\right. \\
&\times \left|f_{ts}(a,c)\right|^q + \zeta\left(a,b;1,-\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)\left|f_{ts}(a,d)\right|^q + \zeta\left(a,b;-1,-\frac{1}{2}\right) \\
&\left. \times\zeta\left(c,d;1,-\frac{1}{2}\right)\left|f_{ts}(b,c)\right|^q + \zeta\left(a,b;-1,-\frac{1}{2}\right)\zeta\left(c,d;-1,-\frac{1}{2}\right)\left|f_{ts}(b,d)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Using the above four inequalities in (2.7) and by resolution, it reveals (2.6) and proof is completed. □

Corollary 1. *Suppose the assumptions of Theorem 1 are met and if $q = 1$,then*

$$\begin{aligned}
\left|\Phi(a,b,c,d;f,h)\right| &\leq \frac{(\ln b - \ln a)(\ln d - \ln c)}{16} \|h\|_\infty \\
&\times \left\{\sigma_1(a,b,c,d;1) + \sigma_2(a,b,c,d;1) + \sigma_3(a,b,c,d;1) + \sigma_4(a,b,c,d;1)\right\}. \quad (2.8)
\end{aligned}$$

Corollary 2. *If we consider* $h(x,y)=\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}$, $(x,y)\in[a,b]\times[c,d]$ *in Theorem 1, then*

$$\begin{aligned}
&\left|\Phi\left(a,b,c,d;f,\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\right)\right| \\
&\leq\left(\frac{1}{4}\right)^{\frac{1}{q}+1}\left\{\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(a,b,c,d;q)\right. \\
&\quad+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(a,b,c,d;q) \\
&\quad+\left[\zeta\left(a,b;0,\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(a,b,c,d;q) \\
&\quad\left.+\left[\zeta\left(a,b;0,-\frac{1}{2}\right)\zeta\left(c,d;0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(a,b,c,d;q)\right\}. \qquad (2.9)
\end{aligned}$$

Theorem 2. *Suppose* $f:\Delta\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a twice partially differentiable mapping on* Δ° *and* $[a,b]\times[c,d]\subseteq\Delta^\circ$ *with* $a<b$ *and* $c<d$. *Further let* $h:[a,b]\times[c,d]\to[0,\infty)$ *be a twice partially differentiable mapping. If* $f_{ts}\in L([a,b]\times[c,d])$ *and* $|f_{ts}|^q$ *is GA-convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q>1$, *then we have inequality of the form:*

$$\begin{aligned}
&|\Phi(a,b,c,d;f,h)|\leq\left(\frac{1}{4}\right)^{1+\frac{1}{q}}(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty \\
&\times\left\{\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(1,1,1,1;q)\right. \\
&+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(1,1,1,1;q) \\
&+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(1,1,1,1;q) \\
&\left.+\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(1,1,1,1;q)\right\}, \qquad (2.10)
\end{aligned}$$

where $\|h\|_\infty=\sup_{(x,y)\in[a,b]\times[c,d]}h(x,y)$ *and* $\zeta(u,v;k,\eta)$ *is defined in Lemma 2.*

Proof. From Lemma 1, we may write

$$\begin{aligned}
&|\Phi(a,b,c,d;f,h)| \\
&\leq\frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty}{4}\left[\int_0^1\int_0^1 L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|\,dsdt\right. \\
&\quad+\int_0^1\int_0^1 U_1(t)L_2(s)|f_{ts}(U_1(t),L_2(s))|\,dsdt \\
&\quad+\int_0^1\int_0^1 L_1(t)U_2(s)|f_{ts}(L_1(t),U_2(s))|\,dsdt \\
&\quad\left.+\int_0^1\int_0^1 U_1(t)U_2(s)|f_{ts}(U_1(t),U_2(s))|\,dsdt\right]. \qquad (2.11)
\end{aligned}$$

Now by using Hölder's inequality for double integrals, Lemma 2 and by the GA-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$, consequently we

have

$$\begin{aligned}
&\int_0^1\int_0^1 L_1(t)\,L_2(s)\,|f_{ts}(L_1(t),L_2(s))|\,dsdt \\
&\leq \left[\int_0^1\int_0^1 (L_1(t)\,L_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(L_1(t),L_2(s))|^q\,dsdt\right]^{\frac{1}{q}} \\
&\qquad\qquad \leq \left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \\
&\qquad \times\left[\frac{9}{16}|f_{ts}(a,c)|^q+\frac{3}{16}|f_{ts}(a,d)|^q+\frac{3}{16}|f_{ts}(b,c)|^q+\frac{1}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

In addition

$$\begin{aligned}
&\int_0^1\int_0^1 U_1(t)\,L_2(s)\,|f_{ts}(U_1(t),L_2(s))|\,dsdt \\
&\leq \left[\int_0^1\int_0^1 (U_1(t)\,L_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(U_1(t),L_2(s))|^q\,dsdt\right]^{\frac{1}{q}} \\
&\qquad\qquad \leq \left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \\
&\qquad \times\left[\frac{3}{16}|f_{ts}(a,c)|^q+\frac{1}{16}|f_{ts}(a,d)|^q+\frac{9}{16}|f_{ts}(b,c)|^q+\frac{3}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1 L_1(t)\,U_2(s)\,|f_{ts}(L_1(t),U_2(s))|\,dsdt \\
&\leq \left[\int_0^1\int_0^1 (L_1(t)\,U_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(L_1(t),U_2(s))|^q\,dsdt\right]^{\frac{1}{q}} \\
&\qquad\qquad \leq \left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \\
&\qquad \times\left[\frac{3}{16}|f_{ts}(a,c)|^q+\frac{9}{16}|f_{ts}(a,d)|^q+\frac{1}{16}|f_{ts}(b,c)|^q+\frac{3}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

equivalently

$$\begin{aligned}
&\int_0^1\int_0^1 U_1(t)\,U_2(s)\,|f_{ts}(U_1(t),U_2(s))|\,dsdt \\
&\leq \left[\int_0^1\int_0^1 (U_1(t)\,U_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left[\int_0^1\int_0^1 |f_{ts}(U_1(t),U_2(s))|^q\,dsdt\right]^{\frac{1}{q}} \\
&\qquad\qquad \leq \left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \\
&\qquad \times\left[\frac{1}{16}|f_{ts}(a,c)|^q+\frac{3}{16}|f_{ts}(a,d)|^q+\frac{3}{16}|f_{ts}(b,c)|^q+\frac{9}{16}|f_{ts}(b,d)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Using the above four inequalities in (2.11) and simplifying, we get the required inequality (2.10). □

Corollary 3. *If we take* $h(x,y) = \frac{1}{(\ln b - \ln a)(\ln d - \ln c)}$, $(x,y) \in [a,b] \times [c,d]$ *in Theorem 2, then*

$$\begin{aligned}
&\left|\Phi\left(a,b,c,d;f,\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\right)\right| \\
&\leq \left(\frac{1}{4}\right)^{1+\frac{1}{q}} \left\{\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \sigma_1(1,1,1,1;q)\right. \\
&\quad + \left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \sigma_2(1,1,1,1;q) \\
&\quad + \left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \sigma_3(1,1,1,1;q) \\
&\left. + \left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \sigma_4(1,1,1,1;q)\right\}. \qquad (2.12)
\end{aligned}$$

We shall use the following notation for the next theorem and its related corollary.

$$\begin{aligned}
\Delta_1(a,b,c,d;q) &= (\theta(q))^{\frac{2}{q}} |f_{ts}(a,c)|^q + (\theta(q))^{\frac{1}{q}} |f_{ts}(a,d)|^q \\
&\quad + (\theta(q))^{\frac{1}{q}} |f_{ts}(b,c)|^q + |f_{ts}(b,d)|^q, \\
\Delta_2(a,b,c,d;q) &= (\theta(q))^{\frac{1}{q}} |f_{ts}(a,c)|^q + |f_{ts}(a,d)|^q \\
&\quad + (\theta(q))^{\frac{2}{q}} |f_{ts}(b,c)|^q + (\theta(q))^{\frac{1}{q}} |f_{ts}(b,d)|^q, \\
\Delta_3(a,b,c,d;q) &= (\theta(q))^{\frac{1}{q}} |f_{ts}(a,c)|^q + (\theta(q))^{\frac{2}{q}} |f_{ts}(a,d)|^q \\
&\quad + |f_{ts}(b,c)|^q + (\theta(q))^{\frac{1}{q}} |f_{ts}(b,d)|^q
\end{aligned}$$

and

$$\begin{aligned}
\Delta_4(a,b,c,d;q) &= |f_{ts}(a,c)|^q + (\theta(q))^{\frac{1}{q}} |f_{ts}(a,d)|^q \\
&\quad + (\theta(q))^{\frac{1}{q}} |f_{ts}(b,c)|^q + (\theta(q))^{\frac{2}{q}} |f_{ts}(b,d)|^q,
\end{aligned}$$

where $\theta(q) = 2^{q+1} - 1$.

Theorem 3. *Let* $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ *be a twice partially differentiable mapping on* Δ° *and* $[a,b] \times [c,d] \subseteq \Delta^\circ$ *with* $a < b$ *and* $c < d$. *Further let* $h : [a,b] \times [c,d] \to [0,\infty)$ *is a twice partially differentiable mapping. If* $f_{ts} \in L([a,b] \times [c,d])$ *and* $|f_{ts}|^q$ *is GA-convex on the co-ordinates on* $[a,b] \times [c,d]$ *for* $q > 1$, *then the following inequality holds:*

$$\begin{aligned}
|\Phi(a,b,c,d;f,h)| &\leq \frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty}{16}\left(\frac{1}{q+1}\right)^{2/q} \\
&\quad \times \left\{\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \Delta_1(a,b,c,d;q)\right. \\
&\quad + \left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \Delta_2(a,b,c,d;q) \\
&\quad + \left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \Delta_3(a,b,c,d;q) \\
&\left. + \left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \Delta_4(a,b,c,d;q)\right\}, \qquad (2.13)
\end{aligned}$$

where $\|h\|_{\infty} = \sup_{(x,y)\in[a,b]\times[c,d]} h(x,y)$, $\zeta(u,v;k,\eta)$ *is defined in Lemma 2.*

Proof. From Lemma 1, we have

$$
\begin{aligned}
|\Phi(a,b,c,d;f,h)| \leq{} & \frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_{\infty}}{4} \\
& \left[\int_0^1\int_0^1 L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|\,dsdt\right. \\
& +\int_0^1\int_0^1 U_1(t)L_2(s)|f_{ts}(U_1(t),L_2(s))|\,dsdt \\
& +\int_0^1\int_0^1 L_1(t)U_2(s)|f_{ts}(L_1(t),U_2(s))|\,dsdt \\
& \left.+\int_0^1\int_0^1 U_1(t)U_2(s)|f_{ts}(U_1(t),U_2(s))|\,dsdt\right]. \qquad (2.14)
\end{aligned}
$$

Now by using the GA-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$, Lemma 2 together with the Hölder's inequality for double integrals, we have

$$
\begin{aligned}
&\int_0^1\int_0^1 L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|\,dsdt \\
&\leq \int_0^1\int_0^1 (L_1(t)L_2(s))\left[\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(a,c)|+\left(\frac{1+t}{2}\right)\right. \\
&\left.\left(\frac{1-s}{2}\right)|f_{ts}(a,d)|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)|f_{ts}(b,c)|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)|f_{ts}(b,d)|\right] \\
&\leq \left[\int_0^1\int_0^1 (L_1(t)L_2(s))^{\frac{q}{q-1}}\,dsdt\right]^{1-\frac{1}{q}}\left\{\left[\int_0^1\int_0^1\left(\frac{1+t}{2}\right)^q\left(\frac{1+s}{2}\right)^q dsdt\right]^{\frac{1}{q}}|f_{ts}(a,c)|\right. \\
&+\left[\int_0^1\int_0^1\left(\frac{1+t}{2}\right)^q\left(\frac{1-s}{2}\right)^q dsdt\right]^{\frac{1}{q}}|f_{ts}(a,d)|+\left[\int_0^1\int_0^1\left(\frac{1-t}{2}\right)^q\left(\frac{1+s}{2}\right)^q dsdt\right]^{\frac{1}{q}} \\
&\quad\left.\times|f_{ts}(b,c)|+\left[\int_0^1\int_0^1\left(\frac{1-t}{2}\right)^q\left(\frac{1-s}{2}\right)^q dsdt\right]^{\frac{1}{q}}|f_{ts}(b,d)|\right\} \\
&=\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\left[\frac{1}{2^q(q+1)}\right]^{2/q}\left[\left(2^{q+1}-1\right)^{2/q}\right. \\
&\left.\times|f_{ts}(a,c)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(a,d)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q\right].
\end{aligned}
$$

Likewise, we have

$$
\begin{aligned}
&\int_0^1\int_0^1 U_1(t)L_2(s)|f_{ts}(U_1(t),L_2(s))|\,dsdt \\
&\leq\left[\zeta\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q}{q-1}},d^{\frac{q}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}} \\
&\times\left[\frac{1}{2^q(q+1)}\right]^{2/q}\left[\left(2^{q+1}-1\right)^{1/q}|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q\right. \\
&\left.+\left(2^{q+1}-1\right)^{2/q}|f_{ts}(b,c)|^q+\left(2^{q+1}-1\right)^{1/q}|f_{ts}(b,d)|^q\right],
\end{aligned}
$$

$$\int_0^1 \int_0^1 L_1(t) U_2(s) |f_{ts}(L_1(t), U_2(s))| \, dsdt$$
$$\leq \left[\zeta \left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}; 0, \frac{1}{2} \right) \zeta \left(c^{\frac{q}{q-1}}, d^{\frac{q}{q-1}}; 0, -\frac{1}{2} \right) \right]^{1-\frac{1}{q}}$$
$$\times \left[\frac{1}{2^q(q+1)} \right]^{2/q} \left[\left(2^{q+1}-1\right)^{1/q} |f_{ts}(a,c)|^q + \left(2^{q+1}-1\right)^{1/q} |f_{ts}(a,d)|^q \right.$$
$$\left. + |f_{ts}(b,c)|^q + \left(2^{q+1}-1\right)^{1/q} |f_{ts}(b,d)|^q \right],$$

and

$$\int_0^1 \int_0^1 U_1(t) U_2(s) |f_{ts}(U_1(t), U_2(s))| \, dsdt$$
$$\leq \left[\zeta \left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}; 0, -\frac{1}{2} \right) \zeta \left(c^{\frac{q}{q-1}}, d^{\frac{q}{q-1}}; 0, -\frac{1}{2} \right) \right]^{1-\frac{1}{q}}$$
$$\times \left[\frac{1}{2^q(q+1)} \right]^{2/q} \left[\left(2^{q+1}-1\right)^{1/q} |f_{ts}(a,c)|^q + \left(2^{q+1}-1\right)^{1/q} |f_{ts}(a,d)|^q \right.$$
$$\left. + |f_{ts}(b,c)|^q + \left(2^{q+1}-1\right)^{1/q} |f_{ts}(b,d)|^q \right].$$

Further employing the above four inequalities in (2.14) and after simplification, we built up the required inequality (2.13). □

Corollary 4. *If we take $h(x,y) = \frac{1}{(\ln b - \ln a)(\ln d - \ln c)}$, $(x,y) \in [a,b] \times [c,d]$ in Theorem 3,then*

$$\left| \Phi \left(a,b,c,d; f, \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \right) \right| \leq \frac{1}{16} \left(\frac{1}{q+1} \right)^{2/q}$$
$$\left\{ \left[\zeta \left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}; 0, \frac{1}{2} \right) \zeta \left(c^{\frac{q}{q-1}}, d^{\frac{q}{q-1}}; 0, \frac{1}{2} \right) \right]^{1-\frac{1}{q}} \Delta_1(a,b,c,d;q) \right.$$
$$+ \left[\zeta \left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}; 0, -\frac{1}{2} \right) \zeta \left(c^{\frac{q}{q-1}}, d^{\frac{q}{q-1}}; 0, \frac{1}{2} \right) \right]^{1-\frac{1}{q}} \Delta_2(a,b,c,d;q)$$
$$+ \left[\zeta \left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}; 0, \frac{1}{2} \right) \zeta \left(c^{\frac{q}{q-1}}, d^{\frac{q}{q-1}}; 0, -\frac{1}{2} \right) \right]^{1-\frac{1}{q}} \Delta_3(a,b,c,d;q)$$
$$\left. + \left[\zeta \left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}; 0, -\frac{1}{2} \right) \zeta \left(c^{\frac{q}{q-1}}, d^{\frac{q}{q-1}}; 0, -\frac{1}{2} \right) \right]^{1-\frac{1}{q}} \Delta_3(a,b,c,d;q) \right\}, \quad (2.15)$$

where $\zeta(u,v;k,\eta)$ is defined in Lemma 2 and $\theta(q) = 2^{q+1} - 1$.

Theorem 4. *Let $f : \Delta \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ be a twice partially differentiable mapping on Δ° and $[a,b] \times [c,d] \subseteq \Delta^\circ$ with $a < b$ and $c < d$. Further let $h : [a,b] \times [c,d] \to [0,\infty)$ is a twice partially differentiable mapping. If $f_{ts} \in L([a,b] \times [c,d])$ and $|f_{ts}|^q$ is GA-convex on the co-ordinates on $[a,b] \times [c,d]$ for $q > 1$ and $q \geq r \geq 0$,*

then we attain the following inequality:

$$\begin{aligned}
|\Phi(a,b,c,d;f,h)| &\le \left(\frac{1}{4}\right)^{\frac{1}{q}+1}(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty\\
&\times\left\{\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(a^r,b^r,c^r,d^r;q)\right.\\
&+\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(a^r,b^r,c^r,d^r;q)\\
&+\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(a^r,b^r,c^r,d^r;q)\\
&\left.+\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(a^r,b^r,c^r,d^r;q)\right\},
\end{aligned}\tag{2.16}$$

where $\|h\|_\infty=\sup_{(x,y)\in[a,b]\times[c,d]}h(x,y)$ *and* $\zeta(u,v;k,\eta)$ *is defined in Lemma 2.*

Proof. From Lemma 1, it follows that

$$\begin{aligned}
&|\Phi(a,b,c,d;f,h)|\\
&\le\frac{(\ln b-\ln a)(\ln d-\ln c)\|h\|_\infty}{4}\left[\int_0^1\int_0^1 L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|\,dsdt\right.\\
&\quad+\int_0^1\int_0^1 U_1(t)L_2(s)|f_{ts}(U_1(t),L_2(s))|\,dsdt\\
&\quad+\int_0^1\int_0^1 L_1(t)U_2(s)|f_{ts}(L_1(t),U_2(s))|\,dsdt\\
&\quad\left.+\int_0^1\int_0^1 U_1(t)U_2(s)|f_{ts}(U_1(t),U_2(s))|\,dsdt\right].
\end{aligned}\tag{2.17}$$

Now by virtue of GA-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$, Lemma 2 and by the Hölder's inequality for double integrals, we have in hand

$$\begin{aligned}
\int_0^1\int_0^1 L_1(t)L_2(s)|f_{ts}(L_1(t),L_2(s))|\,dsdt &\le\left(\int_0^1\int_0^1(L_1(t)L_2(s))^{\frac{q-r}{q-1}}dsdt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\int_0^1(L_1(t)L_2(s))^r|f_{ts}(L_1(t),L_2(s))|^q dsdt\right)^{\frac{1}{q}}\\
&\le\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_1(a^r,b^r,c^r,d^r;q)
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_0^1\int_0^1 U_1(t)L_2(s)|f_{ts}(U_1(t),L_2(s))|\,dsdt &\le\left(\int_0^1\int_0^1(U_1(t)L_2(s))^{\frac{q-r}{q-1}}dsdt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\int_0^1(U_1(t)L_2(s))^r|f_{ts}(U_1(t),L_2(s))|^q dsdt\right)^{\frac{1}{q}}\\
&\le\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_2(a^r,b^r,c^r,d^r;q)
\end{aligned}$$

$$\int_0^1\int_0^1 L_1(t)\,U_2(s)\,|f_{ts}(L_1(t),U_2(s))|\,dsdt \le \left(\int_0^1\int_0^1 (L_1(t)\,U_2(s))^{\frac{q-r}{q-q}}\,dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 (L_1(t)\,U_2(s))^r\,|f_{ts}(L_1(t),U_2(s))|^q\,dsdt\right)^{\frac{1}{q}}$$
$$\le\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_3(a^r,b^r,c^r,d^r;q)$$

and

$$\int_0^1\int_0^1 U_1(t)\,U_2(s)\,|f_{ts}(U_1(t),U_2(s))|\,dsdt \le \left(\int_0^1\int_0^1 (U_1(t)\,U_2(s))^{\frac{q-r}{q-1}}\,dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 (U_1(t)\,U_2(s))^r\,|f_{ts}(U_1(t),U_2(s))|^q\,dsdt\right)^{\frac{1}{q}}$$
$$\le\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[\zeta\left(a^{\frac{q-r}{q-1}},b^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\zeta\left(c^{\frac{q-r}{q-1}},d^{\frac{q-r}{q-1}};0,-\frac{1}{2}\right)\right]^{1-\frac{1}{q}}\sigma_4(a^r,b^r,c^r,d^r;q)$$

Using the above four inequalities in (2.17) and simplifying, we obtained the required inequality (2.16). □

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ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fourteen times annually)

Editor in Chief: George Anastassiou
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Journal of Computational Analysis and Applications
An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
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Effect of RTI drug efficacy on the HIV dynamics with two cocirculating target cells

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Abstract

In this paper, we propose and analyze an HIV dynamics model. The model can be seen as a generalization of many HIV dynamics models presented in the literature since it incorporates (i) two classes of target cells, $CD4^+$ T cells and macrophages, (ii) two types of infected cells, short-lived infected cells and the long-lived chronically infected cells, (iii) intracellular discrete delays, (iv) reverse transcriptase inhibitors (RTIs) drugs with different drug efficacies on $CD4^+$T cells and macrophages. The incidence rate of infection is represented by a general function. A bifurcation parameter, known as the basic reproduction number, R_0 is derived. We established a set of conditions on the general function which are sufficient to determine the global dynamics of the model. Using Lyapunov functionals and LaSalle's invariance principle, the global asymptotic stability of the two equilibria of the model is obtained. An example is presented and some numerical simulations are conducted in order to illustrate the dynamical behavior.

Keywords: Delayed-HIV models; Chronically infected cells; Cocirculating target cells; Immune responses; Lyapunov method.

1 Introduction

Human immunodeficiency virus (HIV) is one of the most dangerous human viruses that destroys the immune system and causes acquired immunodeficiency syndrome (AIDS). During the past decades, several HIV mathematical models have been presented and analyzed (see e.g. [1]-[25]). Global stability of equilibria has become one of the most important features which help us to better understanding of the HIV dynamics. Thus, several researchers have devoted extensive efforts to study the global stability of HIV infection models (see e.g. [7], [8], [9], [11], [25], [14], [15], [16], [17], [22], [23], [19] and [24]). Some of these works assume that HIV infects only the $CD4^+$ T cells ([7], [8], [9], [11], [25], [22], [23], [19] and [24]), while, others assume that HIV infects two types of immune cells, $CD4^+$ T cells and macrophages ([14], [15], [18], [16] and [17]). Callaway and Perelson [3] pointed out that there are two types of infected cells, short-lived infected cells (which produce the most amounts of viruses) and the long-lived chronically infected cells. Moreover, the model presented in [3] incroporates reverse transcriptase inhibitors (RTIs) drugs with different drug efficacies on $CD4^+$T cells and macrophages.

Actually, there exists a time lag between the time the HIV contacts $CD4^+$ T cells or macrophages and the time the production of new infectious HIV particles. Intracellular time delay was first introduced into viral infection model by Herz et al. [5]. Since then, several delayed HIV models have been investigated (see e.g. [6], [7], [8], [9], [11], [25], [14], [17], [18], [22] and [19]). In a very recent work, Elaiw and Almualem [17] have

presented the following delayed HIV model:

$$\begin{aligned}
\dot{x}_1(t) &= \lambda_1 - d_1x_1 - (1-\varepsilon)\bar{\beta}_1x_1v, \qquad (1)\\
\dot{x}_2(t) &= \lambda_2 - d_2x_2 - (1-\chi\varepsilon)\bar{\beta}_2x_2v,\\
\dot{y}_1(t) &= (1-q_1)(1-\varepsilon)\bar{\beta}_1x_1(t-\tau_1)v(t-\tau_1) - \delta_1y_1,\\
\dot{y}_2(t) &= (1-q_2)(1-\chi\varepsilon)\bar{\beta}_2x_2(t-\tau_2)v(t-\tau_2) - \delta_2y_2,\\
\dot{z}_1(t) &= q_1(1-\varepsilon)\bar{\beta}_1x_1(t-\tau_1)v(t-\tau_1) - a_1z_1,\\
\dot{z}_2(t) &= q_2(1-\chi\varepsilon)\bar{\beta}_2x_1(t-\tau_1)v(t-\tau_1) - a_2z_2,\\
\dot{v}(t) &= \sum_{i=1}^{2}\left(N_i\delta_ie^{-n_i\kappa_i}y_i(t-\kappa_i) + M_ia_ie^{-h_i\omega_i}z_i(t-\omega_i)\right) - uv(t) \qquad (2)
\end{aligned}$$

where x_i, y_i, z_i, and v represent the concentrations of uninfected cells, short-lived infected cells, long-lived chronically infected cells and free HIV particles, respectively, where $i = 1$, for the CD4$^+$ T cells and $i = 2$, for the macrophages. The birth and death rates of uninfected cells are given by λ_i and d_ix_i, respectively. Parameter $\bar{\beta}_i$ denotes the infection rate constant. Parameters δ_i and a_i are the death rate constants of the two types of infected cells, and u is the clearance rate of HIV. The uninfected target cells become short-lived infected and long-lived chronically infected cells with fractions $(1-q_i)$ and q_i, respectively, where $q_i.\in(0,1)$. The average number of free viruses produced in the lifetime of the two types of infected cells are given by N_i and M_i, respectively. Parameter τ_i represents for the time between viral contact with an uninfected cell of class i, until it becomes infected but not yet producer cells. The loss of the cells during the delay period $[t-\tau_i, t]$ is given by $e^{-m_i\tau_i}$, where $m_i > 0$. The parameters κ_i and ω_i represent the time necessary for producing new infectious viruses from the short-lived and long-lived chronically infected cells, respectively. The factors $e^{-n_i\kappa_i}$ and $e^{-h_i\omega_i}$ represent the loss of the two types of infected cells during the delay periods $[t-\kappa_i, t]$ and $[t-\omega_i, t]$, where $n_i > 0$ and $h_i > 0$.

The immune system has two main responses to viral infections. The first is based on the Cytotoxic T Lymphocyte (CTL) cells which are responsible to attack and kill the infected cells. The second immune response is based on the antibodies that are produced by the B cells. The function of the antibodies is to attack the viruses [1]. In some infections such as in malaria, the CTL immune response is less effective than the antibody immune response [26]. Several mathematical models have been proposed to consider the antibody immune response into the viral infection models (see [27]-[33])).

All the models presented in [27]-[33] are based on the assumption that, the virus attacks one class of target cells. Moreover, model (1)-(2) did not consider the immune response. Therefore, our aim in this paper is to propose an HIV dynamics model with humoral immunity. Our model generalize model (1)-(2) by taking into account the humoral immune response. We use Lyapunov functionals and LaSalle's invariance principle to prove the global stability of all the equilibria of the models.

2 The model

In this section, we propose and analyze the following HIV model:

$$\begin{aligned}
\dot{x}_i(t) &= \lambda_i - d_ix_i(t) - \phi_i(x_i(t), v(t)), && i = 1,2, \qquad (3)\\
\dot{y}_i(t) &= (1-q_i)e^{-m_i\tau_i}\phi_i((t-\tau_i), v(t-\tau_i)) - \delta_iy_i(t), && i = 1,2, \qquad (4)\\
\dot{z}_i(t) &= q_ie^{-m_i\tau_i}\phi_i((t-\tau_i), v(t-\tau_i)) - a_iz_i(t), && i = 1,2, \qquad (5)\\
\dot{v}(t) &= \sum_{i=1}^{2}\left(N_{y_i}\delta_ie^{-n_i\kappa_i}y_i(t-\kappa_i) + M_{z_i}a_ie^{-r_i\omega_i}z_i(t-\omega_i)\right) - uv(t) - bv(t)f(w(t)), && \qquad (6)\\
\dot{w}(t) &= cv(t) - pw(t). && \qquad (7)
\end{aligned}$$

The incidence rate of infection is given by a general function $\phi_i(x_i, v)$, where $\phi_1(x_1, v) = (1-\varepsilon)\bar{\phi}_1(x_1, v)$, and $\phi_2(x_2, v) = (1-\chi\varepsilon)\bar{\phi}_2(x_2, v)$. In addition, the neutralize rate of viruses is given by a general nonlinear function $f(w)$. Parameter b is the B cells neutralize rate, the antibody response is induced at a rate proportional to the concentration of free viruses. Parameters c and p are the recruited rate and death rate constants of B cells, respectively. All the parameters and variables of the model have the same meanings as given in (1)-(2).

2.1 Initial conditions

The initial conditions for system (3)-(7) take the form

$$\begin{aligned}
x_1(\theta) &= \varphi_1(\theta),\ y_1(\theta) = \varphi_3(\theta),\ z_1(\theta) = \varphi_5(\theta),\\
x_2(\theta) &= \varphi_2(\theta),\ y_2(\theta) = \varphi_4(\theta),\ z_2(\theta) = \varphi_6(\theta),\\
v(\theta) &= \varphi_7(\theta),\ w(\theta) = \varphi_8(\theta)\\
\varphi_j(\theta) &\geq 0,\ \theta \in [-\varrho, 0),\ \varphi_j(0) > 0,\ j = 1, ..., 8,
\end{aligned} \tag{8}$$

where $\varrho = \max\{\tau_1, \tau_2, \kappa_1, \kappa_2, \omega_1, \omega_2\}$ and $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_8(\theta)) \in C([-\varrho, 0], \mathbb{R}^8_{\geq 0})$, where C is the Banach space of continuous functions mapping the interval $[-\varrho, 0]$ into $\mathbb{R}^8_{\geq 0}$. By the fundamental theory of functional differential equations [35], system (3)-(7) has a unique solution satisfying the initial conditions (8).

Assumption A1 Function ϕ_i, is continuously differentiable and satisfies the following:
(i) $\phi_i(x_i, v) > 0$, $\phi_i(x_i, 0) = \phi_i(0, v) = 0$, for all $x_i > 0$, $v > 0$, $i = 1, 2$,
(ii) $\frac{\partial \phi_i(x_i, v)}{\partial v} > 0$, $\frac{\partial \phi_i(x_i, v)}{\partial x_i} > 0$, for any $x_i > 0$, $v > 0$. Furthermore, $\frac{\partial \phi_i(x_i, 0)}{\partial v} > 0$ for any $x_i > 0$, $i = 1, 2$.

Assumption A2 The function $f(\theta)$ is locally Lipschitz on $[0, \infty)$, and satisfies $f(\theta) > 0$ for all $\theta > 0$ and $f(0) = 0$, and $f(\theta)$ is strictly increasing in $[0, \infty)$.

2.2 Non-negativity and boundedness of solutions

In the following, we establish the non-negativity and boundedness of solutions of system (3)-(7) with initial conditions (8).

Proposition 1. Let $(x_1(t), x_2(t), y_1(t), y_2(t), z_1(t), z_2(t), v(t), w(t))$ be any solution of (3)-(7) satisfying the initial conditions (8), then $x_i(t), y_i(t), z_i(t), i = 1, 2, v(t)$ and $w(t)$ are all non-negative for $t \geq 0$ and ultimately bounded.

Proof. First, we prove that $x_i(t) > 0$, $i = 1, 2$, for all $t \geq 0$. Assume that $x_i(t)$ lose its positivity on some local existence interval $[0, l]$ for some constant l and let $t_i^* \in [0, l]$ be such that $x_i(t_i^*) = 0$. From Eq. (3) we have $\dot{x}_i(t_i^*) = \lambda_i > 0$. Hence $x_i(t) > 0$ for some $t \in (t_i^*, t_i^* + \epsilon)$, where $\epsilon > 0$ is sufficiently small. This leads to a contradiction and hence $x_i(t) > 0$, for all $t \geq 0$. Furthermore, from Eqs. (4)-(7) we have

$$\begin{aligned}
y_i(t) &= y_i(0)e^{-\delta_i t} + (1-q_i)e^{-m_i\tau_i}\int_0^t e^{-\delta_i(t-\theta)}\phi(x_i(\theta-\tau_i), v(\theta-\tau_i))d\theta, \qquad i = 1, 2,\\
z_i(t) &= z_i(0)e^{-a_i t} + q_i e^{-m_i\tau_i}\int_0^t e^{-a_i(t-\theta)}\phi(x_i(\theta-\tau_i), v(\theta-\tau_i))d\theta, \qquad i = 1, 2,\\
v(t) &= v(0)e^{-\int_0^t (u+bf(w(\zeta)))d\zeta} + \int_0^t e^{-\int_\theta^t (u+bf(w(\zeta)))d\zeta}\sum_{i=1}^{2}(N_{y_i}\delta_i e^{-n_i\kappa_i}y_i(\theta-\kappa_i) + M_{z_i}a_i e^{-r_i\omega_i}z_i(\theta-\omega_i))d\theta,\\
w(t) &= w(0)e^{-pt} + c\int_0^t e^{-p(t-\theta)}v(\theta)d\theta,
\end{aligned}$$

then $y_i(t) \geq 0$, $z_i(t) \geq 0$, $i = 1, 2$, $v(t) \geq 0$ and $w(t) \geq 0$, for all $t \in [0, \varrho]$. By a recursive argument, we obtain $y_i(t) \geq 0$, $z_i(t) \geq 0$, $v(t) \geq 0$ and $w(t) \geq 0$, $i = 1, 2$, for all $t \geq 0$.

Next we show the boundedness of the solutions. From Eq. (3) we have $\dot{x}_i(t) \leq \lambda_i - d_i x_i(t)$, $i = 1, 2$. This implies that $\limsup_{t\to\infty} x_i(t) \leq \frac{\lambda_i}{d_i}$, $i = 1, 2$. Let $T_i(t) = e^{-m_i\tau_i} x_i(t-\tau_i) + y_i(t) + z_i(t)$, $i = 1, 2$ then

$$\begin{aligned}\dot{T}_i(t) &= e^{-m_i\tau_i}\lambda_i - e^{-m_i\tau_i} d_i x_i(t-\tau_i) - \delta_i y_i(t) - a_i z_i(t) \\ &\leq e^{-m_i\tau_i}\lambda_i - \sigma_i\left(e^{-m_i\tau_i} x_i(t-\tau_i) + y_i(t) + z_i(t)\right) \leq \lambda_i - \sigma_i T_i(t),\end{aligned}$$

where $\sigma_i = \min\{d_i, \delta_i, a_i\}$. Hence, $\limsup_{t\to\infty} T_i(t) \leq L_i$, where $L_i = \dfrac{\lambda_i}{\sigma_i}$. Since $x_i(t)$, $y_i(t)$ and $z_i(t)$ are all non-negative, then $\limsup_{t\to\infty} y_i(t) \leq L_i$, and $\limsup_{t\to\infty} z_i(t) \leq L_i$ for all $t \geq 0$. Moreover,

$$\begin{aligned}\dot{v} &= \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i} y_i(t-\kappa_i) + M_{z_i} a_i e^{-r_i\omega_i} z_i(t-\omega_i)\right) - uv - bv(t)f(w(t)) \\ &\leq \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i} + M_{z_i} a_i e^{-r_i\omega_i}\right) L_i - uv.\end{aligned}$$

Then $\limsup_{t\to\infty} v(t) \leq L_3$, for all $t \geq 0$, where $L_3 = \sum_{i=1}^{2}\frac{\left(N_{y_i}\delta_i e^{-n_i\kappa_i} + M_{z_i} a_i e^{-r_i\omega_i}\right)L_i}{u}$. Furthermore, $\dot{w} = cv - pw \leq cL_3 - pw$, then $\limsup_{t\to\infty} w(t) \leq L_4$, for all $t \geq 0$, where $L_4 = \frac{cL_3}{p}$.Therefore, $x_i(t), y_i(t), z_i(t), v(t)$ and $w(t)$ are ultimately bounded.

2.3 Equilibria

Let Assumptions A1 (i) and A2 be satisfied, then system (3)-(7) has a disease-free equilibrium $E_0 = (x_1^0, x_2^0, 0, 0, 0, 0, 0, 0)$, where $x_i^0 = \frac{\lambda_i}{d_i}$, $i = 1, 2$. The system can also has another positive equilibrium $E_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2, \tilde{z}_1, \tilde{z}_2, \tilde{v}, \tilde{w})$ which is called endemic equilibrium. The coordinates of the endemic equilibrium, if it exists satisfy the equalities:

$$\begin{aligned}&\lambda_i = d_i\tilde{x}_i + \phi_i(\tilde{x}_i, \tilde{v}), \quad \delta_i\tilde{y}_i = (1-q_i)e^{-m_i\tau_i}\phi_i(\tilde{x}_i, \tilde{v}), \quad a_i\tilde{z}_i = q_i e^{-m_i\tau_i}\phi_i(\tilde{x}_i, \tilde{v}), \\ &u\tilde{v} = \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i}\tilde{y}_i + M_{z_i} a_i e^{-r_i\omega_i}\tilde{z}_i\right) - b\tilde{v}f(\tilde{w}), \qquad \tilde{w} = \frac{c}{p}\tilde{v}.\end{aligned}$$

Then the basic infection reproduction number for system (3)-(7) is

$$R_0 = \sum_{i=1}^{2} R_{0i} = \sum_{i=1}^{2}\frac{((1-q_i)N_{y_i}e^{-n_i\kappa_i} + q_i M_{z_i}e^{-r_i\omega_i})e^{-m_i\tau_i}}{u}\frac{\partial\phi_i(x_i^0, 0)}{\partial v}.$$

The term $\partial\phi_i(x_i^0, 0)/\partial v$ represents the maximal average number of target cells of class i that infects by viruses, and R_{01} denotes the basic infection reproduction number of the HIV dynamics with CD4$^+$ T cells (in the absence of macrophages) and R_{02} denotes the basic infection reproduction number of the HIV dynamics with macrophages (in the absence of CD4$^+$T cells), respectively. The parameter R_0 determines whether the infection can be established.

2.4 Global stability analysis

In this subsection, we establish a set of conditions which are sufficient for the global stability of the two equilibria of system (3)-(7) employing Lyapunov method and LaSalle's invariance principle. The following function will be used throughout the paper $H(s) = s - 1 - \ln s$.

Assumption A3 The function ϕ_i, $i = 1, 2$ satisfies:
(i) $\left(\frac{\partial\phi_i(x_i,0)}{\partial v} - \frac{\partial\phi_i(x_i^0,0)}{\partial v}\right)\left(x_i^0 - x_i\right) \leq 0$, for $x_i > 0$,
(ii) $\phi_i(x_i, v) \leq v\frac{\partial\phi_i(x_i,0)}{\partial v}$, for all $x_i, v > 0$.

Theorem 1. Let Assumptions A1-A3 be satisfied and $R_0 \leq 1$, then the disease-free equilibrium E_0 of system (3)-(7) is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = \sum_{i=1}^{2}\gamma_i \left[x_i - x_i^0 - \int_{x_i^0}^{x_i} \lim_{v\to 0^+} \frac{\phi_i(x_i^0, v)}{\phi_i(s,v)} ds + \frac{N_{y_i} e^{-n_i\kappa_i}}{\gamma_i} y_i + \frac{M_{z_i} e^{-r_i\omega_i}}{\gamma_i} z_i \right.$$
$$\left. + \int_0^{\tau_i} \phi_i(x_i(t-\theta), v(t-\theta))d\theta + \frac{e^{-n_i\kappa_i} N_{y_i}\delta_i}{\gamma_i} \int_0^{\kappa_i} y_i(t-\theta)d\theta + \frac{e^{-r_i\omega_i} M_{z_i} a_i}{\gamma_i} \int_0^{\omega_i} z_i(t-\theta)d\theta \right]$$
$$+ v + \frac{b}{c}\int_0^{w} f(\theta)d\theta,$$

where $\gamma_i = e^{-m_i\tau_i}((1-q_i)e^{-n_i\kappa_i}N_{y_i} + q_i e^{-r_i\omega_i}M_{z_i})$, $i = 1, 2$. We calculate $\frac{dW_0}{dt}$ along the trajectories of system (3)-(7) as:

$$\frac{dW_0}{dt} = \sum_{i=1}^{2}\gamma_i \left[\left(1 - \lim_{v\to 0^+} \frac{\phi_i(x_i^0, v)}{\phi_i(x_i, v)}\right)(\lambda_i - d_i x_i - \phi_i(x_i, v)) \right.$$
$$+ \frac{N_{y_i} e^{-n_i\kappa_i}}{\gamma_i}((1-q_i)e^{-m_i\tau_i}\phi_i(x_i(t-\tau_i), v(t-\tau_i)) - \delta_i y)$$
$$+ \frac{M_{z_i} e^{-r_i\omega_i}}{\gamma_i}(q_i e^{-m_i\tau_i}\phi_i(x_i(t-\tau_i), v(t-\tau_i)) - a_i z_i)$$
$$+ \phi_i(x_i, v) - \phi_i(x_i(t-\tau_i), v(t-\tau_i))$$
$$\left. + \frac{e^{-n_i\kappa_i} N_{y_i}\delta_i}{\gamma_i}(y_i - y_i(t-\kappa_i)) + \frac{e^{-r_i\omega_i} M_{z_i} a_i}{\gamma_i}(z_i - z_i(t-\omega_i)) \right]$$
$$+ \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i} y_i(t-\kappa_i) + M_{z_i} a_i e^{-r_i\omega_i} z_i(t-\omega_i)\right) - uv - bvf(w) + \frac{b}{c}f(w)(cv - pw). \tag{9}$$

Collecting terms of Eq. (9) we get

$$\frac{dW_0}{dt} = \sum_{i=1}^{2}\gamma_i \left[\left(1 - \frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right)(\lambda_i - d_i x_i) + \phi_i(x_i, v)\frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right] - uv - \frac{bp}{c}wf(w)$$
$$= \sum_{i=1}^{2}\gamma_i \left[\lambda_i\left(1 - \frac{x_i}{x_i^0}\right)\left(1 - \frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right) + \phi_i(x_i, v)\frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right] - uv - \frac{bp}{c}wf(w)$$
$$= \sum_{i=1}^{2}\gamma_i\lambda_i\left(1 - \frac{x_i}{x_i^0}\right)\left(1 - \frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right) + \sum_{i=1}^{2}\gamma_i\phi_i(x_i, v)\frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v} - uv - \frac{bp}{c}wf(w). \tag{10}$$

Using A3 we get

$$\frac{dW_0}{dt} \le \sum_{i=1}^{2}\gamma_i\lambda_i\left(1 - \frac{x_i}{x_i^0}\right)\left(1 - \frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right) + \sum_{i=1}^{2}\gamma_i v\frac{\partial\phi_i(x_i^0, 0)}{\partial v} - uv - \frac{bp}{c}wf(w)$$
$$= \sum_{i=1}^{2}\gamma_i\lambda_i\left(1 - \frac{x_i}{x_i^0}\right)\left(1 - \frac{\partial\phi_i(x_i^0, 0)/\partial v}{\partial\phi_i(x_i, 0)/\partial v}\right) + (R_0 - 1)uv - \frac{bp}{c}wf(w). \tag{11}$$

By using Assumption A2, the last term is less than or equal zero. Therefore, If $R_0 \le 1$ then $\frac{dW_0}{dt} \le 0$ for all $x_1, x_2, v, w > 0$. We note that, the solutions of the system (3)-(7) converge to Γ, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$. From Eq. (11) we have $\frac{dW_0}{dt} = 0$ iff $x_i = x_i^0$, $i = 1, 2$, $v = 0$ and $w = 0$. The set Γ is invariant and for any element belongs to Γ satisfies $w = 0$, $v = 0$ then $\dot{v} = 0$. We can see from Eq. (19) that

$$\sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i} y_i(t-\kappa_i) + M_{z_i} a_i e^{-r_i\omega_i} z_i(t-\omega_i)\right) = 0.$$

Since y_i and z_i are non-negative for $i = 1, 2$, then $y_1 = y_2 = 0$ and $z_1 = z_2 = 0$. It follows that, $\frac{dW_0}{dt} = 0$ iff $x_i = x_i^0$, $y_i = z_i = v = w = 0$, $i =$ 1,2. From LaSalle's invariance principle, E_0 is GAS.

To establish the global stability of the endemic equilibrium, we need the following condition.

Assumption A4 Function $\phi_i(x_i, v)$ satisfies the following:

$$\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})} - \frac{v}{\tilde{v}}\right)\left(1 - \frac{\phi_i(x_i,\tilde{v})}{\phi_i(x_i,v)}\right) \le 0, \quad x_i, v > 0$$

Theorem 2. Let Assumptions A1, A2 and A4 hold true and the endemic equilibrium E_1 of system (3)-(7) exists, then E_1 is GAS.

Proof. We consider the Lyapunov functional W_1 as:

$$\begin{aligned}
W_1 = & \sum_{i=1}^{2}\gamma_i \left[x_i - \tilde{x}_i - \int_{\tilde{x}_i}^{x_i} \frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(s,\tilde{v})} ds + \frac{N_{y_i} e^{-n_i\kappa_i}}{\gamma_i} \tilde{y}_i H\left(\frac{y_i}{\tilde{y}_i}\right)\right. \\
& + \frac{M_{z_i} e^{-r_i\omega_i}}{\gamma_i} \tilde{z}_i H\left(\frac{z_i}{\tilde{z}_i}\right) + \phi_i(\tilde{x}_i,\tilde{v}) \int_0^{\tau_i} H\left(\frac{\phi_i(x_i(t-\theta), v(t-\theta))}{\phi_i(\tilde{x}_i,\tilde{v})}\right) d\theta \\
& \left. + \frac{e^{-n_i\kappa_i} N_{y_i}\delta_i\tilde{y}_i}{\gamma_i} \int_0^{\kappa_i} H\left(\frac{y_i(t-\theta)}{\tilde{y}_i}\right) d\theta + \frac{e^{-r_i\omega_i} M_{z_i} a_i \tilde{z}_i}{\gamma_i} \int_0^{\omega_i} H\left(\frac{z_i(t-\theta)}{\tilde{z}_i}\right) d\theta \right] + \tilde{v} H\left(\frac{v}{\tilde{v}}\right) \\
& + \frac{b}{c} \int_{\tilde{w}}^{w} (f(\theta) - f(\tilde{w})) d\theta.
\end{aligned}$$

Calculating $\frac{dW_1}{dt}$ along the solutions of system (3)-(7) we obtain

$$\begin{aligned}
\frac{dW_1}{dt} = & \sum_{i=1}^{2}\gamma_i \left[\left(1 - \frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)(\lambda_i - d_i x_i - \phi_i(x_i,v)) \right. \\
& + \frac{N_{y_i} e^{-n_i\kappa_i}}{\gamma_i}\left(1 - \frac{\tilde{y}_i}{y_i}\right)((1-q_i)e^{-m_i\tau_i}\phi_i(x_i(t-\tau_i), v(t-\tau_i)) - \delta_i y_i) \\
& + \frac{M_{z_i} e^{-r_i\omega_i}}{\gamma_i}\left(1 - \frac{\tilde{z}_i}{z_i}\right)(q_i e^{-m_i\tau_i}\phi_i(x_i(t-\tau_i), v(t-\tau_i)) - a_i z_i) \\
& + \phi_i(x_i,v) - \phi_i(x_i(t-\tau_i), v(t-\tau_i)) + \phi_i(\tilde{x}_i,\tilde{v}) \ln\left(\frac{\phi_i(x_i(t-\tau_i), v(t-\tau_i))}{\phi_i(x_i,v)}\right) \\
& + \frac{e^{-n_i\kappa_i} N_{y_i}\delta_i\tilde{y}_i}{\gamma_i}\left(\frac{y_i}{\tilde{y}_i} - \frac{y_i(t-\kappa_i)}{\tilde{y}_i} + \ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right)\right) \\
& \left. + \frac{e^{-r_i\omega_i} M_{z_i} a_i\tilde{z}_i}{\gamma_i}\left(\frac{z_i}{\tilde{z}_i} - \frac{z_i(t-\omega_i)}{\tilde{z}_i} + \ln\left(\frac{z_i(t-\omega_i)}{z_i}\right)\right)\right] \\
& + \left(1 - \frac{\tilde{v}}{v}\right)\left(\sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i} y_i(t-\kappa_i) + M_{z_i} a_i e^{-r_i\omega_i} z_i(t-\omega_i)\right) - uv - bvf(w)\right) \\
& + \frac{b}{c}(f(w) - f(\tilde{w}))(cv - pw). \qquad (12)
\end{aligned}$$

Collecting terms of Eq. (12) we get

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\gamma_i \left[\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)(\lambda_i-d_ix_i)+\phi_i(x_i,v)\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}+\frac{N_{y_i}e^{-n_i\kappa_i}\delta_i}{\gamma_i}\tilde{y}_i+\frac{M_{z_i}e^{-r_i\omega_i}a_i}{\gamma_i}\tilde{z}_i\right.\\
&-\frac{(1-q_i)e^{-m_i\tau_i}N_{y_i}e^{-n_i\kappa_i}}{\gamma_i}\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i}-\frac{q_ie^{-m_i\tau_i}M_{z_i}e^{-r_i\omega_i}}{\gamma_i}\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i}\\
&+\phi_i(\tilde{x}_i,\tilde{v})\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right)\\
&\left.+\frac{e^{-n_i\kappa_i}N_{y_i}\delta_i\tilde{y}_i}{\gamma_i}\ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right)+\frac{e^{-r_i\omega_i}M_{z_i}a_i\tilde{z}_i}{\gamma_i}\ln\left(\frac{z_i(t-\omega_i)}{z_i}\right)\right]\\
&-\sum_{i=1}^{2}N_{y_i}\delta_ie^{-n_i\kappa_i}\frac{\tilde{v}y_i(t-\kappa_i)}{v}-\sum_{i=1}^{2}M_{z_i}a_ie^{-r_i\omega_i}\frac{\tilde{v}z_i(t-\omega_i)}{v}-uv+u\tilde{v}\\
&+b\tilde{v}f(w)-\frac{bp}{c}wf(w)-bvf(\tilde{w})+\frac{bp}{c}wf(\tilde{w}).
\end{aligned}$$

Using the equilibrium conditions for E_1:

$$\begin{aligned}
&\lambda_i=d_i\tilde{x}_i+\phi_i(\tilde{x}_i,\tilde{v}),\quad (1-q_i)e^{-m_i\tau_i}\phi_i(\tilde{x}_i,\tilde{v})=\delta_i\tilde{y}_i,\quad q_ie^{-m_i\tau_i}\phi_i(\tilde{x}_i,\tilde{v})=a_i\tilde{z}_i,\\
&u\tilde{v}=\sum_{i=1}^{2}\left(N_{y_i}\delta_ie^{-n_i\kappa_i}\tilde{y}_i+M_{z_i}a_ie^{-r_i\omega_i}\tilde{z}_i\right)-b\tilde{v}f(\tilde{w}),\quad \tilde{w}=\frac{c}{p}\tilde{v}
\end{aligned}$$

and the following equality

$$uv=u\tilde{v}\frac{v}{\tilde{v}}=\frac{v}{\tilde{v}}\left(\sum_{i=1}^{2}(N_{y_i}\delta_ie^{-n_i\kappa_i}\tilde{y}_i+M_{z_i}a_ie^{-r_i\omega_i}\tilde{z}_i)-b\tilde{v}f(\tilde{w})\right)=\frac{v}{\tilde{v}}\sum_{i=1}^{2}\gamma_i\phi_i(\tilde{x}_i,\tilde{v})-bvf(\tilde{w}),$$

we obtain

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\gamma_i\left[d_i\tilde{x}_i\left(1-\frac{x_i}{\tilde{x}_i}\right)\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\phi_i(\tilde{x}_i,\tilde{v})\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)\right.\\
&+\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}\right)+\frac{2N_{y_i}e^{-n_i\kappa_i}\delta_i}{\gamma_i}\tilde{y}_i+\frac{2M_{z_i}e^{-r_i\omega_i}a_i}{\gamma_i}\tilde{z}_i\\
&-\frac{N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i}{\gamma_i}\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}+\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right)\\
&-\frac{M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i}{\gamma_i}\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}+\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right)\\
&+\frac{N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i}{\gamma_i}\left(\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right)+\ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right)\right)\\
&\left.+\frac{M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i}{\gamma_i}\left(\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right)+\ln\left(\frac{z_i(t-\omega_i)}{z_i}\right)\right)\right]\\
&-b\tilde{v}f(\tilde{w})+b\tilde{v}f(w)-\frac{bp}{c}wf(w)+\frac{bp}{c}wf(\tilde{w}). \qquad (13)
\end{aligned}$$

Using the following equalities

$$\begin{aligned}
\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right) &= \ln\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\ln\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\\
&\quad+\ln\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)+\ln\left(\frac{\tilde{v}y_i}{v\tilde{y}_i}\right),\\
\ln\left(\frac{y_i(t-\kappa_i)}{y_i}\right) &= \ln\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right)+\ln\left(\frac{v\tilde{y}_i}{\tilde{v}y_i}\right),\\
\ln\left(\frac{\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{\phi_i(x_i,v)}\right) &= \ln\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\ln\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\\
&\quad+\ln\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)+\ln\left(\frac{\tilde{v}z_i}{v\tilde{z}_i}\right),\\
\ln\left(\frac{z_i(t-\omega_i)}{z_i}\right) &= \ln\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right)+\ln\left(\frac{v\tilde{z}_i}{\tilde{v}z_i}\right).
\end{aligned}$$

Eq. (13) can be rewritten as

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\Bigg[\gamma_i d_i\tilde{x}_i\left(1-\frac{x_i}{\tilde{x}_i}\right)\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}-1+\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)\\
&-\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}-1-\ln\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)\right)-\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}-1-\ln\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)\right)\\
&-N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}-1-\ln\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\right)\\
&-N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}-1-\ln\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right)\right)\\
&-M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}-1-\ln\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}\right)\right)\\
&-M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}-1-\ln\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right)\right)\Bigg]-\frac{bp}{c}(w-\tilde{w})(f(w)-f(\tilde{w})). \qquad (14)
\end{aligned}$$

Then Eq. (14) becomes,

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\Bigg[\gamma_i d_i\tilde{x}_i\left(1-\frac{x_i}{\tilde{x}_i}\right)\left(1-\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}\right)\left(1-\frac{\phi_i(x_i,\tilde{v})}{\phi_i(x_i,v)}\right)\\
&-\gamma_i\phi_i(\tilde{x}_i,\tilde{v})\left\{H\left(\frac{\phi_i(\tilde{x}_i,\tilde{v})}{\phi_i(x_i,\tilde{v})}\right)+H\left(\frac{v\phi_i(x_i,\tilde{v})}{\tilde{v}\phi_i(x_i,v)}\right)\right\}\\
&-N_{y_i}e^{-n_i\kappa_i}\delta_i\tilde{y}_i\left\{H\left(\frac{\tilde{y}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{y_i\phi_i(\tilde{x}_i,\tilde{v})}\right)+H\left(\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}\right)\right\}\\
&-M_{z_i}e^{-r_i\omega_i}a_i\tilde{z}_i\left\{H\left(\frac{\tilde{z}_i\phi_i(x_i(t-\tau_i),v(t-\tau_i))}{z_i\phi_i(\tilde{x}_i,\tilde{v})}\right)+H\left(\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}\right)\right\}\Bigg]-\frac{bp}{c}(w-\tilde{w})(f(w)-f(\tilde{w})).
\end{aligned}$$

By using Assumption A2, the last term is less than or equal zero. It is easy to see that, if $\tilde{x}_1,\tilde{x}_2,\tilde{y}_1,\tilde{y}_2,\tilde{z}_1,\tilde{z}_2,\tilde{v}$ and $\tilde{w}>0$, then $\frac{dW_1}{dt}\le 0$ for all $x_1,x_2,y_1,y_2,z_1,z_2,v$ and $w>0$. The solutions of the system limit to Γ, the largest invariant subset of $\{\frac{dW_1}{dt}=0\}$. It can be seen that $\frac{dW_1}{dt}=0$ if and only if $x_i=\tilde{x}_i$, $v=\tilde{v}$, $w=\tilde{w}$ and $H=0$ i.e.

$$\frac{\tilde{v}y_i(t-\kappa_i)}{v\tilde{y}_i}=\frac{\tilde{v}z_i(t-\omega_i)}{v\tilde{z}_i}=1 \qquad (15)$$

From Eq. (15), we have $y_i=\tilde{y}_i$ and $z_i=\tilde{z}_i$. It follows that $\frac{dW_1}{dt}$ equal to zero at E_1. LaSalle's invariance principle implies the global stability of E_1.

3 Example and numerical simulations

We introduce the following example:

$$\dot{x}_i(t) = \lambda_i - d_i x_i(t) - \frac{\beta_i x_i^{k_i}(t)v(t)}{(x_i^{k_i}(t)+\rho_i)(v(t)+\varsigma_i)}, \qquad i=1,2, \tag{16}$$

$$\dot{y}_i(t) = (1-q_i)e^{-m_i\tau_i}\frac{\beta_i x_i^{k_i}(t-\tau_i)v(t-\tau_i)}{(x_i^{k_i}(t-\tau_i)+\rho_i)(v(t-\tau_i)+\varsigma_i)} - \delta_i y_i(t), \qquad i=1,2, \tag{17}$$

$$\dot{z}_i(t) = q_i e^{-m_i\tau_i}\frac{\beta_i x_i^{k_i}(t-\tau_i)v(t-\tau_i)}{(x_i^{k_i}(t-\tau_i)+\rho_i)(v(t-\tau_i)+\varsigma_i)} - a_i z_i(t), \qquad i=1,2, \tag{18}$$

$$\dot{v}(t) = \sum_{i=1}^{2}\left(N_{y_i}\delta_i e^{-n_i\kappa_i}y_i(t-\kappa_i) + M_{z_i}a_i e^{-r_i\omega_i}z_i(t-\omega_i)\right) - uv(t) - bv(t)w(t), \tag{19}$$

$$\dot{w}(t) = cv(t) - pw(t). \tag{20}$$

For this example we have

$$\phi_i(x_i,v) = \frac{\beta_i x_i^{k_i}v}{(x_i^{k_i}+\rho_i)(v+\varsigma_i)}, \qquad f(w)=w \tag{21}$$

where $k_i, \rho_i, \varsigma_i > 0$, $i=1,2$. Function ϕ_i satisfies the following:

$$\begin{aligned}
\frac{\partial\phi_i(x_i,v)}{\partial x_i} &= \frac{k_i\rho_i\beta_i x_i^{k_i-1}v}{(x_i^{k_i}+\rho_i)^2(v+\varsigma_i)} > 0, \text{for all } x_i>0,\ v>0,\\
\frac{\partial\phi_i(x_i,v)}{\partial v} &= \frac{\varsigma_i\beta_i x_i^{k_i}}{(x_i^{k_i}+\rho_i)(v+\varsigma_i)^2} > 0, \text{for all } x_i>0,\\
\frac{\partial\phi_i(x_i,0)}{\partial v} &= \frac{\beta_i x_i^{k_i}}{\varsigma_i(x_i^{k_i}+\rho_i)} > 0, \text{for all } x_i>0,\ v>0,\\
\phi_i(x_i,v) &= \frac{\beta_i x_i^{k_i}v}{(x_i^{k_i}+\rho_i)(v+\varsigma_i)} \le \frac{\beta_i x_i^{k_i}v}{\varsigma_i(x_i^{k_i}+\rho_i)} = v\frac{\partial\phi_i(x_i,0)}{\partial v}, \text{for all } x_i>0,\ v>0,\\
\left(\frac{\phi_i(x_i,v)}{\phi_i(x_i,\tilde{v})} - \frac{v}{\tilde{v}}\right)\left(1-\frac{\phi_i(x_i,\tilde{v})}{\phi_i(x_i,v)}\right) &= \frac{-\varsigma_i(v-\tilde{v})^2}{\tilde{v}(\tilde{v}+\varsigma_i)(v+\varsigma_i)} \le 0, \text{for all } x_i,\ v>0.
\end{aligned}$$

Thus Assumption A1-A4 hold true and Theorems 1 and 2 are applicable. The basic reproduction number in this case is given by

$$R_0 = \sum_{i=1}^{2}R_{0i} = \sum_{i=1}^{2}\frac{((1-q_i)e^{-n_i\kappa_i}N_{y_i} + q_i e^{-r_i\omega_i}M_{z_i})e^{-m_i\tau_i}}{u}\frac{\beta_i (x_i^0)^{k_i}}{\varsigma_i((x_i^0)^{k_i}+\rho_i)}.$$

Without loss of generality we let, $\tau_e = \tau_1 = \tau_2 = \kappa_1 = \kappa_2 = \omega_1 = \omega_2$. In Table 1, we present the values of some parameters of model (16)-(20). The effect of the drug efficacy ε and time delay τ_e on the qualitative behavior of the system will be studied below in details. All computations are carried out by MATLAB.

3.1 Evolution of the system state with different initial conditions

We have chosen three different initial conditions as follows:
IC1: $\varphi_1(\theta)=600$, $\varphi_2(\theta)=200$, $\varphi_3(\theta)=1$, $\varphi_4(\theta)=0.5$, $\varphi_5(\theta)=1$, $\varphi_6(\theta)=2$, $\varphi_7(\theta)=1$, $\varphi_8(\theta)=0.02$,
IC2: $\varphi_1(\theta)=700$, $\varphi_2(\theta)=350$, $\varphi_3(\theta)=2$, $\varphi_4(\theta)=2$, $\varphi_5(\theta)=3$, $\varphi_6(\theta)=5$, $\varphi_7(\theta)=6$, $\varphi_8(\theta)=1$
IC3: $\varphi_1(\theta)=800$, $\varphi_2(\theta)=500$, $\varphi_3(\theta)=3.5$, $\varphi_4(\theta)=3.5$, $\varphi_5(\theta)=6$, $\varphi_6(\theta)=8$, $\varphi_7(\theta)=10$, $\varphi_8(\theta)=1.4$,
where $\theta\in[-\varrho,0)$. We will fix the delay parameter $\tau_e = 0.01$ day^{-1}, and using two sets of the parameter ε to get the following two cases.

Case (I): In this case, we choose $\varepsilon=0.8$ then we get $R_0=0.79<1$. Figure 1 shows that, the state of the system eventually approach to the infection-free equilibrium $E_0=(1000,600,0,0,0,0,0,0)$ for the three initial conditions IC1-IC3. This supports the results of Theorem 1 that the infection-free equilibrium E_0 is GAS. In

Table 1: The values of the parameters of model (16)-(20).

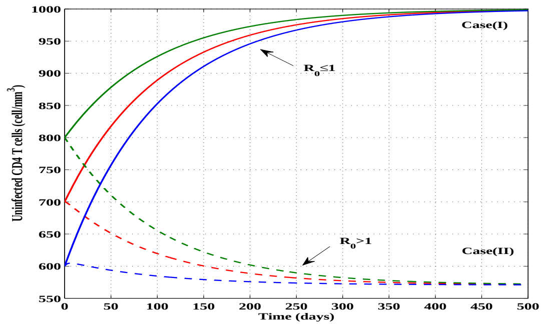
Parameter	Value	Parameter	Value
λ_1	10 cells mm^{-3}day^{-1}	λ_2	6 cells mm^{-3}day^{-1}
$\bar{\beta}_1$	8 cells mm^{-3}day^{-1}	$\bar{\beta}_2$	5 cells mm^{-3}day^{-1}
d_1	0.01 day^{-1}	d_2	0.01 day^{-1}
δ_1	0.5 day^{-1}	δ_2	0.3 day^{-1}
a_1	0.3 day^{-1}	a_2	0.1 day^{-1}
q_1	0.5	q_2	0.5
ς_1	10 virus mm^{-3}	ς_2	10 virus mm^{-3}
k_1	2	k_2	2
N_{y_1}	9 virus cells^{-1}	N_{y_2}	4 virus cells^{-1}
M_{z_1}	4 virus cells^{-1}	M_{z_2}	1 virus cells^{-1}
ρ_1	0.1 cells^{k_1} mm$^{-3k_1}$	ρ_2	0.1 cells^{k_1} mm$^{-3k_1}$
m_1	1 day^{-1}	m_2	1 day^{-1}
n_1	1 day^{-1}	n_2	1 day^{-1}
r_1	1 day^{-1}	r_2	1 day^{-1}
χ	0.5	u	1 day^{-1}
b	1 cells mm^{-3}day^{-1}	p	6 day^{-1}
c	1 day^{-1}	ε	Varied
τ_e	Varied		

this case, the virus particles will be cleared from the body.

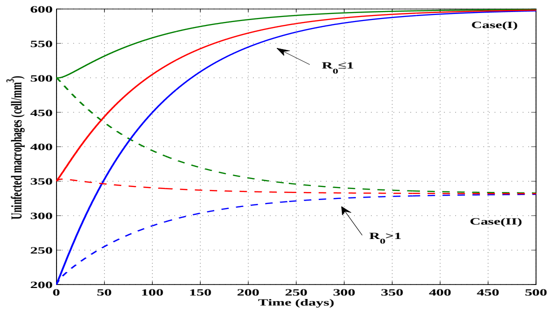
Case (II): In this case, we choose $\varepsilon = 0$ then we calculate $R_0 = 2.13 > 1$. Consequently, the system has two equilibria E_0 and E_1, and based on Theorem 2, E_1 is GAS. From Figure 1 we can see that, our simulation results are consistent with the theoretical results of Theorem 2. We observe that, the state of the system converge the endemic equilibrium $E_1 = (571.06, 332.13, 4.25, 4.43, 7.08, 13.28, 11.58, 1.93)$. for the three initial conditions IC1-IC3. In this case, the infection becomes chronic.

3.2 Effect of the drug efficacy on the dynamical behavior of the system

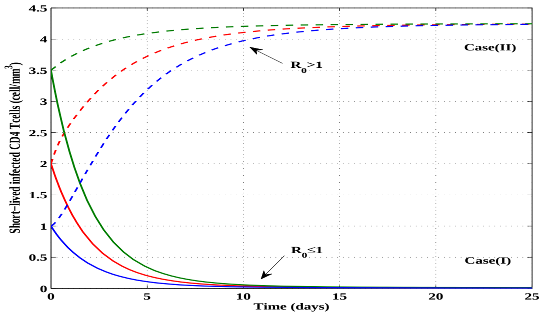
In this case, we will fix the delay parameter $\tau_e = 0.01$ day^{-1}. Figures 2 shows the effect of the parameter ε on the evolution of the uninfected CD4$^+$T cells and macrophages, short-lived infected cells, long-lived chronically infected cells, free virus particles and B cells. When there is no treatment i.e. $\varepsilon = 0$, the trajectory of the system tends to the endemic equilibrium $E_1 = (571.06, 332.13, 4.25, 4.43, 7.08, 13.28, 11.58, 1.93)$. Since E_1 exists, then according to Theorem 2, E_1 is GAS. We can see from the figures that, our simulation results are consistent with the theoretical results of Theorem 2. We observe that, as the drug efficacy is increased from $\varepsilon = 0$ to $\varepsilon = 0.8$, E_1 is still exists and is GAS, moreover, the concentrations of the uninfected CD4$^+$T cells and macrophages are increasing, while the concentrations of the short-lived infected cells, long-lived chronically infected cells, free virus particles and B cells are decreasing. When $\varepsilon = 0.98$, the basic reproduction number is given by $R_0 = 0.73 < 1$, then according to Theorem 1, the disease-free equilibrium E_0 is GAS. We can see that, the concentrations of uninfected CD4$^+$T cells and macrophages are increasing and converge to their normal values $\frac{\lambda_1}{d_1} = 1000$ cells mm^{-3}, $\frac{\lambda_2}{d_2} = 600$ cells mm^{-3}, respectively, while the concentrations of short-lived infected cells, long-lived chronically infected cells, free viruses and B cells are decaying and tend to zero. It means that, the numerical results are also compatible with the results of Theorem 1. In this case, the treatment with such drug



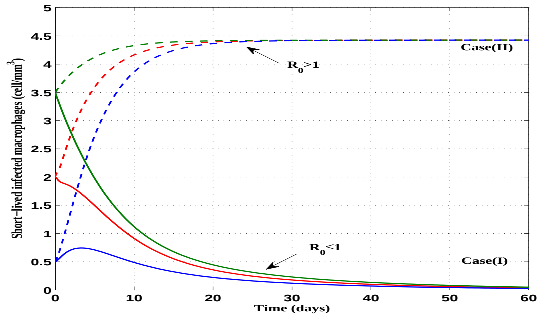
(a) Uninfected CD4$^+$T cells



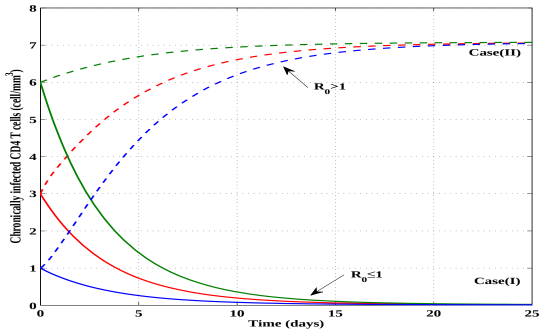
(b) Uninfected macrophages



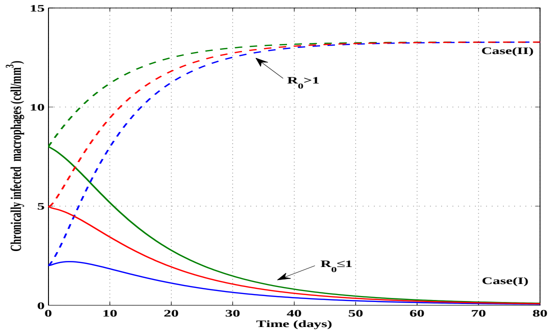
(c) Short-lived infected CD4$^+$T cells



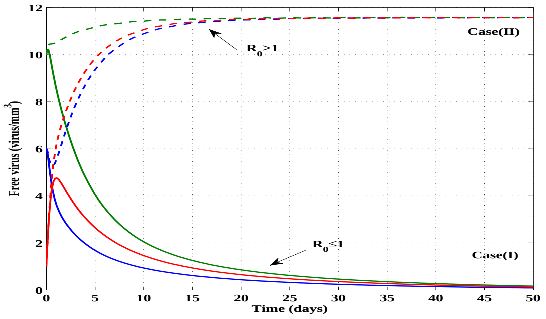
(d) Short-lived infected macrophages



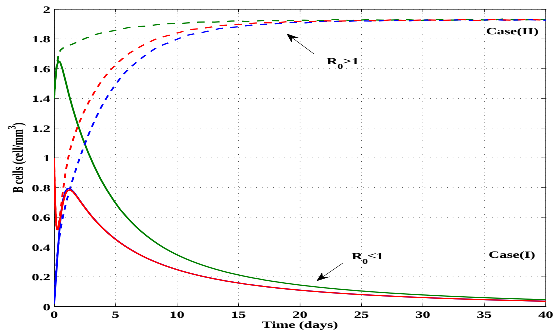
(e) Chronically infected CD4$^+$T cells



(f) Chronically infected macrophages



(g) Free virus



(h) B cells

Figure 1: The evolution of the system state in different initial conditions for model (16) -(20).

efficacy succeeded to eliminate the viruses from the blood.

3.3 Effect of the time delay on the dynamical behavior of the system

In this case, we will fix the drug efficacy $\varepsilon = 0.2$. Figure 3 shows the effect of the parameter τ_e on the evolution of the state variables of the system. When $\tau_e = 0.01$, the trajectory of the system tends to the endemic equilibrium $E_1 = (684.2, 378.23, 3.13, 3.66, 5.2, 10.9, 9.75, 1.62)$. Then E_1 exists and according to Theorem 2 E_1 is GAS. It means that, both the numerical and theoretical results of Theorem 2 are consistent. One can see that, as the time delay is increased from $\tau_e = 0.01$ to $\tau_e = 0.7$, E_1 is still exists and is GAS, in addition, the concentrations of the uninfected CD4$^+$T cells and macrophages are increased, while the concentrations of the short-lived infected cells, long-lived chronically infected cells, free virus particles and B cells are decreased. When $\tau_e = 1$, the basic reproduction number is given by $R_0 = 0.71 < 1$, then according to Theorem 1, E_0 is GAS. We can see that, the concentrations of uninfected CD4$^+$T cells and macrophages are increasing and converge to their normal values $\frac{\lambda_1}{d_1} = 1000$ cells mm^{-3}, $\frac{\lambda_2}{d_2} = 600$ cells mm^{-3}, respectively, while the concentrations of short-lived infected cells, long-lived chronically infected cells, free viruses and B cells are decaying and tend to zero. Figure 3 shows that the numerical results are also compatible with the results of Theorem 1. This shows the effect of time delay on preventing the disease from development.

3.4 Effects of the drug efficacy and the delay on the basic reproduction number:

Figure 4 shows the effect of the parameters ε and τ_e on the basic reproduction number R_0. We note that, $R_0 > 1$ for small values of ε or τ_e, and the endemic equilibrium exists and is GAS, while the disease-free equilibrium is unstable. When $R_0 = 1$ (which is a bifurcation point), both disease-free equilibrium and endemic equilibrium coincide and it is GAS. Moreover, as ε or τ_e is increasing, R_0 is decreasing until it becomes less than one, which makes the endemic equilibrium does not exists and the disease-free equilibrium is GAS. From a biological point of view, the intracellular delay plays a similar role as antiviral treatment in eliminating the virus. We observe that, even if there is no treatment i.e. $\varepsilon = 0$, sufficiently large delay suppress viral replication and clear the virus. This give us some suggestions on new drugs to prolong the increase the intracellular delay period.

3.5 Effects of two types of target cells on the dynamics and controls of HIV infection

In this subsection, we show the effects of two types of target cells on the dynamics and controls of HIV infection. We note that if $R_0 < 1$, then it is sure that $R_{01} < 1$ and $R_{02} < 1$. But if one neglect the presence of the macrophages in the HIV dynamics model, then the HIV model system (16) -(20) will become

$$\dot{x}_1(t) = \lambda_1 - d_1 x_1(t) - \frac{(1-\varepsilon)\bar{\beta}_1 x_1^{k_1}(t) v(t)}{(x_1^{k_1}(t) + \rho_1)(v(t) + \varsigma_1)}, \tag{22}$$

$$\dot{y}_1(t) = (1-q_1)e^{-m_1\tau_1} \frac{(1-\varepsilon)\bar{\beta}_1 x_1^{k_1}(t-\tau_1) v(t-\tau_1)}{(x_1^{k_1}(t-\tau_1) + \rho_1)(v(t-\tau_1) + \varsigma_1)} - \delta_1 y_1(t), \tag{23}$$

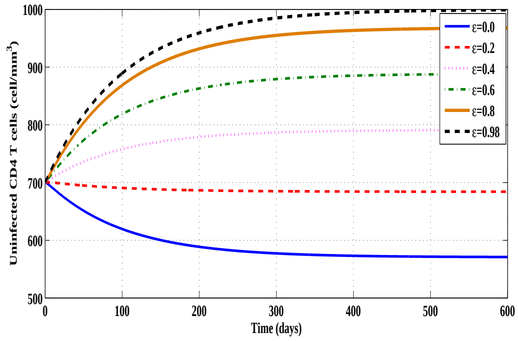
$$\dot{z}_1(t) = q_1 e^{-m_1\tau_1} \frac{(1-\varepsilon)\bar{\beta}_1 x_1^{k_1}(t-\tau_1) v(t-\tau_1)}{(x_1^{k_1}(t-\tau_1) + \rho_1)(v(t-\tau_1) + \varsigma_1)} - a_1 z_1(t), \tag{24}$$

$$\dot{v}(t) = N_{y_1}\delta_1 e^{-n_1\kappa_1} y_1(t-\kappa_1) + M_{z_1} a_1 e^{-r_1\omega_1} z_1(t-\omega_1) - uv(t) - bv(t)w(t), \tag{25}$$

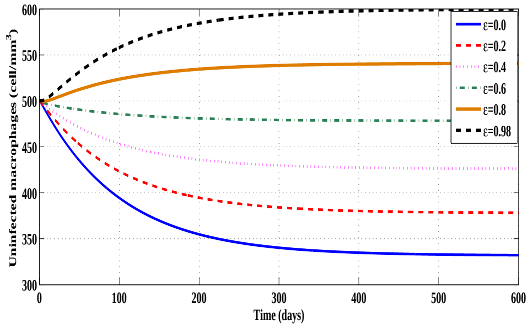
$$\dot{w}(t) = cv(t) - pw(t). \tag{26}$$

The basic reproduction number of model (22)-(26) is given by

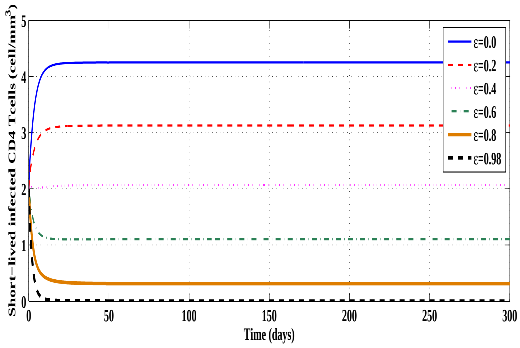
$$R_{01} = \frac{\left((1-q_1)e^{-n_1\kappa_1} N_{y_1} + q_1 e^{-r_1\omega_1} M_{z_1}\right)}{u} \frac{e^{-m_1\tau_1}(1-\varepsilon)\bar{\beta}_1 (x_1^0)^{k_1}}{\varsigma_1((x_1^0)^{k_1} + \rho_1)}.$$



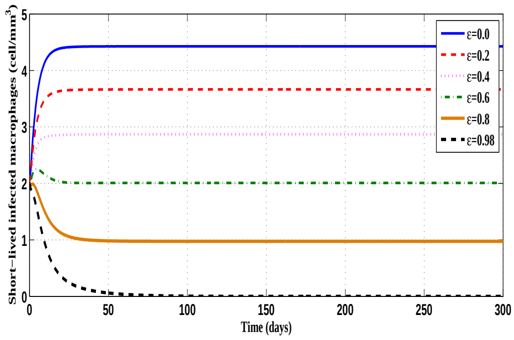
(a) Uninfected CD4$^+$T cells



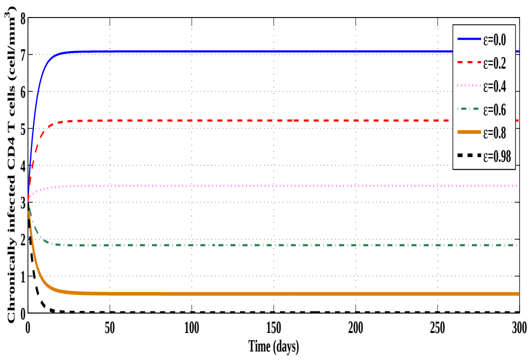
(b) Uninfected macrophages



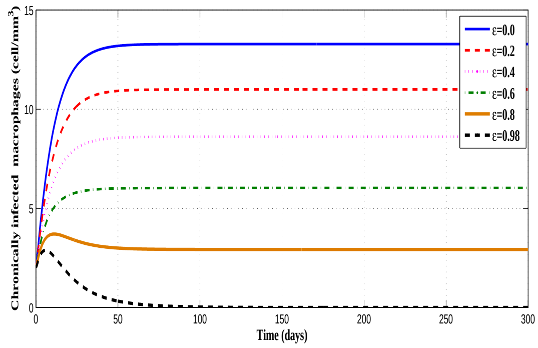
(c) Short-lived infected CD4$^+$T cells



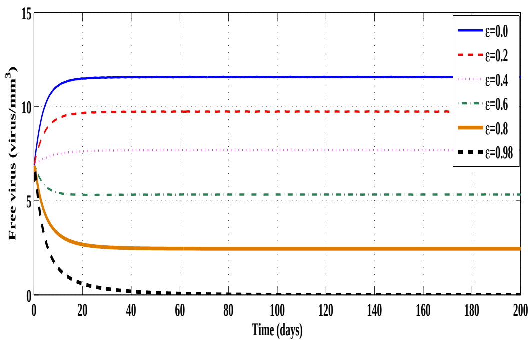
(d) Short-lived infected macrophages



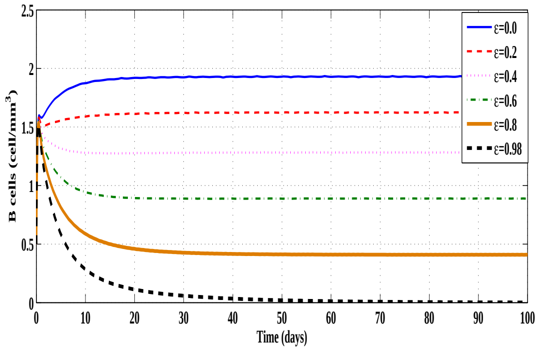
(e) Chronically infected CD4$^+$T cells



(f) Chronically infected macrophages

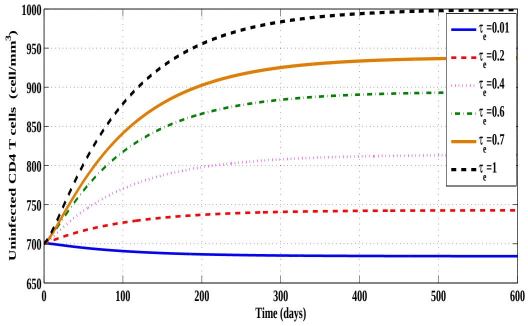


(g) Free virus

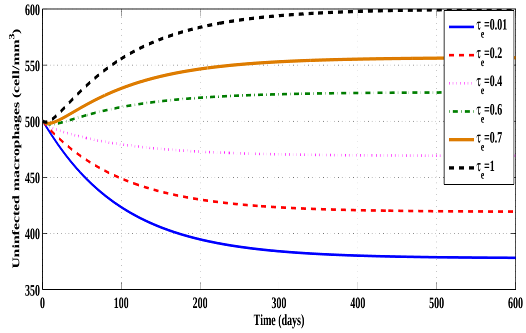


(h) B cells

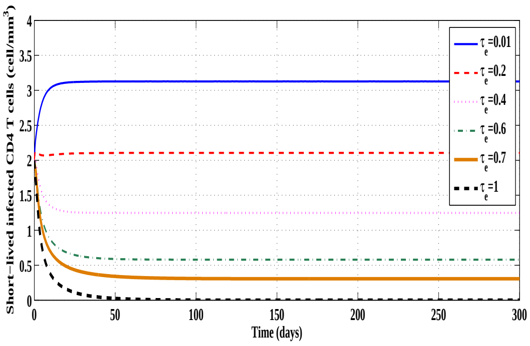
Figure 2: The evolution of the system state with different values of drug efficacy for model (16) -(20).



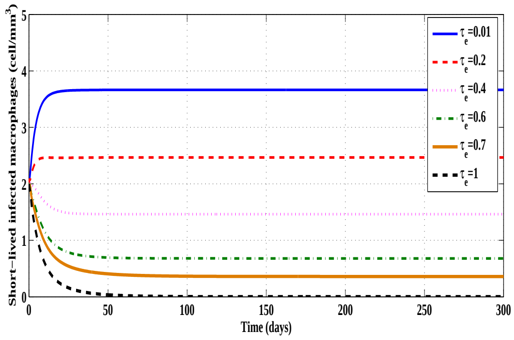
(a) Uninfected CD4$^+$T cells



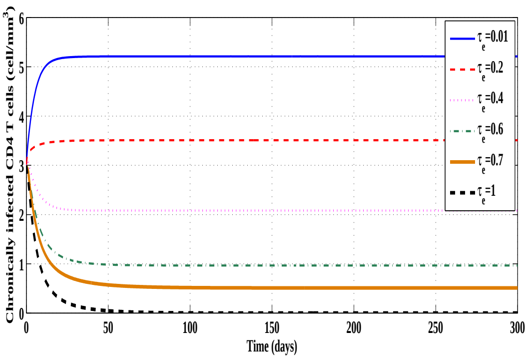
(b) Uninfected macrophages



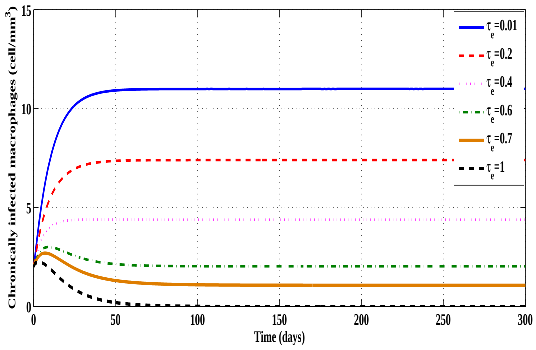
(c) Short-lived infected CD4$^+$T cells



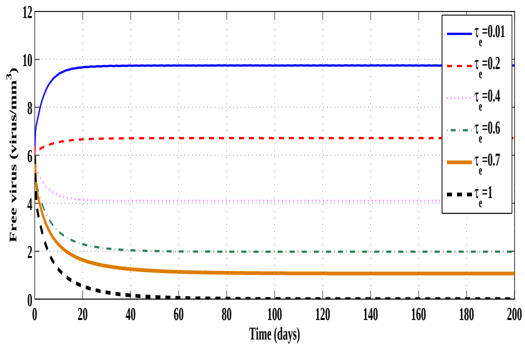
(d) Short-lived infected macrophages



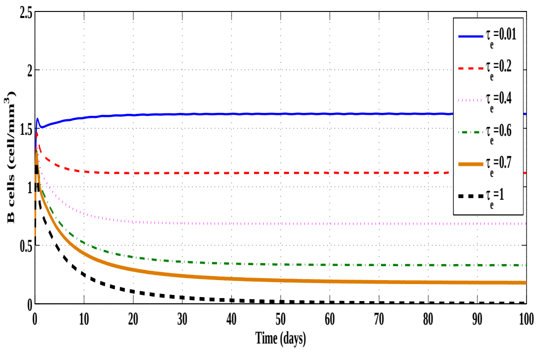
(e) Chronically infected CD4$^+$T cells



(f) Chronically infected macrophages



(g) Free virus



(h) B cells

Figure 3: The evolution of the system state with different values of delayed for model (16) -(20).

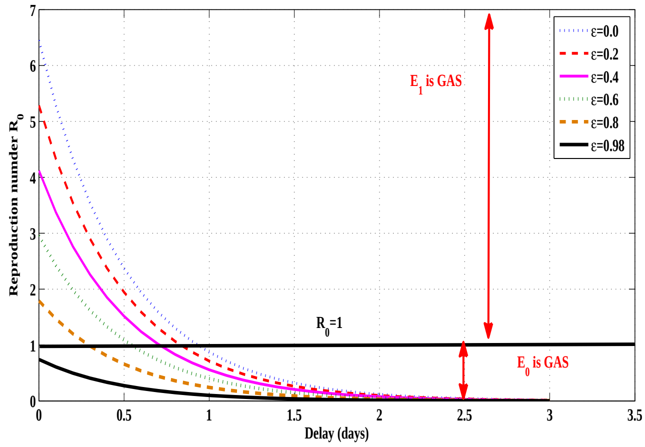


Figure 4: Effects of the drug efficacy and delays on the basis reproduction number of model (3)-(7)

Now we show that there is a number of parameter values for which $R_{01} \leq 1$, but $R_0 > 1$, and in such cases the solutions of system (22)-(26) tend to E_0 (in $\mathbb{R}^5_{\geq 0}$) as $t \to \infty$, while those of (16) -(20) tend to E_1 (in $\mathbb{R}^8_{\geq 0}$) as $t \to \infty$. We calculate the critical drug efficacy for system (16) -(20), E_0 is GAS when $R_0 \leq 1$ i.e.

$$\varepsilon_1^{crit} \leq \varepsilon < 1, \quad \varepsilon_1^{crit} = \max\left\{0, \frac{\overline{R}_0 - 1}{\overline{R}_{01} + \chi \overline{R}_{02}}\right\},$$

where $\overline{R}_0 = R_0 \mid_{\varepsilon=0}$ and $\overline{R}_{0i} = R_{0i} \mid_{\varepsilon=0}$, $i = 1, 2$.
For system (22)-(26), E_0 is GAS when $R_{01} \leq 1$ i.e.

$$\varepsilon_2^{crit} \leq \varepsilon < 1, \quad \varepsilon_2^{crit} = \max\left\{0, \frac{\overline{R}_{01} - 1}{\overline{R}_{01}}\right\}.$$

Clearly, $\varepsilon_1^{crit} > \varepsilon_2^{crit}$. Then, if one design treatment with drug efficacy $\varepsilon_2^{crit} \leq \varepsilon \leq \varepsilon_1^{crit}$, then E_0 is GAS for system (22)-(26) but unstable for system (16) -(20). Using the data in Table 1 and $\tau_e = 0.01$, we have $\varepsilon_1^{crit} = 0.93$ and $\varepsilon_2^{crit} = 0.80$. Let us choose $\varepsilon = 0.88$, then $R_{01} \mid_{\varepsilon=0.88} = 0.62 < 1$, but $R_0 \mid_{\varepsilon=0.88} = 1.31 > 1$. Therefore, more accurate treatment can be designed using the model (16) -(20) than those designed using model (22)-(26). Figure 5 shows the effect of two target cells on dynamics and control of HIV infection. We observe that, if we choose $\varepsilon = 0.88$, then the trajectory of model (16) -(20 tends to the infection-free equilibrium $E_0 = (1000, 0, 0, 0, 0, 0)$, while the trajectory of model (16) -(20) tends to the endemic equilibrium $E_1 = (990.54, 573.24, 0.1, 0.4, 0.15, 1.31, 1.04, 0.17)$.

3.6 Effect of long-lived chronically infected cells on the dynamics and controls of HIV infection

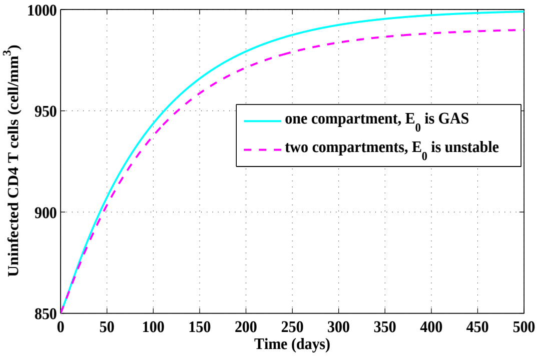
To show the effect of the presence of long-lived chronically infected cells on the dynamics and controls of HIV infection, we write the HIV model without long-lived chronically infected cells as:

$$\dot{x}_i(t) = \lambda_i - d_i x_i(t) - \frac{\beta_i x_i^{k_i}(t) v(t)}{(x_i^{k_i}(t) + \rho_i)(v(t) + \varsigma_i)}, \tag{27}$$

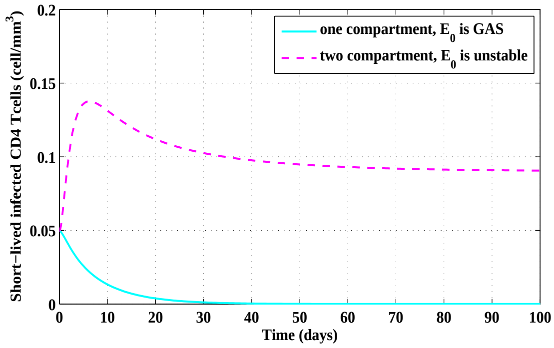
$$\dot{y}_i(t) = \frac{e^{-m_i \tau_i} \beta_i x_i^{k_i}(t - \tau_i) v(t - \tau_i)}{(x_i^{k_i}(t - \tau_i) + \rho_i)(v(t - \tau_i) + \varsigma_i)} - \delta_i y_i(t), \tag{28}$$

$$\dot{v}(t) = \sum_{i=1}^{2} N_{y_i} \delta_i e^{-n_i \kappa_i} y_i(t - \kappa_i) - uv(t) - bv(t)w(t), \tag{29}$$

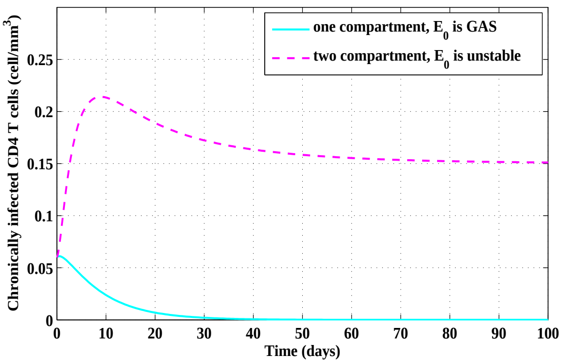
$$\dot{w}(t) = cv(t) - pw(t). \tag{30}$$



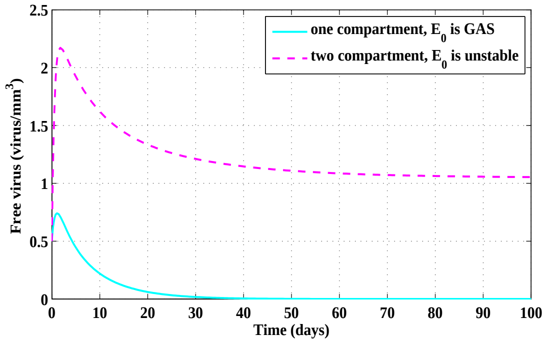
(a) Uninfected CD4$^+$T cells for model (16)-(20) and model (22)-(26).



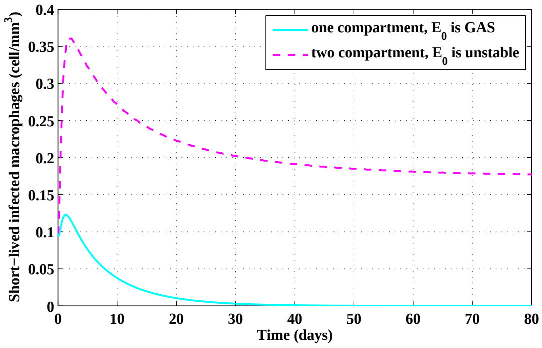
(b) Short-lived CD4$^+$T cells for model (16) -(20) and ((22)-(26).



(c) Chronically infected CD4$^+$T cells for model (16) -(20) and (22)-(26).



(d) Free virus for model (16) -(20)) and (22)-(26).



(e) B cells for model (16) -(20) and (22)-(26).

Figure 5: Effect of two types of target cells on the dynamics and controls of HIV infection

The basic reproduction number for system (27)-(30) is given by

$$\widetilde{R}_0 = \sum_{i=1}^{2} \widetilde{R}_{0i} = \sum_{i=1}^{2} \frac{e^{-n_i\kappa_i} e^{-m_i\tau_i} N_{y_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)},$$

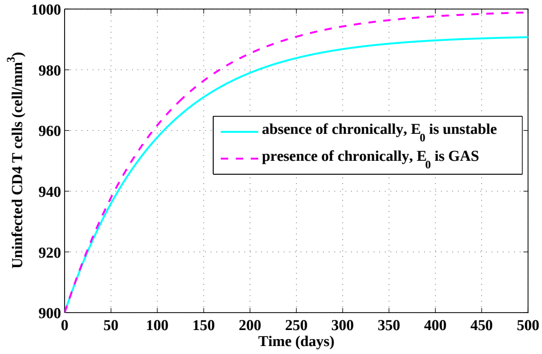
where $\widetilde{R}_0 = R_0 \mid_{q_1=q_2=0}$. Since $e^{-n_i\kappa_i} N_{y_i} > e^{-r_i\omega_i} M_{z_i}$, $i = 1, 2$, then we have

$$\begin{aligned}
R_0 &= \sum_{i=1}^{2} \frac{((1-q_i)e^{-n_i\kappa_i} N_{y_i} + q_i e^{-r_i\omega_i} M_{z_i})}{u} \frac{e^{-m_i\tau_i} \beta_i (x_i^0)^{k_i}}{\varsigma_i((x_i^0)^{k_i} + \rho_i)} \\
&= \sum_{i=1}^{2} \frac{e^{-n_i\kappa_i} e^{-m_i\tau_i} N_{y_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)} - \sum_{i=1}^{2} \frac{(e^{-n_i\kappa_i} N_{y_i} - e^{-r_i\omega_i} M_{z_i}) q_i e^{-m_i\tau_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)} \\
&= \widetilde{R}_0 - \sum_{i=1}^{2} \frac{(e^{-n_i\kappa_i} N_{y_i} - e^{-r_i\omega_i} M_{z_i}) q_i e^{-m_i\tau_i} \beta_i (x_i^0)^{k_i}}{u\varsigma_i((x_i^0)^{k_i} + \rho_i)} < \widetilde{R}_0.
\end{aligned}$$

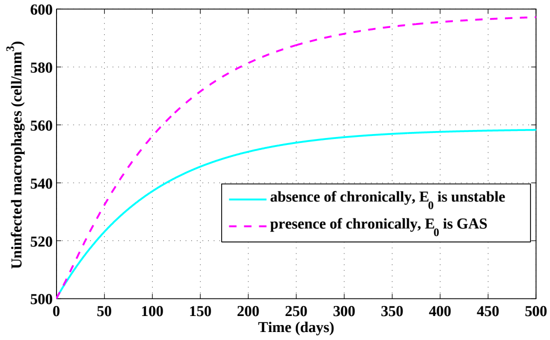
Therefore even without the incorporation of treatment, the long-lived infected cell population decreases the basic reproduction number of the system. Now, we calculate the critical drug efficacy needed in order stabilize the system around the infection-free equilibrium. The critical drug efficacy for systems (16) -(20) and (27)-(30) is given by ε_1^{crit} and ε_3^{crit}, respectively, where,

$$\varepsilon_3^{crit} = \max\left\{0, \frac{\widehat{R}_0 - 1}{\widehat{R}_0}\right\}$$

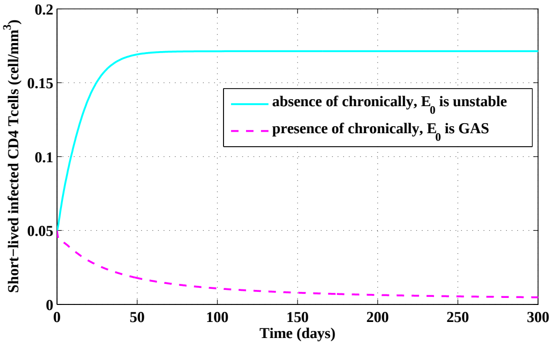
where $\widehat{R}_0 = \widetilde{R}_0 \mid_{\varepsilon=0} = R_0 \mid_{\varepsilon=q_1=q_2=0}$. Using the data given in Table 1 with $\tau_e = 0.01$, we have $\varepsilon_1^{crit} = 0.93$ and $\varepsilon_3^{crit} = 0.99$. Therefore the drug efficacy necessary to drive the system to the infection-free equilibrium is actually less for system (16) -(20) than that for system (27)-(30). Figure 6 shows the effect of chronically infected cells on dynamic and control of HIV infection. We observed that, if we choose $\varepsilon = 0.93$, then the trajectory of model (16)-(20) tends to infection-free equilibrium $E_0 = (1000, 600, 0, 0, 0, 0, 0, 0)$, while in the model (27)-(30), $\widetilde{R}_0 = 1.54 > 1$ and the trajectory tends to the endemic equilibrium with humoral immunity $E_1 = (990.99, 558.53, 0.17, 1.36, 1.83, 0.3)$.



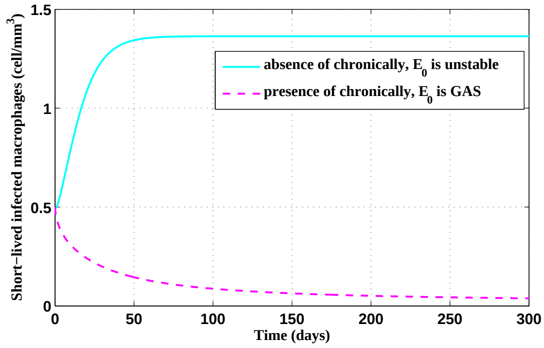
(a) Uninfected CD4$^+$T cells for model (16)-(20) and (27)-(30).



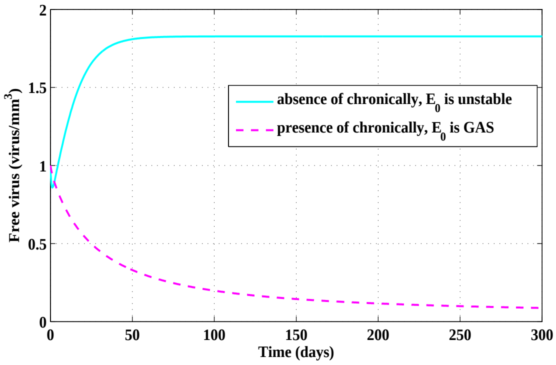
(b) Uninfected macrophages for model (16) -(20) and (27)-(30).



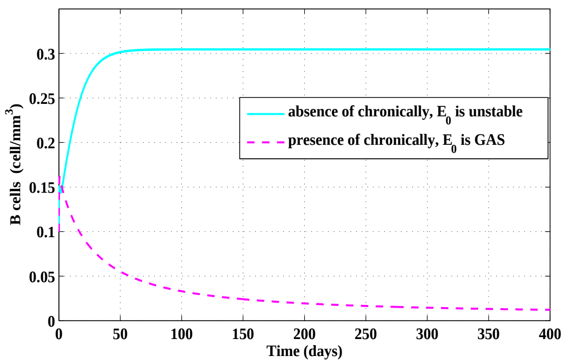
(c) Short-lived CD4$^+$T cells for model (16) -(20) and (27)-(30).



(d) Short-lived macrophages for model (16) -(20) and (27)-(30).



(e) Free virus for model (16) -(20) and (27)-(30).



(f) B cells for model (16) -(20) and (27)-(30).

Figure 6: Effect of long-lived chronically infected cells on the dynamics and controls of HIV infection

4 Acknowledgment

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

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COMPOSITION OPERATORS ON DIRICHLET-TYPE SPACES

LIU YANG, YECHENG SHI*

ABSTRACT. In this note, motivated by [8], under the conditions of weighted function in [10], we characterize bounded and compact composition operator on Dirichlet-type spaces D_K. We also give an equavalent charaterization of composition operator on D_K, if the composition operator on D_K spaces is Hilbert-Schmidt.

Keywords: D_K spaces; composition operators; Hilbert-Schmidt

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. Let $K : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function. The Dirichlet-type spaces D_K, consists of those functions $f \in H(\mathbb{D})$, such that

$$\|f\|_{D_K}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 K(1-|z|^2) dA(z) < \infty.$$

When $K(t) = t^\alpha$, $0 < \alpha < 1$, it give the classical Dirichlet-type space D_α. For more informations on D_α and D_K spaces, we refer to [1], [3], [12], [19], [25].

Let φ be a holomorphic self-map of $\mathbb{D}$. The composition operator C_φ on D_K is defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in D_K.$$

There are many papers study composition operator, we refer to [4], [13], [14], [15], [17], [20], [21], [22], [24], [26]. Recently, Kellay and Lefèvre using Nevanlinna counting function, characterize bounded and compact composition operator on Dirichlet-type space D_K under certain conditions in [13]. Later, Pau and Pèrez studied the essential norm and closed ranged of composition operator on D_α in [17].

2000 *Mathematics Subject Classification.* 30D45, 30D50.

The first author is supported by NSF of China (No. 11471202) and supported by the Special Fund of the Shaanxi Provincial Education Department (grant Nr. 2013JK0567), Shaanxi Provincial Natural Science Foundation (grant Nr. 2014JM1018).

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In this paper, motivated by [8], we generalize Theorem 2.2 of [8] to D_K spaces. We also give a characterizations of boundedness and compactness of composition operator C_φ on D_K spaces by φ^n. Furthermore, equavalent characterizations of composition operator on D_K spaces belong to Hilbert-Schmidt was gave.

Throughout this paper, suppose that $K : [0,\infty) \to [0,\infty)$ is a right-continuous and nondecreasing function. Satisfying

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1.1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{1.2}$$

where

$$\varphi_K(s) = \sup_{0\le t\le 1} K(st)/K(t), \quad 0 < s < \infty.$$

To learn more about weight function K, we refer to [2], [3], [9], [10] and [16].

Throughout this paper, for two functions f and g, $f \asymp g$ means that $g \lesssim f \lesssim g$, that is, there are positive constants C_1 and C_2 depend on K and index s, α, such that $C_1 g \le f \le C_2 g$.

2. Auxiliary results

Before to proof, we need to know some results. The following lemma can be found in Lemma 2.1 of [2].

Lemma 1. *Let (1.1) and (1.2) hold for K. If $2 - \frac{\alpha}{2} < s < 1 + c$, then*

$$\int_{\mathbb{D}} \frac{K\left(1 - |\sigma_a(w)|^2\right)}{(1-|w|^2)^s |1-\overline{w}z|^\alpha} dA(w) \lesssim \frac{K\left(1 - |\sigma_a(z)|^2\right)}{(1-|z|^2)^{s+\alpha-2}}$$

for all $a, z \in \mathbb{D}$, where $\sigma_a(z) = \frac{z-a}{1-\bar{a}z}$.

Lemma 2. *Suppose that K satisfies (1.1) and (1.2). Then*

$$1 + \sum_{n=1}^{\infty} \frac{n+1}{K(\frac{1}{n+1})} t^n \asymp \frac{1}{(1-t)^2 K(1-t)}$$

for all $0 \le t < 1$.

Proof. Without loss of generality, we can assume $1/3 < t < 1$, otherwise, it obvious. Make change of variables $y = \frac{1}{x}$, an easy computation gives

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n+1}{K(\frac{1}{n+1})}t^n &\asymp \sum_{n=1}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}}\frac{t^{\frac{1}{x}}}{x^3K(x)}dx\\ &\asymp \int_0^1\frac{t^{\frac{1}{x}}}{x^3K(x)}dx \asymp \int_1^{\infty}\frac{yt^y}{K(\frac{1}{y})}dy.\end{aligned}$$

Let $y = \frac{\gamma}{-\ln t}$. We can deduce that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n+1}{K(\frac{1}{n+1})}t^n &\asymp \frac{1}{(\ln\frac{1}{t})^2}\int_{-\ln t}^{\infty}\frac{\gamma e^{-\gamma}}{K(\frac{1}{\gamma}\ln\frac{1}{t})}d\gamma\\ &= \frac{1}{(\ln\frac{1}{t})^2K(\ln\frac{1}{t})}\int_{-\ln t}^{\infty}\frac{\gamma e^{-\gamma}K(\ln\frac{1}{t})}{K(\frac{1}{\gamma}\ln\frac{1}{t})}d\gamma\\ &\lesssim \frac{1}{(1-t)^2K(1-t)}\int_{-\ln t}^{\infty}\gamma e^{-\gamma}\varphi_K(\gamma)d\gamma.\end{aligned}$$

By [10], under conditions (1.1) and (1.2), there exists an enough small $c > 0$ only depending on K such that

$$\varphi_K(s) \lesssim s^c,\ 0 < s \le 1$$

and

$$\varphi_K(s) \lesssim s^{1-c},\ s \ge 1.$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n+1}{K(\frac{1}{n+1})}t^n &\lesssim \frac{1}{(1-t)^2K(1-t)}\int_{-\ln t}^{\infty}\gamma e^{-\gamma}\varphi_K(\gamma)d\gamma\\ &\lesssim \frac{1}{(1-t)^2K(1-t)}\left(\int_0^{\infty}e^{-\gamma}\gamma^{2-c}d\gamma + \int_0^{\infty}e^{-\gamma}\gamma^{1+c}d\gamma\right)\\ &\asymp \frac{1}{(1-t)^2K(1-t)}\left(\Gamma(3-c)+\Gamma(2+c)\right),\end{aligned}$$

where $\Gamma(.)$ is the Gamma function. It follows that

$$1+\sum_{n=1}^{\infty}\frac{n+1}{K(\frac{1}{n+1})}t^n \lesssim \frac{1}{(1-t)^2K(1-t)}.$$

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Conversely, since K is nondecreasing, we deduce that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n+1}{K(\frac{1}{n+1})}t^n &\asymp \frac{1}{(\ln\frac{1}{t})^2K(\ln\frac{1}{t})}\int_{-\ln t}^{\infty}\frac{\gamma e^{-\gamma}K(\ln\frac{1}{t})}{K(\frac{1}{\gamma}\ln\frac{1}{t})}d\gamma\\ &\gtrsim \frac{1}{(1-t)^2K(1-t)}\int_{\ln 2}^{\infty}\frac{\gamma e^{-\gamma}K(\ln\frac{1}{t})}{K(\frac{1}{\gamma}\ln\frac{1}{t})}d\gamma\\ &\gtrsim \frac{1}{(1-t)^2K(1-t)}\int_{\ln 2}^{\infty}\gamma e^{-\gamma}d\gamma\\ &\asymp \frac{1}{(1-t)^2K(1-t)}.\end{aligned}$$

The proof is completed. □

The next lemma can be found in Theorem 5 of [23].

Lemma 3. *Let (1.2) hold for K. Then for any $\alpha > 0$ and $0 \le \beta < 1$, we have*

$$\int_0^1 r^{\alpha-1}(\log\frac{1}{r})^{-\beta}K(\log\frac{1}{r})dr \asymp \left(\frac{1-\beta}{\alpha}\right)^{1-\beta}K\left(\frac{1-\beta}{\alpha}\right).$$

3. BOUNDEDNESS AND COMPACTNESS

In this section, motivated by [8], we discuss the boundedness and compactness of compostion operators by a general computation.

Theorem 1. *Suppose that (1.1) and (1.2) hold for K, $s \ge 0$. Suppose $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi \in D_K$. Then C_φ is bounded on D_K if and only if*

$$\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}}K(1-|z|^2)dA(z) < \infty.$$

Proof. Let

$$F_a(z) = \frac{(1-|a|^2)^{1+s}}{\sqrt{K(1-|a|^2)}}\frac{1}{(1-\overline{a}z)^{1+s}},\quad s \ge 0.$$

Using Lemma 1, it is easy to check that $F_a \in D_K$. If C_φ is bounded on D_K, then $\|C_\varphi(F_a)\|_{D_K} < \infty$, that is,

$$\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}}K(1-|z|^2)dA(z) < \infty.$$

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On the other hand, we know that for any pseudohyperbolic discs $D(z,r)$, we have $1-|w| \asymp 1-|z| \asymp |1-\overline{w}z|$, for any $w \in D(z,r)$ (see [27, page 69]). Let $f \in D_K$. Applying sub-mean-property to $|f'|^2$, we have

$$\begin{aligned}|f'(z)|^2 &\le \int_{D(z,r)} \frac{|f'(w)|^2}{|1-\overline{w}z|^2} dA(w)\\ &\asymp \int_{D(z,r)} \frac{|f'(w)|^2(1-|w|^2)^{2+2s}}{|1-\overline{w}z|^{4+2s}} dA(w)\\ &\lesssim \int_{\mathbb{D}} \frac{|f'(w)|^2(1-|w|^2)^{2+2s}}{|1-\overline{w}z|^{4+2s}} dA(w).\end{aligned}$$

Therefore, we get

$$\int_{\mathbb{D}} |f'(\varphi(z))|^2|\varphi'(z)|^2 K(1-|z|^2)dA(z)$$

$$\begin{aligned}&\le \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{|f'(w)|^2}{|1-\overline{w}\varphi(z)|^{4+2s}} (1-|w|^2)^{2+2s} dA(w) \right) |\varphi'(z)|^2 K(1-|z|^2)dA(z)\\ &\le \left(\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^{2+2s}}{K(1-|w|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{w}\varphi(z)|^{4+2s}} K(1-|z|^2)dA(z) \right)\\ &\qquad \times \int_{\mathbb{D}} |f'(w)|^2 K(1-|w|^2)dA(w) < \infty.\end{aligned}$$

The proof is completed. □

Theorem 2. *Suppose that (1.1) and (1.2) hold for K, $s \ge 0$. Suppose $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi \in D_K$. Then C_φ is compact on D_K if and only if*

$$\lim_{|a|\to 1} \frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}} K(1-|z|^2)dA(z) = 0.$$

Proof. Let

$$G(w) = \frac{(1-|w|^2)^{2+2s}}{K(1-|w|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{w}\varphi(z)|^{4+2s}} K(1-|z|^2)dA(z).$$

Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence of D_K such that $f_k \to 0$ weakly. Therefore, $f_k' \to 0$ uniformly on compact sets. From the proof of Theorem 1 and

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dominated convergence theorem, when $k \to \infty$, and $r \to 1$, it follows that

$$\begin{aligned}
&\|C_\varphi(f_k)\|_{D_K}^2 - |f_k(\varphi(0))|^2 \\
&\le \int_{\mathbb{D}} |f_k'(w)|^2 G(w) K(1-|w|^2)\, dA(w) \\
&\le \int_{r\mathbb{D}} |f_k'(w)|^2 G(w) K(1-|w|^2)\, dA(w) \\
&\qquad + \int_{\mathbb{D}\setminus r\mathbb{D}} |f_k'(w)|^2 G(w) K(1-|w|^2)\, dA(w) \to 0.
\end{aligned}$$

Thus, C_φ is compact.

Conversely, if C_φ is compact, let $\{a_k\}_{k=1}^\infty \subseteq \mathbb{D}$, $|a_k| \to 1$,

$$F_{a_k}(z) = \frac{(1-|a_k|^2)^{1+s}}{\sqrt{K(1-|a_k|^2)}} \frac{1}{(1-\overline{a_k}z)^{1+s}}.$$

Then, it is easy to verify that $F_{a_k} \to 0$ uniformly on compact sets. Thus, $\|C_\varphi(F_{a_k})\|_{D_K} \to 0$ as $k \to \infty$. The proof is completed. □

4. φ^n-TYPE CHARACTERIZATIONS

In [24], Wulan, Zheng and Zhu gave an interesting characterizations of compostion operators C_φ by φ^n. In this section, we are going to give an analogy results on D_K spaces.

Theorem 3. *Let (1.1) and (1.2) hold for K. Suppose $\varphi \in D_K$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $C_\varphi : D_K \to D_K$. Then*

(1) *If*
$$\sup_n \frac{1}{K(\frac{1}{n})} \|\varphi^n\|_{D_K}^2 < \infty,$$
then C_φ is bounded;

(2) *If C_φ is bounded, then*
$$\sup_n \frac{1}{nK(\frac{1}{n})} \|\varphi^n\|_{D_K}^2 < \infty.$$

Proof. (1). Let $a, z \in \mathbb{D}$ and $s > 0$. Since

$$|1 - \bar{a}\varphi(z)| \ge 1 - |a||\varphi(z)|$$

and

$$\frac{1}{(|1-|a||\varphi(z)|)^{4+2s}} \asymp \frac{1}{(|1-|a|^2|\varphi(z)|^2)^{4+2s}}.$$

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Note that

$$\frac{1}{(|1-|a|^2|\varphi(z)|^2)^{4+2s}} \asymp \sum_{n=0}^{\infty} \frac{\Gamma(n+4+2s)}{n!\Gamma(4+2s)}|a|^{2n}|\varphi(z)|^{2n},$$

it follows that

$$\begin{aligned}
&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}} K(1-|z|^2)dA(z) \\
\lesssim &\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|a||\varphi(z)|)^{4+2s}} K(1-|z|^2)dA(z) \\
\asymp &\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \sum_{n=0}^{\infty} \frac{\Gamma(n+4+2s)}{n!\Gamma(4+2s)}|a|^{2n}|\varphi(z)|^{2n}|\varphi'(z)|^2 K(1-|z|^2)dA(z).
\end{aligned}$$

By Stirling formula, we get

$$\frac{\Gamma(n+4+2s)}{n!\Gamma(4+2s)} \sim n^{3+2s},\ n\to\infty.$$

Therefore,

$$\begin{aligned}
&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}} K(1-|z|^2)dA(z) \\
\lesssim &\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \sum_{n=0}^{\infty} n^{3+2s}|a|^{2n} \int_{\mathbb{D}} |\varphi(z)|^{2n}|\varphi'(z)|^2 K(1-|z|^2)dA(z) \\
\le &\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \sum_{n=0}^{\infty} (n+1)^{3+2s}|a|^{2n} \int_{\mathbb{D}} |\varphi(z)|^{2n}|\varphi'(z)|^2 K(1-|z|^2)dA(z) \\
\le &\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \sum_{n=0}^{\infty} (n+1)^{1+2s}|a|^{2n}\|\varphi^n\|_{D_K}^2 \\
\lesssim &\sup_n \frac{1}{K(\frac{1}{n})}\|\varphi^n\|_{D_K}^2 \frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \sum_{n=0}^{\infty} (n+1)^{1+2s} K(\frac{1}{n})|a|^{2n}.
\end{aligned}$$

Following the proof of Lemma 2, we have

$$\sum_{n=0}^{\infty} (n+1)^{1+2s} K(\frac{1}{n})|a|^{2n} \asymp \frac{K(1-|a|^2)}{(1-|a|^2)^{2+2s}}.$$

Thus,

$$\begin{aligned}
&\frac{(1-|a|^2)^{2+2s}}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{a}\varphi(z)|^{4+2s}} K(1-|z|^2)dA(z) \\
\lesssim &\sup_n \frac{1}{K(\frac{1}{n})}\|\varphi^n\|_{D_K}^2.
\end{aligned}$$

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Hence, by Theorem 1, we prove (1).

(2). Suppose that C_φ is bounded on D_K. Let $f_n(z) = z^n/\|z^n\|^2_{D_K}$. Then, we have $\|f_n\|^2_{D_K} = 1$. An easy computation gives,

$$\infty > \|C_\varphi f_n\|^2_{D_K} = \frac{\|\varphi^n\|^2_{D_K}}{\|z^n\|^2_{D_K}} \gtrsim \frac{1}{nK(\frac{1}{n})}\|\varphi^n\|^2_{D_K}.$$

The last inequality is deduced by Lemma 3. The proof is completed. □

Theorem 4. *Let (1.1) and (1.2) hold for K. Suppose $\varphi \in D_K$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $C_\varphi : D_K \to D_K$. Then*

(1) *If*

$$\lim_{n\to\infty} \frac{1}{K(\frac{1}{n})}\|\varphi^n\|^2_{D_K} = 0,$$

then C_φ is compact;

(2) *If C_φ is compact, then*

$$\lim_{n\to\infty} \frac{1}{nK(\frac{1}{n})}\|\varphi^n\|^2_{D_K} = 0.$$

Proof. (1). The proof is similar to (1) of Theorem 3.

(2). Let $\{f_n\}$ be a bounded sequence in D_K that convergence to 0 weakly. If C_φ is compact on D_K, then $\|C_\varphi f_n\|_{D_K} \to 0$, as $n \to \infty$. Thus, for any $z \in \mathbb{D}$, we have

$$f_n(\varphi(z)) \to 0, \ \ n \to \infty.$$

Since $\{z^n/\|z^n\|_{D_K}, n \geq 1\}$ is bounded in D_K and it converges to 0 pointwise, the compactness of C_φ on D_K implies that

$$\lim_{n\to\infty} \frac{\|\varphi^n\|^2_{D_K}}{\|z^n\|^2_{D_K}} = \frac{1}{nK(\frac{1}{n})}\|\varphi^n\|^2_{D_K} = 0.$$

The proof is completed. □

5. Hilbert-Schmidt class

Let Hilbert-Schmidt class be the space of all compact operators on Hilbert space with its singular value sequence $\{\lambda_n\} \in l^2$, the 2-summable sequence space (see [27, page 18]). The following theorem give an equavalent charaterizations of composition operator on D_K spaces, when it belong to Hilbert-Schmidt class.

Theorem 5. *Let (1.1) and (1.2) hold for K. Suppose $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi \in D_K$ and C_φ is compact. Then C_φ is Hilbert-Schmidt on D_K if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \frac{K(1-|z|^2)}{K(1-|\varphi(z)|^2)} dA(z) < \infty.$$

Proof. Without loss of generality, we can assume $\{1\} \cup \{\frac{z^n}{\sqrt{n}\sqrt{K(\frac{1}{n})}}\}_{n=1}^{\infty}$ is an orthonormal basis in D_K and $\varphi(0) = 0$. From Theorem 1.22 of [27], C_φ is Hilbert-Schmidt on D_K if and only if

$$\sum_{n=1}^{\infty} \frac{D_K(\varphi^n)}{nK(\frac{1}{n})} < \infty.$$

Applying Lemma 2, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{D_K(\varphi^n)}{nK(\frac{1}{n})} &= \sum_{n=1}^{\infty} \frac{n}{K(\frac{1}{n})} \int_{\mathbb{D}} |\varphi^2(z)|^{n-1} |\varphi'(z)|^2 K(1-|z|^2) dA(z) \\ &= \sum_{n=0}^{\infty} \frac{n+1}{K(\frac{1}{n+1})} \int_{\mathbb{D}} |\varphi^2(z)|^{n} |\varphi'(z)|^2 K(1-|z|^2) dA(z) \\ &\asymp \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \frac{K(1-|z|^2)}{K(1-|\varphi(z)|^2)} dA(z).\end{aligned}$$

The proof is completed. □

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On left multidimensional Riemann-Liouville fractional integral

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Abstract

Here we study some important properties of left multidimensional Riemann-Liouville fractional integral operator, such as of continuity and boundedness.

2010 AMS Subject Classification: 26A33.
Key Words and Phrases: Riemann-Liouville fractional integral, continuity, boundedness.

1 Motivation

From [1], p. 388 we have

Theorem 1 *Let $r > 0$, $F \in L_\infty(a,b)$, and*

$$G(s) = \int_a^s (s-t)^{r-1} F(t)\,dt,$$

all $s \in [a,b]$. Then $G \in AC([a,b])$ (absolutely continuous functions) for $r \geq 1$, and $G \in C([a,b])$, only for $r \in (0,1)$.

2 Main Results

We give

Theorem 2 *Let $f \in L_\infty([a,b] \times [c,d])$, $\alpha_1, \alpha_2 > 0$. Consider the function*

$$F(x_1,x_2) = \int_{a_1}^{x_1} \int_{a_2}^{x_2} (x_1-t_1)^{\alpha_1-1} (x_2-t_2)^{\alpha_2-1} f(t_1,t_2)\,dt_1 dt_2, \qquad (1)$$

1

where $a_1, x_1 \in [a,b]$, $a_2, x_2 \in [c,d] : a_1 \leq x_1,\ a_2 \leq x_2$.

Then F *is continuous on* $[a_1, b] \times [a_2, d]$.

Proof. (I) Let $a_1, b_1, b_1^* \in [a,b]$ with $b_1 > b_1^* > a_1$, and $a_2, b_2, b_2^* \in [c,d]$ with $b_2 > b_2^* > a_2$.

We observe that

$$F(b_1, b_2) - F(b_1^*, b_2^*) =$$
$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 -$$
$$\int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} (b_1^* - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 = \tag{2}$$
$$\int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 -$$
$$\int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} (b_1^* - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$
$$\int_{b_1^*}^{b_1} \int_{a_2}^{b_2^*} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$
$$\int_{a_1}^{b_1^*} \int_{b_2^*}^{b_2} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$
$$\int_{b_1^*}^{b_1} \int_{b_2^*}^{b_2} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2.$$

Call

$$I(b_1^*, b_2^*) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left| (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} - (b_1^* - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} \right| dt_1 dt_2. \tag{3}$$

Thus

$$|F(b_1, b_2) - F(b_1^*, b_2^*)| \leq$$
$$\left\{ I(b_1^*, b_2^*) + \frac{(b_1 - b_1^*)^{\alpha_1}}{\alpha_1} \left[\frac{(b_2 - a_2)^{\alpha_2} - (b_2 - b_2^*)^{\alpha_2}}{\alpha_2} \right] + \right.$$
$$\left. \left(\frac{(b_1 - a_1)^{\alpha_1} - (b_1 - b_1^*)^{\alpha_1}}{\alpha_1} \right) \frac{(b_2 - b_2^*)^{\alpha_2}}{\alpha_2} + \frac{(b_1 - b_1^*)^{\alpha_1}}{\alpha_1} \frac{(b_2 - b_2^*)^{\alpha_2}}{\alpha_2} \right\} \|f\|_\infty. \tag{4}$$

Hence, by (4), it holds

$$\delta := \lim_{\substack{(b_1^*, b_2^*) \to (b_1, b_2) \\ \text{or} \\ (b_1, b_2) \to (b_1^*, b_2^*)}} |F(b_1, b_2) - F(b_1^*, b_2^*)| \leq \Big(\lim_{\substack{(b_1^*, b_2^*) \to (b_1, b_2) \\ \text{or} \\ (b_1, b_2) \to (b_1^*, b_2^*)}} I(b_1^*, b_2^*) \Big) \|f\|_\infty =: \rho. \tag{5}$$

If $\alpha_1 = \alpha_2 = 1$, then $\rho = 0$, proving $\delta = 0$.

If $\alpha_1 = 1$, $\alpha_2 > 0$ we get

$$I\left(b_1^*, b_2^*\right) = \left(b_1^* - a_1\right)\left(\int_{a_2}^{b_2^*} \left|\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1}\right| dt_2\right). \tag{6}$$

Assume $\alpha_2 > 1$, then $\alpha_2 - 1 > 0$. Hence by $b_2 > b_2^*$, then $b_2 - t_2 > b_2^* - t_2 \geq 0$, and $\left(b_2 - t_2\right)^{\alpha_2 - 1} > \left(b_2^* - t_2\right)^{\alpha_2 - 1}$ and $\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1} > 0$.

That is

$$I\left(b_1^*, b_2^*\right) = \left(b_1^* - a_1\right)\left[\left.\frac{\left(b_2 - t_2\right)^{\alpha_2}}{\alpha_2}\right|_{b_2^*}^{a_2} - \frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}\right]$$

$$= \left(b_1^* - a_1\right)\left[\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(b_2 - b_2^*\right)^{\alpha_2} - \left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}\right]. \tag{7}$$

Clearly, then

$$\lim_{\substack{b_2^* \to b_2 \\ \text{or} \\ b_2 \to b_2^*}} I\left(b_1^*, b_2^*\right) = 0. \tag{8}$$

Similarly and symmetrically, we obtain that

$$\lim_{\substack{b_1^* \to b_1 \\ \text{or} \\ b_1 \to b_1^*}} I\left(b_1^*, b_2^*\right) = 0, \tag{9}$$

for the case of $\alpha_2 = 1$, $\alpha_1 > 1$.

If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I\left(b_1^*, b_2^*\right) = \left(b_1^* - a_1\right)\left(\int_{a_2}^{b_2^*} \left(\left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_2 - t_2\right)^{\alpha_2 - 1}\right) dt_2\right) =$$

$$\left(b_1^* - a_1\right)\left[\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2 - a_2\right)^{\alpha_2} + \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2}\right]. \tag{10}$$

Clearly, then

$$\lim_{\substack{b_2^* \to b_2 \\ \text{or} \\ b_2 \to b_2^*}} I\left(b_1^*, b_2^*\right) = 0. \tag{11}$$

Similarly and symmetrically, we derive that

$$\lim_{\substack{b_1^* \to b_1 \\ \text{or} \\ b_1 \to b_1^*}} I\left(b_1^*, b_2^*\right) = 0, \tag{12}$$

for the case of $\alpha_2 = 1$, $0 < \alpha_1 < 1$.

Case now of $\alpha_1, \alpha_2 > 1$, then

$$I\left(b_1^*, b_2^*\right) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left[\left(b_1 - t_1\right)^{\alpha_1 - 1}\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1}\left(b_2^* - t_2\right)^{\alpha_2 - 1}\right] dt_1 dt_2 = \tag{13}$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right)\left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2}\right) - \frac{\left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1}\frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}.$$

That is

$$\lim_{\substack{(b_1, b_2) \to \left(b_1^*, b_2^*\right) \\ \text{or} \\ \left(b_1^*, b_2^*\right) \to (b_1, b_2)}} I\left(b_1^*, b_2^*\right) = 0. \tag{14}$$

Case now of $0 < \alpha_1, \alpha_2 < 1$, then

$$I\left(b_1^*, b_2^*\right) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left[\left(b_1^* - t_1\right)^{\alpha_1 - 1}\left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_1 - t_1\right)^{\alpha_1 - 1}\left(b_2 - t_2\right)^{\alpha_2 - 1}\right] dt_1 dt_2 =$$

$$\frac{\left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1}\frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2} - \left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right)\left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2}\right). \tag{15}$$

That is again, when $0 < \alpha_1, \alpha_2 < 1$,

$$\lim_{\substack{(b_1, b_2) \to \left(b_1^*, b_2^*\right) \\ \text{or} \\ \left(b_1^*, b_2^*\right) \to (b_1, b_2)}} I\left(b_1^*, b_2^*\right) = 0. \tag{16}$$

Next we treat the case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$.

We observe that

$$I\left(b_1^*, b_2^*\right) \leq I^*\left(b_1^*, b_2^*\right) := \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left(b_1 - t_1\right)^{\alpha_1 - 1}\left|\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1}\right| dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left(b_2^* - t_2\right)^{\alpha_2 - 1}\left|\left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1}\right| dt_1 dt_2. \tag{17}$$

Therefore it holds

$$I^*\left(b_1^*, b_2^*\right) = \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left(b_1 - t_1\right)^{\alpha_1 - 1}\left(\left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_2 - t_2\right)^{\alpha_2 - 1}\right) dt_1 dt_2 \tag{18}$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} \left(b_2^* - t_2\right)^{\alpha_2 - 1}\left(\left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1}\right) dt_1 dt_2 =$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right)\left[\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2 - a_2\right)^{\alpha_2} + \left(b_2 - b_2^*\right)^{\alpha_2}}{\alpha_2}\right] +$$

$$\frac{\left(b_2^* - a_2\right)^{\alpha_2}}{\alpha_2}\left[\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1} - \left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1}\right]. \tag{19}$$

So, in case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$, we proved that

$$\lim_{\substack{(b_1,b_2)\to(b_1^*,b_2^*) \\ \text{or} \\ (b_1^*,b_2^*)\to(b_1,b_2)}} I(b_1^*, b_2^*) = 0. \tag{20}$$

Finally, we prove the case of $\alpha_2 > 1$ and $0 < \alpha_1 < 1$. We have that

$$I^*(b_1^*, b_2^*) \overset{(17)}{=} \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} (b_1 - t_1)^{\alpha_1 - 1} \left[(b_2 - t_2)^{\alpha_2 - 1} - (b_2^* - t_2)^{\alpha_2 - 1} \right] dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} (b_2^* - t_2)^{\alpha_2 - 1} \left((b_1^* - t_1)^{\alpha_1 - 1} - (b_1 - t_1)^{\alpha_1 - 1} \right) dt_1 dt_2 = \tag{21}$$

$$\left(\frac{(b_1 - a_1)^{\alpha_1} - (b_1 - b_1^*)^{\alpha_1}}{\alpha_1} \right) \left[\frac{(b_2 - a_2)^{\alpha_2} - (b_2 - b_2^*)^{\alpha_2} - (b_2^* - a_2)^{\alpha_2}}{\alpha_2} \right] +$$

$$\frac{(b_2^* - a_2)^{\alpha_2}}{\alpha_2} \left[\frac{(b_1^* - a_1)^{\alpha_1} - (b_1 - a_1)^{\alpha_1} + (b_1 - b_1^*)^{\alpha_1}}{\alpha_1} \right]. \tag{22}$$

Hence again it holds

$$\lim_{\substack{(b_1,b_2)\to(b_1^*,b_2^*) \\ \text{or} \\ (b_1^*,b_2^*)\to(b_1,b_2)}} I(b_1^*, b_2^*) = 0. \tag{23}$$

We proved $\rho = 0$, and $\delta = 0$ in all cases of this section.

The case of $b_1^* > b_1$ and $b_2^* > b_2$, as symmetric to $b_1 > b_1^*$ and $b_2 > b_2^*$ we treated, it is omitted, a totally similar treatment.

(II) The remaining cases are: let $a_1, b_1, b_1^* \in [a, b]$; $a_2, b_2, b_2^* \in [c, d]$, we can have

(II_1) $b_1 > b_1^*$ and $b_2 < b_2^*$,

or

(II_2) $b_1 < b_1^*$ and $b_2 > b_2^*$.

Notice that (II_1) and (II_2) cases are symmetric, and treated the same way. As such we treat only the case (II_1).

We observe again that

$$F(b_1, b_2) - F(b_1^*, b_2^*) =$$

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2) dt_1 dt_2 -$$

$$\int_{a_1}^{b_1^*} \int_{a_2}^{b_2^*} (b_1^* - t_1)^{\alpha_1 - 1} (b_2^* - t_2)^{\alpha_2 - 1} f(t_1, t_2) dt_1 dt_2 = \tag{24}$$

$$\int_{a_1}^{b_1^*} \int_{a_2}^{b_2} (b_1 - t_1)^{\alpha_1 - 1} (b_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2) dt_1 dt_2 +$$

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$$\int_{b_1^*}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2-$$
$$\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2-$$
$$\int_{a_1}^{b_1^*}\int_{b_2}^{b_2^*}(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2=$$
$$\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left((b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}-(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}\right)f(t_1,t_2)\,dt_1dt_2$$
$$+\int_{b_1^*}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2-$$
$$\int_{a_1}^{b_1^*}\int_{b_2}^{b_2^*}(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}f(t_1,t_2)\,dt_1dt_2. \tag{25}$$

We call

$$I(b_1^*,b_2):=\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}\left|(b_1-t_1)^{\alpha_1-1}(b_2-t_2)^{\alpha_2-1}-(b_1^*-t_1)^{\alpha_1-1}(b_2^*-t_2)^{\alpha_2-1}\right|dt_1dt_2. \tag{26}$$

Hence, we have

$$|F(b_1,b_2)-F(b_1^*,b_2^*)|\leq$$
$$\left\{I(b_1^*,b_2)+\frac{(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}+\frac{(b_1^*-a_1)^{\alpha_1}}{\alpha_1}\frac{(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right\}\|f\|_\infty. \tag{27}$$

Therefore it holds

$$\delta:=\lim_{\substack{|b_1-b_1^*|\to 0,\\ |b_2-b_2^*|\to 0}}|F(b_1,b_2)-F(b_1^*,b_2^*)|\leq \left(\lim_{\substack{|b_1-b_1^*|\to 0,\\ |b_2-b_2^*|\to 0}} I(b_1^*,b_2)\right)\|f\|_\infty=:\theta. \tag{28}$$

We will prove that $\theta=0$, hence $\delta=0$, in all possible cases.

If $\alpha_1=\alpha_2=1$, then $I(b_1^*,b_2)=0$, hence $\theta=0$.

If $\alpha_1=1$, $\alpha_2>0$ we get

$$I(b_1^*,b_2)=(b_1^*-a_1)\left(\int_{a_2}^{b_2}\left|(b_2-t_2)^{\alpha_2-1}-(b_2^*-t_2)^{\alpha_2-1}\right|dt_2\right). \tag{29}$$

Assume $\alpha_2>1$, then $\alpha_2-1>0$. Hence

$$I(b_1^*,b_2)=(b_1^*-a_1)\left(\int_{a_2}^{b_2}\left((b_2^*-t_2)^{\alpha_2-1}-(b_2-t_2)^{\alpha_2-1}\right)dt_2\right)$$
$$=(b_1^*-a_1)\left[\frac{(b_2^*-t_2)^{\alpha_2}\big|_{b_2}^{a_2}-(b_2-a_2)^{\alpha_2}}{\alpha_2}\right]$$

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$$= \left(b_1^* - a_1\right) \left[\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2^* - b_2\right)^{\alpha_2} - \left(b_2 - a_2\right)^{\alpha_2}}{\alpha_2} \right]. \tag{30}$$

Clearly, then

$$\lim_{\substack{\left|b_1 - b_1^*\right| \to 0, \\ \left|b_2 - b_2^*\right| \to 0}} I\left(b_1^*, b_2\right) = 0, \tag{31}$$

hence $\theta = 0$.

Let the case now of $\alpha_2 = 1$, $\alpha_1 > 1$. Then

$$I\left(b_1^*, b_2\right) = \left(b_2 - a_2\right) \left(\int_{a_1}^{b_1^*} \left| \left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1} \right| dt_1 \right)$$

$$= \left(b_2 - a_2\right) \left[\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1} - \left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right]. \tag{32}$$

Then $\theta = 0$.

If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I\left(b_1^*, b_2\right) = \left(b_1^* - a_1\right) \int_{a_2}^{b_2} \left| \left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1} \right| dt_2 =$$

$$\left(b_1^* - a_1\right) \int_{a_2}^{b_2} \left(\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1} \right) dt_2 =$$

$$\left(b_1^* - a_1\right) \left[\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(b_2^* - a_2\right)^{\alpha_2} + \left(b_2^* - b_2\right)^{\alpha_2}}{\alpha_2} \right]. \tag{33}$$

Hence $\theta = 0$.

Let now $\alpha_2 = 1$, $0 < \alpha_1 < 1$. Then

$$I\left(b_1^*, b_2\right) = \left(b_2 - a_2\right) \int_{a_1}^{b_1^*} \left| \left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1} \right| dt_1$$

$$= \left(b_2 - a_2\right) \int_{a_1}^{b_1^*} \left(\left(b_1^* - t_1\right)^{\alpha_1 - 1} - \left(b_1 - t_1\right)^{\alpha_1 - 1} \right) dt_1$$

$$= \left(b_2 - a_2\right) \left[\frac{\left(b_1^* - a_1\right)^{\alpha_1} - \left(b_1 - a_1\right)^{\alpha_1} + \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1} \right]. \tag{34}$$

Hence $\theta = 0$.

We observe that:

$$I\left(b_1^*, b_2\right) \leq \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left| \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \right| dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*} \int_{a_2}^{b_2} \left| \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \right| dt_1 dt_2 =: J\left(b_1^*, b_2\right), \tag{35}$$

i.e.

$$I\left(b_1^*, b_2\right) \leq J\left(b_1^*, b_2\right).$$

Hence it holds

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left|\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1}\right| dt_1 dt_2 \tag{36}$$

$$+ \int_{a_1}^{b_1^*}\int_{a_2}^{b_2} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \left|\left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1}\right| dt_1 dt_2.$$

Case of $\alpha_1, \alpha_2 > 1$. Then

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(\left(b_2^* - t_2\right)^{\alpha_2 - 1} - \left(b_2 - t_2\right)^{\alpha_2 - 1}\right) dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*}\int_{a_2}^{b_2} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \left(\left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1}\right) dt_1 dt_2 =$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right) \left[\left(\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2^* - b_2\right)^{\alpha_2}}{\alpha_2}\right) - \frac{\left(b_2 - a_2\right)^{\alpha_2}}{\alpha_2}\right]$$

$$+ \left(\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2^* - b_2\right)^{\alpha_2}}{\alpha_2}\right) \left[\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right) - \frac{\left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1}\right]. \tag{37}$$

So that $\theta = 0$.

Case of $0 < \alpha_1, \alpha_2 < 1$, then

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1}\right) dt_1 dt_2$$

$$+ \int_{a_1}^{b_1^*}\int_{a_2}^{b_2} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \left(\left(b_1^* - t_1\right)^{\alpha_1 - 1} - \left(b_1 - t_1\right)^{\alpha_1 - 1}\right) dt_1 dt_2 =$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right) \left[\frac{\left(b_2 - a_2\right)^{\alpha_2}}{\alpha_2} - \left(\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2^* - b_2\right)^{\alpha_2}}{\alpha_2}\right)\right] \tag{38}$$

$$+ \left(\frac{\left(b_2^* - a_2\right)^{\alpha_2} - \left(b_2^* - b_2\right)^{\alpha_2}}{\alpha_2}\right) \left[\frac{\left(b_1^* - a_1\right)^{\alpha_1}}{\alpha_1} - \left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(b_1 - b_1^*\right)^{\alpha_1}}{\alpha_1}\right)\right].$$

One more time $\theta = 0$.

Next case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$. We observe that

$$J\left(b_1^*, b_2\right) = \int_{a_1}^{b_1^*}\int_{a_2}^{b_2} \left(b_1 - t_1\right)^{\alpha_1 - 1} \left(\left(b_2 - t_2\right)^{\alpha_2 - 1} - \left(b_2^* - t_2\right)^{\alpha_2 - 1}\right) dt_1 dt_2 \tag{39}$$

$$+ \int_{a_1}^{b_1^*}\int_{a_2}^{b_2} \left(b_2^* - t_2\right)^{\alpha_2 - 1} \left(\left(b_1 - t_1\right)^{\alpha_1 - 1} - \left(b_1^* - t_1\right)^{\alpha_1 - 1}\right) dt_1 dt_2 =$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)\left[\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}-\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\right]$$
$$+\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\left[\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)-\frac{(b_1^*-a_1)^{\alpha_1}}{\alpha_1}\right]. \tag{40}$$

Hence $\theta=0$.

Finally, we prove the case of $\alpha_2>1$ and $0<\alpha_1<1$. In that case it holds

$$J(b_1^*,b_2)=\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha_1-1}\left((b_2^*-t_2)^{\alpha_2-1}-(b_2-t_2)^{\alpha_2-1}\right)dt_1dt_2 \tag{41}$$
$$+\int_{a_1}^{b_1^*}\int_{a_2}^{b_2}(b_2^*-t_2)^{\alpha_2-1}\left((b_1^*-t_1)^{\alpha_1-1}-(b_1-t_1)^{\alpha_1-1}\right)dt_1dt_2=$$
$$\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)\left[-\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}+\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\right]$$
$$+\left(\frac{(b_2^*-a_2)^{\alpha_2}-(b_2^*-b_2)^{\alpha_2}}{\alpha_2}\right)\left[-\left(\frac{(b_1-a_1)^{\alpha_1}-(b_1-b_1^*)^{\alpha_1}}{\alpha_1}\right)+\frac{(b_1^*-a_1)^{\alpha_1}}{\alpha_1}\right]. \tag{42}$$

Hence again $\theta=0$.

We have proved that $\delta=0$, in all possible subcases of (II_1).

We have proved that F is a continuous function over $[a_1,b]\times[a_2,d]$. ■

Now we can state:

Theorem 3 *Let $f\in L_\infty\left(\prod_{i=1}^k[a_i,b_i]\right)$, $\alpha_i>0$, $i=1,...,k\in\mathbb{N}$. Consider the function*

$$F(x_1,...,x_k)=\int_{a_1^*}^{x_1}...\int_{a_k^*}^{x_k}\prod_{i=1}^k(x_i-t_i)^{\alpha_i-1}f(t_1,...,t_k)\,dt_1...dt_k, \tag{43}$$

where $a_i^*,x_i\in[a_i,b_i]$, $a_i^*\le x_i$, $i=1,...,k$.

Then F is continuous on $\prod_{i=1}^k[a_i^*,b_i]$.

Remark 4 *In the setting of Theorem 3: Consider the left multidimensional Riemann-Liouville fractional integral of order $\alpha=(\alpha_1,...,\alpha_k)$:*

$$\left(I_{a_+^*}^{\alpha}f\right)(x)=\frac{1}{\prod_{i=1}^k\Gamma(\alpha_i)}\int_{a_1^*}^{x_1}...\int_{a_k^*}^{x_k}\prod_{i=1}^k(x_i-t_i)^{\alpha_i-1}f(t_1,...,t_k)\,dt_1...dt_k, \tag{44}$$

where $a^*=(a_1^*,...,a_k^*)$, $x=(x_1,...,x_k)$, $a_i^*\le x_i$, $i=1,...,k$. Here Γ denotes the gamma function.

By Theorem 3 we get that $\left(I_{a_+^*}^{\alpha}f\right)(x)$ is a continuous function for every $x\in\prod_{i=1}^k[a_i^*,b_i]$.

We notice that

$$\left|\left(I_{a_+^*}^{\alpha} f\right)(x)\right| \leq \frac{1}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \left(\int_{a_1^*}^{x_1} ... \int_{a_k^*}^{x_k} \prod_{i=1}^{k} (x_i - t_i)^{\alpha_i - 1} dt_1 ... dt_k \right) \|f\|_\infty$$

$$= \frac{\|f\|_\infty}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} \left(\int_{a_i^*}^{x_i} (x_i - t_i)^{\alpha_i - 1} dt_i \right) = \frac{\|f\|_\infty}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} \frac{(x_i - a_i^*)^{\alpha_i}}{\alpha_i} \quad (45)$$

$$= \|f\|_\infty \left(\prod_{i=1}^{k} \frac{(x_i - a_i^*)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right).$$

That is

$$\left|\left(I_{a_+^*}^{\alpha} f\right)(x)\right| \leq \left(\prod_{i=1}^{k} \frac{(x_i - a_i^*)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \|f\|_\infty . \quad (46)$$

In particular we get that

$$\left(I_{a_+^*}^{\alpha} f\right)(a^*) = 0, \quad (47)$$

and

$$\left\|I_{a_+^*}^{\alpha} f\right\|_\infty \leq \left(\prod_{i=1}^{k} \frac{(b_i - a_i^*)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \|f\|_\infty . \quad (48)$$

That is $I_{a_+^*}^{\alpha} f$ *is a bounded linear operator, which here is also a positive operator.*

References

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10

Weak closure operations on ideals of BCK-algebras

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Abstract. Weak closure operation, which is more general form than closure operation, on ideals of BCK-algebras is introduced, and related properties are investigated. Regarding weak closure operation, finite type and (strong) quasi-primeness are considered. Also positive implicative (resp., commutative and implicative) weak closure operations are discussed.

1. Introduction

Semi-prime closure operations on ideals of BCK-algebras are introduced in the paper [1], and a finite type of closure operations on ideals of BCK-algebras are discussed in [2].

In this paper, we consider more general form than closure operations on ideals of BCK-algebras. We introduce the notion of weak closure operations on ideals of BCK-algebras. Regarding weak closure operation, we define finite type and (strong) quasi-primeness, and investigate related properties. We also discuss positive implicative (resp., commutative and implicative) weak closure operations, and provide several examples to illustrate notions and properties.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

[0]**2010 Mathematics Subject Classification**: 06F35; 03G25.
[0]**Keywords**: weak closure operation of finite type; (quasi-prime, positive implicative, commutative, implicative) weak closure operation.
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(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X (see [4]) if it satisfies:

$$0 \in A, \tag{2.1}$$

$$(\forall x \in X)\,(\forall y \in A)\,(x * y \in A\ \Rightarrow\ x \in A)\,. \tag{2.2}$$

For any subset A of X, the ideal generated by A is defined to be the intersection of all ideals of X containing A, and it is denoted by $\langle A\rangle$. If A is finite, then we say that $\langle A\rangle$ is *finitely generated ideal* of X (see [4]).

A subset A of a BCK-algebra X is called a *commutative ideal* of X (see [4]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A)\,((x * y) * z \in A\ \Rightarrow\ x * (y * (y * x)) \in A)\,. \tag{2.3}$$

A subset A of a BCK-algebra X is called a *positive implicative ideal* of X (see [4]) if it satisfies (2.1) and

$$(\forall x, y, z \in X)\,((x * y) * z \in A,\ y * z \in A\ \Rightarrow\ x * z \in A)\,. \tag{2.4}$$

A subset A of a BCK-algebra X is called an *implicative ideal* of X (see [4]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A)\,((x * (y * y)) * z \in A\ \Rightarrow\ x \in A)\,. \tag{2.5}$$

Denote by $\mathcal{I}_{pi}(X)$ (resp., $\mathcal{I}_c(X)$ and $\mathcal{I}_m(X)$) the set of all positive implicative (resp., commutative and implicative) ideals of X.

We refer the reader to the books [3, 4] for further information regarding BCK/BCI-algebras.

3. **Weak Closure operations**

In what follows, let X and $\mathcal{I}(X)$ be a BCK-algebra and a set of all ideals of X, respectively, unless otherwise specified .

Definition 3.1. A mapping $c : \mathcal{I}(X) \to \mathcal{I}(X)$ is called a *weak closure operation* on $\mathcal{I}(X)$ if the following conditions are valid.

$$(\forall A \in \mathcal{I}(X))\,(A \subseteq c(A)), \tag{3.1}$$

$$(\forall A, B \in \mathcal{I}(X))\,(A \subseteq B \;\Rightarrow\; c(A) \subseteq c(B)). \tag{3.2}$$

If a weak closure operation $c : \mathcal{I}(X) \to \mathcal{I}(X)$ satisfies the condition

$$(\forall A \in \mathcal{I}(X))\,(c(c(A)) = c(A)), \tag{3.3}$$

then we say that c is a closure operation on $\mathcal{I}(X)$ (see [2]). In what follows, we use A^{cl} instead of $c(A)$.

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

We have 8 ideals of X, and they are $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 4\}$, $A_4 = \{0, 1, 4\}$, $A_5 = \{0, 1, 2, 3\}$, $A_6 = \{0, 2, 4\}$, and $A_7 = X$. Define a mapping $c : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_4$, $A_2^{cl} = A_5$, $c(A_3) = A_6$, and $c(A_4) = c(A_5) = c(A_6) = c(A_7) = A_7$. Then c is a weak closure operation on $\mathcal{I}(X)$. But it is not a closure operation on $\mathcal{I}(X)$ since $c(A_2^{cl}) = c(A_5) = A_7$.

In a BCK-algebra X, let $x \wedge y$ denote the greatest lower bound of x and y. Note that $0 \wedge x = 0$ for all $x \in X$. For any element x of X, consider the following condition

$$(\exists\, y \in X \setminus \{0\})\,(x \wedge y = 0). \tag{3.4}$$

In the following example, we know that there are two kinds of element. One is an element x satisfying the condition (3.4). The other is an element x which does not satisfy the condition (3.4).

Example 3.3. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	4	4	4	0

Then X is a BCK-algebra. We know that 1 and 2 satisfy the condition (3.4), but 3 and 4 do not satisfy the condition (3.4).

On the basis of this consideration, we define the zeromeet element in a BCK-algebra.

Definition 3.4. An element x of X is called a *zeromeet element* of X if the condition (3.4) is valid. Otherwise, x is called a non-zeromeet element of X.

Denote by $Z(X)$ the set of all zeromeet elements of X, that is,

$$Z(X) = \{x \in X \mid x \wedge y = 0 \text{ for some nonzero element } y \in X\}.$$

Obviously, $0 \in Z(X)$. We know that $0, 1, 2 \in Z(X)$ and $3, 4 \notin Z(X)$ in Example 3.3.

Lemma 3.5. *For any $x, y \in X$, if $x, y \notin Z(X)$, then $x \wedge y \notin Z(X)$, that is, the set $X \setminus Z(X)$ is closed under the operation $\wedge$.*

Proof. Let $x, y \in X \setminus Z(X)$ and assume that $x \wedge y \in Z(X)$. Then $x \wedge (y \wedge a) = (x \wedge y) \wedge a = 0$ for some nonzero element $a \in X$. Since $x \notin Z(X)$, it follows that $y \wedge a = 0$ and so that $a = 0$ since $y \notin Z(X)$. This is a contradiction, and thus $x \wedge y \notin Z(X)$. □

For any subsets A and B of X, we define

$$A \wedge B := \langle\{a \wedge b \mid a \in A,\ b \in B\rangle.$$

We use $x \wedge A$ instead of $\{x\} \wedge A$, that is, $x \wedge A := \langle\{x \wedge a \mid a \in A\}\rangle$.

Definition 3.6. A weak closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is said to be *quasi-prime* if it satisfies:

$$(\forall a \in X \setminus Z(X))\, (\forall A \in \mathcal{I}(X))\, \left(a \wedge A^{cl} \subseteq (a \wedge A)^{cl}\right). \tag{3.5}$$

Example 3.7. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

We know that $Z(X) = \{0\}$ and there are four ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$ and $A_3 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0, A_1^{cl} = A_2, A_2^{cl} = A_3$ *and* $A_3^{cl} = A_3$. Then "*cl*" is a weak closure operation on $\mathcal{I}(X)$. For $1, 2, 3 \in X \setminus Z(X)$, we have

$1 \wedge A_0^{cl} = 1 \wedge A_0 = \langle\{0\}\rangle = A_0 = A_0^{cl} = (1 \wedge A_0)^{cl}$,
$1 \wedge A_1^{cl} = 1 \wedge A_2 = \langle\{0, 1\}\rangle = A_1 \subseteq A_2 = A_1^{cl} = (1 \wedge A_1)^{cl}$,
$1 \wedge A_2^{cl} = 1 \wedge A_3 = \langle\{0, 1\}\rangle = A_1 \subseteq A_2 = A_1^{cl} = (1 \wedge A_2)^{cl}$,
$1 \wedge A_3^{cl} = 1 \wedge A_3 = \langle\{0, 1\}\rangle = A_1 \subseteq A_2 = A_1^{cl} = (1 \wedge A_3)^{cl}$,

$2 \wedge A_0^{cl} = 2 \wedge A_0 = \langle\{0\}\rangle = A_0 = A_0^{cl} = (2 \wedge A_0)^{cl}$,
$2 \wedge A_1^{cl} = 2 \wedge A_2 = \langle\{0,1,2\}\rangle = A_2 = A_1^{cl} = (2 \wedge A_1)^{cl}$,
$2 \wedge A_2^{cl} = 2 \wedge A_3 = \langle\{0,1,2\}\rangle = A_2 \subseteq A_3 = A_2^{cl} = (2 \wedge A_2)^{cl}$,
$2 \wedge A_3^{cl} = 2 \wedge A_3 = \langle\{0,1,2\}\rangle = A_2 \subseteq A_3 = A_2^{cl} = (2 \wedge A_3)^{cl}$,
$3 \wedge A_0^{cl} = 3 \wedge A_0 = \langle\{0\}\rangle = A_0 = A_0^{cl} = (3 \wedge A_0)^{cl}$,
$3 \wedge A_1^{cl} = 3 \wedge A_2 = \langle\{0,1,2\}\rangle = A_2 = A_1^{cl} = (3 \wedge A_1)^{cl}$,
$3 \wedge A_2^{cl} = 3 \wedge A_3 = \langle\{0,1,2,3\}\rangle = A_3 = A_2^{cl} = (3 \wedge A_2)^{cl}$,
$3 \wedge A_3^{cl} = 3 \wedge A_3 = \langle\{0,1,2,3\}\rangle = A_3 = A_3^{cl} = (3 \wedge A_3)^{cl}$,
Therefore "cl" is a quasi-prime weak closure operation on $\mathcal{I}(X)$.

Definition 3.8. A weak closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is said to be *strong quasi-prime* if it satisfies:

$$(\forall a \in X \setminus Z(X))\,(\forall A \in \mathcal{I}(X))\,\left(a \wedge A^{cl} = (a \wedge A)^{cl}\right). \tag{3.6}$$

Example 3.9. Consider a BCK-algebra $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

We know that $Z(X) = \{0,1,2\}$ and there are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0,1\}$, $A_2 = \{0,2\}$, $A_3 = \{0,1,2\}$, $A_4 = \{0,1,2,3\}$ and $A_5 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl} = A_1$, $A_1^{cl} = A_2^{cl} = A_3$, $A_3^{cl} = A_4^{cl} = A_4$ and $A_5^{cl} = A_5$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$. For $3, 4 \in X \setminus Z(X)$, we have
$3 \wedge A_0^{cl} = 3 \wedge A_1 = \langle\{0,1\}\rangle = A_1 = A_0^{cl} = (3 \wedge A_0)^{cl}$,
$3 \wedge A_1^{cl} = 3 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_1^{cl} = (3 \wedge A_1)^{cl}$,
$3 \wedge A_2^{cl} = 3 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_2^{cl} = (3 \wedge A_2)^{cl}$,
$3 \wedge A_3^{cl} = 3 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_3^{cl} = (3 \wedge A_3)^{cl}$,
$3 \wedge A_4^{cl} = 3 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (3 \wedge A_4)^{cl}$,
$3 \wedge A_5^{cl} = 3 \wedge A_5 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (3 \wedge A_5)^{cl}$,
$4 \wedge A_0^{cl} = 4 \wedge A_1 = \langle\{0,1\}\rangle = A_1 = A_0^{cl} = (4 \wedge A_0)^{cl}$,
$4 \wedge A_1^{cl} = 4 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_1^{cl} = (4 \wedge A_1)^{cl}$,
$4 \wedge A_2^{cl} = 4 \wedge A_3 = \langle\{0,1,2\}\rangle = A_3 = A_2^{cl} = (4 \wedge A_2)^{cl}$,
$4 \wedge A_3^{cl} = 4 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_3^{cl} = (4 \wedge A_3)^{cl}$,
$4 \wedge A_4^{cl} = 4 \wedge A_4 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (4 \wedge A_4)^{cl}$,
$4 \wedge A_5^{cl} = 4 \wedge A_5 = \langle\{0,1,2,3\}\rangle = A_4 = A_4^{cl} = (4 \wedge A_5)^{cl}$.
Therefore "cl" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$.

Given an ideal A of X and an operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ on $\mathcal{I}(X)$, we consider the following set:

$$K := \cup\{B^{cl} \mid B \subseteq A,\ B \in \mathcal{I}_f(X)\} \tag{3.7}$$

where $\mathcal{I}_f(X)$ is the set of all finitely generated ideals of X. The following example shows that the set K in (3.7) may not be an ideal of X in general.

Example 3.10. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$, $A_3 = \{0, 1, 3\}$ and $A_4 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl} = A_3$, $A_1^{cl} = A_2$, $A_2^{cl} = A_0$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_3$. For the ideal A_2 of X, we have

$$\cup\{B^{cl} \mid B \subseteq A,\ B \in \mathcal{I}_f(X)\} = A_0^{cl} \cup A_1^{cl} \cup A_2^{cl} = \{0, 1, 2, 3\}$$

which is not an ideal of X.

We provide a condition for the set K in (3.7) to be an ideal of X.

Theorem 3.11. *If $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is a weak closure operation on $\mathcal{I}(X)$, then the set K in (3.7) is an ideal of X for any ideal A of X.*

Proof. Obviously, $0 \in K$. Let $x, y \in X$ such that $x * y \in K$ and $y \in K$. Then there exist B_x, $B_y \in \mathcal{I}_f(X)$ such that $B_x \subseteq A$, $B_y \subseteq A$, $x * y \in B_x^{cl}$ and $y \in B_y^{cl}$. Since $B_x, B_y \subseteq B_x + B_y = \langle B_x \cup B_y \rangle$, we have $x * y \in B_x^{cl} \subseteq (B_x + B_y)^{cl}$ and $y \in B_y^{cl} \subseteq (B_x + B_y)^{cl}$, which imply that $x \in (B_x + B_y)^{cl}$. Since B_x, $B_y \in \mathcal{I}_f(X)$, we get $B_x + B_y \in \mathcal{I}_f(X)$ and $B_x + B_y \subseteq A$. Therefore $x \in K$, and K is an ideal of X. □

Corollary 3.12. *If $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is a closure operation on $\mathcal{I}(X)$, then the set K in (3.7) is an ideal of X for any ideal A of X.*

Lemma 3.13 ([4])**.** (Extension property) *Let A and B be ideals of X such that $A \subseteq B$. If A is a positive implicative (resp., commutative and implicative) ideal, then so is B.*

Using Lemma 3.13 and (3.1), we have the following theorem.

Theorem 3.14. *Let "cl" be a weak closure operation on $\mathcal{I}(X)$. If A is a positive implicative (resp., commutative and implicative) ideal of X, then so is A^{cl}.*

The following example shows that the converse of Theorem 3.14 is not true in general.

Example 3.15. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$ and $A_4 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $(A_0)^{cl} = A_0$, $(A_1)^{cl} = (A_2)^{cl} = A_3$, and $(A_3)^{cl} = (A_4)^{cl} = A_4$. Then "$cl$"" is a weak closure operation on $\mathcal{I}(X)$. The ideal $A_2 = \{0, 2\}$ is not positive implicative (resp., commutative and implicative) ideal, but $(A_2)^{cl} = A_3 = \{0, 1, 2\}$ is a positive implicative (resp., commutative and implicative) ideal of X.

Theorem 3.16. *An operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ on $\mathcal{I}(X)$ defined by*

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl} = \cap\{I_\lambda \mid I_\lambda \in \mathcal{I}_\Gamma(X),\, A \subseteq I_\lambda, \lambda \in \Lambda\} \right) \tag{3.8}$$

is a weak closure operation on $\mathcal{I}(X)$ where $\mathcal{I}_\Gamma(X) \in \{\mathcal{I}_{pi}(X), \mathcal{I}_c(X), \mathcal{I}_m(X)\}$ and Λ is any index set.

Proof. Obviously, $A \subseteq A^{cl}$ for every $A \in \mathcal{I}(X)$. Let $A, B \in \mathcal{I}(X)$ be such that $A \subseteq B$. Then

$$\begin{aligned} B^{cl} &= \cap\{I_\lambda \mid I_\lambda \in \mathcal{I}_\Gamma(X),\, B \subseteq I_\lambda, \lambda \in \Lambda\} \\ &\supseteq \cap\{I_\lambda \mid I_\lambda \in \mathcal{I}_\Gamma(X),\, A \subseteq I_\lambda, \lambda \in \Lambda\} \\ &= A^{cl}, \end{aligned}$$

and so "cl" is a weak closure operation on $\mathcal{I}(X)$. □

The following example illustrates Theorem 3.16.

Example 3.17. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1, 3\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 2, 4\}$ and $A_5 = X$.

(1) Define a mapping $cl_1 : \mathcal{I}(X) \to \mathcal{I}(X)$ by

$$A^{cl_1} = \cap\{B \mid A \subseteq B \text{ and } B \in \mathcal{I}_{pi}(X)\}.$$

Then we have
$A_0^{cl_1} = A_1 \cap A_3 \cap A_5 = A_1$, $A_1^{cl_1} = A_1 \cap A_3 \cap A_5 = A_1$,
$A_2^{cl_1} = A_3 \cap A_5 = A_3$, $A_3^{cl_1} = A_3 \cap A_5 = A_3$, $A_4^{cl_1} = A_5 = A_5^{cl_1}$.
We can check that "cl_1" is a weak closure operation on $\mathcal{I}(X)$.

(2) We define an operation "cl_2" on $\mathcal{I}(X)$ by

$$A^{cl_2} = \cap\{B \mid A \subseteq B \text{ and } B \in \mathcal{I}_c(X)\}.$$

Then we have
$A_0^{cl_2} = A_2 \cap A_3 \cap A_4 \cap A_5 = A_2$, $A_1^{cl_2} = A_3 \cap A_5 = A_3$,
$A_2^{cl_2} = A_2 \cap A_3 \cap A_4 \cap A_5 = A_2$, $A_3^{cl_2} = A_3 \cap A_5 = A_3$,
$A_4^{cl_2} = A_4 \cap A_5 = A_4$, $A_5^{cl_2} = A_5$.
It is routine to verify that "cl_2" is a weak closure operation on $\mathcal{I}(X)$.

(3) We define an operation "cl_3" on $\mathcal{I}(X)$ by

$$A^{cl_3} = \cap\{B \mid A \subseteq B \text{ and } B \in \mathcal{I}_m(X)\}.$$

Then we have
$A_0^{cl_3} = A_3 \cap A_5 = A_3$, $A_1^{cl_3} = A_3 \cap A_5 = A_3$,
$A_2^{cl_3} = A_3 \cap A_5 = A_3$, $A_3^{cl_3} = A_3 \cap A_5 = A_3$,
$A_4^{cl_3} = A_5$, $A_5^{cl_3} = A_5$.
It is easy to show that "cl_3" is weak closure operation on $\mathcal{I}(X)$.

Let $\{cl_\lambda \mid \lambda \in \Lambda\}$ be a collection of operations on $\mathcal{I}(X)$. We define the intersection of cl_λ's, denoted by $\underset{\lambda\in\Lambda}{\cap} cl_\lambda$, as follows:

$$\underset{\lambda\in\Lambda}{\cap} cl_\lambda : \mathcal{I}(X) \to \mathcal{I}(X),\ A \mapsto \underset{\lambda\in\Lambda}{\cap} A^{cl_\lambda}.$$

Note that if cl_λ is a weak closure operation on $\mathcal{I}(X)$ for all $\lambda \in \Lambda$, then $\underset{\lambda\in\Lambda}{\cap} cl_\lambda$ is a weak closure operation on $\mathcal{I}(X)$ (see [2]). But the following example shows that the union of weak closure operations may not be a weak closure operation.

Example 3.18. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	1	4	1	0

There are four ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1, 4\}$, $A_2 = \{0, 2\}$ and $A_3 = X$. Define a mapping $cl_1 : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl_1} = A_1$, $A_1^{cl_1} = A_3$, $A_2^{cl_1} = A_3$, $A_3^{cl_1} = A_3$. Then "cl_1" is a weak closure operation on $\mathcal{I}(X)$. Also, define a mapping $cl_2 : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl_2} = A_2$, $A_1^{cl_2} = A_3$, $A_2^{cl_2} = A_3$, $A_3^{cl_2} = A_3$. Then "cl_2" is a weak closure operation on $\mathcal{I}(X)$.

Now if we define "cl_3" by $A^{cl_3} = A^{cl_1} \cup A^{cl_2}$, then "$cl_3$" is not a weak closure operation on $\mathcal{I}(X)$ because for an ideal A_0 of X, we have

$$A_0^{cl_3} = A_0^{cl_1} \cup A_0^{cl_2} = A_1 \cup A_2 = \{0, 1, 2, 4\}$$

which is not an ideal of X. Thus "cl_3" is not a weak closure operation on $\mathcal{I}(X)$.

Definition 3.19. Given a (weak) closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ on $\mathcal{I}(X)$, we define a new operation $cl_f : \mathcal{I}(X) \to \mathcal{I}(X)$ by

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A,\ B \in \mathcal{I}_f(X)\} \right), \tag{3.9}$$

where $\mathcal{I}_f(X)$ is the set of all finitely generated ideals of X.

Definition 3.20. A (weak) closure operation cl on $\mathcal{I}(X)$ is said to be of *finite type* if the following assertion is valid.

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl} = A^{cl_f} \right). \tag{3.10}$$

Note that every weak closure operation on a finite BCK-algebra is of finite type.

Example 3.21. Let X be a BCK-algebra of infinite order. Define an operation "cl'' on $\mathcal{I}(X)$ as follows:

$$A^{cl} = \begin{cases} X & \text{if } A \text{ is a maximal ideal or } A = X, \\ M & \text{otherwise,} \end{cases} \tag{3.11}$$

where M is a maximal ideal of X containing A. We can easily check that "cl" is a weak closure operation. Now let A be a maximal ideal of X which is not finitely generated. Then

$$A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A \text{ and } B \in \mathcal{I}_f(X)\} \subseteq M \subsetneq X = A^{cl},$$

and thus "cl" is a weak closure operation which is not of finite type.

For two operations "cl_1" and "cl_2" on $\mathcal{I}(X)$, we say that "cl_1" is *weaker* than "cl_2", denoted by $cl_1 \leq cl_2$, if $A^{cl_1} \subseteq A^{cl_2}$ for every $A \in \mathcal{I}(X)$.

Theorem 3.22. *Given an operation "cl" on $\mathcal{I}(X)$, we have*

(i) *If "cl" is a weak closure operation of finite type, then so is "cl_f", and it is largest in the set of weak closure operations which are weaker than "cl".*

(ii) *If "cl" is a (strong) quasi-prime weak closure operation, then so is "cl_f".*

Proof. (i) Let "cl" be a weak closure operation of finite type. Then "cl_f" is a weak closure operation on $\mathcal{I}(X)$ (see [2]). To prove that "cl_f" is of finite type, we should prove that $A^{cl_f} = A^{(cl_f)_f}$ for every ideal A of X. Clearly, we have $A^{cl_f} \subseteq A^{(cl_f)_f}$. Suppose that $x \in A^{(cl_f)_f}$. Then there exists a finitely generated ideal B such that $B \subseteq A$ and $x \in B^{cl_f}$. Since "cl" is a weak closure operation of finite type, we have $B^{cl} = B^{cl_f}$. Thus $x \in B^{cl}$, $B \subseteq A$ and B is finitely generated ideal. Therefore $x \in A^{cl_f}$ and $A^{cl_f} = A^{(cl_f)_f}$ which means that "cl_f" is a weak closure

operation on $\mathcal{I}(X)$ of finite type. Now let c be a weak closure operation on $\mathcal{I}(X)$ of finite type which is weaker than "*cl*". Let A be an ideal of X and $a \in A^c$. Then there exists a finitely generated ideal B of X such that $B \subseteq A$ and $a \in B^c$. It follows from $c \leq cl$ that $a \in B^{cl}$. Therefore $a \in A^{cl_f}$, and so $c \leq cl_f$.

(ii) Suppose that "*cl*" be a quasi prime weak closure operation on $\mathcal{I}(X)$. To prove that "cl_f" is a quasi prime weak closure operation, it is enough to show that $a \wedge A^{cl_f} \subseteq (a \wedge A)^{cl_f}$. Now let $x \in a \wedge A^{cl_f} = \langle \{a \wedge \alpha \mid \alpha \in A^{cl_f}\} \rangle$. Then there exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in A^{cl_f}$ such that

$$(\cdots((x \wedge (a \wedge \alpha_1)) * (a \wedge \alpha_2)) * \cdots) * (a \wedge \alpha_n) = 0.$$

Since $\alpha_i \in A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A \text{ and } B \in \mathcal{I}_f(X)\}$ for each $1 \leq i \leq n$, we have $\alpha_i \in A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A \text{ } and \text{ } B \in \mathcal{I}_f(X)\}$, and so there exists a finitely generated ideal B such that $\alpha_i \in B^{cl}$ and $B \subseteq A$. Since $\alpha_i \in B^{cl}$, we have

$$a \wedge \alpha_i \in \{a \wedge \beta \mid \beta \in B\} \subseteq \langle \{a \wedge \beta \mid \beta \in B\} \rangle = a \wedge B^{cl},$$

which implies that $a \wedge \alpha_i \in a \wedge B$ and

$$(\cdots((x \wedge (a \wedge \alpha_1)) * (a \wedge \alpha_2)) * \cdots) * (a \wedge \alpha_n) = 0.$$

This means that $x \in a \wedge B^{cl}$. Since "*cl*" is a quasi prime weak closure operation on $\mathcal{I}(X)$, it follows that

$$x \in a \wedge B^{cl} \subseteq (a \wedge B)^{cl} \subseteq (a \wedge A)^{cl} \subseteq (a \wedge A)^{cl_f}.$$

Therefore $x \in (a \wedge A)^{cl_f}$ and "cl_f" is a quasi-prime weak closure operation on $\mathcal{I}(X)$. Similarly, we can check that if "*cl*" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_f" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$. □

Definition 3.23. An operation $\alpha : \mathcal{I}(X) \to \mathcal{I}(X)$ is called a *positive implicative* (resp. *commutative* and *implicative*) *weak closure operation* if the following conditions are valid.

(i) For any $A, B \in \mathcal{I}_{pi}(X)$ (resp. $\mathcal{I}_c(X)$ and $\mathcal{I}_m(X)$),

$$A \subseteq A^\alpha, \tag{3.12}$$

$$A \subseteq B \;\Rightarrow\; A^\alpha \subseteq B^\alpha. \tag{3.13}$$

(ii) $(\forall A \notin \mathcal{I}_{pi}(X)(\text{resp., } \mathcal{I}_c(X) \text{ and } \mathcal{I}_m(X)))\,(A^\alpha = A)$.

Obviously, every positive implicative (resp., commutative and implicative) weak closure operation is a weak closure operation, but the converse is not true in general as seen in the following example.

Example 3.24. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0,1,3\}$, $A_2 = \{0,2\}$, $A_3 = \{0,1,2,3\}$, $A_4 = \{0,2,4\}$ and $A_5 = X$. Note that A_1, A_3 and A_5 are positive implicative ideals and A_0, A_2 and A_4 are not positive implicative ideals. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ as follows: $A_0^{cl} = A_0$ $A_1^{cl} = A_3$, $A_2^{cl} = A_2$, $A_3^{cl} = A_5$, $A_4^{cl} = A_4$ and $A_5^{cl} = X$. Then "cl" is a positive implicative weak closure operation on $\mathcal{I}(X)$. Now we define an operation "cl_1" on $\mathcal{I}(X)$ as follows:

$$A_0^{cl_1} = A_1,\ A_1^{cl_1} = A_3,\ A_2^{cl_1} = A_4,\ A_3^{cl_1} = A_5,\ A_4^{cl_1} = A_5 \text{ and } A_5^{cl_1} = X.$$

Then "cl_1" is a weak closure operation on $\mathcal{I}(X)$, but it is not positive implicative because the ideal A_2 is not a positive implicative ideal and $A_2^{cl_1} = A_4 \neq A_2$.

Example 3.25. Consider a BCK-algebra $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0,1\}$, $A_2 = \{0,1,2\}$, $A_3 = \{0,1,2,3\}$ and $A_4 = X$ where A_3 and A_4 are commutative ideals and A_0, A_1 and A_2 are not commutative ideals. Now define "cl" as follows:

$$A_0^{cl} = A_1,\ A_1^{cl} = A_2,\ A_2^{cl} = A_3,\ A_3^{cl} = A_4 \text{ and } A_4^{cl} = X$$

Then "cl" is a weak closure operation on $\mathcal{I}(X)$, but it is not commutative since the ideal A_2 is not a commutative ideal and $A_2^{cl} = A_3 \neq A_2$.

Example 3.26. Let $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

Then X is a BCK-algebra with seven ideals $A_0 = \{0\}$, $A_1 = \{0,1\}$, $A_2 = \{0,1,2\}$, $A_3 = \{0,1,4\}$ $A_4 = \{0,1,2,3\}$, $A_5 = \{0,1,2,4\}$ and $A_6 = X$. Note that A_2, A_4, A_5 and A_6 are implicative

ideals and A_0, A_1 and A_3 are not implicative ideals. Now we define an operation define "*cl*" on $\mathcal{I}(X)$ by

$$A_0^{cl} = A_1,\ A_1^{cl} = A_2,\ A_2^{cl} = A_5,\ A_3^{cl} = A_5,\ A_4^{cl} = A_6,\ A_5^{cl} = A_6 \text{ and } A_6^{cl} = X.$$

Then "*cl*" is a weak closure operation on $\mathcal{I}(X)$, but it is not implicative since the ideal A_3 is not an implicative ideal and $A_3^{cl} = A_5 \neq A_3$.

Given a weak closure operation, we kame a positive implicative weak closure operation.

Theorem 3.27. *Given $A \in \mathcal{I}(X)$, let "cl" be a weak closure operation on $\mathcal{I}(X)$ and "cl_{pi}" be an operation on $\mathcal{I}(X)$ such that $cl \leq cl_{pi}$ and*

(i) $(\forall C \in \mathcal{I}(X)) \left(A \subseteq C \;\Rightarrow\; C^{cl_{pi}} = C^{cl}\right)$.

(ii) $(\forall C \in \mathcal{I}(X)) \left(C \subsetneq A \;\Rightarrow\; C^{cl_{pi}} = C\right)$.

(iii) *For any $C \in \mathcal{I}(X)$, if A and C have no inclusion relation, then $C^{cl_{pi}} = C$.*

If A is positive implicative (resp., commutative and implicative) ideals of X, then "cl_{pi}" is a positive implicative (resp., commutative and implicative) weak closure operation on $\mathcal{I}(X)$.

Proof. Let A and C be ideals of X such that $A \subseteq C$. Suppose that A is a positive implicative (resp., commutative and implicative) ideal of X. Then C is a positive implicative (resp., commutative and implicative) ideal of X by Lemma 3.13. Let A and C be ideals of X such that $C \subseteq A$. If A is not a positive implicative (resp., commutative and implicative) ideal of X, then C is not a positive implicative (resp., commutative and implicative) ideal of X. Therefore "*cl*" is a positive implicative (resp., commutative and implicative) weak closure operation on $\mathcal{I}(X)$. □

The following examples illustrate Theorem 3.27.

Example 3.28. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table,

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 4\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 1, 4\}$ and $A_5 = X$ in which A_1, A_3, A_4 and A_5 are positive implicative ideals and A_0 and A_2 are not positive implicative ideals. Now define "*cl*" as follows:

$$A_0^{cl} = A_0, A_1^{cl} = A_3, A_2^{cl} = A_4, A_3^{cl} = A_3, A_4^{cl} = A_5 \text{ and } A_5^{cl} = X.$$

Then "*cl*" is a weak closure operation. Now let $A = \{0, 4\} = A_2$ which is not a positive implicative ideal. By using Theorem 3.27 we have "cl_{pi} as follows:

$$A_0^{cl_{pi}} = A_0,\ A_1^{cl_{pi}} = A_1,\ A_2^{cl_{pi}} = A_4,\ A_3^{cl_{pi}} = A_3,\ A_4^{cl_{pi}} = A_5 \text{ and } A_5^{cl_{pi}} = X.$$

Clearly, $cl \leq cl_{pi}$. But, "cl_{pi}" is not a positive implicative weak closure operation because $A_2^{cl_{pi}} = A_4 \neq A_2$. Now let $A = \{0, 1\} = A_1$ which is a positive implicative ideal. By using Theorem 3.27 we have "cl_{pi}" as follows:

$$A_0^{cl_{pi}} = A_0,\ A_1^{cl_{pi}} = A_3,\ A_2^{cl_{pi}} = A_2,\ A_3^{cl_{pi}} = A_3,\ A_4^{cl_{pi}} = A_5 \text{ and } A_5^{cl_{pi}} = X.$$

Clearly, $cl \leq cl_{pi}$ and "cl_{pi}" is a positive implicative weak closure operation.

Example 3.29. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	4	4	4	0

There are five ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2, 3\}$, and $A_4 = X$ in which A_3 and A_4 are commutative ideals and A_0, A_1 and A_2 are not commutative ideals. Now define "cl" as follows:

$$A_0^{cl} = A_1, A_1^{cl} = A_3, A_2^{cl} = A_3, A_3^{cl} = A_4 \text{ and } A_4^{cl} = X.$$

Then "cl" is a weak closure operation. Now let $A = \{0, 1\} = A_1$ which is not a commutative ideal. By using Theorem 3.27 we have "cl_c as follows:

$$A_0^{cl_c} = A_0, A_1^{cl_c} = A_3, A_2^{cl_c} = A_2, A_3^{cl_c} = A_4 \text{ and } A_4^{cl_c} = X.$$

Clearly, $cl \leq cl_c$. But, "cl_c" is not a commutative weak closure operation because $A_1^{cl_c} = A_3 \neq A_1$. Now let $A = \{0, 1, 2, 3\} = A_3$ which is a commutative ideal. By using Theorem 3.27 we have "cl_c" as follows:

$$A_0^{cl_c} = A_0, A_1^{cl_c} = A_1, A_2^{cl_c} = A_2, A_3^{cl_c} = A_4 \text{ and } A_4^{cl_c} = X.$$

Clearly, $cl \leq cl_c$ and "cl_c" is a commutative weak closure operation.

Example 3.30. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	4	4	4	0

There are six ideals in X, that is, $A_0 = \{0\}$, $A_1 = \{0, 2\}$, $A_2 = \{0, 1\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 1, 4\}$ and $A_5 = X$ in which A_2, A_3, A_4 and A_5 are implicative ideals and A_0 and A_1 are not implicative ideals. Now define "cl" as follows:

$$A_0^{cl} = A_1, A_1^{cl} = A_3, A_2^{cl} = A_4, A_3^{cl} = A_5, A_4^{cl} = A_4 \text{ and } A_5^{cl} = X.$$

Then "cl" is a weak closure operation. Now let $A = \{0, 2\} = A_1$ which is not an implicative ideal. By using Theorem 3.27 we have "cl_m" as follows:

$$A_0^{cl_m} = A_0, A_1^{cl_m} = A_3, A_2^{cl_m} = A_2, A_3^{cl_m} = A_5, A_4^{cl_m} = A_4 \text{ and } A_5^{cl_m} = X.$$

Clearly, $cl \leq cl_m$. But, "cl_m" is not an implicative weak closure operation because $A_1^{cl_m} = A_3 \neq A_1$. Now let $A = \{0, 1\} = A_2$ which is an implicative ideal. By using Theorem 3.27 we have "cl_m" as follows:

$$A_0^{cl_m} = A_0, A_1^{cl_m} = A_1, A_2^{cl_m} = A_4, A_3^{cl_m} = A_5, A_4^{cl_m} = A_4 \text{ and } A_5^{cl_m} = X.$$

Clearly, $cl \leq cl_m$ and "cl_m" is an implicative weak closure operation.

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Communication between relation information systems*

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December 23, 2015

Abstract: Communication between information systems is considered as an important issue in granular computing. A relation information system is the generalization of an information system. This paper investigates communication between relation information systems and obtain some invariant characterizations of relation information systems under homomorphism.

Keywords: Relation information system; Reduction; Consistent function; Relation mapping; Homomorphism.

1 Introduction

Rough set theory, proposed by Pawlak [17], is an important tool for dealing with fuzzyness and uncertainty of knowledge and has become an active branch of information science. With more than thirty years development, rough set theory has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [13, 14, 15, 16].

Communication between information systems is a very important topic in the field of artificial intelligence. In mathematics, it can be explained as a mapping between information systems. The approximations and reductions in the original system can be regarded as encoding while the image system is seen as an interpretive system. The concept of homomorphisms as a kind of tool to study relationships between information systems with rough sets was introduced by Grzymala-Busse [1, 2]. A homomorphism can be viewed as a special communication between two information systems. As explained in [23], homomorphisms allow one to translate the information contained in one granular world into the

*This work was supported by the National Natural Science Foundation of China (61473181, 11501436).

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granularity of another granular world and thus provide a communication mechanism for exchanging information with other granular worlds. Li et al. [5] studied invariant characters of information systems under some homomorphism. Wang et al. [20, 21] introduced the notions of consistent functions, relation mappings and relation information systems which are the generalization of information systems. By using these notions, they proposed the homomorphisms as a mechanism for communicating between relation information systems. Zhu et al. [26] obtained some improved results on communication between relation information systems. Li et al. [12] investigated communication between knowledge bases. It should be pointed out that some other related works investigating information systems through homomorphisms [1, 2, 3, 5, 25] are based on equivalence relations or other particular relations and are quite different from [20, 21, 26].

The purpose of this paper is to investigate some invariant characterizations of relation information systems under homomorphisms.

2 Preliminaries

In this section, we recall some basic concepts on consistent functions, relation mappings and relation information systems.

Throughout this paper, U denotes a non-empty finite set called the universe, 2^U denotes the family of all subsets of U, $2^{U\times U}$ denotes the family of all binary relations on U, All mappings are assumed to be surjective.

For $R\in 2^{U\times U}$, the successor neighborhood of $x\in U$ with respect to R will be denoted by $R_s(x)$, that is, $R_s(x)=\{y\in U: xRy\}$ ([22]). Denote

$$U/R=\{R_s(x): x\in U\}.$$

If R is an equivalence relation on U, then $\forall\ x\in U$, $R_s(x)=[x]_R$.

For $\mathcal{R}\subseteq 2^{U\times U}$, denote $ind(\mathcal{R})=\bigcap\limits_{R\in\mathcal{R}} R$.

2.1 Consistent functions

Definition 2.1 ([20, 21]). *Let U and V be two finite nonempty universes, $f\colon U\to V$ a mapping and $R\in 2^{U\times U}$. Define*

$$[x]_f=\{u\in U: f(u)=f(x)\},$$

$$(x)_R=\{u\in U: R_s(u)=R_s(x)\}.$$

Then $\{[x]_f: x\in U\}$ and $\{(x)_R: x\in U\}$ are two partitions on U. If $[x]_f\subseteq R_s(u)$ or $[x]_f\cap R_s(u)\neq\emptyset$ for any $x,u\in U$, then f is called a type-1 consistent function with respect to R on U. If $[x]_f\subseteq (x)_R$ for any $x\in U$, then f is called a type-2 consistent function with respect to R on U.

Remark 2.2. (1) $\forall\ x\in U$, $[x]_f=f^{-1}(f(x))$.

(2) *If R is an equivalence relation on U, then $\forall\ x\in U$, $(x)_R=[x]_R$.*

(3) *If f is type-2 consistent with respect to R on U and $f(u)=f(x)$, then $R_s(u)=R_s(x)$.*

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Obviously,

f is type-1 $\Longleftrightarrow$ If $[x]_f \cap R_s(y) \neq \emptyset$, then $[x]_f \subseteq R_s(y)$
$\Longleftrightarrow$ If $[x]_f \not\subseteq R_s(y)$, then $[x]_f \cap R_s(y) = \emptyset$,
f is type-2 $\Longleftrightarrow$ If $f(x_1) = f(x_2)$, then $R_s(x_1) = R_s(x_2)$.

2.2 Relation mappings

Definition 2.3 ([20, 21])**.** *Let $f : U \to V$ be a mapping. Define*

$$\hat{f} : 2^{U\times U} \to 2^{V\times V},\ R| \to \hat{f}(R) = \bigcup_{x\in U} (\{f(x)\} \times f(R_s(x)));$$

$$\hat{f}^{-1} : 2^{V\times V} \to 2^{U\times U},\ T| \to \hat{f}^{-1}(T) = \bigcup_{y\in V} (\{f^{-1}(y)\} \times f^{-1}(T_s(y))).$$

Then $\hat{f}$ and $\hat{f}^{-1}$ are called the relation mapping and inverse relation mapping induced by f, respectively.

Obviously,

$$y_1\hat{f}(R)y_2 \Longleftrightarrow\ \exists\ x_1, x_2 \in U,\ y_1 = f(x_1),\ y_2 = f(x_1) \text{ and } x_1Rx_2,$$

$$x_1\hat{f}^{-1}(T)x_2 \Longleftrightarrow \exists\ y_1, y_2 \in V,\ y_1 = f(x_1), y_2 = f(x_1) \text{ and } y_1Ty_2.$$

For $\mathcal{R} \subseteq 2^{U\times U}$, denote

$$\hat{f}(\mathcal{R}) = \{\hat{f}(R) \mid R \in \mathcal{R}\}.$$

Proposition 2.4 ([20])**.** *If $f : U \to V$ is both type-1 and type-2 consistent with respect to $R \in 2^{U\times U}$, then*

$$\hat{f}^{-1}(\hat{f}(R)) = R.$$

2.3 Relation information systems

Definition 2.5 ([13])**.** *An information system is a pair (U, A) of non-empty finite sets U and A, where U is a set of objects and A is a set of attributes; each attribute $a \in A$ is a function $a : U \to V_a$, where V_a is the set of values (called domain) of attribute a.*

If (U, A) is an information system and $B \subseteq A$, then an equivalence relation (or indiscernibility relation) R_B can be defined by

$$(x, y) \in R_B \iff a(x) = a(y),\ \forall\ a \in B.$$

Definition 2.6 ([20])**.** *A pair $(U, \mathcal{R})$ is called a relation information system, if $\mathcal{R} \subseteq 2^{U\times U}$.*

Definition 2.7. *Let (U, A) be an information system. Put*

$$\mathcal{R} = \{R_{\{a\}} :\ a \in A\}.$$

Then the pair $(U, \mathcal{R})$ is called the relation information system induced by (U, A).

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Definition 2.8 ([20])**.** *Let $f\colon U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. If f is type-1 (resp. type-2) consistent with respect to R on U for every $R \in \mathcal{R}$, then f is called type-1 (resp. type-2) consistent with respect to $\mathcal{R}$ on U.*

Proposition 2.9 ([20])**.** *Let $f : U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. If f is both type-1 and type-2 consistent with respect to $\mathcal{R}$, then $\hat{f}(ind(\mathcal{R})) = ind(\hat{f}(\mathcal{R}))$.*

Proposition 2.10 ([20])**.** *Let $f : U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. If f is both type-1 and type-2 consistent with respect to $\mathcal{R}$, then $\hat{f}^{-1}(\hat{f}(ind(\mathcal{R})) = ind(\mathcal{R})$.*

Definition 2.11 ([20])**.** *Let $f : U \to V$ be a mapping and $\mathcal{R} \subseteq 2^{U\times U}$. Then the pair $(V, \hat{f}(\mathcal{R}))$ is called an f-induced relation information system of $(U, \mathcal{R})$.*

Definition 2.12 ([20])**.** *Let $(U, \mathcal{R})$ be a relation information system and $(V, \hat{f}(\mathcal{R}))$ an f-induced relation information system of $(U, \mathcal{R})$. If f is both type-1 and type-2 consistent with respect to $\mathcal{R}$ on U, then f is called a homomorphism from $(U, \mathcal{R})$ to $(V, \hat{f}(\mathcal{R}))$. We write*

$$(U, \mathcal{R}) \ \sim_f \ (V, \hat{f}(\mathcal{R})).$$

We often consider reductions in a relation information system by deleting unrelated or unimportant elements with the requirement of keeping the ability of classification.

Definition 2.13 ([20])**.** *Let $(U, \mathcal{R})$ be a relation information system and $\mathcal{P} \subseteq \mathcal{R}$.*

(1) $\mathcal{P}$ is called a coordination subfamily of $\mathcal{R}$, if $ind(\mathcal{P}) = ind(\mathcal{R})$.

(2) $R \in \mathcal{P}$ is called independent in $\mathcal{P}$, if $ind(\mathcal{P} - \{R\}) \neq ind(\mathcal{P})$; $\mathcal{P}$ is called a independent subfamily of $\mathcal{R}$, if $\forall\ R \in \mathcal{P}$, R is independent in $\mathcal{P}$.

(3) $\mathcal{P}$ is called a reductions of $\mathcal{R}$, if $\mathcal{P}$ is both coordination and independent.

In this paper, the set of all coordination subfamilies (resp., all reductions) of $\mathcal{R}$ is denoted by $co(\mathcal{R})$ (resp., $red(\mathcal{R})$).

Obviously,

$\mathcal{P} \in red(\mathcal{R}) \ \Leftrightarrow \mathcal{P} \in co(\mathcal{R})$ and $\forall\ \mathcal{Q} \subset \mathcal{P}$, $\mathcal{Q} \notin co(\mathcal{R})$.

3 Some results on reductions in relation information systems

Proposition 3.1. *Let $(U, \mathcal{R})$ be a relation information system. Then $red(\mathcal{R}) \neq \emptyset$.*

Proof. Suppose $\forall\ R \in \mathcal{R}$, $\mathcal{R} - \{R\} \notin co(\mathcal{R})$. Then $\mathcal{R} \in red(\mathcal{R})$.

Suppose $\exists\ R_1 \in \mathcal{R}$, $\mathcal{R} - \{R_1\} \in co(\mathcal{R})$. Then, we consider $\mathcal{R} - \{R_1\}$. Again suppose $\forall\ R \in \mathcal{R} - \{R_1\}$, $(\mathcal{R} - \{R_1\}) - \{R\} \notin co(\mathcal{R})$. Then $\mathcal{R} - \{R_1\} \in red(\mathcal{R})$. Again suppose $\exists\ R_2 \in \mathcal{R} - \{R_1\}$, $(\mathcal{R} - \{R_1\}) - \{R_2\} \in co(\mathcal{R})$. Then, we consider

$\mathcal{R} - \{R_1, R_2\}$. Repeat this process. Since $\mathcal{R}$ is finite, we can find a reductions of $\mathcal{R}$.

Thus $red(\mathcal{R}) \neq \emptyset$. □

Definition 3.2. *Let $(U, \mathcal{R})$ be a relation information system. Put*

$$\mathcal{D}(x, y) = \{R \in \mathcal{R} | (x, y) \notin R\} \ (x, y \in U).$$

Then

(1) $\mathcal{D}(x, y)$ is called is called the discernibility subfamily of $\mathcal{R}$ on x and y.

(2) $\mathfrak{D}(\mathcal{R}) = (d_{ij})_{n\times n}$ is called the discernibility matrix of $\mathcal{R}$ where $U = \{x_1, x_2, \cdots, x_n\}$ and $d_{ij} = \mathcal{D}(x_i, x_j)$ $(1 \le i, j \le n)$.

Example 3.3. *Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. We consider the relation information system $(U, \mathcal{R})$ where $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$ and*

$U/R_1 = \{\{x_1, x_2, x_5\}, \{x_3, x_4, x_6\}\}$,
$U/R_2 = \{\{x_1, x_6\}, \{x_2, x_3, x_4, x_5\}\}$,
$U/R_3 = \{\{x_1, x_2, x_5, x_6\}, \{x_3, x_4\}\}$,
$U/R_4 = \{\{x_1, x_2, x_5\}, \{x_3, x_4, x_6\}\}$.

We can obtain the discernibility matrix $\mathfrak{D}(\mathcal{R})$ as follows:

$$\begin{pmatrix} \emptyset & \{R_2\} & \mathcal{R} & \mathcal{R} & \{R_2\} & \{R_1, R_4\} \\ \{R_2\} & \emptyset & \{R_1, R_3, R_4\} & \{R_1, R_3, R_4\} & \emptyset & \{R_1, R_2, R_4\} \\ \mathcal{R} & \{R_1, R_3, R_4\} & \emptyset & \emptyset & \{R_1, R_3, R_4\} & \{R_2, R_3\} \\ \mathcal{R} & \{R_1, R_3, R_4\} & \emptyset & \emptyset & \{R_1, R_3, R_4\} & \{R_2, R_3\} \\ \{R_2\} & \emptyset & \{R_1, R_3, R_4\} & \{R_1, R_3, R_4\} & \emptyset & \{R_1, R_2, R_4\} \\ \{R_1, R_4\} & \{R_1, R_2, R_4\} & \{R_2, R_3\} & \{R_2, R_3\} & \{R_1, R_2, R_4\} & \emptyset \end{pmatrix}$$

Discernibility family can expediently judge coordination families and reductions.

Proposition 3.4. *Let $(U, \mathcal{R})$ be a relation information system. Then*

$$\mathcal{P} \in co(\mathcal{R}) \iff If\ (x, y) \notin ind(\mathcal{R}),\ then\ \mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset.$$

Proof. (1) "$\Longrightarrow$". Let $(x, y) \notin ind(\mathcal{R})$. Since $\mathcal{P} \in co(\mathcal{R})$, we have $ind(\mathcal{P}) = ind(\mathcal{R})$. Then $(x, y) \notin ind(\mathcal{P})$. It follows $(x, y) \notin P\ for\ some\ P \in \mathcal{P}$.

$(x, y) \notin P$ implies $P \in \mathcal{D}(x, y)$. Then $P \in \mathcal{P} \cap \mathcal{D}(x, y)$.

Thus $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$.

"$\Longleftarrow$". Suppose $\mathcal{P} \notin co(\mathcal{R})$. Then $ind(\mathcal{P}) \neq ind(\mathcal{R})$. It follows $ind(\mathcal{P}) - ind(\mathcal{R}) \neq \emptyset$. Pick

$$(x, y) \in ind(\mathcal{P}) - ind(\mathcal{R}).$$

Since $(x, y) \notin ind(\mathcal{R})$, we have $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$.

Note that $(x, y) \in ind(\mathcal{P})$. Then $\forall\ P \in \mathcal{P}$, $(x, y) \in P$. So $P \notin \mathcal{D}(x, y)$. Thus $\mathcal{P} \cap \mathcal{D}(x, y) = \emptyset$. This is a contradiction.

Thus $\mathcal{P} \in co(\mathcal{R})$. □

Theorem 3.5. *Let $(U, \mathcal{R})$ be a relation information system. Then $\mathcal{P} \in red(\mathcal{R})$ $\Longleftrightarrow$ (1) If $(x, y) \notin ind(\mathcal{R})$, then $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$;*
(2) $\forall\ R \in \mathcal{P}$, $\exists\ (x_R, y_R) \in ind(\mathcal{R})$, $(\mathcal{P} - \{R\}) \cap \mathcal{D}(x_R, y_R) = \emptyset$.

Proof. This holds by Proposition 3.4. □

Definition 3.6. *Let $(U, \mathcal{R})$ be a relation information system. Put*

$$core(\mathcal{R}) = \bigcap_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P}.$$

Then $core(\mathcal{R})$ is called the core of $\mathcal{R}$. Moreover,
(1) $R \in \mathcal{R}$ is called necessary, if $R \in core(\mathcal{R})$.
(2) $R \in \mathcal{R}$ is called relatively necessary, if $R \in \bigcup_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P} - core(\mathcal{R})$.
(3) $R \in \mathcal{R}$ is called unnecessary, if $R \in \mathcal{R} - \bigcup_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P}$.

Discernibility family can easily determine the core.

Proposition 3.7. *Let $(U, \mathcal{R})$ be a relation information system. The following are equivalent:*
(1) R is necessary;
(2) R is independent in $\mathcal{R}$;
(3) $\exists\ x, y \in U$, $\mathcal{D}(x, y) = \{R\}$.

Proof. (1) $\Longrightarrow$ (2). Suppose that R is not independent in $\mathcal{R}$. Then

$$ind(\mathcal{R} - \{R\}) = ind(\mathcal{R}).$$

It follows $\mathcal{R} - \{R\} \in co(\mathcal{R})$. Consider $\mathcal{R} - \{R\}$. By Proposition 3.1, $\exists\ \mathcal{P} \subseteq \mathcal{R} - \{R\}$, $\mathcal{P} \in red(\mathcal{R})$.

$\mathcal{P} \subseteq \mathcal{R} - \{R\}$ implies $R \notin \mathcal{P}$. Then R is not necessary. This is a contradiction.

(2) $\Longrightarrow$ (1). Suppose that R is not necessary. Then $\exists\ \mathcal{P} \in red(\mathcal{R})$, $R \notin \mathcal{P}$. So $\mathcal{P} \subseteq \mathcal{R} - \{R\} \subseteq \mathcal{R}$. It follows

$$ind(\mathcal{P}) \supseteq ind(\mathcal{R} - \{R\}) \supseteq ind(\mathcal{R}).$$

By $\mathcal{P} \in red(\mathcal{R})$, $ind(\mathcal{P}) = ind(\mathcal{R})$. Then $ind(\mathcal{R} - \{R\}) = ind(\mathcal{R})$. So R is not independent in $\mathcal{R}$. This is a contradiction.

(2) $\Longrightarrow$ (3). Since R is independent in $\mathcal{R}$, we have $ind(\mathcal{R} - \{R\}) \neq ind(\mathcal{R})$. Then $ind(\mathcal{R} - \{R\}) - ind(\mathcal{R}) \neq \emptyset$. Pick

$$(x, y) \in ind(\mathcal{R} - \{R\}) - ind(\mathcal{R}).$$

Denote $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$. Then $R = R_j$ for some $j \leq n$. So

$$(x, y) \in \bigcap_{1 \leq i \leq n, i \neq j} R_i - \bigcap_{1 \leq i \leq n} R_i.$$

It follows $(x, y) \notin R_j$ and $(x, y) \in R_i$ $(i \neq j)$.

Thus $\mathcal{D}(x, y) = \{R\}$.

(3) $\Longrightarrow$ (2). Since $\exists\ x, y \in U$, $\mathcal{D}(x, y) = \{R\}$, we have

$$(x, y) \notin R,\ (x, y) \in R^{'}\ (R^{'} \neq R).$$

Then $(x, y) \in ind(\mathcal{R} - \{R\})$. But $(x, y) \notin ind(\mathcal{R})$.

Thus $ind(\mathcal{R} - \{R\}) \neq ind(\mathcal{R})$.

Hence R is independent in $\mathcal{R}$. □

Proposition 3.8. *Let $(U, \mathcal{R})$ be a relation information system. Denote*

$$R^{\star} = \bigcup_{\mathcal{P} \in co(\mathcal{R})} ind(\mathcal{P} - \{R\}).$$

Then the following are equivalent.

(1) *R is unnecessary;*

(2) $\forall\ \mathcal{P} \in co(\mathcal{R})$, $\mathcal{P} - \{R\} \in co(\mathcal{R})$;

(3) *$R^{\star} = ind(\mathcal{R})$;*

(4) *$R^{\star} \subseteq R$.*

Proof. (1) $\Longrightarrow$ (2). By Proposition 3.1, $\exists\ \mathcal{Q} \subseteq \mathcal{P}$, $\mathcal{Q} \in red(\mathcal{R})$. Since R is unnecessary, we have $R \notin \mathcal{Q}$. It follows $\mathcal{Q} \subseteq \mathcal{R} - \{R\}$. Then

$$\mathcal{Q} \subseteq \mathcal{P} \cap (\mathcal{R} - \{R\}) = \mathcal{P} - \{R\} \subseteq \mathcal{P}.$$

We have

$$ind(\mathcal{Q}) \supseteq ind(\mathcal{R} - \{R\}) \supseteq ind(\mathcal{P}).$$

Note that $\mathcal{P} \in co(\mathcal{R})$ and $\mathcal{Q} \in red(\mathcal{R})$. Then $ind(\mathcal{P}) = ind(\mathcal{R}) = ind(\mathcal{Q})$.

Thus $ind(\mathcal{P} - \{R\}) = ind(\mathcal{R})$. This shows $\mathcal{P} - \{R\} \in co(\mathcal{R})$.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (1). Suppose $\exists\ \mathcal{P} \in red(\mathcal{R})$, $R \in \mathcal{P}$. Then $\mathcal{P} - \{R\} \subset \mathcal{P}$. Since $\mathcal{P} \in red(\mathcal{R})$, we have $\mathcal{P} - \{R\} \notin co(\mathcal{R})$. Then $ind(\mathcal{P} - \{R\}) - ind(\mathcal{R}) \neq \emptyset$. $\mathcal{P} \in red(\mathcal{R})$ implies $ind(\mathcal{P}) = ind(\mathcal{R})$. Then

$$ind(\mathcal{P} - \{R\}) - ind(\mathcal{P}) \neq \emptyset.$$

Pick $(x, y) \in ind(\mathcal{P} - \{R\}) - ind(\mathcal{P})$. Note that $ind(\mathcal{P}) = ind(\mathcal{P} - \{R\}) \cap R$. Then $(x, y) \notin R$.

Since $\mathcal{P} \in co(\mathcal{R})$ and $R^{\star} \subseteq R$, we have $ind(\mathcal{P} - \{R\}) \subseteq R$. Then $(x, y) \in R$. This is a contradiction.

Thus R is unnecessary. □

Theorem 3.9. *Let $(U, \mathcal{R})$ be a relation information system. Then*

(1) *R is necessary $\Leftrightarrow$ $\mathcal{R} - \{R\} \notin co(\mathcal{R})$.*

(2) *R is relatively necessary $\Leftrightarrow$ $\mathcal{R} - \{R\} \in co(\mathcal{R})$ and $R^{\star} \not\subseteq R$.*

(3) *R is unnecessary $\Leftrightarrow$ $R^{\star} \subseteq R$.*

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Proof. This holds by Proposition 3.7 and Proposition 3.8. □

Example 3.10. *In Example 3.3, we have*
(1) R_2 *is necessary.*
(2) R_1 *and* R_4 *are relatively necessary.*
(3) R_3 *is unnecessary.*
(4) $red(\mathcal{R}) = \{\{R_1, R_2\}, \{R_2, R_4\}\}$, $core(\mathcal{R}) = \{R_2\}$.

4 Communication between relation information systems

Proposition 4.1. *Let* $(U, \mathcal{R}) \sim_f (V, \hat{f}(\mathcal{R}))$. *Then*
(1) $\mathcal{P} \in co(\mathcal{R}) \iff \hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.
(2) $co(\hat{f}(\mathcal{R})) = \hat{f}(co(\mathcal{R}))$.

Proof. (1) "$\Longrightarrow$". Since $\mathcal{P} \in co(\mathcal{R})$, we have $ind(\mathcal{P}) = ind(\mathcal{R})$. Then

$$\hat{f}(ind(\mathcal{P})) = \hat{f}(ind(\mathcal{R})).$$

By Proposition 2.6,

$$ind(\hat{f}(\mathcal{P})) = ind(\hat{f}(\mathcal{R})).$$

Thus $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.
"$\Longleftarrow$". Since $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$, we have

$$ind(\hat{f}(\mathcal{P})) = ind(\hat{f}(\mathcal{R})).$$

By Proposition 2.6,

$$\hat{f}(ind(\mathcal{P})) = \hat{f}(ind(\mathcal{R})).$$

Then

$$\hat{f}^{-1}(\hat{f}(ind(\mathcal{P}))) = \hat{f}^{-1}(\hat{f}(ind(\mathcal{R}))).$$

By Proposition 2.7, $ind(\mathcal{P}) = ind(\mathcal{R})$.
Thus $\mathcal{P} \in co(\mathcal{R})$.
(2) By (1),

$$\begin{aligned}\hat{f}(co(\mathcal{R})) &= \{\hat{f}(\mathcal{P}) | \mathcal{P} \in co(\mathcal{R})\} \\ &= \{\hat{f}(\mathcal{P}) | \hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))\} \\ &= co(\hat{f}(\mathcal{R})).\end{aligned}$$

□

Theorem 4.2. *Let* $(U, \mathcal{R}) \sim_f (V, \hat{f}(\mathcal{R}))$. *Then*
(1) $\mathcal{P} \in red(\mathcal{R}) \iff \hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$.
(2) $red(\hat{f}(\mathcal{R})) = \hat{f}(red(\mathcal{R}))$.

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Proof. (1) "$\Longrightarrow$". Since $\mathcal{P} \in red(\mathcal{R})$, we have $\mathcal{P} \in co(\mathcal{R})$. By Proposition 4.1, $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.

$\forall\ \mathcal{T} \subset \hat{f}(\mathcal{P})$. Pick $\mathcal{Q} \subseteq \mathcal{R}$, $\mathcal{T} = \hat{f}(\mathcal{Q})$. Then $\hat{f}(\mathcal{Q}) \subset \hat{f}(\mathcal{P})$. By Proposition 2.4,

$$\mathcal{Q} = \hat{f}^{-1}(\hat{f}(\mathcal{Q})) \subseteq \hat{f}^{-1}(\hat{f}(\mathcal{P})) = \mathcal{P}.$$

Suppose $\mathcal{Q} = \mathcal{P}$. Then $\mathcal{T} = \hat{f}(\mathcal{Q}) = \hat{f}(\mathcal{P})$. This is a contradiction.

Thus $\mathcal{Q} \subset \mathcal{P}$.

Since $\mathcal{P} \in red(\mathcal{R})$, we have $\mathcal{Q} \notin co(\mathcal{R})$. By Proposition 4.1, $\mathcal{T} = \hat{f}(\mathcal{Q}) \notin co(\hat{f}(\mathcal{R}))$.

Hence $\hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$.

"$\Longleftarrow$". Since $\hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$, we have $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$. By Proposition 4.1, $\mathcal{P} \in co(\mathcal{R})$.

$\forall\ \mathcal{Q} \subset \mathcal{P}$, $\hat{f}(\mathcal{Q}) \subseteq \hat{f}(\mathcal{P})$. Suppose $\hat{f}(\mathcal{Q}) = \hat{f}(\mathcal{P})$. By Proposition 2.4,

$$\mathcal{Q} = \hat{f}^{-1}(\hat{f}(\mathcal{Q})) = \hat{f}^{-1}(\hat{f}(\mathcal{P})) = \mathcal{P}.$$

This is a contradiction. Thus $\hat{f}(\mathcal{Q}) \subset \hat{f}(\mathcal{P})$.

Since $\hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))$, we have $\hat{f}(\mathcal{Q}) \notin co(\hat{f}(\mathcal{R}))$. By Proposition 4.1, $\mathcal{Q} \notin co(\mathcal{R})$.

Hence $\mathcal{P} \in red(\mathcal{R})$.

(2) By (1),

$$\begin{aligned}\hat{f}(red(\mathcal{R})) &= \{\hat{f}(\mathcal{P}) | \mathcal{P} \in red(\mathcal{R})\} \\ &= \{\hat{f}(\mathcal{P}) | \hat{f}(\mathcal{P}) \in red(\hat{f}(\mathcal{R}))\} \\ &= red(\hat{f}(\mathcal{R})).\end{aligned}$$

□

Remark 4.3. *Theorem 3.20(1) is Theorem 4.4 in [20]. We just prove this result from another angle.*

Lemma 4.4. *Let* $(U, \mathcal{R}) \ \sim_f \ (V, \hat{f}(\mathcal{R}))$. *Then*

$$\hat{f}(\mathcal{R} - \{R\}) = \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}.$$

Proof. $\forall\ S \in \mathcal{R} - \{R\}$, $S \neq R$. By Proposition 2.4, $\hat{f}(S) \neq \hat{f}(R)$. It follows $\hat{f}(S) \in \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}$. Thus

$$\hat{f}(\mathcal{R} - \{R\}) \subseteq \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}.$$

On the other hand, $\forall\ T \in \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}$, $T = \hat{f}(S)$ for some $S \in \mathcal{R}$. $T \notin \{\hat{f}(R)\}$ implies $\hat{f}(S) \neq \hat{f}(R)$. Then $S \neq R$. So $S \in \mathcal{R} - \{R\}$. It follows $T \in \hat{f}(\mathcal{R} - \{R\})$. Thus

$$\hat{f}(\mathcal{R} - \{R\}) \supseteq \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}.$$

Hence $\hat{f}(\mathcal{R} - \{R\}) = \hat{f}(\mathcal{R}) - \{\hat{f}(R)\}$. □

Theorem 4.5. *Let* $(U,\mathcal{R}) \sim_f (V,\hat{f}(\mathcal{R}))$*. Then*

$$R \in core(\mathcal{R}) \iff \hat{f}(R) \in core(\hat{f}(\mathcal{R})).$$

Proof. This holds by Theorem 3.9(1), Proposition 4.1(1) and Lemma 4.4. □

Theorem 4.6. *Let* $(U,\mathcal{R}) \sim_f (V,\hat{f}(\mathcal{R}))$*. Then*

$$\hat{f}(core(\mathcal{R})) = core(\hat{f}(\mathcal{R})).$$

Proof. By Theorem 3.23,

$$\begin{aligned}\hat{f}(core(\mathcal{R})) &= \{\hat{f}(R)|R \in core(\mathcal{R})\}\\ &= \{\hat{f}(R)|\hat{f}(R) \in core(\hat{f}(\mathcal{R}))\}\\ &= core(\hat{f}(\mathcal{R})).\end{aligned}$$

□

Theorem 4.7. *Let* $(U,\mathcal{R}) \sim_f (V,\hat{f}(\mathcal{R}))$*. Then*

$$R \textit{ is unnecessary} \Leftrightarrow \hat{f}(R) \textit{ is unnecessary.}$$

Proof. "$\Longrightarrow$". $\forall\, \mathcal{T} \in co(\hat{f}(\mathcal{R}))$, pick $\mathcal{P} \subseteq \mathcal{R}$, $\mathcal{T} = \hat{f}(\mathcal{P})$. Then $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$. By Proposition 3.19(1), $\mathcal{P} \in co(\mathcal{R})$.

Since R is unnecessary, by Proposition 3.8, we have $\mathcal{P} - \{R\} \in co(\mathcal{R})$. Then $ind(\mathcal{P} - \{R\}) = ind(\mathcal{R})$. By Proposition 2.6 and Lemma 4.4,

$$ind(\hat{f}(\mathcal{P}) - \{\hat{f}(R)\}) = ind(\hat{f}(\mathcal{P} - \{R\}) = \hat{f}(ind(\mathcal{P} - \{R\})),$$

$$ind(\hat{f}(\mathcal{R})) = \hat{f}(ind(\mathcal{R})).$$

Then $ind(\mathcal{T} - \{\hat{f}(R)\}) = ind(\hat{f}(\mathcal{R}))$. This implies $\mathcal{T} - \{\hat{f}(R)\} \in co(\hat{f}(\mathcal{R}))$.

By Proposition 3.8, $\hat{f}(R)$ is unnecessary.

"$\Longleftarrow$". $\forall\, \mathcal{P} \in co(\mathcal{R})$, by Proposition 4.1(1), $\hat{f}(\mathcal{P}) \in co(\hat{f}(\mathcal{R}))$.

Since $\hat{f}(R)$ is unnecessary, by Proposition 3.8, we have

$$\hat{f}(\mathcal{P}) - \{\hat{f}(R)\} \in co(\hat{f}(\mathcal{R})).$$

Then

$$ind(\hat{f}(\mathcal{P}) - \{\hat{f}(R)\}) = ind(\hat{f}(\mathcal{R})).$$

By Proposition 2.6 and Lemma 4.4,

$$\hat{f}(ind(\mathcal{P} - \{R\})) = ind(\hat{f}(\mathcal{P} - \{R\}) = ind(\hat{f}(\mathcal{P}) - \{\hat{f}(R)\}),$$

$$\hat{f}(ind(\mathcal{R})) = ind(\hat{f}(\mathcal{R})).$$

Then $\hat{f}(ind(\mathcal{P} - \{R\})) = \hat{f}(ind(\mathcal{R}))$.

By Proposition 2.7,

$$ind(\mathcal{P} - \{R\}) = \hat{f}^{-1}(\hat{f}(ind(\mathcal{P} - \{R\}))) = \hat{f}^{-1}(\hat{f}(ind(\mathcal{R}))) = ind(\mathcal{R}).$$

Then $\mathcal{P} - \{R\} \in co(\mathcal{R})$.

By Proposition 3.8, R is unnecessary. □

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Corollary 4.8. *Let* $(U, \mathcal{R}) \sim_f (V, \hat{f}(\mathcal{R}))$. *Then*

$$R \text{ is relatively necessary} \iff \hat{f}(R) \text{ is relatively necessary.}$$

Proof. This hods by Theorem 4.5 and Theorem 4.7. □

Example 4.9. *Let* $U = \{x_i | 1 \leq i \leq 15\}$. *We consider the relation information system* $(U, \mathcal{R})$ *where* $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$,

$U/R_1 = \{\{x_1, x_2, x_4, x_7, x_8, x_9, x_{10}, x_{11}\}, \{x_3, x_5, x_6, x_{12}, x_{13}, x_{14}, x_{15}\}\}$,
$U/R_2 = \{\{x_1, x_4, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\}, \{x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\}\}$,
$U/R_3 = \{\{x_1, x_2, x_4, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\}, \{x_3, x_5, x_6\}\}$,
$U/R_4 = \{\{x_1, x_2, x_4, x_7, x_8, x_9, x_{10}, x_{11}\}, \{x_3, x_5, x_6, x_{12}, x_{13}, x_{14}, x_{15}\}\}$.

Let $V = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. *Define a mapping as follows:*

x_1, x_4, x_{11}	x_2, x_8	x_3, x_6	x_5	x_7, x_9, x_{10}	$x_{12}, x_{13}, x_{14}, x_{15}$
y_1	y_2	y_3	y_4	y_5	y_6

Let $(V, \hat{f}(\mathcal{R}))$ *be the* f*-induced relation information system of* (U, R). *It is very easy to verify that* f *is a homomorphism from* (U, R) *to* $(V, \hat{f}(\mathcal{R}))$.

We have $\hat{f}(\mathcal{R}) = \{\hat{f}(R_1), \hat{f}(R_3), \hat{f}(R_3), \hat{f}(R_4)\}$ *where*

$V/\hat{f}(R_1) = \{\{y_1, y_2, y_5\}, \{y_3, y_4, y_6\}\}$,
$V/\hat{f}(R_2) = \{\{y_1, y_6\}, \{y_2, y_3, y_4, y_5\}\}$,
$V/\hat{f}(R_3) = \{\{y_1, y_2, y_5, y_6\}, \{y_3, y_4\}\}$,
$V/\hat{f}(R_4) = \{\{y_1, y_2, y_5\}, \{y_3, y_4, y_6\}\}$.

By Example 3.10,

$$red(\hat{f}(\mathcal{R})) = \{\{\hat{f}(R_1), \hat{f}(R_2)\}, \{\hat{f}(R_2), \hat{f}(R_4)\}\}, \quad core(\hat{f}(\mathcal{R})) = \{\hat{f}(R_2)\}.$$

By Proposition 2.4, Theorem 4.2(2) and Theorem 4.6,

$$red(\mathcal{R}) = \{\{R_1, R_2\}, \{R_2, R_4\}\}, \quad core(\mathcal{R}) = \{R_2\}.$$

5 Conclusions

In this paper, we have investigated the original relation information system and image relation information system, and obtained some invariant characterizations of relation information systems under homomorphism. These results will be significant for establishing a framework of granular computing in knowledge bases and may have potential applications to knowledge discovery, decision making and reasoning about data. In the future, we will consider concrete applications of our results.

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Global stability in a discrete Lotka-Volterra competition model

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Abstract

We consider the Euler difference scheme for two-dimensional Lotka-Volterra competition equations and show that the difference scheme has positive and bounded solutions. In addition, we present sufficient conditions that the solutions of the scheme converge to the equilibrium points of the scheme. The convergence is shown based on the two approaches: first, partition of the domain used for the boundedness of the solutions and second, calculation of the movement of the species started in each partitioned region. Numerical examples are presented to verify the results.

Key words: Euler difference scheme, positivity, global stability, competition model

1. Introduction

The competition model in the two-dimensional case represents two species which are competing for a common resource; an additional term is included within the logistic prey growth Lotka-Volterra model to incorporate this interspecific competition for some limiting resource. This limiting resource can be anything for which supply is smaller than demand. The classic two-dimensional competition model is given by

$$\frac{dx}{dt} = x(t)(r_1 - a_{11}x(t) - a_{12}y(t)), \ \frac{dy}{dt} = y(t)(r_2 - a_{21}x(t) - a_{22}y(t)), \tag{1}$$

where $r_i > 0$ and $a_{ij} > 0$. Here $x(t)$ and $y(t)$ denote the population sizes or population density in the species x and y at time t; the parameters r_i's are the intrinsic growth rates for the two species x and y; a_{ii}'s measure the inhibiting effect on the two species; a_{12} and a_{21} are the interspecific acting coefficients.

The species x in the model (1) acts on y with functional response of type $a_{12}x(t)y(t)$. However other types of functional responses including Holling types [1–5], Beddington-DeAngelis type [6–8], Crowley-Martin type [9–11], and Ivlev-type of functional responses [12–14] have been applied to many population models

The dynamics of the model (1) is well-known [15–17] ; the solutions of (1) are positive and bounded, and the stability of the system (1) has been studied. There are a number of works on investigating continuous time Lokta-Volterra models, but relatively few theoretical papers are published on their discretized models [18–21]. The author in [22] has

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introduced a method to present global stability in discrete Lokta-Volterra predator-prey models for the case that all species coexist at a unique equilibrium. In [23], the authors have shown the global stability of the Euler difference scheme for a three-dimensional predator-prey model using a new approach.

As far as we know, there is no theoretical research on the global stability of the discrete-time competition model of (1), so that we consider the Euler difference scheme

$$x_{n+1} = F_{y_n}(x_n),\ y_{n+1} = G_{x_n}(y_n),\ n \geq 0 \tag{2}$$

with

$$\begin{aligned} F_y(x) &= x\left\{1 + (r_1 - a_{11}x - a_{12}y)\Delta t\right\}, & (3)\\ G_x(y) &= y\left\{1 + (r_2 - a_{21}x - a_{22}y)\Delta t\right\}, & (4)\end{aligned}$$

where Δt is a time step size, $x_n = x_0 + n\Delta t$ and $y_n = y_0 + n\Delta t$ with $(x_0, y_0) = (x(0), y(0))$.

The paper is organized as follows. Section 2 gives the positivity and boundedness of solutions of (2). In Section 3, we partition the domain used for the boundedness of the discrete solutions and find the geometric properties of the movement of the solutions starting in the partitioned regions. Using the properties, we present sufficient conditions that the solutions converge to equilibrium points of (2). In Section 4, some numerical examples are presented to verify our results.

2. Positivity of the discrete solutions

In this section, we consider the positivity and boundedness of the solutions of (2). Note that if τ_1 and τ_2 are positive constants satisfying

$$U_1(\tau_2) = \frac{1 + r_1\Delta t - a_{12}\tau_2\Delta t}{2a_{11}\Delta t} > 0,\quad U_2(\tau_1) = \frac{1 + r_2\Delta t - a_{21}\tau_1\Delta t}{2a_{22}\Delta t} > 0, \tag{5}$$

then

$$F_{\tau_2}(x), G_{\tau_1}(y) \text{ are increasing on } 0 \leq x \leq U_1(\tau_2),\ 0 \leq y \leq U_2(\tau_1). \tag{6}$$

For the positivity and boundedness of the solutions (x_n, y_n) we assume

$$\max\{r_1, r_2\} < 1/\Delta t \tag{7}$$

and consider constants x^* and y^* such that

$$r_1 a_{11}^{-1} \leq x^* \leq U_1(y^*),\ r_2 a_{22}^{-1} \leq y^* \leq U_2(x^*). \tag{8}$$

Remark 1. For every point (x^*, y^*) satisfying

$$\frac{r_1}{a_{11}} \leq x^* \leq \min\left\{\frac{1 + r_1\Delta t}{4a_{11}\Delta t},\ \frac{1 + r_2\Delta t}{2a_{21}\Delta t}\right\},\quad \frac{r_2}{a_{22}} \leq y^* \leq \min\left\{\frac{1 + r_1\Delta t}{2a_{12}\Delta t},\ \frac{1 + r_2\Delta t}{4a_{22}\Delta t}\right\}, \tag{9}$$

the two conditions in (8) hold since

$$\begin{aligned}
U_1(y^*) &= \frac{1+r_1\Delta t - a_{12}y^*\Delta t}{2a_{11}\Delta t} \geq \frac{1+r_1\Delta t - a_{12}\min\left\{\frac{1+r_1\Delta t}{2a_{12}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\}\Delta t}{2a_{11}\Delta t} \\
&= \frac{1+r_1\Delta t}{2a_{11}\Delta t} - \frac{a_{12}}{2a_{11}}\min\left\{\frac{1+r_1\Delta t}{2a_{12}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \\
&= \max\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_1\Delta t}{2a_{11}\Delta t} - \frac{a_{12}}{2a_{11}}\frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \geq \frac{1+r_1\Delta t}{4a_{11}\Delta t} \\
&\geq \min\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{2a_{21}\Delta t}\right\} \geq x^*
\end{aligned}$$

and

$$\begin{aligned}
U_2(x^*) &= \frac{1+r_2\Delta t - a_{21}x^*\Delta t}{2a_{22}\Delta t} \geq \frac{1+r_2\Delta t - a_{21}\min\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{2a_{21}\Delta t}\right\}\Delta t}{2a_{22}\Delta t} \\
&= \frac{1+r_2\Delta t}{2a_{22}\Delta t} - \frac{a_{21}}{2a_{22}}\min\left\{\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{2a_{21}\Delta t}\right\} \\
&= \max\left\{\frac{1+r_2\Delta t}{2a_{22}\Delta t} - \frac{a_{21}}{2a_{22}}\frac{1+r_1\Delta t}{4a_{11}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \geq \frac{1+r_2\Delta t}{4a_{22}\Delta t} \\
&\geq \min\left\{\frac{1+r_1\Delta t}{2a_{12}\Delta t},\ \frac{1+r_2\Delta t}{4a_{22}\Delta t}\right\} \geq y^*.
\end{aligned}$$

Using x^* and y^* in (8), we can obtain the positivity and boundedness of (x_n, y_n).

Theorem 1. *Let (x_n, y_n) be the solution of (2). Assume that (7) and (8) hold.*

If $(x_0, y_0) \in (0, x^*) \times (0, y^*)$, then $(x_n, y_n) \in (0, x^*) \times (0, y^*)$ for all n.

Proof. Using the condition in this theorem and (5), we have

$$0 < x_0 < x^* \leq U_1(y^*) < U_1(y_0),\ 0 < y_0 < y^* \leq U_2(x^*) < U_2(x_0), \tag{10}$$

and then the increasing property (6) gives the positivity of x_1 and y_1:

$$x_1 = F_{y_0}(x_0) > F_{y_0}(0) = 0, \quad y_1 = G_{x_0}(y_0) > G_{x_0}(0) = 0. \tag{11}$$

Now, we claim that $x_1 < x^*$ and $y_1 < y^*$. If $r_1 - a_{11}x_0 - a_{12}y_0 \leq 0$, then

$$x_1 = F_{y_0}(x_0) \leq x_0 < x^*.$$

Otherwise, we get

$$0 < x_0 < (r_1 - a_{12}y_0)a_{11}^{-1} < (1 + r_1\Delta t - a_{12}y_0\Delta t)(2a_{11}\Delta t)^{-1} = U_1(y_0),$$

where the last inequality is obtained from $r_1\Delta t < 1$ in (7). Hence (6) and (8) imply the boundedness of x_1:

$$x_1 = F_{y_0}(x_0) < F_{y_0}\left((r_1 - a_{12}y_0)a_{11}^{-1}\right) = (r_1 - a_{12}y_0)a_{11}^{-1} < r_1 a_{11}^{-1} \leq x^*. \tag{12}$$

Similarly if $r_2 - a_{21}x_0 - a_{22}y_0 \le 0$, then $y_1 = G_{x_0}(y_0) \le y_0 < y^*$. Otherwise, we have

$$0 < y_0 < (r_2 - a_{21}x_0)a_{22}^{-1} < (1 + r_2\Delta t - a_{21}x_0\Delta t)(2a_{22}\Delta t)^{-1} = U_2(x_0),$$

where the last inequality is obtained from $r_2\Delta t < 1$ in (7). Thus (6) and (8) imply the boundedness of y_1 that

$$y_1 = G_{x_0}(y_0) < G_{x_0}\left((r_2 - a_{21}x_0)a_{22}^{-1}\right) = (r_2 - a_{21}x_0)a_{22}^{-1} < r_2a_{22}^{-1} \le y^*. \tag{13}$$

Hence using (11), (12) and (13), we have that

$$\text{if } (x_0, y_0) \in (0, x^*) \times (0, y^*), \text{ then } (x_1, y_1) \in (0, x^*) \times (0, y^*).$$

Therefore, using the mathematical induction, we can obtain the desired result. □

Remark 2. Due to (9), we can choose sufficiently large values of x^* and y^* when letting Δt be sufficiently small, so that the area of $(0, x^*) \times (0, y^*)$ for the initial state (x_0, y_0) in Theorem 1 can be taken large.

3. Stability of the discrete solutions

Let $\mathcal{D} = (0, x^*) \times (0, y^*)$ for x^* and y^* defined in (8). In order to discuss the stability of the Euler scheme (2) for each initial position (x_0, y_0) contained in $\mathcal{D}$, we partition $\mathcal{D}$ into the four regions

$$\begin{aligned} \mathrm{I} &= \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) \ge 0,\ g(\mathbf{x}) > 0\}, \quad \mathrm{II} = \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) < 0,\ g(\mathbf{x}) \ge 0\}, \\ \mathrm{III} &= \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) \le 0,\ g(\mathbf{x}) < 0\}, \quad \mathrm{IV} = \{\mathbf{x} \in \mathcal{D} \mid f(\mathbf{x}) > 0,\ g(\mathbf{x}) \le 0\}, \end{aligned} \tag{14}$$

where $\mathbf{x} = (x, y)$ and

$$f(x, y) = r_1 - a_{11}x - a_{12}y,\ g(x, y) = r_2 - a_{21}x - a_{22}y. \tag{15}$$

Since the location of the regions depends on the x and y-intercepts of the two lines $f(x, y) = 0$ and $g(x, y) = 0$, we partition $\mathcal{D}$ by using the four categories $\mathcal{C}_i (1 \le i \le 4)$ as in Figure 1; we use the symbol $\mathcal{C}_1$ for the two conditions $r_1a_{11}^{-1} < r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} < r_2a_{22}^{-1}$, the symbol $\mathcal{C}_2$ for $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$, the symbol $\mathcal{C}_3$ for $r_1a_{11}^{-1} < r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} > r_2a_{22}^{-1}$, and finally the symbol $\mathcal{C}_4$ for $r_1a_{11}^{-1} > r_2a_{21}^{-1}$ and $r_1a_{12}^{-1} < r_2a_{22}^{-1}$. The magenta circles in Figure 1 denote the stable points of the difference model (2) in the categories, which will be proved.

Remark 3. In the case of $\mathcal{C}_1$

$$r_1a_{11}^{-1} < r_2a_{21}^{-1},\ r_1a_{12}^{-1} < r_2a_{22}^{-1}, \tag{16}$$

the region IV is empty. In order to prove this emptiness, suppose, on the contrary, that there exists $(x, y) \in \mathrm{IV}$, which means, from (14), that

$$r_1 - a_{11}x - a_{12}y > 0,\ r_2 - a_{21}x - a_{22}y \le 0. \tag{17}$$

Eliminating x and y from (17), we have the two inequalities, respectively:

$$-r_1a_{21} + r_2a_{11} < (a_{11}a_{22} - a_{12}a_{21})y, \tag{18}$$

$$-r_1a_{22} + r_2a_{12} < (a_{12}a_{21} - a_{11}a_{22})x. \tag{19}$$

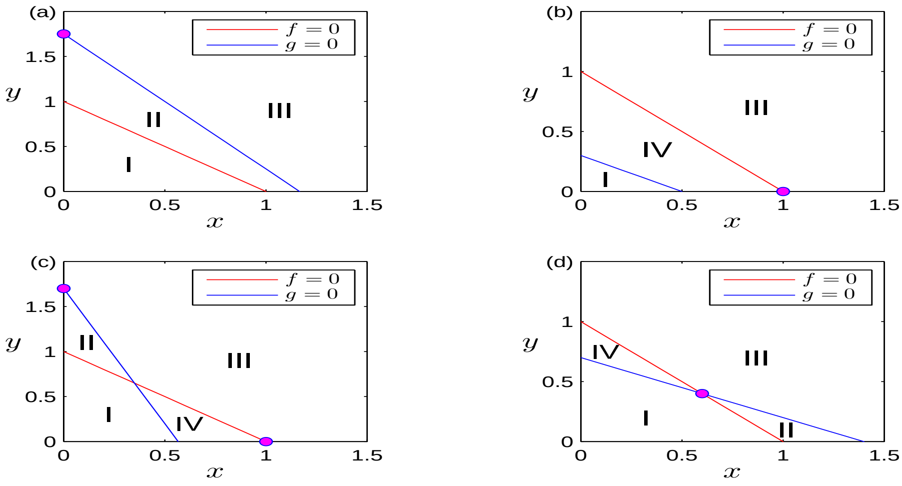

Figure 1: Two lines $f = 0$ and $g = 0$ and regions with stable points. (a) $r_2 = 3.5, a_{21} = 3.0, a_{22} = 2$ (b) $r_2 = 1.5, a_{21} = 3, a_{22} = 5$ (c) $r_2 = 1.7, a_{21} = 3, a_{22} = 1$ (d) $r_2 = 3.5, a_{21} = 2.5, a_{22} = 5$

We find a contradiction by using the following three cases:
Case 1. Let $a_{11}a_{22} - a_{12}a_{21} = 0$.
In this case, (18) becomes $-r_1a_{21} + r_2a_{11} < 0$, which contradicts (16).
Case 2. Let $a_{11}a_{22} - a_{12}a_{21} < 0$.
Using the positivity of y, (18) becomes $-r_1a_{21} + r_2a_{11} < 0$, which contradicts (16).
Case 3. Let $a_{11}a_{22} - a_{12}a_{21} > 0$.
Using the positivity of x, (19) becomes $-r_1a_{22} + r_2a_{12} < 0$, which contradicts (16).
Therefore it follows from Cases 1, 2 and 3 that the region IV is empty and then

$$\mathcal{D} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \text{ for } \mathcal{C}_1 \tag{20}$$

as in Figure 1-(a). Similarly we can obtain

$$\mathcal{D} = \mathrm{I} \cup \mathrm{III} \cup \mathrm{IV} \text{ for } \mathcal{C}_2 \tag{21}$$

as in Figure 1-(b).

For convenience, we use the difference equations

$$x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\}, \tag{22}$$
$$y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} \tag{23}$$

as well as (2), where $f(x, y)$ and $g(x, y)$ are defined in (15).
For the stability we need to assume

$$1 > \Delta t\,(a_{11}x^* + a_{22}y^* + x^*y^*|a_{12}a_{21} - a_{11}a_{22}|\Delta t)\,. \tag{24}$$

Lemma 1. *Let (x_n, y_n) be the solution of (2). Assume that (7), (8) and (24) hold. If $(x_k, y_k) \in \mathrm{I}$ for some k, then (x_{k+1}, y_{k+1}) is not contained in* III.

Proof. The condition $(x_k, y_k) \in \mathrm{I}$ gives

$$g(x_k, y_k) > 0. \tag{25}$$

Suppose, on the contrary, that (x_{k+1}, y_{k+1}) is contained in III, which means

$$f(x_{k+1}, y_{k+1}) \leq 0 \text{ and } g(x_{k+1}, y_{k+1}) < 0.$$

Then (22) and (23) give

$$\begin{aligned} 0 \geq f(x_{k+1}, y_{k+1}) &= f\left(x_k + x_k f(x_k, y_k)\Delta t,\ y_k + y_k g(x_k, y_k)\Delta t\right) \\ &= f(x_k, y_k) + (-a_{11})x_k f(x_k, y_k)\Delta t + (-a_{12})y_k g(x_k, y_k)\Delta t \end{aligned} \tag{26}$$

and

$$\begin{aligned} 0 > g(x_{k+1}, y_{k+1}) &= g(x_k + x_k f(x_k, y_k)\Delta t,\ y_k + y_k g(x_k, y_k)\Delta t) \\ &= g(x_k, y_k) + (-a_{21})x_k f(x_k, y_k)\Delta t + (-a_{22})y_k g(x_k, y_k)\Delta t. \end{aligned} \tag{27}$$

We write (26) and (27) as

$$\begin{aligned} f(x_k, y_k)(1 - a_{11}x_k\Delta t) &\leq a_{12}y_k g(x_k, y_k)\Delta t, \\ g(x_k, y_k)(1 - a_{22}y_k\Delta t) &< a_{21}x_k f(x_k, y_k)\Delta t. \end{aligned} \tag{28}$$

Combining (24) and Theorem 1 gives

$$0 < 1 - a_{11}x^*\Delta t < 1 - a_{11}x_k\Delta t$$

and so (28) implies

$$g(x_k, y_k)(1 - a_{22}y_k\Delta t) < a_{21}x_k\Delta t \frac{a_{12}y_k g(x_k, y_k)\Delta t}{(1 - a_{11}x_k\Delta t)}. \tag{29}$$

Using (24) and (25), we can simplify (29) as follows.

$$\begin{aligned} 1 &< \Delta t\{a_{11}x_k(1 - a_{22}y_k\Delta t) + a_{22}y_k + a_{12}y_k a_{21}x_k\Delta t\} \\ &\leq \Delta t\{a_{11}x_k + a_{22}y_k + x_k y_k|a_{12}a_{21} - a_{11}a_{22}|\Delta t\}, \end{aligned} \tag{30}$$

where the last inequality contradicts (24). Hence (x_{k+1}, y_{k+1}) is not contained in III. □

Remark 4. Similarly to Lemma 1 under the same assumption, we can obtain that

$$\text{if } (x_k, y_k) \in \text{ III for some } k, \text{ then } (x_{k+1}, y_{k+1}) \text{ is not contained in I} \tag{31}$$

as follows. The condition $(x_k, y_k) \in \mathrm{III}$ gives

$$g(x_k, y_k) < 0. \tag{32}$$

Suppose, on the contrary, that

$$f(x_{k+1}, y_{k+1}) \geq 0 \text{ and } g(x_{k+1}, y_{k+1}) > 0. \tag{33}$$

Using (33) instead of $f(x_{k+1}, y_{k+1}) \leq 0$ and $g(x_{k+1}, y_{k+1}) < 0$ in the proof of Lemma 1 and following the proof of Lemma 1 with (32), we have

$$g(x_k, y_k)(1 - a_{22}y_k\Delta t) > a_{21}x_k\Delta t \frac{a_{12}y_k g(x_k, y_k)\Delta t}{(1 - a_{11}x_k\Delta t)}$$

and then obtain the contradiction (30) due to (32). Therefore we obtain (31).

Lemma 2. *Let (x_n, y_n) be the solution of (2). Assume that (7), (8) and (24) hold.*

$$\text{If } (x_k, y_k) \in \text{II for some } k, \text{ then } (x_n, y_n) \in \text{II for all } n \geq k.$$

Proof. Let $(x_k, y_k) \in$ II, which implies $f(x_k, y_k) < 0 \leq g(x_k, y_k)$ and then

$$x_{k+1} < x_k, \quad y_{k+1} \geq y_k. \tag{34}$$

It follows from Theorem 1, (34) and (10) that

$$0 < x_{k+1} < x_k < U_1(y_k), \; 0 < y_k \leq y_{k+1} < y^* < U_2(x_k). \tag{35}$$

Using the decreasing function $F_y(x)$ of y and combining (6) with (35), we have

$$x_{k+2} = F_{y_{k+1}}(x_{k+1}) \leq F_{y_k}(x_{k+1}) < F_{y_k}(x_k) = x_{k+1} \tag{36}$$

and then (22) gives

$$f(x_{k+1}, y_{k+1}) < 0. \tag{37}$$

Similarly, the strictly decreasing function $G_x(y)$ of x with (6) and (35) gives

$$y_{k+2} = G_{x_{k+1}}(y_{k+1}) > G_{x_k}(y_{k+1}) \geq G_{x_k}(y_k) = y_{k+1}. \tag{38}$$

Substituting (23) into (38) yields

$$g(x_{k+1}, y_{k+1}) > 0,$$

with which (37) gives

$$f(x_{k+1}, y_{k+1}) < 0 < g(x_{k+1}, y_{k+1}).$$

This implies

$$(x_{k+1}, y_{k+1}) \in \text{II}.$$

Hence

$$\text{if } (x_k, y_k) \in \text{II, then } (x_{k+1}, y_{k+1}) \in \text{II}.$$

Therefore using mathematical induction, we can obtain the desired result. □

Remark 5. Similarly to Lemma 2 under the same assumption, we can obtain that

$$\text{if } (x_k, y_k) \in \text{IV for some } k, \text{ then } (x_n, y_n) \in \text{IV for all } n \geq k \tag{39}$$

as follows. Let $(x_k, y_k) \in$ IV, which implies

$$f(x_k, y_k) > 0 \geq g(x_k, y_k). \tag{40}$$

Then replacing $f(x_k, y_k) < 0 \leq g(x_k, y_k)$ in the proof of Lemma 2 with (40) and following the proof of Lemma 2, we have

$$f(x_{k+1}, y_{k+1}) > 0 > g(x_{k+1}, y_{k+1}),$$

which implies

$$(x_{k+1}, y_{k+1}) \in \text{IV}.$$

Hence mathematical induction gives (39).

In the following theorem, we show the global stability of the solutions of (2) for the category $\mathcal{C}_1$ as in Figure 1-(a); we present the condition that the species y always outcompetes the species x.

Theorem 2. *Assume that (7), (8) and (24) hold.*

$$\text{If } r_1 a_{11}^{-1} < r_2 a_{21}^{-1} \text{ and } r_1 a_{12}^{-1} < r_2 a_{22}^{-1}, \text{ then } \left(0, r_2 a_{22}^{-1}\right) \text{ is globally stable.}$$

Proof. The condition in this theorem is corresponding to $\mathcal{C}_1$, so that $\mathcal{D}$ is partitioned into the three regions I, II and III due to (20). We claim the global stability for $(x_0, y_0) \in \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$ by using mathematical induction as follows.
Case 2-1. Let $(x_0, y_0) \in \mathrm{II}$.
Using Lemma 2 and Theorem 1, we have that

$$0 < x_{n+1} < x_n, \ 0 < y_n \leq y_{n+1} < y^*, \tag{41}$$

which give the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively.
Note that the increasing property of $\{y_n\}$ gives $\omega_2 > 0$.
In addition, the limit ω_1 is zero, which can be obtained by indirect proof. Suppose, on the contrary, that ω_1 is nonzero. Taking the limit of (2) and using $\omega_i > 0$ $(i = 1, 2)$, we have

$$\left(a_{11}a_{22} - a_{12}a_{21}\right)\left(\omega_1, \omega_2\right) = \left(r_1 a_{22} - r_2 a_{12}, -r_1 a_{21} + r_2 a_{11}\right). \tag{42}$$

Since $r_1 a_{22} - r_2 a_{12} < 0$ and $-r_1 a_{21} + r_2 a_{11} > 0$ from the conditions in this theorem, the equality (42) with $\omega_i > 0$ gives

$$0 > a_{11}a_{22} - a_{12}a_{21} > 0, \tag{43}$$

which is a contradiction. Consequently, ω_1 is zero.
Taking the limit of the second equation in (2) with $\omega_1 = 0$ and $\omega_2 > 0$, we have $\omega_2 = r_2 a_{22}^{-1}$, which completes the proof for Case 2-1.
Case 2-2. Let $(x_0, y_0) \in \mathrm{I}$.
This case implies that $f(x_0, y_0) \geq 0$ and $g(x_0, y_0) > 0$. We use the following three steps to prove this theorem in this case.

Step 1. There exists a positive integer m_1 such that $(x_{m_1}, y_{m_1}) \notin \mathrm{I}$.
Suppose, on the contrary, that $(x_n, y_n) \in \mathrm{I}$ for all n, which means $f(x_n, y_n) \geq 0$ and $g(x_n, y_n) > 0$ for all n. Then

$$x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\} \geq x_n > 0, \quad y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} > y_n > 0$$

and hence the boundedness of (x_n, y_n) in Theorem 1 gives the convergence of the increasing sequences $\{x_n\}$ and $\{y_n\}$, which have positive limits ω_1 and ω_2, respectively. Therefore we have a contradiction by using (42)–(43).

Step 2. There exists a positive integer m such that $(x_m, y_m) \in \mathrm{II}$.
Using $(x_0, y_0) \in \mathrm{I}$ and Step 1, there exists a positive integer m_1 such that $(x_{m_1-1}, y_{m_1-1}) \in \mathrm{I}$ and $(x_{m_1}, y_{m_1}) \in \mathcal{D}-\mathrm{I}$. Since $\mathcal{D}-\mathrm{I} = \mathrm{II} \cup \mathrm{III}$, we have

$$(x_{m_1}, y_{m_1}) \in \mathrm{II} \text{ or } (x_{m_1}, y_{m_1}) \in \mathrm{III}. \tag{44}$$

Applying Lemma 1 with $(x_{m_1-1}, y_{m_1-1}) \in \mathrm{I}$, it is not true that $(x_{m_1}, y_{m_1}) \in \mathrm{III}$ and then $(x_{m_1}, y_{m_1}) \in \mathrm{II}$. Taking $m = m_1$ gives the desired result.

Step 3. If $(x_0, y_0) \in \mathrm{I}$, then $(x_m, y_m) \in \mathrm{II}$ for some positive integer m due to Step 2. Therefore the proof for Case 2-1 completes the proof for Case 2-2.

Case 2-3. Let $(x_0, y_0) \in \mathrm{III}$.

This case implies that $f(x_0, y_0) \le 0$ and $g(x_0, y_0) < 0$. We use the following two steps to prove this theorem in this case.

Step 1. If $(x_n, y_n) \in \mathrm{III}$ for all n, then $\lim_{n\to\infty}(x_n, y_n) = (0, r_2 a_{22}^{-1})$.

Assume that $(x_n, y_n) \in \mathrm{III}$ for all n, which implies

$$f(x_n, y_n) \le 0,\ \ g(x_n, y_n) < 0 \tag{45}$$

for all n. The assumption gives the decreasing property

$$0 < x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\} \le x_n,\ \ 0 < y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} < y_n$$

and then Theorem 1 gives the convergence of $\{x_n\}$ and $\{y_n\}$ with the nonnegative limits ω_1 and ω_2, respectively. It is only possible that $\omega_1 = 0$ and $\omega_2 > 0$ as follows.

If $\omega_1 > 0$ and $\omega_2 > 0$, then (42)–(43) give a contradiction.

If $\omega_1 > 0$ and $\omega_2 = 0$, then $\omega_1 = r_1 a_{11}^{-1}$. This is impossible due to the unstability of $(r_1 a_{11}^{-1}, 0)$ since the linearized system of (2) at $(r_1 a_{11}^{-1}, 0)$ has the eigenvalue

$$1 + \Delta t a_{11}^{-1}\left(r_2 a_{11} - r_1 a_{21}\right) > 1$$

under the condition $a_{21} a_{11}^{-1} < r_2 {r_1}^{-1}$. Therefore $\{(x_n, y_n)\}$ cannot have the limit $(r_1 a_{11}^{-1}, 0)$.

If $\omega_1 = 0$ and $\omega_2 = 0$, then

$$\lim_{n\to\infty} f\left(x_n, y_n\right) = r_1 > 0, \quad \lim_{n\to\infty} g\left(x_n, y_n\right) = r_2 > 0,$$

which are contradictory to (45).

Therefore it remains that $\omega_1 = 0$ and $\omega_2 > 0$, which gives $(\omega_1, \omega_2) = (0, r_2 a_{22}^{-1})$.

Step 2. If $(x_m, y_m) \notin \mathrm{III}$ for some m, then $\lim_{n\to\infty}(x_n, y_n) = (0, r_2 a_{22}^{-1})$.

Since $(x_m, y_m) \in \mathcal{D}-\mathrm{III}$ and $\mathcal{D}-\mathrm{III} = \mathrm{I} \cup \mathrm{II}$, we have

$$(x_m, y_m) \in \mathrm{I} \text{ or } (x_m, y_m) \in \mathrm{II}.$$

However it is not true that $(x_m, y_m) \in \mathrm{I}$ due to Remark 4 and so we have $(x_m, y_m) \in \mathrm{II}$. Therefore, following the proof for Case 2-1, we obtain $\lim_{n\to\infty}(x_n, y_n) = (0, r_2 a_{22}^{-1})$.

Finally, we obtain the desired result from the proofs for Cases 2-1, 2-2 and 2-3. □

In the following theorem, we show the global stability of (2) for $\mathcal{C}_2$ as in Figure 1-(b) and present the condition that the species x always outcompetes the species y.

Theorem 3. *Assume that (7), (8) and (24) hold.*

$$\textit{If } r_1 a_{11}^{-1} > r_2 a_{21}^{-1} \textit{ and } r_1 a_{12}^{-1} > r_2 a_{22}^{-1}, \textit{ then } \left(r_1 a_{11}^{-1}, 0\right) \textit{ is globally stable.}$$

Proof. The condition in this theorem is corresponding to $\mathcal{C}_1$ and so $\mathcal{D}$ is partitioned into the three regions I, III and IV due to (21). We claim the global stability for $(x_0, y_0) \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{IV}$ by using mathematical induction as follows.

Case 3-1. Let $(x_0, y_0) \in \mathrm{IV}$.

In this case, (39) gives $(x_n, y_n) \in \mathrm{IV}$ for all n, with which (22) and (23) give $x_n < x_{n+1}$ and $y_{n+1} \le y_n$. Then Theorem 1 gives

$$0 < x_n < x_{n+1} < x^*,\ \ 0 < y_{n+1} \le y_n, \tag{46}$$

which imply the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively. The increasing property of $\{x_n\}$ gives $\omega_1 > 0$.
In addition, the limit ω_2 is zero, which can be obtained by indirect proof as in Case 2-1. Suppose, on the contrary, that ω_2 is nonzero. Taking the limit of (2) and using the positivity of ω_1 and ω_2, we have (42). Applying the conditions $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ to (42) yields the contradiction (43) Consequently, ω_2 is zero.
Taking the limit of the first equation in (2) with $\omega_1 > 0$ and $\omega_2 = 0$, we have $\omega_1 = r_1 a_{11}^{-1}$, which completes the proof for Case 3-1.
Case 3-2. Let $(x_0, y_0) \in \mathrm{I}$.
In this case we have $f(x_0, y_0) \geq 0$ and $g(x_0, y_0) > 0$, and use the following three steps.

Step 1. There exists a positive integer m_1 such that $(x_{m_1}, y_{m_1}) \notin \mathrm{I}$.
Suppose, on the contrary, that $(x_n, y_n) \in \mathrm{I}$ for all n, which means $f(x_n, y_n) \geq 0$ and $g(x_n, y_n) > 0$ for all n. Then

$$x_{n+1} = x_n\{1 + f(x_n, y_n)\Delta t\} \geq x_n > 0, \quad y_{n+1} = y_n\{1 + g(x_n, y_n)\Delta t\} > y_n > 0,$$

and hence the boundedness of (x_n, y_n) in Theorem 1 gives the convergence of the increasing sequences $\{x_n\}$ and $\{y_n\}$, which have positive limits ω_1 and ω_2, respectively. Therefore we have the contradiction (43) as in Case 3-1.

Step 2. There exists a positive integer m such that $(x_m, y_m) \in \mathrm{IV}$.
Using $(x_0, y_0) \in \mathrm{I}$ and Step 1, there exists a positive integer m_1 such that $(x_{m_1-1}, y_{m_1-1}) \in \mathrm{I}$ and $(x_{m_1}, y_{m_1}) \in \mathcal{D}-\mathrm{I}$ for some m_1. Since $\mathcal{D}-\mathrm{I} = \mathrm{III} \cup \mathrm{IV}$, we have

$$(x_{m_1}, y_{m_1}) \in \mathrm{III} \text{ or } (x_{m_1}, y_{m_1}) \in \mathrm{IV}.$$

Applying Lemma 1 with $(x_{m_1-1}, y_{m_1-1}) \in \mathrm{I}$, it is not true that $(x_{m_1}, y_{m_1}) \in \mathrm{III}$ and then $(x_{m_1}, y_{m_1}) \in \mathrm{IV}$. Taking $m = m_1$ gives $(x_m, y_m) \in \mathrm{IV}$.

Step 3. If $(x_0, y_0) \in \mathrm{I}$, then $(x_m, y_m) \in \mathrm{IV}$ for some positive integer m due to Step 2. Therefore the proof used in Case 3-1 completes the proof for Case 3-2.
Case 3-3. Let $(x_0, y_0) \in \mathrm{III}$.
In this case we have $f(x_0, y_0) \leq 0$ and $g(x_0, y_0) < 0$,and use the following two steps.

Step 1. If $(x_n, y_n) \in \mathrm{III}$ for all n, then $\lim_{n\to\infty}(x_n, y_n) = (r_1 a_{11}^{-1}, 0)$.
As in Step 1 of Case 2-3 in Theorem 2, $\{(x_n, y_n)\}$ has the limit (ω_1, ω_2). It is only possible that $\omega_1 > 0$ and $\omega_2 = 0$ as follows.
If $\omega_1 > 0$ and $\omega_2 > 0$, then (46)–(**??**) give a contradiction.
If $\omega_1 = 0$ and $\omega_2 > 0$, then $\omega_2 = r_2 a_{22}^{-1}$. This is impossible due to the unstability of $(0, r_2 a_{22}^{-1})$ since the linearized system of (2) at $(0, r_2 a_{22}^{-1})$ has the eigenvalue

$$1 + \Delta t a_{22}^{-1} (r_1 a_{22} - r_2 a_{12}) > 1$$

under the condition $a_{22} a_{12}^{-1} > r_2 {r_1}^{-1}$. Therefore $\{(x_n, y_n)\}$ cannot have the limit $(r_1 a_{11}^{-1}, 0)$.
If $\omega_1 = 0$ and $\omega_2 = 0$, then

$$\lim_{n\to\infty} f(x_n, y_n) = r_1 > 0, \quad \lim_{n\to\infty} g(x_n, y_n) = r_2 > 0,$$

which are contradictory to (45).
It remains that $\omega_1 > 0$ and $\omega_2 = 0$, which yields the desired result $(\omega_1, \omega_2) = (r_1 a_{11}^{-1}, 0)$.

Step 2. If $(x_m, y_m) \notin \mathrm{III}$ for some m, then $\lim_{n\to\infty}(x_n, y_n) = (r_1 a_{11}^{-1}, 0)$.
Since $(x_m, y_m) \in \mathcal{D}-\mathrm{III} = \mathrm{I} \cup \mathrm{IV}$, we have

$$(x_m, y_m) \in \mathrm{I} \text{ or } (x_m, y_m) \in \mathrm{IV}.$$

However it is not true that $(x_m, y_m) \in \mathrm{I}$ due to Remark 4. Therefore, we have $(x_m, y_m) \in \mathrm{IV}$ and then $\lim_{n\to\infty}(x_n, y_n) = (r_1a_{11}^{-1}, 0)$ by following the proof for Case 3-1.

Finally, we obtain the desired result from the proofs for Cases 3-1 and 3-2. □

In the following theorem, we show the convergence of the solutions of (2) for the category $\mathcal{C}_3$ as in Figure 1-(c) and the dependence of the limit on the region in which the initial state is located.

From now on, in the case that $a_{11}a_{22} - a_{12}a_{21} \neq 0$, we use the symbol (θ_1, θ_2) to mean

$$(\theta_1, \theta_2) = (a_{11}a_{22} - a_{12}a_{21})^{-1} (r_1a_{22} - r_2a_{12}, -r_1a_{21} + r_2a_{11}), \tag{47}$$

where (θ_1, θ_2) satisfies

$$f(\theta_1, \theta_2) = g(\theta_1, \theta_2) = 0. \tag{48}$$

Theorem 4. *Let the conditions (7), (8) and (24) hold. Assume that*

$$r_1a_{11}^{-1} > r_2a_{21}^{-1} \text{ and } r_1a_{12}^{-1} < r_2a_{22}^{-1}.$$

(a) *If* $(x_0, y_0) \in \mathrm{II}$, *then* $\lim_{n\to\infty}(x_n, y_n) = (0, r_2a_{22}^{-1})$.
(b) *If* $(x_0, y_0) \in \mathrm{IV}$, *then* $\lim_{n\to\infty}(x_n, y_n) = (r_1a_{11}^{-1}, 0)$.
(c) *If* $(x_0, y_0) \in \mathrm{I} \cup \mathrm{III}$, *then* $\{(x_n, y_n)\}$ *converges with the limit* $(r_1a_{11}^{-1}, 0)$ *or* $(0, r_2a_{22}^{-1})$.

Proof. For the proof of (a), let $(x_0, y_0) \in \mathrm{II}$. We have from Lemma 2 and Theorem 1 that

$$0 < x_{n+1} < x_n,\ 0 < y_n \leq y_{n+1} < y^*, \tag{49}$$

which gives the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively. The increasing property of $\{y_n\}$ gives $\omega_2 > 0$.

In addition, the limit ω_1 is zero, which can be obtained by indirect proof. Suppose, on the contrary, that ω_1 is nonzero. Taking the limit of (2) and using the positivity of ω_1 and ω_2, we have

$$(a_{11}a_{22} - a_{12}a_{21})\omega_1 = r_1a_{22} - r_2a_{12}. \tag{50}$$

Since $(x_0, y_0) \in \mathrm{II}$, the definition of the region II gives

$$f(x_0, y_0) < 0 \leq g(x_0, y_0). \tag{51}$$

Solving (51) for x_0, we obtain

$$(r_1a_{22} - r_2a_{12}) - (a_{11}a_{22} - a_{12}a_{21})x_0 < 0. \tag{52}$$

The conditions $a_{21}a_{11}^{-1} > r_2{r_1}^{-1} > a_{22}a_{12}^{-1}$ in this theorem give

$$a_{11}a_{22} - a_{12}a_{21} < 0. \tag{53}$$

Applying (53) into both (52) and (50) yields

$$\omega_1 > x_0. \tag{54}$$

Combining (54) with (49), we have that for all n

$$\omega_1 > x_0 > x_n,$$

which is contradictory to $\lim_{n\to\infty} x_n = \omega_1$. Consequently, ω_1 is zero.
Taking the limit of the second equation in (2) with $\omega_1 = 0$ and $\omega_2 > 0$, we have $\omega_2 = r_2 a_{22}^{-1}$, which completes the proof of (a).

For the proof of (b), let $(x_0, y_0) \in \mathrm{IV}$. Using (46), we have the convergence of $\{x_n\}$ and $\{y_n\}$ with limits ω_1 and ω_2, respectively. The increasing property of $\{x_n\}$ gives $\omega_1 > 0$. In addition, the limit ω_2 is zero, which can be obtained by indirect proof. Suppose, on the contrary, that ω_2 is nonzero. Taking the limit of (2) and using the positivity of ω_1 and ω_2, we have

$$(a_{11}a_{22} - a_{12}a_{21})\omega_2 = -r_1 a_{21} + r_2 a_{11}. \tag{55}$$

Since $(x_0, y_0) \in \mathrm{IV}$, the definition of the region IV gives

$$f(x_0, y_0) > 0 \geq g(x_0, y_0). \tag{56}$$

Solving (56) for y_0, we obtain

$$(r_1 a_{21} - r_2 a_{11}) + (a_{11}a_{22} - a_{12}a_{21})y_0 > 0. \tag{57}$$

Applying (53) into (57) yields

$$\omega_2 > y_0. \tag{58}$$

Combining (58) with (46), we have that for all n

$$\omega_2 > y_0 > y_n,$$

which is contradictory to $\lim_{n\to\infty} y_n = \omega_2$. Consequently, ω_2 is zero.
Taking the limit of the first equation in (2) with $\omega_1 > 0$ and $\omega_2 = 0$, we have $\omega_1 = r_1 a_{11}^{-1}$, which completes the proof of (b).

For the proof of (c), we consider the following two cases.
Case 4-1. Let $(x_0, y_0) \in \mathrm{I}$.
We use the following three steps to obtain the desired result in this case.

Step 1. There exists a positive constant m such that $(x_m, y_m) \notin \mathrm{I}$.
Suppose, on the contrary, that $(x_n, y_n) \in \mathrm{I}$ for all n. Then $\{x_n\}$ and $\{y_n\}$ have the positive limits (θ_1, θ_2) defined in (47) by applying (53) and the approach used in Step1 of Case 2-2 in Theorem 2. However the system (2) under the condition $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ is unstable at the point (θ_1, θ_2) since the linearized system at (θ_1, θ_2) has the eigenvalue $1 + \Delta t a_{11}^{-1} (r_2 a_{11} - r_1 a_{21})$ greater than 1. Therefore $\{x_n\}$ and $\{y_n\}$ cannot have the positive limits θ_1 and θ_2, respectively, which is contradictory.

Step 2. There exists a positive constant m such that $(x_m, y_m) \in \mathrm{II} \cup \mathrm{IV}$.
Since $(x_0, y_0) \in \mathrm{I}$, Step 1 gives the existence of a positive integer m such that

$$(x_{m-1}, y_{m-1}) \in \mathrm{I} \text{ and } (x_m, y_m) \notin \mathrm{I},$$

which implies $(x_m, y_m) \in \mathrm{II} \cup \mathrm{IV}$ due to Lemma 1 and $\mathcal{D} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$.

Step 3. It follows from (a), (b) and Step 2 in this theorem that (x_n, y_n) converges and has the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$.
Case 4-2. Let $(x_0, y_0) \in \mathrm{III}$.
We use the following two steps to obtain the desired result in this case.

Step 1. If $(x_n, y_n) \in \text{III}$ for all n, then $\{(x_n, y_n)\}$ converges with the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$. To prove this, note that we have the convergence of $\{(x_n, y_n)\}$ with the limit (ω_1, ω_2) by following the proof of Step 1 of Case 2-3 in Theorem 2.
If $\omega_1 > 0$ and $\omega_2 > 0$, then $(\omega_1, \omega_2) = (\theta_1, \theta_2)$. This is impossible due to the unstability of (θ_1, θ_2) since the linearized system of (2) at (θ_1, θ_2) has the eigenvalue greater than 1:

$$1 + 0.5\Delta t \left\{ -\left(a_{11}\theta_1 + a_{22}\theta_2\right) + \sqrt{\left(a_{11}\theta_1 + a_{22}\theta_2\right)^2 + \alpha} \right\} > 1$$

since $\alpha = 4\left(a_{12}a_{21} - a_{11}a_{22}\right)\theta_1\theta_2 > 0$ under the condition $a_{21}a_{11}^{-1} > r_2 r_1{}^{-1} > a_{22}a_{12}^{-1}$. Therefore it is not possible that $\omega_1 > 0$ and $\omega_2 > 0$.
If $\omega_1 = 0$ and $\omega_2 = 0$, then we have the contradictions to to (45):

$$\lim_{n\to\infty} f\left(x_n, y_n\right) = r_1 > 0, \quad \lim_{n\to\infty} g\left(x_n, y_n\right) = r_2 > 0.$$

Therefore the remaining signs of ω_1 and ω_2 are

$$(+, 0) \text{ and } (0, +),$$

which give the desired result

$$(\omega_1, \omega_2) = (r_1 a_{11}^{-1}, 0) \text{ and } (0, r_2 a_{22}^{-1}),$$

respectively, by taking the limit of (2) and using the signs of ω_1 and ω_2.

Step 2. If $(x_m, y_m) \notin \text{III}$ for some m, then $\{(x_n, y_n)\}$ converges with the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$. To prove this, we follow the proof used in Step 2 of Case 4-1.
Since $(x_0, y_0) \in \text{III}$, using the condition $(x_m, y_m) \notin \text{III}$ for some m, we can assume that there exists a positive constant m_1 such that

$$(x_{m_1-1}, y_{m_1-1}) \in \text{III and } (x_{m_1}, y_{m_1}) \notin \text{III},$$

which implies

$$(x_{m_1}, y_{m_1}) \in \text{II} \cup \text{IV} \tag{59}$$

due to $\mathcal{D} = \text{I} \cup \text{II} \cup \text{III} \cup \text{IV}$ and Lemma 1. Therefore, using (59) and (a) and (b) in this theorem, we have that (x_n, y_n) converges and has the limit $(r_1 a_{11}^{-1}, 0)$ or $(0, r_2 a_{22}^{-1})$.

Finally, we obtain the desired result from the proofs for Cases 4-1 and 4-2. □

In the following theorem, we show the global stability of the solutions of (2) for the category $\mathcal{C}_4$ as in Figure 1-(d) where each component of the equilibrium point is positive.

Theorem 5. *Let the conditions (7), (8) and (24) hold. Assume that*

$$r_1 a_{11}^{-1} < r_2 a_{21}^{-1} \text{ and } r_1 a_{12}^{-1} > r_2 a_{22}^{-1}.$$

Then for (θ_1, θ_2) *defined in (47)*

$$(\theta_1, \theta_2) \text{ is globally stable.}$$

Proof. Note that the conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ in this theorem give

$$a_{11}a_{22} - a_{12}a_{21} > 0. \tag{60}$$

We prove this theorem by using the four cases and mathematical induction.
Case 5-1. Let $(x_0, y_0) \in \text{II}$.
Lemma 2 and Theorem 1 give (49). Then we have

$$\lim_{n\to\infty}(x_n, y_n) = (\omega_1, \omega_2),\ \omega_2 > 0$$

and

$$f(x_n, y_n) < 0 \leq g(x_n, y_n). \tag{61}$$

Solving (61) for x_n as in (51) and (52) and using (60), we have that for all n

$$0 < \theta_1 < x_n$$

and then $\omega_1 \geq \theta_1 > 0$. Since ω_1 and ω_2 are positive, we have

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Case 5-2. Let $(x_0, y_0) \in \mathrm{IV}$.
Using Remark 5 and Theorem 1, we have

$$0 < x_n < x_{n+1} < x^*,\ 0 < y_{n+1} \leq y_n \tag{62}$$

and

$$\lim_{n\to\infty}(x_n, y_n) = (\omega_1, \omega_2),\ \omega_1 > 0.$$

The inequalities (62) implies

$$f(x_n, y_n) > 0 \geq g(x_n, y_n). \tag{63}$$

Solving (63) for y_n as in (56) and (57), we have that for all n

$$0 < \theta_2 < y_n$$

and then $\omega_2 \geq \theta_2 > 0$. Since ω_1 and ω_2 are positive, we have

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Case 5-3. Let $(x_0, y_0) \in \mathrm{I}$.
If $(x_m, y_m) \notin \mathrm{I}$ for some m, then

$$(x_m, y_m) \in \mathcal{D} - \mathrm{I} = \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$$

and further

$$(x_m, y_m) \in \mathrm{II} \cup \mathrm{IV}$$

due to Lemma 1. By Case 5-1 and 5-2, we have

$$\lim_{n\to\infty}(x_n, y_n) = (\theta_1, \theta_2).$$

On the other hand, if $(x_n, y_n) \in \mathrm{I}$ for all n, then we have the positive limits ω_1 and ω_2 of $\{x_n\}$ and $\{y_n\}$, respectively, due to the definition of I and Theorem 1. Taking the limit of (2) and using ω_i $(i = 1, 2)$, we have

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Case 5-4. Let $(x_0, y_0) \in \mathrm{III}$.
Replacing I in the proof of Case 5-3 with III, we can obtain

$$(\omega_1, \omega_2) = (\theta_1, \theta_2).$$

Finally, we obtain the desired result from the proofs for Cases 5-1 to 5-4. □

4. Numerical examples

In this section, we provide simulations that illustrate our results in Theorem 2 to Theorem 5 for the difference scheme (2) with $\Delta t = 0.001$ and $(x^*, y^*) = (r_1 a_{11}^{-1} + 50, r_2 a_{22}^{-1} + 50)$. The values of parameters used in the following four examples satisfy the three conditions (7), (8) and (24).

Example 1. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 2, 3.5, 3, 2)$, which satisfies the two conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ in Theorem 2. Then the solutions (x_n, y_n) of (2) converge to $(0, r_2 a_{22}^{-1} = 1.75)$ as displayed in Figure 2-(a).

Example 2. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 1.5, 3, 5)$, which satisfies the two conditions $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ in Theorem 3. Then the solutions (x_n, y_n) of (2) converge to $(r_1 a_{11}^{-1} = 1, 0)$ as displayed in Figure 2-(b).

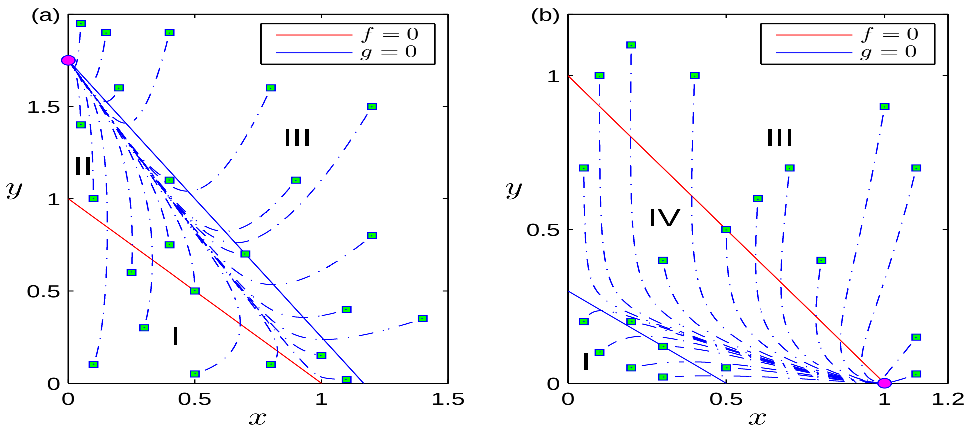

Figure 2: (a) Trajectories for different initial points in the regions I, II, III with $r_1 = 1, a_{11} = 1, a_{12} = 2, r_2 = 3.5, a_{21} = 3, a_{22} = 2$ in the category $\mathcal{C}_1$. (b) Trajectories for different initial points in the regions I, III, IV with $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 1.5, a_{21} = 3, a_{22} = 5$ in the category $\mathcal{C}_2$. The box and circle symbols denote initial and equilibrium points, respectively.

Example 3. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 1.7, 3, 1)$, which satisfies the two conditions $r_1 a_{11}^{-1} > r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} < r_2 a_{22}^{-1}$ in Theorem 4. Then as displayed in Figure 3-(a), we obtain the results in Theorem 4. If $(x_0, y_0) \in \mathrm{II}$, then the solutions (x_n, y_n) of (2) converge to $(0, r_2 a_{22}^{-1}) = (0, 1.7)$. If $(x_0, y_0) \in \mathrm{IV}$, then $\lim_{n\to\infty}(x_n, y_n) = (r_1 a_{11}^{-1}, 0) = (1, 0)$. If $(x_0, y_0) \in \mathrm{I} \cup \mathrm{III}$, then $\{(x_n, y_n)\}$ converges with the limit $(r_1 a_{11}^{-1}, 0) = (1, 0)$ or $(0, r_2 a_{22}^{-1}) = (0, 1.7)$. Especially, Figure 3-(a) shows that there exist at least two initial points contained in I converging to $(r_1 a_{11}^{-1}, 0) = (1, 0)$ and $(0, r_2 a_{22}^{-1}) = (0, 1.7)$, respectively. In the region III, the same phenomenon happens. The outcome depends on the initial abundances of the two species.

Example 4. Let $(r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}) = (1, 1, 1, 3.5, 2.5, 5)$, which satisfies the two conditions $r_1 a_{11}^{-1} < r_2 a_{21}^{-1}$ and $r_1 a_{12}^{-1} > r_2 a_{22}^{-1}$ in Theorem 5. Then the solutions x_n and y_n of (2) converge to

$$(r_1 a_{22} - r_2 a_{12})(a_{11} a_{22} - a_{12} a_{21})^{-1} = 0.6$$

and

$$(-r_1a_{21} + r_2a_{11})(a_{11}a_{22} - a_{12}a_{21})^{-1} = 0.4,$$

respectively, as displayed in Figure 3-(b). Although the outcome in Example 3 depends on the initial abundances of the two species, the outcome in Example 4 is independent of the initial abundances.

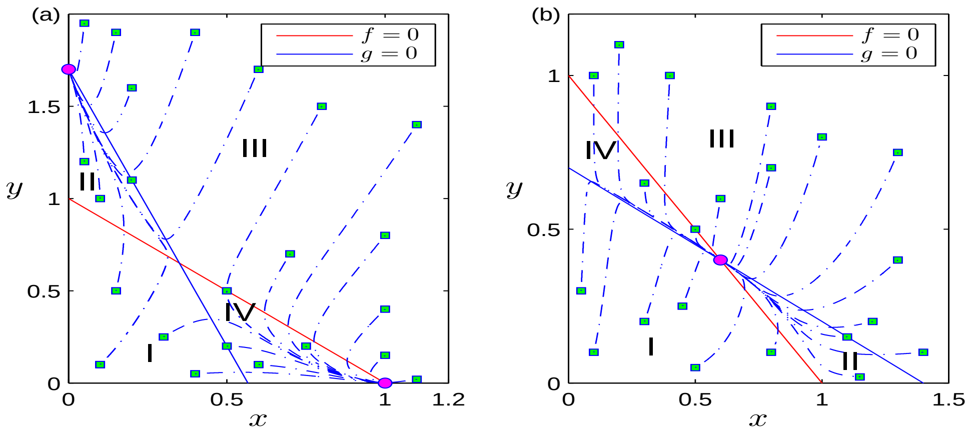

Figure 3: Trajectories for different initial points in the regions I, II, III and IV. The values of the parameters are (a) $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 1.7, a_{21} = 3, a_{22} = 1$ in the category $\mathcal{C}_3$. (b) $r_1 = 1, a_{11} = 1, a_{12} = 1, r_2 = 3.5, a_{21} = 2.5, a_{22} = 5$ in the category $\mathcal{C}_4$. The box and circle symbols denote initial and equilibrium points, respectively.

5. Conclusions and future work

In this paper, we have studied the Euler difference scheme for a two-dimensional Lotka-Volterra competition model and presented sufficient conditions for the global stability of the fixed points of a discrete competition model with two species. The main idea of our approach is to divide the domain used for the boundedness of solutions of the discrete model and to describe how to trace the trajectories with respect to each partition. Although we have applied our method for the two-dimensional discrete model, this method can be utilized to two-dimensional and other higher dimensional discrete models.

Acknowledgments

The present research has been conducted by the Research Grant of Kwangwoon University in 2015.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.2, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Weighted Composition Operators from Bloch spaces into Zygmund spaces*

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Abstract

In this paper we characterize the boundedness and compactness of the weighted composition operator from the classical Bloch space β to the Zygmund space $\mathcal{Z}$, and from the little Bloch space β_0 to the little Zygmund space $\mathcal{Z}_0$, respectively.

Keywords Bloch space, Zygmund space; Weighted composition operator; Boundedness; Compactness

2010 MR Subject Classification 47B38, 30D99, 30H05

1 Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane and $H(D)$ denote the set of all analytic functions on D. Let $u, \varphi \in H(D)$, where φ is an analytic self-map of D. Then the well-known *weighted composition operator* uC_φ on $H(D)$ is defined by $uC_\varphi(f)(z) = u(z)\cdot(f\circ\varphi(z))$ for $f \in H(D)$ and $z \in D$. Weighted composition operators can be regarded as a generalization of multiplication operators and composition operators. In 2001, Ohno and Zhao studied the weighted composition operators on the classical Bloch space β in [14], which has led many researchers to study this operator on other Banach spaces of analytic functions. The boundedness and compactness of it have been studied on various Banach spaces of analytic functions, such as Hardy, Bergman, BMOA, Bloch-type spaces, see, e.g. [2, 4, 8, 18, 27].

In 2006, the boundedness of composition operators on the Zygmund space $\mathcal{Z}$ was first studied by Choe, Koo, and Smith in [1]. Later, many researchers have studied composition operators and weighted composition operators acting on the Zygmund space $\mathcal{Z}$. Li and Stević in [9] studied the boundedness and compactness of the generalized composition operators on Zygmund spaces and Bloch type spaces. They in [11] considered the boundedness and compactness of the weighted composition operators from Zygmund spaces to Bloch spaces. Ye and Hu in [22] characterized boundedness and compactness of weighted composition operators on the Zygmund space $\mathcal{Z}$. Esmaeili and Lindström in [7] studied weighted composition operators from Zygmund type spaces to Bloch type spaces and their essential norms. Sanatpour and Hassanlou in [17] gave the essential norms of this operators between Zygmund-type spaces and Bloch-type spaces. See also

*The research was supported by the National Natural Science Foundation of China (Grant No. 11571217) and the Natural Science Foundation of Fujian Province, China(Grant No. 2015J01005).

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[5, 15, 16, 19, 20, 21, 23, 24, 25, 26] for corresponding results for weighted composition operators from one Banach space of analytic functions to another. It is well-known that $\mathcal{Z} \subset \beta$. It is more interesting to characterize u, φ such that this operator uC_φ has the pull-back properly, that is, $uC_\varphi f \in \mathcal{Z}$ whenever $f \in \beta$. In this paper we consider this question.

Now we give a detailed definition of these spaces. A function f analytic on the unit disk is said to belong to the *Bloch* space β if

$$b(f) = \sup_{z\in\mathbb{D}}\{(1-|z|^2)|f'(z)|\} < \infty,$$

and to the little *Bloch* space β_0 if $f \in \beta$ and

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

It is well known that β is a Banach space under the norm

$$\|f\|_\beta = |f(0)| + b(f),$$

and β_0 is a closed subspace of β.

The Zygmund space $\mathcal{Z}$ consists of all analytic functions f defined on D such that

$$z(f) = \sup\{(1-|z|^2)|f''(z)| : z \in D\}, \quad 0 < \alpha < +\infty.$$

From a theorem of Zygmund (see [29, vol. I, p. 263] or [6, Theorem 5.3]), we see that $f \in \mathcal{Z}$ if and only if f is continuous in the close unit disk $\overline{D} = \{z : |z| \le 1\}$ and the boundary function $f(e^{i\theta})$ such that

$$\sup_{h>0,\theta} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$

An analytic function $f \in H(D)$ is said to belong to the little Zymund space $\mathcal{Z}_0$ consists of all $f \in \mathcal{Z}$ satisfying $\lim_{|z|\to 1^-}(1-|z|^2)|f''(z)| = 0$. It can easily proved that $\mathcal{Z}$ is a Banach space under the norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + z(f)$$

and the polynomials are norm-dense in closed subspace $\mathcal{Z}_0$ of $\mathcal{Z}$. For some other information on this space and some operators on it, see, for example, [9, 10, 11].

Throughout this paper, constants are denoted by C, they are positive and only depending on p, and may differ from one occurrence to the other.

2 Auxiliary results

In order to prove the main results of this paper. we need some auxiliary results. The first part of the following lemma is a well known.

Lemma 2.1 *Suppose that* $f \in \beta$, *then*

(i) $|f(z)| \le \log \dfrac{e}{(1-|z|^2)}\|f\|_\beta$ *for every* $z \in D$;

(ii) $|f''(z)| \le \dfrac{8}{(1-|z|^2)^2} b(f)$ *for every* $z \in D$.

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Proof For any $f \in \beta$. Fix $z \in D$ and let $\rho = \dfrac{1+|z|}{2}$, by the Cauchy integral formula, we obtain that

$$|f''(z)| = |\frac{1}{2\pi i}\int_{|\xi|=\rho} \frac{f'(\xi)}{(\xi - z)^2}\, d\xi| \le \frac{b(f)}{1-\rho^2}\frac{1}{2\pi}\int_0^{2\pi} \frac{\rho\, d\theta}{|\rho e^{i\theta} - z|^2} = \frac{\|f\|_\infty}{1-\rho^2}\frac{\rho}{\rho^2 - |z|^2} \le \frac{8}{(1-|z|^2)^2}.$$

Hence (ii) holds.

Lemma 2.2 *[28] Suppose that* $f \in \beta_0$, *then*

(i) $\lim\limits_{|z|\to 1^-} \dfrac{|f(z)|}{\log(e/(1-|z|^2))} = 0$;
(ii) $\lim\limits_{|z|\to 1^-} (1-|z|^2)^2|f''(z)| = 0$.

Lemma 2.3 *Suppose* $uC_\varphi : \beta_0 \to \mathcal{Z}_0$ *is a bounded operator, then* $uC_\varphi : \beta \to \mathcal{Z}$ *is a bounded operator.*

The proof is similar to that of Lemma 2.3 in [21]. The details are omitted.

Lemma 2.4 *Suppose that* uC_φ *be a bounded operator from* β *to* $\mathcal{Z}$, *then* uC_φ *is compact if and only if for any bounded sequence* $\{f_n\}$ *in* β *which converges to* 0 *uniformly on compact subsets of* D. *We have* $\|uC_\varphi(f_n)\|_{\mathcal{Z}} \to 0$, *as* $n \to \infty$.

The proof is similar to that of Proposition 3.11 in [3] . The details are omitted.

Lemma 2.5 *Let* $U \subset \mathcal{Z}_0$. *Then* U *is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in U}(1-|z|^2)|f''(z)| = 0.$$

The proof is similar to that of Lemma 1 in [12], we omit it.

3 Main results

Theorem 3.1 *Let* u *be an analytic function on the unit disc* D, *and* φ *an analytic self-map of* D. *Then* uC_φ *is a bounded operator from the classical space* β *to the Zygmund space* $\mathcal{Z}$ *if and only if the following are satisfied:*

$$\sup_{z\in D}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2} < \infty; \tag{3.1}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{1-|\varphi(z)|^2} < \infty; \tag{3.2}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} < \infty. \tag{3.3}$$

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Proof Suppose uC_φ is bounded from the Bloch space β to the Zygmund space $\mathcal{Z}$. Then we can easily obtain the following results by taking $f(z) = 1$ and $f(z) = z$ in β respectively:

$$u \in \mathcal{Z}; \tag{3.4}$$

$$\sup_{z\in D}(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z) + \varphi(z)u''(z)| < +\infty. \tag{3.5}$$

By (3.4), (3.5) and the boundedness of the function $\varphi(z)$, we get

$$K_1 = \sup_{z\in D}(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)| < +\infty. \tag{3.6}$$

Let $f(z) = z^2$ in β again, in the same way we have

$$\sup_{z\in D}(1-|z|^2)|4\varphi(z)\varphi'(z)u'(z) + \varphi^2(z)u''(z) + 2u(z)(\varphi(z)\varphi''(z) + (\varphi'(z))^2)| < \infty.$$

Using these facts and the boundedness of the function $\varphi(z)$ again, we get

$$K_2 = \sup_{z\in D}(1-|z|^2)|(\varphi'(z))^2u(z)| < +\infty. \tag{3.7}$$

Fix $a \in D$, we take the test functions

$$f_a(z) = 3\log\frac{e}{1-\bar{a}z} + \frac{3}{\log\frac{e}{1-|a|^2}}(\log\frac{e}{1-\bar{a}z})^2 - \frac{1}{\log^2\frac{e}{1-|a|^2}}(\log\frac{e}{1-\bar{a}z})^3 \tag{3.8}$$

for $z \in D$. By a directly calculation we obtain that $f_a \in \beta$ and $\sup_a \|f_a\|_\beta \le C < \infty$, where C is not depended on a. Since $f_a(a) = 5\log\frac{e}{1-|a|^2}$, $f_a'(a) = 0$, $f_a''(a) = 0$, we have

$$\begin{aligned} C\|f_a\|_\beta &\ge \|uC_\varphi f_a\|_{\mathcal{Z}} \ge \sup_{z\in D}(1-|z|^2)|(uC_\varphi f_a)''(z)| \\ &= \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)f_a'(\varphi(z)) \\ &+ f_a''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)f_a(\varphi(z))|. \end{aligned}$$

Let $a = \varphi(\lambda)$, it follows that

$$\begin{aligned} C\|f_a\|_\beta &\ge (1-|\lambda|^2)^\alpha|\big(2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda)\big)f_{\varphi(\lambda)}'(\varphi(\lambda)) \\ &+ f_{\varphi(\lambda)}''(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda) + u''(\lambda)f_{\varphi(\lambda)}(\varphi(\lambda))| \\ &= 5(1-|\lambda|^2)^\alpha|u''(\lambda)\log\frac{e}{1-|\varphi(\lambda)|^2}|. \end{aligned}$$

Hence (3.1) holds.

Next, we will show that (3.2) holds. Fix $a \in D$ with $|a| > \frac{1}{2}$, we take another test functions:

$$g_a(z) = \frac{8(1-|a|^2)^2}{(1-\bar{a}z)^2} - \frac{14(1-|a|^2)^3}{(1-\bar{a}z)^3} + \frac{6(1-|a|^2)^4}{(1-\bar{a}z)^4} \tag{3.9}$$

for $z \in D$. By a directly calculation we obtain that $g_a \in \beta$ and $\sup_a \|g_a\|_\beta \le C < \infty$, where C is not depended on a. Since $g_a(a) = 0$, $g_a'(a) = \dfrac{-2\bar{a}}{1-|a|^2}$, $g_a''(a) = 0$, it follows that for all $\lambda \in D$ with $|\varphi(\lambda)| > \frac{1}{2}$, we have

$$\begin{aligned} C\|g_a\|_\beta &\ge \|uC_\varphi g_a\|_{\mathcal{Z}} \ge \sup_{z\in D}(1-|z|^2)|(uC_\varphi g_a)''(z)| \\ &= \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)g_a'(\varphi(z)) \\ &+ g_a''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)g_a(\varphi(z))|. \end{aligned}$$

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Let $a = \varphi(\lambda)$, it follows that

$$\begin{aligned} C\|g_a\|_\beta &\geq (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)g'_{\varphi(\lambda)}(\varphi(\lambda)) \\ &+ g''_{\varphi(\lambda)}(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda)+u''(\lambda)g_{\varphi(\lambda)}(\varphi(\lambda))| \\ &= (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)\frac{-2\overline{\varphi(\lambda)^2}}{1-|\varphi(\lambda)|^2}| \\ &\geq \frac{1}{2}\frac{(1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)|}{1-|\varphi(\lambda)|^2}. \end{aligned}$$

For $\forall\lambda \in D$ with $|\varphi(\lambda)| \leq \frac{1}{2}$, by (3.6), we have

$$\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)|}{1-|\varphi(\lambda)|^2} \leq \frac{4}{3}\sup_{\lambda\in D}(1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)| < +\infty.$$

Hence (3.2) holds.

Finally we will show (3.3) holds. Fix $a \in D$ with $|a| > \frac{1}{2}$, we take the test functions:

$$h_a(z) = -\frac{3(1-|a|^2)^2}{(1-\bar{a}z)^2} + \frac{6(1-|a|^2)^3}{(1-\bar{a}z)^3} - \frac{3(1-|a|^2)^4}{(1-\bar{a}z)^4} \tag{3.10}$$

for $z \in D$. It is easily proved that $\sup_{\frac{1}{2}<|a|<1}\|h_a\|_\beta \leq C < \infty$, where C is not depended on a. For $w \in D$, let $a = \varphi(w)$, since

$$h_{\varphi(w)}(\varphi(w)) = 0, \quad h'_{\varphi(w)}(\varphi(w)) = 0, \quad h''_{\varphi(w)}(\varphi(w)) = \frac{-6(\overline{\varphi(w)})^2}{(1-|\varphi(w)|^2)^2},$$

then, for all $w \in D$ with $|\varphi(w)| > \frac{1}{2}$, we obtain that

$$C\|h_a\|_\beta \geq \|uC_\varphi g_a\|_{\mathcal{Z}} \geq (1-|w|^2)\frac{|6u(w)(\varphi'(w))^2(\overline{\varphi(w)})^2|}{(1-|\varphi(w)|^2)^2}.$$

Then, by (3.7), we have

$$\begin{aligned} &\sup_{w\in D}\frac{(1-|w|^2)|u(w)(\varphi'(w))^2|}{(1-|\varphi(w)|^2)^2} \leq \sup_{|\varphi(w)|>\frac{1}{2}}\frac{(1-|w|^2)|u(w)(\varphi'(w))^2|}{(1-|\varphi(w)|^2)^2} \\ &+ \sup_{|\varphi(w)|\leq\frac{1}{2}}\frac{(1-|w|^2)|u(w)(\varphi'(w))^2|}{(1-|\varphi(w)|^2)^2} \\ &\leq 4\sup_{|\varphi(w)|>\frac{1}{2}}(1-|w|^2)|\frac{|u(w)(\varphi'(w))^2(\overline{\varphi(w)})^2|}{(1-|\varphi(w)|^2)^2} + \frac{16}{9}\sup_{|\varphi(w)|\leq\frac{1}{2}}(1-|w|^2)|u(w)(\varphi'(w))^2| \\ &< \infty. \end{aligned}$$

Hence (3.3) holds.

Conversely, suppose that (3.1), (3.2), and (3.2) hold. For $f \in \beta$, by Lemma 2.1, we have the

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following inequality:

$$\begin{aligned}(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))\\ &+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))|\\ &\leq (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))|\\ &+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)|+(1-|z|^2)|u''(z)f(\varphi(z))|\\ &\leq \frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2}b(f)\\ &+ 8\frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^2}b(f)+(1-|z|^2)|u''(z)|\log(\frac{e}{1-|\varphi(z)|^2})\|f\|_\beta\\ &\leq C\|f\|_\beta,\end{aligned}$$

and

$$\begin{aligned}&|u(0)f(\varphi(0))|+|u'(0)f(\varphi(0))|+|u(0)f'(\varphi(0))\varphi'(0)|\\ &\leq \big((|u(0)|+|u'(0)|)\log(\frac{e}{1-|\varphi(0)|^2})+\frac{|u(0)\varphi'(0)|}{1-|\varphi(0)|^2}\big)\|f\|_\beta.\end{aligned}$$

This shows that uC_φ is bounded. This completes the proof of Theorem 3.1.

Corollary 3.1 *Let φ be an analytic self-map of D. Then C_φ is a bounded operator from the Bloch space β to the Zygmund space $\mathcal{Z}$ if and only if*

$$\sup_{z\in D}\frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2}<\infty \quad\text{and}\quad \sup_{z\in D}\frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2}<\infty.$$

In the formulation of lemma, we use the notation M_u on $H(D)$ defined by $M_u f = uf$ for $f\in H(D)$.

Corollary 3.2 *The pointwise multiplier $M_u:\beta\to\mathcal{Z}$ is a bounded operator if and only if $u=0$.*

Theorem 3.2 *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Then uC_φ is a compact operator from β to $\mathcal{Z}$ if and only if uC_φ is a bounded operator and the following are satisfied:*

$$\lim_{|\varphi(z)|\to 1^-}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2}=0; \tag{3.11}$$

$$\lim_{|\varphi(z)|\to 1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2}=0; \tag{3.12}$$

$$\lim_{|\varphi(z)|\to 1^-}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2}=0. \tag{3.13}$$

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Proof Suppose that uC_φ is compact from β to the Zygmund space $\mathcal{Z}$. Let $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. If such a sequence does not exist then (3.11), (3.12) and (3.13) are automatically satisfied. Without loss of generality we may suppose that $|\varphi(z_n)| > \frac{1}{2}$ for all n. We take the test functions

$$f_n(z) = \frac{6}{a_n}\log^2\frac{e}{1-\overline{\varphi(z_n)}z} - \frac{8}{a_n^2}\log^3\frac{e}{1-\overline{\varphi(z_n)}z} + \frac{3}{a_n^3}\log^4\frac{e}{1-|\varphi(z_n)|^2}. \tag{3.14}$$

where $a_n = \log\dfrac{e}{1-|\varphi(z_n)|^2}$. By a directly calculation, we may easily prove that $\{f_n\}$ converges to 0 uniformly on compact subsets of D and $\sup_n \|f_n\|_\beta \le C < \infty$. Then $\{f_n\}$ is a bounded sequence in β which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty}\|uC_\varphi(f_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. Note that

$$f_n(\varphi(z_n)) = a_n, \quad f_n'(\varphi(z_n)) = 0, \quad f_n''(\varphi(z_n)) = 0.$$

It follows that

$$\|uC_\varphi f_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))f_n'(\varphi(z_n))$$

$$+u(z_n)f_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)f_n(\varphi(z_n))|$$

$$= (1-|z_n|^2)|u''(z_n)|\log\frac{e}{1-|\varphi(z_n)|^2}.$$

Then

$$\lim_{n\to\infty}(1-|z_n|^2)|u''(z_n)|\log\frac{e}{1-|\varphi(z_n)|^2} = 0.$$

Next, let

$$g_n(z) = \frac{8(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^2} - \frac{14(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^3} + \frac{6(1-|\varphi(z_n)|^2)^4}{(1-\overline{\varphi(z_n)}z)^4}.$$

By a directly calculation we obtain that $g_n \rightrightarrows 0$ $(n \to \infty)$ on compact subsets of D and $\sup_n \|g_n\|_\beta \le C < \infty$. Consequently, $\{g_n\}$ is a bounded sequence in β which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty}\|uC_\varphi(g_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. Note that $g_n(\varphi(z_n)) \equiv 0$, $g_n''(\varphi(z_n)) \equiv 0$ and $g_n'(\varphi(z_n)) = \dfrac{-2\overline{\varphi(z_n)}}{1-|\varphi(z_n)|^2}$.

It follows that

$$\|uC_\varphi g_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))g_n'(\varphi(z_n))$$

$$+u(z_n)g_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)g_n(\varphi(z_n))|$$

$$= 2(1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))\frac{\overline{\varphi(z_n)}}{1-|\overline{\varphi(z_n)}|^2}|.$$

Then $\lim\limits_{n\to\infty}\dfrac{(1-|z_n|^2)|2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)|}{1-|\varphi(z_n)|^2} = 0.$

Finally, let

$$h_n(z) = -\frac{3(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^2} + \frac{6(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^3} - \frac{3(1-|\varphi(z_n)|^2)^4}{(1-\overline{\varphi(z_n)}z)^4}.$$

By a directly calculation we obtain that $h_n \rightrightarrows 0$ $(n \to \infty)$ on compact subsets of D and $\sup_n \|h_n\|_{\mathcal{Z}} \le C < \infty$. Consequently, $\{h_n\}$ is a bounded sequence in $\mathcal{Z}$ which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty} \|uC_\varphi(h_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. Note that $h_n(\varphi(z_n)) \equiv 0$, $h_n'(\varphi(z_n)) \equiv 0$ and $h_n''(\varphi(z_n)) = \dfrac{-6(\overline{\varphi(z_n)})^2}{(1-|\varphi(z_n)|^2)^2}$. It follows that

$$\|uC_\varphi h_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))h_n'(\varphi(z_n))$$

$$+u(z_n)h_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)h_n(\varphi(z_n))|$$

$$= 6(1-|z_n|^2)|u(z_n)(\varphi'(z_n))^2|\frac{|\overline{\varphi(z_n)}|^2}{(1-|\varphi(z_n)|^2)^2}.$$

Then $\lim\limits_{n\to\infty}(1-|z_n|^2)\dfrac{|u(z_n)(\varphi'(z_n))^2|}{(1-|\varphi(z_n)|^2)^2} = 0$. The proof of the necessary is completed.

Conversely, suppose that (3.11), (3.12), and (3.13) hold. Since uC_φ is a bounded operator, by Theorem 3.1, we have

$$M_1 \triangleq \sup_{z\in D}(1-|z|^2)|u''(z)|\log\frac{1}{1-|\varphi(z)|^2} < \infty, \qquad M_3 \triangleq \sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} < \infty,$$

and

$$M_2 \triangleq \sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{1-|\varphi(z)|^2} < \infty.$$

Let $\{f_n\}$ be a bounded sequence in β with $\|f_n\|_\beta \le 1$ and $f_n \to 0$ uniformly on compact subsets of D. We only prove $\lim\limits_{n\to\infty}\|uC_\varphi(f_n)\|_{\mathcal{Z}} = 0$ by Lemma 2.4. By the assumption, for any $\epsilon > 0$, there is a constant δ, $0 < \delta < 1$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} < \epsilon, \qquad (1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2} < \epsilon,$$

and

$$\frac{(1-|z|^2)|2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{1-|\varphi(z)|^2} < \epsilon.$$

Let $K = \{w \in D : |w| \le \delta\}$. Noting that K is a compact subset of D, we get that

$$\begin{aligned}
z(uC_\varphi f_n) &= \sup_{z\in D}(1-|z|^2)|(uC_\varphi f_n)''(z)| \\
&\le \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)f_n'(\varphi(z))| \\
&+ \sup_{z\in D}(1-|z|^2)|f_n''(\varphi(z))(\varphi'(z))^2u(z)| + \sup_{z\in D}(1-|z|^2)|u''(z)f_n(\varphi(z))| \\
&\le 10\epsilon + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|\big(2\varphi'(z)u'(z) + \varphi''(z)u(z)\big)f_n'(\varphi(z))| \\
&+ \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|f_n''(\varphi(z))(\varphi'(z))^2u(z)| + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|u''(z)f_n(\varphi(z))| \\
&\le 10\epsilon + M_2\sup_{w\in K}|f_n'(w)| + M_3\sup_{w\in K}|f_n''(w)| + M_1\sup_{w\in K}|f_n(w)|.
\end{aligned}$$

As $n \to \infty$, $\|uC_\varphi f_n\|_{\mathcal{Z}} \to 0$. Hence uC_φ is compact. This completes the proof of Theorem 3.2.

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Corollary 3.3 *Let φ be an analytic self-map of D. Then C_φ is a compact operator from the Bloch space β to the Zygmund space $\mathcal{Z}$ if and only if C_φ is bounded,*

$$\lim_{|\varphi(z)|\to 1^-}\frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2}=0 \quad \text{and} \quad \lim_{|\varphi(z)|\to 1^-}\frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2}=0.$$

Theorem 3.3 *Let u be an analytic function on the unit disc D, and φ an analytic self-map of D. Then $uC_\varphi:\beta_0\to\mathcal{Z}_0$ is a bounded operator if and only if $u\in\mathcal{Z}_0$, (3.1), (3.2), and (3.3) hold, and the following are satisfied:*

$$\lim_{|z|\to 1^-}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|=0. \tag{3.15}$$

$$\lim_{|z|\to 1^-}(1-|z|^2)|u(z)(\varphi'(z))^2|=0; \tag{3.16}$$

Proof Suppose that uC_φ is bounded from the little Bloch space β_0 to the little Zygmund type spaces $\mathcal{Z}_0$. Then $u=uC_\varphi 1\in\mathcal{Z}_0$. Also $u\varphi=uC_\varphi z\in\mathcal{Z}_0$, thus

$$(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)+\varphi(z)u''(z)|\longrightarrow 0 \quad (|z|\to 1^-).$$

Since $|\varphi|\le 1$ and $u\in\mathcal{Z}_0$, we have $\lim\limits_{|z|\to 1^-}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|=0$. Hence (3.15) holds.

Similarly, $uC_\varphi z^2\in\mathcal{Z}_0$, then

$$(1-|z|^2)|4\varphi(z)\varphi'(z)u'(z)+\varphi^2(z)u''(z)+2u(z)(\varphi(z)\varphi''(z)+(\varphi'(z))^2)|\longrightarrow 0 \quad (|z|\to 1^-).$$

By (3.15), $|\varphi|\le 1$ and $u\in\mathcal{Z}_0$, we get that $\lim\limits_{|z|\to 1^-}(1-|z|^2)|u(z)(\varphi'(z))^2|=0$, i. e. that (3.16) holds. On the other hand, from Lemma 2.3 and Theorem 3.1, we obtain that (3.1), (3.2), and (3.3) hold.

Conversely, for $\forall f\in\beta_0$, we have both $(1-|z|^2)^2|f''(z)|\to 0$ and $\dfrac{|f(z)|}{\ln\frac{e}{1-|z|^2}}\to 0$ as $|z|\to 1^-$ by Lemma 2.2. Given $\epsilon>0$ there is a $0<\delta<1$ such that $(1-|z|^2)|f'(z)|<\dfrac{\epsilon}{3M_2}$, $(1-|z|^2)^2|f''(z)|<\dfrac{\epsilon}{3M_3}$ and $\dfrac{|f(z)|}{\log\frac{e}{1-|z|^2}}<\dfrac{\epsilon}{3M_1}$ for all z with $\delta<|z|<1$, where $M_1,\ M_2,\ M_3$ are defined in above.

If $|\varphi(z)|>\delta$, it follows that

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|[2\varphi'(z)u'(z)+\varphi''(z)u(z)]f'(\varphi(z))\\
&+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))|\\
&\le (1-|z|^2)|[2\varphi'(z)u'(z)+\varphi''(z)u(z)]f'(\varphi(z))|\\
&+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)|+(1-|z|^2)|u''(z)f(\varphi(z))|\\
&\le M_2(1-|\varphi(z)|^2)|f'(\varphi(z))|+M_3(1-|\varphi(z)|^2)^2|f''(\varphi(z))|+M_1\frac{|f(\varphi(z))|}{\log\frac{e}{1-|\varphi(z)|^2}}\\
&< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.
\end{aligned}$$

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We know that there exists a constant M_4 such that $|f(z)| \le M_3$, $|f'(z)| \le M_4$ and $|f''(z)| \le M_4$ for all $|z| \le \delta$.

If $|\varphi(z)| \le \delta$, it follows that

$$\begin{aligned}(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|[2\varphi'(z)u'(z)+\varphi''(z)u(z)]f'(\varphi(z)) \\ &+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))| \\ &\le M_4(1-|z|^2)||2\varphi'(z)u'(z)+\varphi''(z)u(z)| \\ &+ M_4(1-|z|^2)|(\varphi'(z))^2u(z)|+M_4(1-|z|^2)|u''(z)|.\end{aligned}$$

Thus we conclude that $(1-|z|^2)|(uC_\varphi(f))''(z)| \to 0$ as $|z| \to 1^-$. Hence $uC_\varphi f \in \mathcal{Z}_0$ for all $f \in \beta_0$. On the other hand, uC_φ is a bounded operator from β to $\mathcal{Z}$ by Theorem 3.1. Hence uC_φ is a bounded operator from the little Bloch space β_0 to the little Zygmund space $\mathcal{Z}_0$.

Corollary 3.4 *Let φ be an analytic self-map of D. Then C_φ is a bounded operator from β_0 to $\mathcal{Z}_0$ if and only if C_φ is a bounded operator from β to $\mathcal{Z}$ and $\varphi \in \mathcal{Z}_0$.*

Proof By Theorem 3.3 we have that C_φ is a bounded operator from β_0 to $\mathcal{Z}_0$ if and only $C_\varphi : \beta \to \mathcal{Z}$ is bounded, $\varphi \in \mathcal{Z}_0$, and

$$\lim_{|z|\to 1^-}(1-|z|^2)|(\varphi'(z))^2| = 0.$$

However, That $\varphi \in \mathcal{Z}_0$ means $\varphi' \in \beta_0$. Then we have that $|\varphi'(z)| \le \log\frac{e}{1-|z|^2}\|\varphi'\|_\beta$ by Lemma 2.1. It follows that

$$(1-|z|^2)|(\varphi'(z))^2| \le (1-|z|^2)\log^2\frac{e}{1-|z|^2}\|\varphi'\|_\beta^2 \to 0,$$

as $|z| \to 1^-$.

Theorem 3.4 *Let u be an analytic function on the unit disc D, and φ an analytic self-map of D. Then uC_φ is compact from β_0 to $\mathcal{Z}_0$ if and only if the following are satisfied:*

$$\lim_{|z|\to 1^-}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2} = 0; \tag{3.17}$$

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2} = 0; \tag{3.18}$$

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} = 0. \tag{3.19}$$

Proof Assume (3.17), (3.18), and (3.19) hold. From Theorem 3.3, we know that uC_φ is bounded from β_0 to $\mathcal{Z}_0$. Suppose that $f \in \beta_0$ with $\|f\|_\beta \le 1$. We obtain that

$$\begin{aligned}&(1-|z|^2)|(uC_\varphi f)''(z)| \le (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))| \\ &+(1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)|+(1-|z|^2)|u''(z)f(\varphi(z))| \\ &\le (1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|\frac{1}{1-|\varphi(z)|^2}b(f) \\ &+8\frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^2}b(f)+(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2}\|f\|_\beta,\end{aligned}$$

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thus

$$\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f\in\beta_0, \|f\|_\beta\le 1\}$$

$$\le (1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|\frac{1}{1-|\varphi(z)|^2}$$

$$+\frac{8(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^2}+(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2},$$

and it follows that

$$\lim_{|z|\to 1^-}\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f\in\beta_0, \|f\|_\beta\le 1\}=0,$$

hence $uC_\varphi:\beta_0\to\mathcal{Z}_0$ is compact by Lemma 2.5.

Conversely, suppose that $uC_\varphi:\beta_0\to\mathcal{Z}_0$ is compact.

First, it is obvious $uC_\varphi:\beta_0\to\mathcal{Z}_0$ is bounded, then by Theorem 3.3, we have $u\in\mathcal{Z}_0$ and that (3.15) and (3.16) hold. On the other hand, by Lemma 2.5 we have

$$\lim_{|z|\to 1^-}\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f\in\beta_0, \|f\|_\beta\le M\}=0,$$

for some $M>0$.

Next, noting that the proof of Theorem 3.1 and the fact that the functions given in (3.8) are in β_0 and have norms bounded independently of a, we obtain that

$$\lim_{|z|\to 1^-}(1-|z|^2)|u''(z)|\log\frac{e}{1-|\varphi(z)|^2}=0.$$

Similarly, noting that the functions given in (3.9) are in β_0 and have norms bounded independently of a, we obtain that

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2}=0 \tag{3.20}$$

for $|\varphi(z)|>\frac{1}{2}$. However, if $|\varphi(z)|\le\frac{1}{2}$, by (3.15), we easily have

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{1-|\varphi(z)|^2}$$

$$\le\frac{4}{3}\lim_{|z|\to 1^-}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|=0.$$

Thus (3.18) holds.

Also, the third statement, that (3.19), is proved similarly. We omitted it here. This completes the proof of Theorem 4.2.

Corollary 3.5 *Let φ be an analytic self-map of D. Then C_φ is a compact operator from β_0 to $\mathcal{Z}_0$ if and only if*

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2}=0$$

and

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2}=0.$$

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Approximate homomorphisms and derivations on non-Archimedean Lie JC^*-algebras

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Abstract. In this paper, by using the fixed point method, we prove the Hyers-Ulam stability of homomorphisms in non-Archimedean Lie JC^*-algebras and derivations on non-Archimedean Lie JC^*-algebras associated with the following additive mapping:

$$\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\cdots,i_{n-k+1}}^{n}x_i-\sum_{r=1}^{n-k+1}x_{i_r}\Big)$$
$$+f\Big(\sum_{i=1}^{n}x_i\Big)=2^{n-1}f(x_1)$$

for a fixed positive integer n with $n\geqslant 2$.

1. Introduction

In 1896, Hensel [4] introduced a field with a valuation in which does not have the Archimedean property. Let $\mathcal{K}$ be a field. A non-Archimedean absolute value on$\mathcal{K}$ is a function $|\cdot|:\mathcal{K}\to[0,+\infty)$ such that, for any $a,b\in\mathcal{K}$, the following conditions are satisfying

(i) $|a|\geqslant 0$ and equality holds if and only if $a=0$,

(ii) $|ab|=|a||b|$,

(iii) $|a+b|\leqslant\max\{|a|,|b|\}$ (the strict triangle inequality).

Note that $|1|=|-1|=1$ and $|n|\leqslant 1$ for each integer n. We always assume, in addition, that $|\cdot|$ is non-trival, i.e., there exists an $a_0\neq 0,1$. A function $\|.\|:X\to[0,\infty)$ is called a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii) for any $r\in K,x\in X,\|rx\|=|r|\|x\|$;

(iii) the strong triangle inequality holds, namely,

$$\|x+y\|\leqslant\max\{\|x\|,\|y\|\}\quad(x,y\in X).$$

Then $(X,\|\cdot\|)$ is called a non-Archimedean normed space. From the fact that

$$\|x_n-x_m\|\leqslant\max\{\|n_n-x_m\|:m\leqslant j\leqslant n-1\}\quad(n>m),$$

[0]**2010 Mathematics Subject Classification**: 39B52, 39B72, 46L05, 47H10, 46B03.

[0]**Keywords**: Hyers-Ulam stability; additive functional equation; fixed point; non-Archimedean space; homomorphisms in a non-Archimedean Lie JC^*-algebras; derivations in a non-Archimedean Lie JC^*-algebras.

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holds, a sequence $\{x_n\}$ is Cauchy if and only if $\{x_n - x_m\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

A non-Archimedean Banach algebra is a complete non-Archimedean algebra $\mathcal{A}$ which satisfies $\|ab\| \leqslant \|a\| \cdot \|b\|$ for all $a, b \in \mathcal{A}$. For more detailed definitions of non-Archimedean Banach algebras, we refer the reader to [15].

If $\mathcal{U}$ is a non-Archimedean Banach algebra, then an involution on $\mathcal{U}$ is mapping $t \to t^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

$(i)\, t^{**} = t$ for $t \in \mathcal{U}$;

$(ii)\, (\alpha s + \beta t)^* = \bar{\alpha} s^* + \bar{\beta} t^*$;

$(iii)\, (st)^* = t^* s^*$ for all $s, t \in \mathcal{U}$.

If, in addition, $\|t^* t\| = \|t\|^2$ for $t \in \mathcal{U}$, then $\mathcal{U}$ is a non-Archimedean C^*-algebra.

The stability problem of functional equations originated from a question of Ulam [16] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond)$ be a metric group (a metric is defined on a set with group property) with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $h(x * y) = h(x) * h(y)$ is stable (see also [3, 5, 9, 10, 12, 13, 14]).

For explicitly later use, we recall a fundamental result in fixed point theory.

Theorem 1.1. *(The fixed point alternative theorem [2]) Let (Ω, d) be a complete generalized metric space and $J : \Omega \to \Omega$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$, that is,*

$$d(Jx, Jy) \leqslant L d(x, y), \quad x, y \in \Omega.$$

Then, for each given $x \in \Omega$, either

$$d(J^n x, J^{n+1} x) = \infty, \quad \forall n \geqslant 0,$$

or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geqslant n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $\Delta = \{y \in \Omega : d(J^{n_0} x, y) < \infty\}$;*

(4) $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ *for all $y \in \Delta$.*

A non-Archimedean C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy - yx}{2}$ and endowed with *anticommutator product* (Jordan product) $x \circ y := \frac{xy + yx}{2}$ on $\mathcal{C}$, is called a non-Archimedean Lie JC^*-algebra (see [6, 7, 8]).

Jordan algebras as coordinates for Lie algebras were created to illuminate a particular aspect of physics, quantum-mechanical observables, but turned out to have illuminating connections with many areas of mathematics.

In this paper, we prove the Hyers-Ulam stability of homomorphisms and derivations in non-Archimedean Lie JC^*-algebras associated with the following additive functional equation:

$$\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k+1}}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\cdots,i_{n-k+1}}^{n}x_i-\sum_{r=1}^{n-k+1}x_{i_r}\Big)$$
$$+f\Big(\sum_{i=1}^{n}x_i\Big)=2^{n-1}f(x_1) \tag{1.1}$$

for a fixed positive integer n with $n\geqslant 2$.

2. Stability of homomorphisms in non-Archimedean Lie JC^*-algebras

Definition 2.1. [7] *Let $\mathcal{A}$ and $\mathcal{B}$ be non-Archimedean Lie JC^*-algebras. A $\mathbb{C}$-linear mapping $H:\mathcal{A}\to\mathcal{B}$ is called a (non-Archimedean Lie JC^*-algebra) homomorphism if H satisfies*

$$H([x,y])=[H(x),H(y)],$$
$$H(x\circ y)=H(x)\circ h(y),$$
$$H(x^*)=H(x)^*$$

for all $x,y\in\mathcal{A}$.

Throughout this section, assume that A and B are two non-Archimedean Lie JC^*-algebras, respectively, with norm $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$.

For a given mapping $f:\mathcal{A}\to\mathcal{B}$, we define

$$D_\mu f(x_1,\cdots,x_n):=\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k+1}}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\cdots,i_{n-k+1}}^{n}\mu x_i-\sum_{r=1}^{n-k+1}\mu x_{i_r}\Big)$$
$$+f\Big(\sum_{i=1}^{n}\mu x_i\Big)-2^{n-1}f(\mu x_1)$$

for all $\mu\in\mathbb{T}^1:=\{\lambda\in\mathbb{C}:|\lambda|=1\}$ and all $x_1,\cdots,x_n\in\mathcal{A}$.

We recall the following needed lemmas in this paper.

Lemma 2.2. [11] *Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces and $f:\mathcal{V}\to\mathcal{W}$ be an additive mapping such that $f(\mu x)=\mu f(x)$ for all $x\in\mathcal{V}$ and $\mu\in\mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.3. [7] *A mapping $f:\to\mathcal{A}\to\mathcal{B}$ with $f(0)=0$ satisfies the functional equation (1.1) if and only if $f:\mathcal{A}\to\mathcal{B}$ is additive.*

We prove the Hyers-Ulam stability of homomorphisms in non-Archimedean Lie JC^*-algebras for the functional equation $D_\mu f(x_1,\cdots,x_n)=0$.

Theorem 2.4. *Let $f:\mathcal{A}\to\mathcal{B}$ be a mapping for which there are functions $\varphi:\mathcal{A}^n\to[0,\infty),\psi:\mathcal{A}^2\to[0,\infty)$, and $\eta:\mathcal{A}\to[0,\infty)$ such that $|2|<1$ is far from zero and*

$$\lim_{m\to\infty}\frac{1}{|2|^m}\varphi(2^m x_1,2^m x_2,\cdots,2^m x_n)=0, \tag{2.1}$$

$$\lim_{m\to\infty}\frac{1}{|2|^{2m}}\psi(2^m x,2^m y)=0, \tag{2.2}$$

$$\lim_{m\to\infty} \frac{1}{|2|^m}\eta(2^m x) = 0, \tag{2.3}$$

$$\|D_\mu f(x_1, \cdots, x_n)\|_{\mathcal{B}} \leqslant \varphi(x_1, \cdots, x_n), \tag{2.4}$$

$$\|f([x, y]) - [f(x), f(y)]\|_{\mathcal{B}} \leqslant \psi(x, y), \tag{2.5}$$

$$\|f(x \circ y) - f(x) \circ f(y)\|_{\mathcal{B}} \leqslant \psi(x, y), \tag{2.6}$$

$$\|f(x^*) - f(x)^*\|_{\mathcal{B}} \leqslant \eta(x), \tag{2.7}$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{A}$ *and* $\mu \in \mathbb{T}^1$. *If there exists a constant* $0 < L < 1$ *such that* $\varphi(x_1, x_2, \cdots, x_n) \leqslant \alpha L\varphi(\frac{x_1}{2}, \frac{x_2}{2}, \cdots, \frac{x_n}{2})$ *for all* $x_1, x_2, \cdots, x_n \in \mathcal{A}$, *where* $\alpha = |2|^{n-1}$, *then there exists a unique homomorphism* $H : \mathcal{A} \to \mathcal{B}$ *such that*

$$\|f(x) - H(x)\| \leqslant \frac{L}{1-L}\varphi(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0) \tag{2.8}$$

for all $x \in \mathcal{A}$.

Proof. Let $\mu = 1$. Using the following relation

$$1 + \sum_{k=1}^{n-k} \binom{n-k}{k} = \sum_{k=0}^{n-k} \binom{n-k}{k} = 2^{n-k} \tag{2.9}$$

for all $n > k$ and putting $x_1 = x_2 = x$ and $x_3 = x_4 = \cdots = x_n = 0$ in (2.4), we obtain

$$\|\frac{\alpha}{2} f(2x) - \alpha f(x)\|_{\mathcal{B}} \leqslant \varphi(x, x, 0, \cdots, 0)$$

for all $x \in \mathcal{A}$. So

$$\|\frac{1}{2} f(2x) - f(x)\|_{\mathcal{B}} \leqslant \frac{1}{\alpha}\varphi(x, x, 0, \cdots, 0) \leqslant L\varphi\Big(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\Big) \tag{2.10}$$

for all $x \in \mathcal{A}$. Let define $\Omega := \{g : \mathcal{A} \to \mathcal{B}\}$ and introduce a generalized metric on Ω as follows

$$d(g, h) = \inf\{k \in (0, \infty) : \|g(x) - h(x)\|_{\mathcal{B}} < k\varphi\Big(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\Big), \forall x \in \mathcal{A}\}.$$

It is easy to show that (Ω, d) is a generalized complete metric space (see [1]).

Now we consider the function $J : \Omega \to \Omega$ define by $Jg(x) = \frac{1}{|2|}g(2x)$ for all $x \in \mathcal{A}$ and $g \in \Omega$. Let for all $g, h \in \Omega$ and an arbitrary constant $k \in [0, \infty)$ with $d(x, y) \leqslant k$, we have

$$\|g(x) - h(x)\|_{\mathcal{B}} \leqslant k\varphi\Big(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\Big)$$

for all $x \in \mathcal{A}$. Then we can write

$$\|Jg(x) - Jh(x)\|_{\mathcal{B}} = \frac{1}{|2|}\|g(2x) - h(2x)\|_{\mathcal{B}} \leqslant \frac{k}{|2|}\varphi(x, x, 0, \cdots, 0) \leqslant \frac{\alpha k L}{|2|}\varphi\Big(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\Big)$$

for all $x \in \mathcal{A}$. So we conclude that $d(Jg, Jh) \leqslant \frac{\alpha}{|2|}Ld(g, h)$ for all $g, h \in \Omega$. It follows from (2.9) that $d(Jf, f) \leqslant L$, that is, J is a self-function of Ω with the Lipchitz constant L. Therefore, from Theorem 1.1, there eists a fixed point H of J set $\Omega_1 = \{h \in X : d(f, h) < \infty\}$ such that

$$H(x) = \lim_{m\to\infty} \frac{1}{|2|^m} f(2^m x) \tag{2.11}$$

for all $x \in \mathcal{A}$, since $\lim_{m\to\infty} d(J^n f, H) = 0$. Also $2H(\frac{x}{2}) = H(x)$ for all $x \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is the unique fixed point of J in Ω_1 such that

$$d(H, f) \leqslant \frac{1}{1-L} d(Jf, f) \leqslant \frac{L}{1-L},$$

i.e., H satisfies (2.8) for all $x \in \mathcal{A}$. It follows from the definition of H, (2.1) and (2.4) that

$$\sum_{k=2}^{n} \Big(\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k+1}}^{n} \Big) H \Big(\sum_{i=1, i\neq i_1, \cdots, i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r} \Big) \\ + H\Big(\sum_{i=1}^{n} x_i\Big) = 2^{n-1} H(x_1)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$. Since $H(0) = 0$, by Lemma 2.3, the mapping H is additive.

Put $x_1 = x$ and $x_2 = x_3 = \cdots = 0$ in (2.4). It follows from (2.9) that

$$\|f(\mu x) - \mu f(x)\| \leqslant \frac{1}{\alpha} \varphi(x, 0, \cdots, 0) \tag{2.12}$$

for all $x \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Also we conclude

$$\|\frac{1}{2^m}(f(\mu 2^m x) - \mu f(2^m x))\|_{\mathcal{B}} \leqslant \frac{1}{\alpha |2|^m} \varphi(2^m x, 0, \cdots, 0)$$

for all $x \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. The right hand side of the above inequality tends to zero as $m \to \infty$, and so we obtain

$$H(\mu x) = \lim_{m\to\infty} \frac{1}{|2|^m} f(\mu 2^m x) = \lim_{m\to\infty} \frac{1}{|2|^m} \mu f(2^m x) = \mu H(x)$$

for all $x \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Hence by Lemma 2.2, the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

It follows from (2.2), (2.5), (2.6) and (2.11) that

$$\begin{aligned} \|H([x, y]) - [H(x), H(y)]\|_{\mathcal{B}} &= \lim_{m\to\infty} \frac{1}{|2|^{2m}} \|f([2^m x, 2^m y]) - [f(2^m x), f(2^m y)]\|_{\mathcal{B}} \\ &\leqslant \lim_{m\to\infty} \frac{1}{|2|^{2m}} \psi(2^m x, 2^m y) = 0 \end{aligned}$$

and

$$\begin{aligned} \|H(x \circ y) - H(x) \circ H(y)\|_{\mathcal{B}} &= \lim_{m\to\infty} \frac{1}{|2|^{2m}} \|f(2^m x \circ 2^m y) - f(2^m x) \circ f(2^m y)\|_{\mathcal{B}} \\ &\leqslant \lim_{m\to\infty} \frac{1}{|2|^{2m}} \psi(2^m x, 2^m y) = 0 \end{aligned}$$

for all $x, y \in \mathcal{A}$. So

$$H([x, y]) = [H(x), H(y) \quad \text{and} \quad H(x \circ y) = H(x) \circ H(y)$$

for all $x, y \in \mathcal{A}$.

Similarly, by (2.3), (2.7) and (2.11), we have

$$\|H(x^*) - H(x)^*\|_{\mathcal{B}} = \lim_{m\to\infty} \frac{1}{|2|^m} \|f(2^m x^*) - f(2^m x)^*\|_{\mathcal{B}} \leqslant \lim_{m\to\infty} \frac{1}{|2|^m} \eta(2^m x) = 0$$

and so $H(x^*) = H(x)^*$ for all $x, y \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is the desired homomorphism satisfying (2.8). □

Corollary 2.5. *Let $r > 1$ and θ be nonnegative real number, and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\begin{aligned}
\|D_\mu f(x_1, x_2, \cdots, x_n)\|_{\mathcal{B}} &\leqslant \theta(\||x_1\|^r_{\mathcal{A}} + \||x_2\|^r_{\mathcal{A}} + \cdots + \||x_n\|^r_{\mathcal{A}}), \\
\|f([x, y]) - [f(x), f(y)]\|_{\mathcal{B}} &\leqslant \theta \cdot \||x\|^r_{\mathcal{A}} \cdot \||y\|^r_{\mathcal{A}}, \\
\|f(x \circ y) - f(x) \circ f(y)\|_{\mathcal{B}} &\leqslant \theta \cdot \||x\|^r_{\mathcal{A}} \cdot \||y\|^r_{\mathcal{A}}, \\
\|f(x^*) - f(x)^*\|_{\mathcal{B}} &\leqslant \theta \cdot \||x\|^r_{\mathcal{A}},
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and $x, y, x_1, \cdots, x_n \in \mathcal{A}$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - H(x)\|_{\mathcal{B}} \leqslant \frac{|2|\theta}{|2| - |2|^r} \|x\|^r_{\mathcal{A}}$$

for all $x \in \mathcal{A}$.

Proof. The proof follows from Theorem 2.4 by taking

$$\begin{aligned}
\varphi(x_1, x_2, \cdots, x_n) &:= \theta(\||x_1\|^r_{\mathcal{A}} + \||x_2\|^r_{\mathcal{A}} + \cdots + \||x_n\|^r_{\mathcal{A}}), \\
\psi(x, y) &:= \theta \cdot \||x\|^r_{\mathcal{A}} \cdot \||y\|^r_{\mathcal{A}}, \\
\eta(x) &:= \theta \cdot \||x\|^r_{\mathcal{A}}
\end{aligned}$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{A}$ and $L = |2|^{r-1}$. □

3. Stability of derivations on non-Archimedean Lie JC^*-algebras

Definition 3.1. [7] *Let $\mathcal{A}$ be a non-Archimedean Lie JC^*-algebra. A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a (non-Archimedean Lie JC^*-algebra) derivation if δ satisfies*

$$\begin{aligned}
\delta([x, y]) &= [\delta(x), y] + [x, \delta(y)], \\
\delta(x \circ y) &= \delta(x) \circ y + x \circ \delta(y), \\
\delta(x^*) &= \delta(x)^*
\end{aligned}$$

for all $x \in \mathcal{A}$.

Throughout this section, assume that $\mathcal{A}$ is a non-Archimedean Lie JC^*-algebra with norm $\| \cdot \|_{\mathcal{A}}$.

We prove the Hyers-Ulam stability of derivation on non-Archimedean Lie JC^*-algebras for the functional equation $D_\mu f(x_1, \cdots, x_n) = 0$.

Theorem 3.2. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are function $\varphi : \mathcal{A}^n \to [0, \infty)$, $\psi : \mathcal{A}^2 \to [0, \infty)$ and $\eta : \mathcal{A} \to [0, \infty)$ such that (2.1), (2.2), (2.3). (2.4) and (2.7) hold and*

$$\|f([x, y]) - [f(x), y]0[x, f(y)]\|_{\mathcal{A}} \leqslant \psi(x, y), \tag{3.1}$$

$$\|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_{\mathcal{A}} \leqslant \psi(x, y) \tag{3.2}$$

for all $x, y \in \mathcal{A}$. If there exists a constant $0 < L < 1$ such that $\varphi(x_1, x_2, \cdots, x_n) \leqslant \alpha L \varphi(\frac{x_1}{2}, \frac{x_2}{2}, \cdots, \frac{x_n}{2})$ for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$, where $\alpha = |2|^{n-1}$, then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \delta(x)\| \leqslant \frac{L}{1-L} \varphi\Big(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\Big) \tag{3.3}$$

for all $x \in \mathcal{A}$.

Proof. By the same reasoning as in the proof of Theorem 2.4, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying in the desired inequality (3.3) and the mapping $\delta : \mathcal{A} \to \mathcal{A}$ is defined by

$$\delta(x) = \lim_{m\to\infty} \frac{1}{|2|^m} f(2^m x) \tag{3.4}$$

for all $x \in \mathcal{A}$.

It follows from (2.2), (3.1), (3.3) and (3.4) that

$$\begin{aligned} &\|\delta([x, y]) - [\delta(x), y] - [x, \delta(y)]\|_{\mathcal{A}} \\ &\quad = \lim_{m\to\infty} \frac{1}{|2|^{2m}} \|f([2^m x, 2^m y]) - [f(2^m x), 2^m y] - [2^m x, f(2^m y)]\|_{\mathcal{A}} \\ &\quad \leqslant \lim_{m\to\infty} \frac{1}{|2|^{2m}} \psi(2^m x, 2^m y) = 0 \end{aligned}$$

and

$$\begin{aligned} &\|\delta(x \circ y) - \delta(x) \circ y - x \circ \delta(y)\|_{\mathcal{A}} \\ &\quad = \lim_{m\to\infty} \frac{1}{|2|^{2m}} \|f(2^m x \circ 2^m y) - f(2^m x) \circ 2^m y - 2^m x \circ f(2^m y)\|_{\mathcal{A}} \\ &\quad \leqslant \lim_{m\to\infty} \frac{1}{|2|^{2m}} \psi(2^m x, 2^m y) = 0 \end{aligned}$$

for all $x, y \in \mathcal{A}$. So

$$\begin{aligned} \delta([x, y]) &= [\delta(x), y] + [x, \delta(y)], \\ \delta(x \circ y) &= \delta(x) \circ y + x \circ \delta(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$.

Similarly, as in the proof of Theorem 2.4, one can show $\delta(x^*) = \delta(x)^*$ for all $x \in \mathcal{A}$. Therefore, $\delta : \mathcal{A} \to \mathcal{A}$ is a non-Archimedean Lie JC^*-algebra derivation satisfying (3.4). □

Corollary 3.3. *Let $r > 1$ and θ be nonnegative and real number, and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping such that*

$$\begin{aligned} \|D_\mu f(x_1, x_2, \cdots, x_n)\|_{\mathcal{B}} &\leqslant \theta(\|x_1\|_{\mathcal{A}}^r + \|x_2\|_{\mathcal{A}}^r + \cdots, \|x_n\|_{\mathcal{A}}^r), \\ \|f([x, y]) - [f(x), y] - [x, f(y)]\|_{\mathcal{B}} &\leqslant \theta \cdot \|x\|_{\mathcal{A}}^r \cdot \|y\|_{\mathcal{A}}^r, \\ \|f(x \circ y) - f(x) \circ y - x \circ f(y)\|_{\mathcal{B}} &\leqslant \theta \cdot \|x\|_{\mathcal{A}}^r \cdot \|y\|_{\mathcal{A}}^r, \\ \|f(x^*) - f(x)^*\|_{\mathcal{B}} &\leqslant \theta \cdot \|x\|_{\mathcal{A}}^r \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ *and* $x, y, x_1, \cdots, x_n \in \mathcal{A}$. *Then there exists a unique homomorphism* $H : \mathcal{A} \to \mathcal{A}$ *such that*

$$\|f(x) - \delta(x)\|_{\mathcal{B}} \leqslant \frac{|2|\theta}{|2| - |2|^r} \|x\|_{\mathcal{A}}^r$$

for all $x \in \mathcal{A}$.

Proof. The proof follows from Theorem 3.2 by taking

$$\begin{aligned} \varphi(x_1, x_2, \cdots, x_n) &:= \theta.(\|x_1\|_{\mathcal{A}}^r + \|x_2\|_{\mathcal{A}}^r + \cdots + \|x_n\|_{\mathcal{A}}^r), \\ \psi(x, y) &:= \theta.(\|x\|_{\mathcal{A}}^r.\|y\|_{\mathcal{A}}^r), \\ \eta(x) &:= \theta.\|x\|_{\mathcal{A}}^r \end{aligned}$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{A}$ and $L = |2|^{r-1}$. □

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ON DISTRIBUTION AND PROBABILITY DENSITY FUNCTIONS OF ORDER STATISTICS ARISING FROM INDEPENDENT BUT NOT NECESSARILY IDENTICALLY DISTRIBUTED RANDOM VECTORS

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ABSTRACT

In this study, joint probability density and distribution functions of any d order statistics of *innid* continuous random vectors are expressed. Then, some results connecting distributions of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors are given.

Keywords: Order Statistics, Distribution Function, Probability Density Function, Continuous Random Variable.

MSC 2010: 62G30, 62E15.

1. Introduction

Several identities and recurrence relations for probability density function (*pdf*) and distribution function (*df*) of order statistics of independent and identically distributed (*iid*) random variables were established by numerous authors including (Arnold et al., 1992; Balasubramanian, Beg, 2003; David, 1981; Reiss, 1989). Furthermore, (Arnold et al., 1992; David, 1981; Gan, Bain, 1995; Khatri, 1962) obtained the probability function (*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. (Corley, 1984) defined a multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution. (Goldie, Maller, 1999) derived expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df*. (Guilbaud, 1982) expressed the probability of the functions of independent but not necessarily identically distributed (*innid*) random vectors as a linear combination of probabilities of the functions of *iid* random vectors and thus also for order statistics of random variables.

(Cao, West, 1997) obtained recurrence relationships among the distribution functions of order statistics arising from *innid* random variables. (Vaughan, Venables, 1972) derived the joint *pdf* and marginal *pdf* of order statistics of *innid* random variables by means of permanents. (Balakrishnan, 2007; Bapat, Beg, 1989) obtained the joint *pdf* and *df* of order statistics of *innid* random variables by means of permanents. (Childs, Balakrishnan, 2006) obtained, using multinomial arguments, the *pdf* of $X_{r:n+1}$ $(1 \le r \le n+1)$ by adding another independent random variable to the original n variables $X_1, X_2, ..., X_n$. Also,

(Balasubramanian et al.,1994) established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators.

In this paper, joint *df* and *pdf* of order statistics from *innid* continuous random vectors are obtained.

As far as we know, these approaches have not been considered in the framework of order statistics from *innid* continuous random vectors.

From now on, subscripts and superscripts are defined in first place in which they are used and these definitions will be valid unless they are redefined.

Consider $\mathrm{x}=(x^{(1)},x^{(2)},...,x^{(b)})$ and $\mathrm{y}=(y^{(1)},y^{(2)},...,y^{(b)})$, then it can be written as;

$\mathrm{x}\le\mathrm{y}$ if $x^{(v)}\le y^{(v)}$ (v=1, 2, …, b) and $\mathrm{x}+\mathrm{y}=(x^{(1)}+y^{(1)},x^{(2)}+y^{(2)},...,x^{(b)}+y^{(b)})$.

Let $\xi_i=(\xi_i^{(1)},\xi_i^{(2)},...,\xi_i^{(b)})$ (i=1,2,…,n) be n *innid* continuous random vectors which components of ξ_i are independent.

$$X_{r:n}^{(v)}=Z_{r:n}(\ \xi_1^{(v)},\xi_2^{(v)},...,\xi_n^{(v)}) \tag{1.1}$$

is stated as rth order statistic of vth components of $\xi_1,\xi_2,\ \dots,\ \xi_n$.

From (1.1), ordered values of vth components of $\xi_1,\xi_2,\ \dots,\ \xi_n$ are expressed as

$$X_{1:n}^{(v)}\le X_{2:n}^{(v)}\le...\le X_{n:n}^{(v)}. \tag{1.2}$$

From (1.2), we can write $\mathrm{X}_{r:n}=(X_{r:n}^{(1)},X_{r:n}^{(2)},...,X_{r:n}^{(b)})\quad(1\le r\le n)$.

Also, $\mathrm{x}_w=(x_w^{(1)},x_w^{(2)},\dots,x_w^{(b)})$, $x_w^{(v)}\in R$ ($w=1,2,...,\mathrm{d};\quad d=1,2,...,\mathrm{n}$).

Let F_i and f_i be *df* and *pdf* of $\xi_i^{(v)}$, respectively.

Moreover, $X_{1:n}^{(v),s},X_{2:n}^{(v),s},...,X_{n:n}^{(v),s}$ are order statistics of *iid* continuous random variables with *df* F^s and *pdf* f^s, respectively, defined by

$$F^s=\frac{1}{n_s}\sum_{i\in s}F_i \tag{1.3}$$

and

$$f^s=\frac{1}{n_s}\sum_{i\in s}f_i\ . \tag{1.4}$$

Here, s is a subset of integers {1, 2,…, n} with $n_s\ge1$ elements.

In follows, *df* and *pdf* of $\mathrm{X}_{r_1:n}, \mathrm{X}_{r_2:n}, \ldots, \mathrm{X}_{r_d:n}$ ($1 \le r_1 < r_2 < \ldots < r_d \le n$) are given. Let $\mathrm{X}^{(v)} = (X^{(v)}_{r_1:n}, X^{(v)}_{r_2:n}, \ldots, X^{(v)}_{r_d:n})$ and $\mathrm{x}^{(v)} = (x_1^{(v)}, x_2^{(v)}, \ldots, x_d^{(v)})$. For notational convenience we write $\sum\sum$ and $\sum\limits_{m_d,\ldots,m_2,m_1}^{n,\ldots,m_3,m_2}$ instead of $\sum\limits_{\kappa=1}^{n}(-1)^{n-\kappa}\dfrac{\kappa^n}{n!}\sum\limits_{n_s=\kappa}$ and $\sum\limits_{m_d=r_d}^{n}\cdots\sum\limits_{m_2=r_2}^{m_3}\sum\limits_{m_1=r_1}^{m_2}$ in the expressions below, respectively.

2. Distribution function of order statistics from *innid* random vectors

In this section, *df* of $\mathrm{X}_{r_1:n}, \mathrm{X}_{r_2:n}, \ldots, \mathrm{X}_{r_d:n}$ and its results are given. The results connect *df* of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors using (1.3).

Now, we give the following theorem for establish joint *df* of *d* order statistics of *innid* continuous random vectors.

Theorem 2.1.

$$F_{r_1,r_2,\ldots,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_d) = \prod_{v=1}^{b}\{\sum_{m_d,\ldots,m_2,m_1}^{n,\ldots,m_3,m_2} C\sum_{P}\prod_{w=1}^{d+1}\prod_{l=m_{w-1}+1}^{m_w}[F_{j_l}(x_w^{(v)}) - F_{j_l}(x_{w-1}^{(v)})]\}, \tag{2.1}$$

$\mathrm{x}_1 < \mathrm{x}_2 < \ldots < \mathrm{x}_d$, where $C = [\prod\limits_{w=1}^{d+1}(m_w - m_{w-1})!]^{-1}$, $m_0 = 0$, $m_{d+1} = n$, $\sum\limits_{P}$ denotes sum over all n! permutations $(j_1, j_2, \ldots, j_n)$ of (1,2,…,n), $F_{j_l}(x_0^{(v)}) = 0$ and $F_{j_l}(x_{d+1}^{(v)}) = 1$.

Proof. It can be written

$$\begin{aligned} F_{r_1,r_2,\ldots,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_d) &= P\{\mathrm{X}_{r_1:n} \le \mathrm{x}_1, \mathrm{X}_{r_2:n} \le \mathrm{x}_2, \ldots, \mathrm{X}_{r_d:n} \le \mathrm{x}_d\} \\ &= P\{\mathrm{X}^{(1)} \le \mathrm{x}^{(1)}, \mathrm{X}^{(2)} \le \mathrm{x}^{(2)}, \ldots, \mathrm{X}^{(b)} \le \mathrm{x}^{(b)}\} \\ &= \prod_{v=1}^{b} P\{\mathrm{X}^{(v)} \le \mathrm{x}^{(v)}\} \\ &= \prod_{v=1}^{b} P\{X^{(v)}_{r_1:n} \le x_1^{(v)}, X^{(v)}_{r_2:n} \le x_2^{(v)}, \ldots, X^{(v)}_{r_d:n} \le x_d^{(v)}\}\ . \end{aligned} \tag{2.2}$$

(2.2) can be expressed as

$$F_{r_1,r_2,\ldots,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_d) =$$
$$\prod_{v=1}^{b}\{\sum_{m_d,\ldots,m_2,m_1}^{n,\ldots,m_3,m_2} C\sum_{P}\left(\prod_{l=1}^{m_1}F_{j_l}(x_1^{(v)})\right)\left(\prod_{l=m_1+1}^{m_2}[F_{j_l}(x_2^{(v)}) - F_{j_l}(x_1^{(v)})]\right)\cdots\prod_{l=m_d+1}^{n}[1 - F_{j_l}(x_d^{(v)})]\}.$$

Thus, (2.1) is obtained.

The approach in Theorem 2.1 can also be adapted to Theorem 2.2 for *iid* case.

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Theorem 2.2.

$$F_{r_1, r_2, ..., r_d : n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) = \prod_{v=1}^{b} \{\sum\sum \sum_{m_d, ..., m_2, m_1}^{n, ..., m_3, m_2} n! C \prod_{w=1}^{d+1} [F^s(x_w^{(v)}) - F^s(x_{w-1}^{(v)})]^{m_w - m_{w-1}}\}. \tag{2.3}$$

Proof. (2.2) can be expressed as

$$F_{r_1, r_2, ..., r_d : n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) = \prod_{v=1}^{b} [\sum\sum \ P\{X_{r_1:n}^{(v),s} \le x_1^{(v)}, X_{r_2:n}^{(v),s} \le x_2^{(v)}, ..., X_{r_d:n}^{(v),s} \le x_d^{(v)}\}]. \tag{2.4}$$

(2.3) is obtained from (2.1) and (2.4).

We now obtain the following three results for ***df*** of order statistics of ***innid*** continuous random vectors from the above theorems.

Result 2.1.

$$F_{r_1:n}(x_1^{(1)}) = \sum_{m_1=r_1}^{n} \frac{1}{m_1!(n-m_1)!} \sum_P \left(\prod_{l=1}^{m_1} \left(F_{j_l}(x_1^{(1)})\right) \right) \prod_{l=m_1+1}^{n} [1 - F_{j_l}(x_1^{(1)})]$$

$$= \sum\sum \sum_{m_1=r_1}^{n} \binom{n}{m_1} [F^s(x_1^{(1)})]^{m_1} [1 - F^s(x_1^{(1)})]^{n-m_1} . \tag{2.5}$$

Proof. In (2.1) and (2.3), if $b = 1$, $d = 1$, (2.5) is obtained.

In addition,

$$F_{r_1:n}(x_1^{(1)}) = \sum_{m_1=r_1}^{n} \frac{1}{m_1!(n-m_1)!} \sum_P \left(\prod_{l=1}^{m_1} F_{j_l}(x_1^{(1)}) \right) \prod_{l=m_1+1}^{n} [1 - F_{j_l}(x_1^{(1)})]$$

$$= \sum_{m_1=r_1}^{n} \frac{1}{m_1!(n-m_1)!} \sum_P \left(\prod_{l=1}^{m_1} F_{j_l}(x_1^{(1)}) \right) \sum_{t=m_1}^{n} (-1)^{n-t} \sum_{n_\tau = n-t} \prod_{l=1}^{n-t} F_{\tau_l}(x_1^{(1)}),$$

where $\sum_{n_\tau = n-t}$ denotes sum over all $\binom{n-m_1}{n-t}$ subsets $\tau = \{\tau_1, \tau_2, ..., \tau_{n-t}\}$ of $\{j_{m_1+1}, j_{m_1+2}, ..., j_n\}$.

Result 2.2.

$$F_{1:n}(x_1^{(1)}) = 1 - \frac{1}{n!} \sum_P \prod_{l=1}^{n} [1 - F_{j_l}(x_1^{(1)})]$$

$$= \sum\sum [1 - (1 - F^s(x_1^{(1)}))^n]. \tag{2.6}$$

Proof. In (2.5), if $r_1 = 1$, (2.6) is obtained.

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Result 2.3.

$$F_{n:n}(x_1^{(1)}) = \frac{1}{n!}\sum_P \prod_{l=1}^{n} F_{j_l}(x_1^{(1)})$$

$$= \sum\sum [F^s(x_1^{(1)})]^n . \tag{2.7}$$

Proof. In (2.5), if $r_1 = n$, (2.7) is obtained.

3. Probability density function of order statistics from *innid* random vectors

In this section, *pdf* of $\mathrm{X}_{r_1:n}, \mathrm{X}_{r_2:n}, ..., \mathrm{X}_{r_d:n}$ and its results are given. The results connect *pdf* of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors using (1.3) and (1.4).

Joint *pdf* of *d* order statistics of *innid* continuous random vectors is expressed in the following theorem.

Theorem 3.1.

$$f_{r_1,r_2,...,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) = \prod_{v=1}^{b} \{D \sum_P \left(\prod_{w=1}^{d+1} \prod_{l=r_{w-1}+1}^{r_w - 1} [F_{j_l}(x_w^{(v)}) - F_{j_l}(x_{w-1}^{(v)})] \right) \prod_{w=1}^{d} f_{j_{r_w}}(x_w^{(v)})\}, \tag{3.1}$$

$\mathrm{x}_1 < \mathrm{x}_2 < ... < \mathrm{x}_d$, where $D = [\prod_{w=1}^{d+1} (r_w - r_{w-1} - 1)!]^{-1}$, $r_0 = 0$ and $r_{d+1} = n+1$.

Proof. Let $\delta \mathrm{x}_w = (\delta x_w^{(1)}, \delta x_w^{(2)}, ..., \delta x_w^{(b)})$ and $\delta\, \mathrm{x}^{(v)} = (\delta x_1^{(v)}, \delta x_2^{(v)}, ..., \delta x_d^{(v)})$.

Consider

$\{\mathrm{x}_1 < \mathrm{X}_{r_1:n} \le \mathrm{x}_1 + \delta \mathrm{x}_1, \mathrm{x}_2 < \mathrm{X}_{r_2:n} \le \mathrm{x}_2 + \delta \mathrm{x}_2, ..., \mathrm{x}_d < \mathrm{X}_{r_d:n} \le \mathrm{x}_d + \delta \mathrm{x}_d\}$.

It can be written

$$P\{\mathrm{x}_1 < \mathrm{X}_{r_1:n} \le \mathrm{x}_1 + \delta\,\mathrm{x}_1, \mathrm{x}_2 < \mathrm{X}_{r_2:n} \le \mathrm{x}_2 + \delta\,\mathrm{x}_2, ..., \mathrm{x}_d < \mathrm{X}_{r_d:n} \le \mathrm{x}_d + \delta\,\mathrm{x}_d\}$$

$$= P\{\mathrm{x}^{(1)} < \mathrm{X}^{(1)} \le \mathrm{x}^{(1)} + \delta\,\mathrm{x}^{(1)}, \mathrm{x}^{(2)} < \mathrm{X}^{(2)} \le \mathrm{x}^{(2)} + \delta\,\mathrm{x}^{(2)}, ..., \mathrm{x}^{(b)} < \mathrm{X}^{(b)} \le \mathrm{x}^{(b)} + \delta\,\mathrm{x}^{(b)}\}$$

$$= \prod_{v=1}^{b} P\{\mathrm{x}^{(v)} < \mathrm{X}^{(v)} \le \mathrm{x}^{(v)} + \delta\,\mathrm{x}^{(v)}\}$$

$$= \prod_{v=1}^{b} P\{x_1^{(v)} < X_{r_1:n}^{(v)} \le x_1^{(v)} + \delta\, x_1^{(v)}, x_2^{(v)} < X_{r_2:n}^{(v)} \le x_2^{(v)} + \delta x_2^{(v)}, ..., x_d^{(v)} < X_{r_d:n}^{(v)} \le x_d^{(v)} + \delta x_d^{(v)}\}. \tag{3.2}$$

Dividing (3.2) by $\prod_{v=1}^{b}\prod_{w=1}^{d} \delta x_w^{(v)}$ and then letting $\delta x_1^{(v)}, \delta x_2^{(v)}, ..., \delta x_d^{(v)}$ tend to zero, we obtain

$$f_{r_1,r_2,...,r_d:n}(\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_d) = \prod_{v=1}^{b} \{D \sum_P F_{j_1}(x_1^{(v)}) ... F_{j_{r_1-1}}(x_1^{(v)}) f_{j_{r_1}}(x_1^{(v)}) [F_{j_{r_1+1}}(x_2^{(v)}) - F_{j_{r_1+1}}(x_1^{(v)})]$$

$$...[F_{j_{r_2}-1}(x_2^{(v)})-F_{j_{r_2}-1}(x_1^{(v)})]f_{j_{r_2}}(x_2^{(v)})...f_{j_{r_d}}(x_d^{(v)})[1-F_{j_{r_d+1}}(x_d^{(v)}]...[1-F_{j_n}(x_d^{(v)})]\}. \qquad (3.3)$$

From (3.3), we can write

$$f_{r_1,r_2,...,r_d:n}(\mathrm{x}_1,\mathrm{x}_2,...,\mathrm{x}_d)=\prod_{v=1}^{b}\{D\sum_{p}\left(\prod_{l=1}^{r_1-1}[F_{j_l}(x_1^{(v)})]\right)f_{j_{r_1}}(x_1^{(v)})$$

$$\cdot\left(\prod_{l=r_1+1}^{r_2-1}[F_{j_l}(x_2^{(v)})-F_{j_l}(x_1^{(v)})]\right)f_{j_{r_2}}(x_2^{(v)})...f_{j_{r_d}}(x_d^{(v)})\prod_{l=r_d+1}^{n}[1-F_{j_l}(x_d^{(v)})]\}. \qquad (3.4)$$

Thus, (3.1) is obtained.

Next theorem shows that *pdf* of *d* order statistics of *innid* continuous random vectors can be expressed in terms of *pdf* of *d* order statistics of *iid* continuous random vectors.

Theorem 3.2.

$$f_{r_1,r_2,...,r_d:n}(\mathrm{x}_1,\mathrm{x}_2,...,\mathrm{x}_d)=\prod_{v=1}^{b}\{\sum\sum n!D\left(\prod_{w=1}^{d+1}[F^s(x_w^{(v)})-F^s(x_{w-1}^{(v)})]^{r_w-r_{w-1}-1}\right)\prod_{w=1}^{d}f^s(x_w^{(v)})\}. \qquad (3.5)$$

Proof. (3.2) can be expressed as

$$\prod_{v=1}^{b}[\sum\sum P\{x_1^{(v)}<X_{r_1:n}^{(v),s}\le x_1^{(v)}+\delta x_1^{(v)},x_2^{(v)}<X_{r_2:n}^{(v),s}\le x_2^{(v)}+\delta x_2^{(v)},...,x_d^{(v)}<X_{r_d:n}^{(v),s}\le x_d^{(v)}+\delta x_d^{(v)}\}]. \qquad (3.6)$$

Dividing (3.6) by $\prod_{v=1}^{b}\prod_{w=1}^{d}\delta x_w^{(v)}$ and then letting $\delta x_1^{(v)},\delta x_2^{(v)},...,\delta x_d^{(v)}$ tend to zero, (3.5) is obtained.

The following five results of which first three are belong to *pdf* of single order statistic and last two are belong to joint *pdf* of *d* order statistics of *innid* continuous random vectors can be written from last two theorems.

Result 3.1.

$$f_{r_1:n}(x_1^{(1)})=\frac{1}{(r_1-1)!(n-r_1)!}\sum_{P}\left(\prod_{l=1}^{r_1-1}F_{j_l}(x_1^{(1)})\right)\left(\prod_{l=r_1+1}^{n}[1-F_{j_l}(x_1^{(1)})]\right)f_{j_{r_1}}(x_1^{(1)})$$

$$=\sum\sum r_1\binom{n}{r_1}[F^s(x_1^{(1)})]^{r_1-1}[1-F^s(x_1^{(1)})]^{n-r_1}f^s(x_1^{(1)}). \qquad (3.7)$$

Proof. In (3.1) and (3.5), if $b=1$, $d=1$, (3.7) is obtained.

Result 3.2.

$$f_{1:n}(x_1^{(1)})=\frac{1}{(n-1)!}\sum_{P}\left(\prod_{l=2}^{n}[1-F_{j_l}(x_1^{(1)})]\right)f_{j_1}(x_1^{(1)})$$

$$=\sum\sum n[1-F^s(x_1^{(1)})]^{n-1}f^s(x_1^{(1)}). \qquad (3.8)$$

Proof. In (3.7), if $r_1=1$, (3.8) is obtained.

Result 3.3.

$$f_{n:n}(x_1^{(1)}) = \frac{1}{(n-1)!}\sum_P\left(\prod_{l=1}^{n-1}F_{j_l}(x_1^{(1)})\right)f_{j_n}(x_1^{(1)})$$

$$= \sum\sum n[F^s(x_1^{(1)})]^{n-1} f^s(x_1^{(1)}). \tag{3.9}$$

Proof. In (3.7), if $r_1 = n$, (3.9) is obtained.

Result 3.4.

$$f_{1,n:n}(x_1^{(1)}, x_2^{(1)}) = \frac{1}{(n-2)!}\sum_P\left(\prod_{l=2}^{n-1}[F_{j_l}(x_2^{(1)}) - F_{j_l}(x_1^{(1)})]\right)f_{j_1}(x_1^{(1)})f_{j_n}(x_2^{(1)})$$

$$= \sum\sum n(n-1)[F^s(x_2^{(1)}) - F^s(x_1^{(1)})]^{n-2} f^s(x_1^{(1)}) f^s(x_2^{(1)}). \tag{3.10}$$

Proof. In (3.1) and (3.5), if $b=1$, $d=2$ and $r_1 = 1$, $r_2 = n$, (3.10) is obtained.

Result 3.5.

$$f_{1,2,\ldots,k:n}(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_k) = \prod_{v=1}^{b}\{\frac{1}{(n-k)!}\sum_P\left(\prod_{l=k+1}^{n}[1 - F_{j_l}(x_k^{(v)})]\right)f_{j_1}(x_1^{(v)})f_{j_2}(x_2^{(v)})\ldots f_{j_k}(x_k^{(v)})\}$$

$$= \prod_{v=1}^{b}\{\sum\sum\frac{n!}{(n-k)!}[1 - F^s(x_k^{(v)})]^{n-k} f^s(x_1^{(v)}) f^s(x_2^{(v)})\ldots f^s(x_k^{(v)})\}. \tag{3.11}$$

Proof. In (3.1) and (3.5), if $d = k$ and $r_1 = 1$, $r_2 = 2, \ldots, r_k = k$, (3.11) is obtained.

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Stability of homomorphisms and derivations in non-Archimedean random C^*-algebras via fixed point method

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Abstract. In this paper, using the fixed point method, we investigate the Hyers-Ulam stability of homomorphisms in non-Archimedean random C^*-algebras and non-Archimedean random Lie JC^*-algebras and of derivations on non-Archimedean random C^*-algebras and non-Archimedean random Lie JC^*-algebras related to the generalized Cauchy-Jensen additive functional equation.

1. Introduction

A non-Archimedean field is a field like $\mathcal{K}$ equipped is a function $|\cdot| : \mathcal{K} \to [0, +\infty)$ such that $|a| = 0$ if and only if $a = 0$, $|ab| = |a||b|$ and $|a + b| \leqslant \max\{|a|, |b|\}$ for all $a, b \in \mathcal{K}$. Note that $|1| = |-1| = 1$ and $|n| \leqslant 1$ for each integer n. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. We always assume, in addition, that $|\cdot|$ is non-trivial, i.e., there exists an $a_0 \neq 0, 1$. A function $\|.\| : X \to [0, \infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) for any $r \in K, x \in X, \|rx\| = |r|\|x\|$;

(iii) the strong triangle inequality holds; namely,

$$\|x + y\| \leqslant \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|.\|)$ is called a non-Archimedean normed space. From the fact that

$$\|x_n - x_m\| \leqslant \max\{\|n_n - x_m\| : m \leqslant j \leqslant n - 1\} \quad (n > m)$$

holds, a sequence $\{x_n\}$ is Cauchy if and only if $\{x_n - x_m\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b} p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.

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[0]**2000 Mathematics Subject Classification**: Primary 39B52; 39B72; 46L05; 47H10; 46B03.

[0]**Keywords**: Hyers-Ulam stability; additive functional equation; fixed point; non-Archimedean random space; homomorphisms in non-Archimedean random C^*-algebras and non-Archimedean random Lie JC^*-algebras; derivations on random C^*-algebras and non-Archimedean random Lie JC^*-algebras.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.2, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

A non-Archimedean Banach algebra is a complete non-Archimedean algebra $\mathcal{A}$ which satisfies $\|ab\| \leqslant \|a\| \cdot \|b\|$ for all $a, b \in \mathcal{A}$. For more detailed definitions of non-Archimedean Banach algebras, we refer the reader to [25].

If $\mathcal{U}$ is a non-Archimedean Banach algebra, then an involution on $\mathcal{U}$ is mapping $t \to t^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies

$(i)\, t^{**} = t$ for $t \in \mathcal{U}$;

$(ii)\, (\alpha s + \beta t)^* = \bar{\alpha} s^* + \bar{\beta} t^*$;

$(iii)\, (st)^* = t^* s^*$ for all $s, t \in \mathcal{U}$.

If, in addition, $\|t^* t\| = \|t\|^2$ for $t \in \mathcal{U}$, then $\mathcal{U}$ is a non-Archimedean C^*-algebra.

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond)$ be a metric group (a metric is defined on a set with group property) with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we we would say that the equation of homomorphism $h(x * y) = h(x) * h(y)$ is stable (see also [10, 11, 14, 18, 19, 20, 21, 22]).

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

For explicitly later use, we recall a fundamental result in fixed point theory.

Theorem 1.1. [9] *Let (Ω, d) be a complete generalized metric space and $J : \Omega \to \Omega$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given $x \in \Omega$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative n or there exists a positive integer n_0 such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geqslant n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $\Delta = \{y \in \Omega : d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ for all $y \in \Delta$.*

A C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy - yx}{2}$ and endowed with *anticommutator product* (Jordan product) $x \circ y := \frac{xy + yx}{2}$ on $\mathcal{C}$, is called a Lie JC^*-algebra (see [15, 16, 17]).

Jordan algebras as coordinates for Lie algebras were created to illuminate a particular aspect of physics, quantum-mechanical observables, but turned out to have illuminating connections with many areas of mathematics.

In this paper, using the fixed point method, we prove the Hyers-Ulam stability of homomorphisms and derivations in non-Archimedean random C^*-algebras and non-Archimedean random Lie JC^*-algebras associated with $f : X \to Y$ satisfying the following functional equation (see [1])

$$\sum_{\substack{1 \leqslant i_1 < \cdots < i_m \leqslant n \\ 1 \leqslant k_l (\neq i_j, \forall j \in \{1, \cdots, m\}) \leqslant n}} f\left(\frac{\sum_{j=1}^{m} x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) = \frac{(n-m+1)}{n} \binom{n}{m} \sum_{i=1}^{n} f(x_i) \quad (1.1)$$

for all $x_1, \cdots, x_n \in X$, where $m, n \in \mathbb{N}$ are fixed integer with $n \geqslant 2$, $1 \leqslant m \leqslant n$. In particular, it is shown that in the case $m = 1$, (1.1) yields the Cauchy additive equation $f(\sum_{l=1}^n x_{k_l}) = \sum_{l=1}^n f(x_i)$ and also in the case $m = n$, (1.1) yields the Jensen additive equation $f(\frac{\sum_{j=1}^n x_j}{n}) = \frac{1}{n}\sum_{l=1}^n f(x_i)$. Then (1.1) is a generalized form of the Cauchy-Jensen additive equation, and thus every solution of the equation (1.1) may be analogously called general (m,n)-Cauchy-Jensen additive. For each m with $1 \leqslant m \leqslant n$, a mapping $f : X \to Y$ satisfies (1.1) for all $n \geqslant 2$ if and only if $f(x) - f(0) = A(x)$ is Cauchy additive, where $f(0) = 0$ if $m < n$. In particular, we have $f((n-m+1)x) = (n-m+1)f(x)$ and $f(mx) = mf(x)$ for all $x \in X$.

2. **Random spaces**

In this section, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces as in [2, 3, 6, 7, 8]. Throughout this paper, Δ^+ is the space of distribution functions, that is the space of all mapping $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. And D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leqslant G$ if and only if $F(t) \leqslant G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \leqslant 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Definition 2.1. [23] *A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm norm (briefly, a continuous t-norm) if T satisfies the following conditions:*
$(a)\, T$ is commutative and associative;
$(b)\, T$ is continuous;
$(c)\, T(a,1) = a$ for all $a \in [0,1]$;
$(d)\, T(a,b) \leqslant T(c,d)$ whenever $a \leqslant c$ and $b \leqslant d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1, 0)$ (the Lukasiewicz t-norm).

Definition 2.2. [24] *A non-Archimedean random normed space (briefly, NA-RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:*
$(RN1)\, \mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
$(RN2)\, \mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X, \alpha \neq 0$.
$(RN3)\, \mu_{x+y}(t) \geqslant T(\mu_x(t), \mu_y(t))$ for all $x, y \in X$ and all $t \geqslant 0$.

Every normed space $(X, \|\cdot\|)$ defines a non-Archimedean random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 2.3. [12] *A non-Archimedean random normed algebra* (X, μ, T, T') *is a non-Archimedean random normed space* (X, μ, T) *with an algebraic structure such that*

(RN4) $\mu_{xy}(t) \geqslant T'(\mu_x(t), \mu_y(t))$ *for all* $x, y \in X$ *and all* $t > 0$*, in which* T' *is a continuous* t*-norm.*

Every non-Archimedean normed algebra $(X, \|\cdot\|)$ defines a non-Archimedean random normed algebra (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ if and only if

$$\|xy\| \leqslant \|x\| \, \|y\| + t\|x\| + t\|y\| \ \ (x, y \in X; t > 0).$$

This space is called an induced non-Archimedean random normed algebra.

Definition 2.4. *Let* (X, μ, T_M) *and* (Y, μ, T_M) *be non-Archimedean random normed algebras.*

(1) *An* $\mathbb{R}$*-linear mapping* $f : X \to Y$ *is called a* homomorphism *if* $f(xy) = f(x)f(y)$ *for all* $x, y \in X$*.*
(2) *An* $\mathbb{R}$*-linear mapping* $f : X \to Y$ *is called a* derivation *if* $f(xy) = f(x)y + xf(y)$ *for all* $x, y \in X$*.*

Definition 2.5. *Let* $(\mathcal{U}, \mu, T)$ *be a non-Archimedean random Banach algebra. Then an involution on* $\mathcal{U}$ *is mapping* $u \to u^*$ *from* $\mathcal{U}$ *into* $\mathcal{U}$ *which satisfies*

(i) $u^{**} = u$ *for* $u \in \mathcal{U}$*;*
(ii) $(\alpha u + \beta v)^* = \bar{\alpha} u^* + \bar{\beta} v^*$*;*
(iii) $(uv)^* = v^* u^*$ *for all* $u, v \in \mathcal{U}$*.*

If, in addition, $\mu_{u^*u}(t) = T'(\mu_u(t), \mu_u(t))$ *for* $u \in \mathcal{U}$*, then* $\mathcal{U}$ *is a non-Archimedean random* C^**-algebra.*

Definition 2.6. *Let* (X, μ, T) *be an NA-RN-space.*

(1) *A sequence* $\{x_n\}$ *in* X *is said be* convergent *to* x *in* X *if, for every* $\epsilon > 0$ *and* $\lambda > 0$*, there exists a positive integer* N *such that* $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ *whenever* $n \geqslant N$*.*
(2) *A sequence* $\{x_n\}$ *in* X *is called a* Cauchy sequence *if, for every* $\epsilon > 0$ *and* $\lambda > 0$*, there exists a positive integer* N *such that* $\mu_{x_n - x_{n+1}}(\epsilon) > 1 - \lambda$ *whenever* $n \geqslant m \geqslant N$*.*
(3) *An RN-space* (X, μ, T) *is said to be* complete *if and only if every Cauchy sequence in* X *is convergent to a point in* X*.*

3. Stability of homomorphisms and derivations in non-Archimedean random C^*-algebras

Throughout this section, we suppose that $\mathcal{A}$ and $\mathcal{B}$ are non-Archimedean random C^*-algebras, respectively, with norms $\mu^{\mathcal{A}}_{\cdot}$ and $\mu^{\mathcal{B}}_{\cdot}$.

We use the following abbreviation for a given mapping $f : \mathcal{A} \to \mathcal{B}$:

$$D_\lambda f(x_1, \cdots, x_n) := \sum_{\substack{1 \leqslant i_1 < \cdots < i_m \leqslant n \\ 1 \leqslant k_l (\neq i_j, \forall j \in \{1, \cdots, m\}) \leqslant n}} f\left(\frac{\sum_{j=1}^m \lambda x_{i_j}}{m} + \sum_{l=1}^{n-m} \lambda x_{k_l}\right) - \frac{(n-m+1)\binom{n}{m} \sum_{i=1}^n \lambda f(x_i)}{n}$$

for all $\lambda \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$ and all $x_1, \cdots, x_n \in \mathcal{A}$.

It is well-known that a $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a *random homomorphism* in non-Archimedean random C^*-algebras if H satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in \mathcal{A}$.

We prove the Hyers-Ulam stability oh homomorphisms in non-Archimedean random C^*-algebras for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 3.1. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+, \psi : \mathcal{A}^2 \to D^+$, and $\eta : \mathcal{A} \to D^+$ such that $|\mathcal{M}| = |n - m + 1| < 1$ and $|\mathcal{N}| = |(n - m + 1)\binom{n}{m}| < 1$ are far from zero and*

$$\mu^{\mathcal{B}}_{D_\lambda f(x_1, \cdots, x_n)}(t) \geqslant \varphi_{x_1, \cdots, x_n}(t), \tag{3.1}$$

$$\mu^{\mathcal{B}}_{f(xy) - f(x)f(y)}(t) \geqslant \psi_{x,y}(t), \tag{3.2}$$

$$\mu^{\mathcal{B}}_{f(x^*) - f(x)^*}(t) \geqslant \eta_x(t), \tag{3.3}$$

for all $\lambda \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$ and all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ such that

$$\varphi_{\mathcal{M}x_1, \cdots, \mathcal{M}x_n}(|\mathcal{M}|Lt) \geqslant \varphi_{x_1, \cdots, x_n}(t), \tag{3.4}$$

$$\psi_{\mathcal{M}x, \mathcal{M}y}(|\mathcal{M}|^2 Lt) \geqslant \psi_{x,y}(t), \tag{3.5}$$

$$\eta_{\mathcal{M}x}(|\mathcal{M}|Lt) \geqslant \eta_x(t), \tag{3.6}$$

for all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$, then there exists a unique random homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x) - H(x)}(t) \geqslant \varphi_{x, \cdots, x}((|\mathcal{N}| - |\mathcal{N}|L)t) \tag{3.7}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof. It follows from (3.4), (3.5), (3.6), and $L < 1$ that

$$\lim_{m \to \infty} \varphi_{\mathcal{M}^m x_1, \cdots, \mathcal{M}^m x_n}(|\mathcal{M}|^m t) = 1, \tag{3.8}$$

$$\lim_{m \to \infty} \psi_{\mathcal{M}^m x, \mathcal{M}^m y}(|\mathcal{M}|^{2m} t) = 1, \tag{3.9}$$

$$\lim_{m \to \infty} \eta_{\mathcal{M}^m x}(|\mathcal{M}|^m t) = 1, \tag{3.10}$$

for all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$.

Now we define $\Omega := \{g : \mathcal{A} \to \mathcal{B}; g(0) = 0\}$ and introduce a generalized metric on Ω as following:

$$d(g, h) = \inf\{k \in (0, \infty) : \mu^{\mathcal{B}}_{g(x) - h(x)}(kt) > \varphi_{x, x, \cdots, x}(t), \forall x \in \mathcal{A},\ t > 0\}$$

where $\inf \emptyset = +\infty$. By the same technique as in the proof of [13, Theorem 3.2], we can show that (Ω, d) is a complete generalized metric space. We define $J : \Omega \to \Omega$ by $Jg(x) = \frac{1}{\mathcal{M}} g(\mathcal{M}x)$ for all $x \in \mathcal{A}$ and $g \in \Omega$. Note that for all $g, h \in \Omega$, from (3.4), we have

$$\begin{aligned} d(g,h) \leqslant k &\Rightarrow \mu^{\mathcal{B}}_{g(x)-h(x)}(kt) > \varphi_{x,\cdots,x}(t) \\ &\Rightarrow \mu^{\mathcal{B}}_{\frac{1}{\mathcal{M}}g(\mathcal{M}x)-\frac{1}{\mathcal{M}}h(\mathcal{M}x)}(kt) > \varphi_{\mathcal{M}x,\cdots,\mathcal{M}x}(|\mathcal{M}|t) \\ &\Rightarrow \mu^{\mathcal{B}}_{\frac{1}{\mathcal{M}}g(\mathcal{M}x)-\frac{1}{\mathcal{M}}h(\mathcal{M}x)}(kLt) > \varphi_{x,\cdots,x}(t) \\ &\Rightarrow d(Jg, Jh) < kL. \end{aligned}$$

Then one can show that $d(Jg, Jh) \leqslant Ld(g,h)$ for all $g, h \in \Omega$ and so J is self-function of Ω with the the Lipschitz constant L.

Letting $\lambda = 1$ and putting $x_1 = x_2 = \cdots = x_n = x$ in (3.1), we obtain

$$\mu^{\mathcal{B}}_{\binom{n}{m}f((n-m+1)x)-\binom{n}{m}(n-m+1)f(x)}(t) \geqslant \varphi_{x,x,\cdots,x}(t)$$

for all $x \in \mathcal{A}$ and $t > 0$. Then

$$\mu^{\mathcal{B}}_{f(x)-\frac{1}{\mathcal{M}}f(\mathcal{M}x)}(t) \geqslant \varphi_{x,x,\cdots,x}(|\mathcal{N}|t)$$

for all $x \in \mathcal{A}$ and $t > 0$. This implies that $d(Jf, f) \leqslant \frac{1}{|\mathcal{N}|} < \infty$. By The fixed point alternative theorem, Theorem 1.1, J has a unique fixed point $H : \mathcal{A} \to \mathcal{B}$ in $\Omega_0 := \{h \in \Omega : d(h, f) < \infty\}$ such that

$$H(x) = \lim_{m\to\infty} \frac{1}{|\mathcal{M}|^m} f(\mathcal{M}^m x) \tag{3.11}$$

for all $x \in \mathcal{A}$, since $\lim_{m\to\infty} d(J^m f, H) = 0$.

On the other hand, it follows from (3.1), (3.8) and (3.11) that

$$\begin{aligned} \mu^{\mathcal{B}}_{D_\lambda H(x_1,\cdots,x_n)}(t) &= \lim_{m\to\infty} \mu_{\frac{1}{\mathcal{M}^m}D_\lambda f(\mathcal{M}^m x_1,\cdots,\mathcal{M}^m x_n)}(t) \\ &\geqslant \lim_{m\to\infty} \varphi_{\mathcal{M}^m x_1,\cdots,\mathcal{M}^m x_n}(|\mathcal{M}|^m t) = 1. \end{aligned}$$

By a similar method to the above, we can get $\lambda H(\mathcal{M}x) = H(\lambda\mathcal{M}x)$ for all $\lambda \in \mathbb{T}$ and all $x \in \mathcal{A}$. Then by using the same technique as in the proof of [10, Theorem 2.1], we can show that H is $\mathbb{C}$-linear.

It follows from (3.2), (3.9) and (3.11) that

$$\begin{aligned} \mu^{\mathcal{B}}_{H(xy)-H(x)H(y)}(t) &= \lim_{m\to\infty} \mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}xy)-f(\mathcal{M}^m x)f(\mathcal{M}^m y)}\left(|\mathcal{M}|^{2m}t\right) \\ &\geqslant \lim_{m\to\infty} \psi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right) = 1 \end{aligned}$$

for all $x, y \in \mathcal{A}$. Therefore, we conclude that $H(xy) = H(x)H(y)$ for all $x, y \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is a homomorphism satisfying (3.7).

By same method as above, from (3.3),(3.10) and (3.11), we can write

$$\begin{aligned} \mu^{\mathcal{B}}_{H(x^*)-H(x)^*}(t) &= \lim_{m\to\infty} \mu^{\mathcal{B}}_{\frac{1}{\mathcal{M}^m}(f(\mathcal{M}^m x^*)-f(\mathcal{M}^m x)^*)}(t) \\ &\geqslant \lim_{m\to\infty} \eta_{\mathcal{M}^m x}\left(|\mathcal{M}|^m t\right) = 1 \end{aligned}$$

for all $x \in \mathcal{A}$ and all $t > 0$. Then we conclude that $H(x^*) = H(x)^*$ and the proof is complete, as desired. □

Corollary 3.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\mu^{\mathcal{B}}_{D_\lambda f(x_1,\cdots,x_n)}(t) \geqslant \frac{t}{t + \theta(\|x_1\|^r_{\mathcal{A}} + \|x_2\|^r_{\mathcal{A}} + \cdots + \|x_n\|^r_{\mathcal{A}})},$$
$$\mu^{\mathcal{B}}_{f(xy)-f(x)f(y)}(t) \geqslant \frac{t}{t + \theta(\|x\|^r_{\mathcal{A}}.\|y\|^r_{\mathcal{A}})},$$
$$\mu^{\mathcal{B}}_{f(x^*)-f(x)^*}(t) \geqslant \frac{t}{t + \theta\|x\|^r_{\mathcal{A}}}$$

for all $\lambda \in \mathbb{T}^1$, all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$. Then there exists a unique random homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x)-H(x)}(t) \geqslant \frac{(|\mathcal{N}| - |\mathcal{N}|^r)t}{(|\mathcal{N}| - |\mathcal{N}|^r)t + n\theta\|x\|^r_{\mathcal{A}}}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof. Letting

$$\varphi_{x_1,\cdots,x_n}(t)\frac{t}{t + \theta(\|x_1\|^r_{\mathcal{A}} + \|x_2\|^r_{\mathcal{A}} + \cdots + \|x_n\|^r_{\mathcal{A}})},$$
$$\psi_{x,y}(t) = \frac{t}{t + \theta(\|x\|^r_{\mathcal{A}}.\|y\|^r_{\mathcal{A}})},$$
$$\eta_x(t) = \frac{t}{t + \theta\|x\|^r_{\mathcal{A}}}$$

for all $x_1, \cdots, x_n, x, y \in \mathcal{A}$, $L = |\mathcal{N}|^{r-1}$ and $t > 0$ in Theorem 3.1, we get the desired result. □

In the following theorem, we investigate the Hyers-Ulam stability of derivations on non-Archimedean random C^*-algebras for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 3.3. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$, $\psi : \mathcal{A}^2 \to D^+$, satisfying (3.1), (3.3), and $\eta : \mathcal{A} \to D^+$ such that $|\mathcal{M}| < 1$ and $|\mathcal{N}| < 1$ are far from zero and*

$$\mu^{\mathcal{A}}_{f(xy)-f(x)y-xf(y)}(t) \geqslant \psi_{x,y}(t), \tag{3.12}$$

for all $\lambda \in \mathbb{T}^1$ and all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ such that (3.4), (3.5) and (3.6) hold, then there exists a unique random derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\mu^{\mathcal{A}}_{f(x)-\delta(x)}(t) \geqslant \varphi_{x,\cdots,x}\left((|\mathcal{N}| - |\mathcal{N}|L)\right) \tag{3.13}$$

for all $x \in^{\mathcal{A}}$ and $t > 0$.

Proof. By the same argument as in the proof of Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying (3.13). The mapping δ is given by

$$\delta(x) = \lim_{m\to\infty} \frac{1}{|\mathcal{M}|^m} f(\mathcal{M}^m x) \tag{3.14}$$

for all $x \in \mathcal{A}$.

It follows from (3.12), (3.9) and (3.14) that

$$\begin{aligned}\mu^{\mathcal{B}}_{\delta(xy)-\delta(x)y-x\delta(y)}(t) &= \lim_{m\to\infty} \mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}xy)-f(\mathcal{M}^m x)\mathcal{M}^m y-\mathcal{M}^m x f(\mathcal{M}^m y)}\left(|\mathcal{M}|^{2m}t\right)\\ &\geqslant \lim_{m\to\infty} \psi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right) = 1\end{aligned}$$

for all $x, y \in \mathcal{A}$. Therefore, we conclude that $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. The remainder of the proof is similar to the proof of Theorem 3.1. □

4. Stability of homomorphisms and derivations in non-Archimedean random Lie JC^*-algebras

A non-Archimedean random C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] := \frac{xy-yx}{2}$and endowed with *anticommutator product* (Jordan product) $x \circ y := \frac{xy+yx}{2}$ on $\mathcal{C}$, is called a non-Archimedean random Lie JC^*-algebra.

Definition 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be non-Archimedean random Lie JC^*-algebras. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a random Lie JC^*-algebra homomorphism if H satisfies*

$$\begin{aligned}&H([x, y]) = [H(x), H(y)],\\ &H(x \circ y) = H(x) \circ H(y),\\ &H(x^*) = H(x)^*\end{aligned}$$

for all $x, y \in \mathcal{A}$.

Throughout this section, assume that $\mathcal{A}$ and $\mathcal{B}$ are two non-Archimedean random Lie JC^*-algebras respectively with norm $\mu^{\mathcal{A}}_{.}$ and $\mu^{\mathcal{B}}_{.}$.

In the following theorem, we prove the Hyers-Ulam stability of homomorphisms in non-Archimedean random Lie JC^*-algebra for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 4.2. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$ and $\psi : \mathcal{A}^2 \to D^+$ satisfying (3.1), (3.3) and*

$$\mu^{\mathcal{B}}_{f([x,y])-[f(x),f(y)]}(t) \geqslant \psi_{x,y}(t), \tag{4.1}$$

$$\mu^{\mathcal{B}}_{H(x\circ y)-H(x)\circ H(y)}(t) \geqslant \phi_{x,y}(t) \tag{4.2}$$

for all $\lambda \in \mathbb{T}^1$, all $x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ such that (3.4), (3.5) and (3.6) hold, and also

$$\phi_{\mathcal{M}x,\mathcal{M}y}(|\mathcal{M}|^2 Lt) \geqslant \phi_{x,y}(t), \tag{4.3}$$

for all $x, y \in \mathcal{A}$ and $t > 0$, then there exists a unique random Lie JC^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.7).

Proof. It follows from (4.3) and $L < 1$ that

$$\lim_{m\to\infty} \phi_{\mathcal{M}^m x,\mathcal{M}^m y}(|\mathcal{M}|^{2m}t) = 1, \tag{4.4}$$

for all $x, y \in \mathcal{A}$ and $t > 0$.

By the same argument as in the proof of Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.7). The mapping H is given by

$$H(x) = \lim_{m\to\infty} \frac{f(\mathcal{M}^m x)}{|\mathcal{M}|^m} \tag{4.5}$$

for all $x \in \mathcal{A}$. It follows from (3.9), (4.4) and (4.5) that

$$\begin{aligned}\mu^{\mathcal{B}}_{H([x,y])-[H(x),H(y)]}(t) &= \lim_{m\to\infty} \mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}[x,y])-[f(\mathcal{M}^m x),f(\mathcal{M}^m y)]}\left(|\mathcal{M}|^{2m}t\right)\\ &\geqslant \lim_{m\to\infty} \psi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right) = 1\end{aligned}$$

and

$$\begin{aligned}\mu^{\mathcal{B}}_{H(x\circ y)-H(x)\circ H(y)} &= \lim_{m\to\infty} \mu^{\mathcal{B}}_{f(\mathcal{M}^{2m}(x\circ y))-f(\mathcal{M}x)\circ f(\mathcal{M}y)}\left(|\mathcal{M}|^{2m}t\right)\\ &\geqslant \lim_{m\to\infty} \phi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m}t\right) = 1\end{aligned}$$

for all $x, y \in \mathcal{A}$ and $t > 0$, then it is concluded that

$$H([x,y]) = [H(x), H(y)] \quad ; \quad H(x\circ y) = H(x)\circ H(y)$$

for all $x, y \in \mathcal{A}$. Therefore, $H : \mathcal{A} \to \mathcal{B}$ is the unique random Lie JC^*-algebra homomorphism satisfying (3.7). □

Corollary 4.3. *Let $r > 1$ and θ be nonnegative real numbers, and $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\begin{aligned}&\mu^{\mathcal{B}}_{D_\lambda f(x_1,\cdots,x_n)}(t) \geqslant \frac{t}{t+\theta(\|x_1\|^r_{\mathcal{A}}+\cdots+\|x_n\|^r_{\mathcal{A}})},\\ &\mu^{\mathcal{B}}_{f([x,y])-[f(x),f(y)]} \geqslant \frac{t}{t+\theta(\|x\|^r_{\mathcal{A}}.\|y\|^r_{\mathcal{A}})},\\ &\mu^{\mathcal{B}}_{f(x^8)-f(x)^*}(t) \geqslant \frac{t}{t+\theta.\|x\|^r_{\mathcal{A}}}\end{aligned}$$

for all $\lambda \in \mathbb{T}^1$, all $x_1, \cdots, x_n, x, y \in \mathcal{A}$ and $t > 0$. Then there exists a unique random Lie JC^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^{\mathcal{B}}_{f(x)-H(x)} \geqslant \frac{(|\mathcal{N}|-|\mathcal{N}|^r)t}{(|\mathcal{N}|-|\mathcal{N}|^r)t+n\theta\|x\|^r_{\mathcal{A}}}$$

for all $x \in \mathcal{A}$ and $t > 0$.

Proof. By the same reasoning as in the proof of Theorem 4.2 and a technique similar to Corollary 3.2, by putting $L = |\mathcal{N}|^{r-1}$, the proof will be completed. □

Definition 4.4. *Let $\mathcal{A}$ be a non-Archimedean random Lie JC^*-algebra. A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a random Lie JC^*-algebra derivation if δ satisfies*

$$\begin{aligned}&\delta([x,y]) = [\delta(x), y] + [x, \delta(y)],\\ &\delta(x\circ y) = \delta(x)\circ y + x\circ\delta(y),\\ &\delta(x^*) = \delta(x)^*\end{aligned}$$

for all $x, y \in \mathcal{A}$.

In the following theorem, we prove the Hyers-Ulam stability of derivation on non-Archimedean random Lie JC^*-algebras for the functional equation $D_\lambda f(x_1, \cdots, x_n) = 0$.

Theorem 4.5. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there are functions $\varphi : \mathcal{A}^n \to D^+$ and $\psi : \mathcal{A}^2 \to D^+$ such that (3.1) and (3.3) hold and*

$$\mu^{\mathcal{A}}_{f([x,y])-[f(x),y]-[x,f(y)]}(t) \geqslant \psi_{x,y}(t), \tag{4.6}$$

$$\mu^{\mathcal{A}}_{f(x\circ y)-f(x)\circ y-x\circ f(y)}(t) \geqslant \phi_{x,y}(t) \tag{4.7}$$

for all $x, y \in \mathcal{A}$. If there exists an $L < 1$ and (3.4), (3.5), (3.6) and (4.3) hold, then there exists a unique random Lie JC^*-algebra derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that (3.13) holds.

Proof. By the same argument as in the proof of Theorem 4.2, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying (3.13), and is given by

$$\delta(x) = \lim_{m\to\infty} \frac{f(\mathcal{M}^m x)}{|\mathcal{M}|^m} \tag{4.8}$$

for all $x \in \mathcal{A}$.

It follows from (3.9), (4.4) and (4.8) that

$$\begin{aligned}\mu_{\delta([x,y])-[\delta(x),y]-[x,\delta(y)]}(t) &= \lim_{m\to\infty} \mu^{\mathcal{A}}_{f(\mathcal{M}^{2m}[x,y])-[f(\mathcal{M}^m x),\mathcal{M}^m y]-[\mathcal{M}^m x,f(\mathcal{M}^m y)]}\left(|\mathcal{M}|^{2m} t\right) \\ &\geqslant \lim_{m\to\infty} \psi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m} t\right) = 1\end{aligned}$$

and

$$\begin{aligned}\mu^{\mathcal{A}}_{\delta(x\circ y)-\delta(x)\circ y-x\circ\delta(y)}(t) &= \lim_{m\to\infty} \mu^{\mathcal{A}}_{f(\mathcal{M}^{2m}(x\circ y))-f(\mathcal{M}^m x)\circ y-x\circ f(\mathcal{M}^m y)}\left(|\mathcal{M}|^{2m} t\right) \\ &\geqslant \lim_{m\to\infty} \phi_{\mathcal{M}^m x,\mathcal{M}^m y}\left(|\mathcal{M}|^{2m} t\right) = 1\end{aligned}$$

for all $x, y \in \mathcal{A}$ and $t > 0$, and so we conclude that

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)], \quad \delta(x \circ y) = \delta(x) \circ y + x \circ \delta(y)$$

for all $x, y \in \mathcal{A}$. Therefore, $\delta : \mathcal{A} \to \mathcal{A}$ is the unique desired random Lie JC^*-algebra derivation satisfying (3.13). □

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ON THE FUZZY STABILITY PROBLEMS OF GENERALIZED SEXTIC MAPPINGS

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Abstract. We introduce a fuzzy anti-β-norm and generalized sextic mapping and then investigate the Hyers-Ulam-Rassias stability in quasi β-Banach space and the fuzzy stability by using a fixed point in fuzzy anti-β Banach space for the generalized sextic function.

1. Introduction

The concept of stability problem of a functional equation was first posed by Ulam [33] concerning the stability of group homomorphisms. In the next year, Hyers [14] gave a partial answer to the question of Ulam. Hyers' theorem was generalized in various directions. The very first author who generalized Hyers' theorem to the case of unbounded control functions was Aoki [1]. Rassias [28] succeeded in extending the result of Hyers' theorem by weakening the condition for the Cauchy difference operator $CDf(x, y) = f(x + y) - [f(x) + f(y)]$ to be controlled by $\varepsilon(||x||^p + ||y||^p)$. Rassias' paper [28] has provided a lot of influence in the development of Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. In 1996, Isac and Rassias [16] were first to provide applications of new fixed point theorems for the proof of stability theory of functional equations. By using fixed point methods the stability problems of several functional equations have been extensively investigated by a number of authors; see [6], [7], [25] and [26]. Recently, the stability problem of functional equations was investigated by using shadowing properties; see [20] and [31].

During the last three decades, several stability problems of a large variety of functional equations have been extensively studied and generalized by a number of authors [9], [12], [15], [28], and [2]. In particular, Xu and et al. [37] introduced the sextic functional equation

$$f(x+3y)+f(x-3y)-6[f(x+2y)+f(x-2y)]+15[f(x+y)+f(x-y)] = 20f(x)+720f(y)\,. \tag{1.1}$$

In fact, Xu and et al. [37] and Gordji and et al. [13] introduced a quintic mapping and sextic mapping.

In this paper, we deal with the following functional equation

$$\begin{aligned} &f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay) \\ &= a^2(a^2+1)[f(x+y)+f(x-y)]+2(a^2-1)(a^4-1)[f(x)+f(y)] \end{aligned} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 39B52.

Key words and phrases. Hyers-Ulam-Rassias stability, sextic mapping, quasi-β-mormed space, fixed point, fuzzy anti-normed space, fuzzy anti-β-normed space.

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holds for all $x,\, y \in X$ and all $a \in \mathbb{Z}\,(a \neq 0, \pm 1)$.

We will use the following definition to prove Hyers-Ulam-Rassias stability for the generalized sextic functional equation in the quasi β-normed space. Let β be a real number with $0 < \beta \leq 1$ and $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. *Let X be a linear space over a field $\mathbb{K}$. A quasi β-norm $||\cdot||$ is a real-valued function on X satisfying the following statements:*

(1) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$.

(2) $||\lambda x|| = |\lambda|^{\beta} \cdot ||x||$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $||x+y|| \leq K(||x||+||y||)$ for all $x,\, y \in X$.

The pair $(X, ||\cdot||)$ is called a *quasi β-normed space* if $||\cdot||$ is a quasi β-norm on X. The smallest possible K is called the *modulus of concavity* of $||\cdot||$. A *quasi β-Banach space* is a complete quasi-β-normed space.

A quasi β-norm $||\cdot||$ is called a $(\beta,\, p)$-norm $(0 < p \leq 1)$ if (3) takes the form $||x+y||^p \leq ||x||^p + ||y||^p$ for all $x,\, y \in X$. In this case, a quasi β-Banach space is called a (β, p)-Banach space; see [5], [29] and [27].

In 1984, Katsaras [18] and Wu and Fang [35] independently introduced a notion of a fuzzy norm. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view; see [3], [11], [19], [36] and [23]. In 2003, Bag and Samanta [3] modified the definition of Cheng and Mordeson [8]. Bag and Samanta [3] introduced the following definition of fuzzy normed spaces. The notion of fuzzy stability of functional equations was given in the paper [24]. Jebril and Samanta [17] introduced a fuzzy anti-norm linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [4] and investigated their important properties.

We will use the definition of fuzzy anti-normed spaces to investigate a fuzzy version of Hyers-Ulam-Rassias stability in the fuzzy anti-normed algebra setting.

Definition 1.2. [17] *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,\, 1]$ is called a fuzzy anti-norm on X if for all $x,\, y \in X$ and all $s,\, t \in \mathbb{R}$,*

(aN1) $N(x,t) = 1$ *for* $t \leq 0$

(aN2) $N(x,t) = 0$ *if and only if* $x = 0$ *for all* $t > 0$

(aN3) $N(cx,t) = N(x, \frac{t}{|c|})$ *for* $c \neq 0$

(aN4) $N(x+y, s+t) \leq max\{N(x,s)\,, N(y,t)\}$

(aN5) $N(x,t)$ *is a non-increasing function of* $t \in \mathbb{R}$ *and* $\lim_{t\to\infty} N(x,t) = 0$,

(aN6) *for* $x \neq 0$, $N(x,\,\cdot)$ *is continuous on* $\mathbb{R}$.

The pair (X, N) is called a fuzzy anti-normed space.

The property (aN3) implies that $N(-x,t) = N(x,t)$ for all $x \in X$ and $t > 0$. It is easy to show that (aN4) is equivalent the following condition:

$$N(x+y,t) \leq \max\{N(x,t)\,, N(y,t)\}\,, \text{ for all } x, y \in X \text{ and } t \in \mathbb{R}.$$

Definition 1.3. *Let X be a real vector space. A fuzzy anti-norm $N : X \times \mathbb{R} \to [0,\, 1]$ is called a fuzzy anti-β-norm on X if (aN_3) in Definition 1.2 takes the form*

$$(aN_3') \quad N(cx,\, t) = N(x,\, \frac{t}{|c|^{\beta}}) \quad (c \neq 0,\, 0 < \beta \leq 1).$$

Example 1.4. *Let $(X,\, ||\cdot||)$ be a β-normed space. Define*

$$N(x,t) = \begin{cases} \frac{||x||}{t+||x||} & when\ t > 0,\, t \in \mathbb{R} \\ 1 & when\ t \leq 0\,, \end{cases}$$

where $x \in X$. We note that

$$N(cx, t) = \frac{||cx||}{t + ||cx||} = \frac{||x||}{\frac{t}{|c|^\beta} + ||x||} = N(x, \frac{t}{|c|^\beta}),$$

for all $x \in X$ and $c \in \mathbb{R}\,(c \neq 0, 0 < \beta \leq 1)$. Then (X, N) is a fuzzy anti-β-normed space induced by the β-norm $||\cdot||$.

Definition 1.5. *Let (X, N) be a fuzzy anti-β-normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 0$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.*

Definition 1.6. *Let (X, N) be a fuzzy anti-β-normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all integer $d > 0$, we have $N(x_{n+d} - x_n, t) < \varepsilon$.*

It is well-known that every convergent sequence in a fuzzy anti-β-normed vector space is Cauchy. If each Cauchy sequaence is convergent, then the fuzzy anti-β-normed space is said to be *fuzzy anti-β complete* and the fuzzy anti-β-normed vector space is called a *fuzzy anti-β Banach space.*

Now, we will state the theorem, the alternative of fixed point in a generalized metric space.

Definition 1.7. *Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies*

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.8 (The alternative of fixed point [21], [30]). *Suppose that we are given a complete generalized metric space (X, d) and a strictly contractive mapping $J : X \to X$ with Lipschitz constant $0 < L < 1$. Then for each given $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty \text{ for all } n \geq 0,$$

or there exists a natural number n_0 such that

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set*

$$Y = \{y \in X | d(J^{n_0}x, y) < \infty\};$$

(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In this paper, we investigate the Hyers-Ulam-Rassias stability in quasi β-normed space and then the fuzzy stability by using a fixed point in fuzzy anti-β Banach space for the generalized sextic function $f : X \to Y$ satisfying the equation (1.2). Let us fix some notations which will be used throughout this paper. Let $a \in \mathbb{Z}\,(a \neq 0, \pm 1)$.

2. A sextic functional equation

In this section let X and Y be real vector spaces and we investigate the general solution of the functional equation (1.2). Before we proceed, we would like to introduce some basic definitions concerning n-additive symmetric mappings and key concepts which are found in [32] and [34]. A function $A : X \to Y$ is said to be

additive if $A(x+y)=A(x)+A(y)$ for all $x\,,y\in X$. Let n be a positive integer. A function $A_n : X^n \to Y$ is called n*-additive* if it is additive in each of its variables. A function A_n is said to be *symmetric* if $A_n(x_1\,,\cdots,x_n)=A_n(x_{\sigma(1)}\,,\cdots,x_{\sigma(n)})$ for every permutation $\{\sigma(1)\,,\cdots,\sigma(n)\}$ of $\{1\,,2\,,\cdots,n\}$. If $A_n(x_1\,,x_2\,,\cdots,x_n)$ is an n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x\,,x\,,\cdots,x)$ and $A^n(rx)=r^nA^n(x)$ for all $x\in X$ and all $r\in\mathbb{Q}$. such a function $A^n(x)$ will be called a *monomial function* of degree n (assuming $A^n\not\equiv 0$). Furthermore the resulting function after substitution $x_1=x_2=\cdots=x_s=x$ and $x_{s+1}=x_{s+2}=\cdots=x_n=y$ in $A_n(x_1\,,x_2\,,\cdots,x_n)$ will be denoted by $A^{s,n-s}(x\,,y)$.

Theorem 2.1. *A function $f : X\to Y$ is a solution of the functional equation (1.2) if and only if f is of the form $f(x)=A^6(x)$ for all $x\in X$, where $A^6(x)$ is the diagonal of the* 6*-additive symmetric mapping $A_6 : X^6\to Y$.*

Proof. Assume that f satisfies the functional equation (1.2). Letting $x=y=0$ in the equation (1.2), we have

$$2a^2(2a^2+1)(a^2-1)f(0)=0\,,$$

that is, $f(0)=0$. Let $y=0$ in the equation (1.2). Then we get

$$f(ax)=a^6f(x) \tag{2.1}$$

for all $x\in X$. Putting $x=0$ in the equation (1.2), we get

$$(a^4-1)(a^2-1)\Big(f(y)-f(-y)\Big)=0 \tag{2.2}$$

for all $y\in X$. Hence we have $f(y)=f(-y)$, for all $y\in X$. That is, f is even. We can rewrite the functional equation (1.2) in the form

$$\begin{aligned}
&f(x)-\frac{1}{2(a^2-1)(a^4-1)}f(ax+y)-\frac{1}{2(a^2-1)(a^4-1)}f(ax-y)\\
&-\frac{1}{2(a^2-1)(a^4-1)}f(x+ay)-\frac{1}{2(a^2-1)(a^4-1)}f(x-ay)\\
&+\frac{a^2(a^2+1)}{2(a^2-1)(a^4-1)}f(x+y)+\frac{a^2(a^2+1)}{2(a^2-1)(a^4-1)}f(x-y)+f(y)=0
\end{aligned}$$

for all $x\,,y\in X$ and an integer $a(a\neq 0\,,\pm1)$. By Theorem 3.5 and 3.6 in [34], f is a generalized polynomial function of degree at most 6, that is, f is of the form

$$f(x)=A^6(x)+A^5(x)+A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0(x) \tag{2.3}$$

for all $x\in X$, where $A^0(x)=A^0$ is an arbitrary element of Y , and $A^i(x)$ is the diagonal of the i-additive symmetric mapping $A_i : X^i\to Y$ for $i=1,2,3,4,5,6$. By $f(0)=0$ and $f(-x)=f(x)$ for all $x\in X$, we get $A^0(x)=A^0=0\,,A^5(x)=0\,,A^3(x)=0$ and $A^1(x)=0$. It follows that

$$f(x)=A^6(x)+A^4(x)+A^2(x)$$

for all $x\in X$. By (2.1) and $A^n(rx)=r^nA^n(x)$ for all $x\in X$ and $r\in\mathbb{Q}$, we obtain that $A^2(x)=-\frac{a^2}{a^2+1}A^4(x)$ for all $x\in X$ and an integer $a\,(a\neq 0,\pm1)$. Hence we get $A^4(x)=A^2(x)=0$, for all $x\in X$. Thus we have $f(x)=A^6(x)$ for all $x\in X$.

Conversely, assume that $f(x)=A^6(x)$ for all $x\in X$, where $A^6(x)$ is the diagonal of a 6-additive symmetric mapping $A_6 : X^6\to Y$. Note that

$$\begin{aligned}
A^6(qx+ry) &= q^6A^6(x)+6q^5rA^{5,1}(x,y)+15q^4r^2A^{4,2}(x,y)+20q^3r^3A^{3,3}(x,y)\\
&+ 15q^2r^4A^{2,4}(x,y)+6qr^5A^{1,5}(x,y)+r^6A^6(y)
\end{aligned}$$

$$c^s A^{s,t}(x,y) = A^{s,t}(cx,y)\,, \quad c^t A^{s,t}(x,y) = A^{s,t}(x,cy)$$

where $1 \le s,t \le 5$ and $c \in \mathbb{Q}$. Thus we may conclude that f satisfies the equation (1.2). $\square$

We note that a mapping $f : X \to Y$ is called *generalized sextic* if f satisfies the functional equation (1.2).

3. Hyers-Ulam-Rassias stability over a quasi β-Banach space

Throughout this section, let X be a real linear space and let Y be a quasi β-Banach space with a quasi β-norm $||\cdot||_Y$. Let K be the modulus of concavity of $||\cdot||_Y$. We will investigate the Hyers-Ulam-Rassias stability for the functional equation (1.2); see also the paper [10].

For a given mapping $f : X \to Y$ and all fixed integer a ($a \neq 0, \pm 1$), let

$$\begin{aligned} D_a f(x,y) := {} & f(ax+y) + f(ax-y) + f(x+ay) + f(x-ay) \\ & -a^2(a^2+1)\Big(f(x+y)+f(x-y)\Big) - 2(a^2-1)(a^4-1)\Big(f(x)+f(y)\Big) \end{aligned} \tag{3.1}$$

for all $x, y \in X$.

Theorem 3.1. *Suppose that there exists a mapping $\phi : X^2 \to [0,\infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$||D_a f(x,y)||_Y \le \phi(x,y) \tag{3.2}$$

and the series $\sum_{j=0}^{\infty} \left(\frac{K}{|a|^{6\beta}}\right)^j \phi(a^j x, a^j y)$ converges for all $x, y \in X$. Then there exists a unique generalized sextic mapping $S : X \to Y$ satisfying the equation (1.2) and the inequality

$$||f(x) - S(x)||_Y \le \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=0}^{\infty} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0)\,, \tag{3.3}$$

for all $x \in X$.

Proof. By letting $y = 0$ in inequality (3.2), since $f(0) = 0$ we have

$$\begin{aligned} ||D_a f(x,0)||_Y &= ||2f(ax) + 2f(x) - 2a^2(a^2+1)f(x) - 2(a^2-1)(a^4-1)f(x)||_Y \\ &= 2^\beta |a|^{6\beta} ||f(x) - \frac{1}{a^6} f(ax)||_Y \le \phi(x,0)\,, \end{aligned}$$

that is,

$$||f(x) - \frac{1}{a^6} f(ax)||_Y \le \frac{1}{2^\beta |a|^{6\beta}}\, \phi(x,0)\,, \tag{3.4}$$

for all $x \in X$.

We note that putting $x = ax$ and multiplying $\frac{1}{|a|^{6\beta}}$ in the inequality (3.4), we get

$$\frac{1}{|a|^{6\beta}} ||f(ax) - \frac{1}{a^6} f(a^2 x)||_Y \le \frac{1}{2^\beta |a|^{6\beta}}\, \frac{1}{|a|^{6\beta}} \phi(ax,0)\,, \tag{3.5}$$

for all $x \in X$.

Combining two inequalities (3.4) and (3.5), we have

$$||f(x) - \Big(\frac{1}{a^6}\Big)^2 f(a^2 x)||_Y \le \frac{K}{2^\beta |a|^{6\beta}} \Big(\phi(x,0) + \frac{1}{|a|^{6\beta}} \phi(ax,0)\Big)\,, \tag{3.6}$$

for all $x \in X$.

Since $K \geq 1$, inductively using the previous note we have the following inequalities

$$||f(x) - \Big(\frac{1}{a^6}\Big)^k f(a^k x)||_Y \leq \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=0}^{k-1} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0)\,, \tag{3.7}$$

for all $x \in X$, $k \in \mathbb{N}$ and also

$$||\Big(\frac{1}{a^6}\Big)^k f(a^k x) - \Big(\frac{1}{a^6}\Big)^t f(a^t x)||_Y \leq \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=k}^{t-1} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0)\,, \tag{3.8}$$

for all $x \in X$ and k , $t \in \mathbb{N}\,(k < t)$.

Since the right-hand side of the previous inequality (3.8) tends to 0 as $t \to \infty$, hence $\{\big(\frac{1}{a^6}\big)^n f(a^n x)\}$ is a Cauchy sequence in the quasi β-Banach space Y . Thus we may define

$$S(x) = \lim_{n\to\infty} \Big(\frac{1}{a^6}\Big)^n f(a^n x)\,,$$

for all $x \in X$. Since $K \geq 1$, replacing x and y by $a^n x$ and $a^n y$ respectively and dividing by $|a|^{6\beta n}$ in the inequality (3.2) , we have

$$\begin{aligned}
&\Big(\frac{1}{|a|^{6\beta}}\Big)^n ||D_a f(a^n x, a^n y)||_Y \\
&= \Big(\frac{1}{|a|^{6\beta}}\Big)^n ||f(a^n(ax+y)) + f(a^n(ax-y)) + f(a^n(x+ay)) + f(a^n(x-ay)) \\
&-a^2(a^2+1)\Big(f(a^n(x+y)) + f(a^n(x-y))\Big) \\
&-2(a^2-1)(a^4-1)\Big(f(a^n x) + f(a^n y)\Big)||_Y \\
&\leq \Big(\frac{K}{|a|^{6\beta}}\Big)^n \phi(a^n x, a^n y)
\end{aligned}$$

for all x, $y \in X$.

By taking $n \to \infty$, the definition of S implies that S satisfies (1.2) for all x, $y \in X$, that is, S is the generalized sextic mapping. Also, the inequality (3.7) implies the inequality (3.3).

Now, it remains to show the uniqueness. Assume that there exists $T : X \to Y$ satisfying (1.2) and (3.3). Then

$$\begin{aligned}
||T(x) - S(x)||_Y &= \Big(\frac{1}{|a|^{6\beta}}\Big)^n ||T(a^n x) - S(a^n x)||_Y \\
&\leq \Big(\frac{1}{|a|^{6\beta}}\Big)^n K\Big(||T(a^n x) - f(a^n x)||_Y + ||f(a^n x) - S(a^n x)||_Y\Big) \\
&\leq \frac{2K^2}{2^\beta |a|^{6\beta} K^n} \sum_{j=n}^{\infty} \Big(\frac{K}{|a|^{6\beta}}\Big)^j \phi(a^j x, 0)
\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$, we immediately have the uniqueness of S . □

Corollary 3.2. *Let $\theta \geq 0$, $p < 6$ be a real number and X be a normed linear space with norm $||\cdot||$. Suppose $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and*

$$||D_a f(x,y)||_Y \leq \theta(||x||^p + ||y||^p) \tag{3.9}$$

for all $x,\, y \in X$ *and all* $t > 0$. *Then* $S(x) := N\text{-}\lim_{n\to\infty} \frac{1}{a^{6n}} f(a^n\, x)$ *exists for each* $x \in X$ *and defines a generalized sextic mapping* $S : X \to Y$ *such that*

$$||f(x) - S(x)||_Y \leq \frac{\theta\, K\, ||x||^p}{2^\beta(|a|^{6\beta} - K|a|^{p\beta})}$$

for all $x \in X$ *and all* $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking $\phi(x,\, y) = \theta(||x||^p + ||y||^p)$ for all $x\,, y \in X$. □

4. Fuzzy fixed point stability over a Fuzzy Banach space

Let us fix some notations which will be used throughout this section. We assume X is a vector space and $(Y,\, N)$ is a fuzzy anti-β Banach space. Using fixed point method, we will prove the Hyers-Ulam stability of the functional equation satisfying equation (1.2) in fuzzy anti-β Banach space.

Theorem 4.1. *Let* $\phi : X^2 \to [0, \infty)$ *be a function such that there exists an* $0 < L < 1$ *with*

$$\phi(x,y) \leq \frac{L}{|a|^{6\beta}}\, \phi(ax, ay) \tag{4.1}$$

for all $x,\, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and*

$$N(D_a f(x,y),\, t) \leq \frac{\phi(x,y)}{t + \phi(x,y)} \tag{4.2}$$

for all $x,\, y \in X$ *and all* $t > 0$. *Then* $S(x) := N\text{-}\lim_{n\to\infty} a^{6n} f\Big(\frac{x}{a^n}\Big)$ *exists for each* $x \in X$ *and defines a generalized sextic mapping* $S : X \to Y$ *such that*

$$N(f(x) - S(x),\, t) \leq \frac{L\, \phi(x,0)}{2^\beta |a|^{6\beta}(1-L)\, t + L\, \phi(x,0)} \tag{4.3}$$

for all $x \in X$ *and all* $t > 0$.

Proof. By letting $y = 0$ in the inequality (4.2), we have

$$N\Big(2f(ax) - 2a^6 f(x),\, t\Big) \;\leq\; \frac{\phi(x,\, 0)}{t + \phi(x,\, 0)} \tag{4.4}$$

for all $x \in X$ and all $t > 0$.

We note that by letting $x = \frac{x}{a}$ in the inequality (4.4) we have

$$N\Big(2f(x) - 2a^6 f\Big(\frac{x}{a}\Big),\, t\Big) \;\leq\; \frac{\phi(\frac{x}{a},\, 0)}{t + \phi(\frac{x}{a},\, 0)}\,.$$

The inequality (4.1) implies that

$$N\Big(f(x) - a^6 f\Big(\frac{x}{a}\Big),\, \frac{t}{2^\beta}\Big) \;\leq\; \frac{\frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}{t + \frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}\,.$$

By putting $t = \frac{L}{|a|^{6\beta}}\, t$, we have

$$N\Big(f(x) - a^6 f\Big(\frac{x}{a}\Big),\, \frac{L}{2^\beta |a|^{6\beta}}\, t\Big) \;\leq\; \frac{\frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}{\frac{L}{|a|^{6\beta}}\, t + \frac{L}{|a|^{6\beta}}\, \phi(x,\, 0)}\,,$$

that is,

$$N\Big(f(x) - a^6 f\Big(\frac{x}{a}\Big),\, \frac{L}{2^\beta |a|^{6\beta}}\, t\Big) \;\le\; \frac{\phi(x,\,0)}{t+\phi(x,\,0)}\,, \tag{4.5}$$

for all $x \in X$ and all $t > 0$.

We consider the set

$$F := \{g : X \to X\}$$

and the mapping d defined on $F \times F$ by

$$d(g,\,h) = \inf\{\mu \in \mathbb{R}^+ \mid N\Big(g(x) - h(x),\, \mu t\Big) \le \frac{\phi(x,\,0)}{t+\phi(x,\,0)}\,, \forall x \in X \text{ and } t > 0\}$$

where $\inf \emptyset = +\infty$, as usual. Then $(F,\,d)$ is a complete generalized metric space; see [22, Lemma 2.1]. Now let's consider the linear mapping $J : F \to F$ such that

$$Jg(x) := a^6 g\Big(\frac{x}{a}\Big)$$

for all $x \in X$. Let $g\,, h \in F$ be given such that $d(g\,, h) = \varepsilon$. Then

$$N\Big(g(x) - h(x),\, \varepsilon t\Big) \le \frac{\phi(x,\,0)}{t+\phi(x,\,0)}$$

for all $x \in X$ and all $t > 0$.

$$\begin{aligned} N\Big(Jg(x) - Jh(x),\, L\varepsilon t\Big) &= N\Big(a^6 g\Big(\frac{x}{a}\Big) - a^6 h\Big(\frac{x}{a}\Big),\, L\varepsilon t\Big) \\ = \quad N\Big(g\Big(\frac{x}{a}\Big) - h\Big(\frac{x}{a}\Big),\, \frac{L}{|a|^{6\beta}}\varepsilon t\Big) &\le \frac{\phi(\frac{x}{a},\,0)}{\frac{L}{|a|^{6\beta}}t + \phi(\frac{x}{a},\,0)} \\ \le \quad \frac{\frac{L}{|a|^{6\beta}}\phi(x,\,0)}{\frac{L}{|a|^{6\beta}}t + \frac{L}{|a|^{6\beta}}\phi(x,\,0)} &= \frac{\phi(x,\,0)}{t+\phi(x,\,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. $d(g,\,h) = \varepsilon$ implies that $d(Jg,\,Jh) \le L\varepsilon$. Hence we get

$$d(Jg,\,Jh) \le L\, d(g,\,h)$$

for all $g,\,h \in F$. The inequality (4.5) implies that $d(f,\,Jf) \le \frac{L}{2^\beta |a|^{6\beta}}$. By Theorem 1.8, there exists a mapping $S : X \to Y$ such that

(1) S is a fixed point of J, that is,

$$S\Big(\frac{x}{a}\Big) = \frac{1}{a^6}\, S(x) \tag{4.6}$$

for all $x \in X$. The mapping S is a unique fixed point of J in the set $M = \{g \in F \,|\, d(f,\,g) < \infty\}$. This means that S is a unique mapping satisfying the equation (4.6) such that there exists a $\mu \in (0,\,\infty)$ satisfying

$$N\Big(f(x) - S(x),\, \mu t\Big) \le \frac{\phi(x,\,0)}{t+\phi(x,\,0)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f,\,S) \to 0$ as $n \to \infty$. This implies the following equality

$$\text{N-}\lim_{n\to\infty} a^{6n} f\Big(\frac{x}{a^n}\Big) = S(x)$$

for all $x \in X$;

(3) $d(f, S) \le \frac{1}{1-L}\, d(f, Jf)$, which implies the inequality

$$d(f, S) \le \frac{1}{1-L} \cdot \frac{L}{2^{\beta}|a|^{6\beta}} = \frac{L}{2^{\beta}|a|^{6\beta}(1-L)}\,.$$

It implies that

$$N\Big(f(x) - S(x),\, \frac{L}{2^{\beta}|a|^{6\beta}(1-L)}t\Big) \le \frac{\phi(x,\,0)}{t + \phi(x,\,0)}$$

for all $x \in X$ and all $t > 0$. By replacing t by $\frac{2^{\beta}|a|^{6\beta}(1-L)}{L}\, t$, we have

$$N\Big(f(x) - S(x),\, t\Big) \le \frac{L\phi(x,\,0)}{2^{\beta}|a|^{6\beta}(1-L)\,t + L\phi(x,\,0)}$$

for all $x \in X$ and all $t > 0$. That is, the inequality (4.3) holds. By letting $x = \frac{x}{a^n}$ and $y = \frac{y}{a^n}$ in the inequality (4.2), we have

$$N\Big(a^{6n} D_a f\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big),\, |a|^{6\beta n}\, t\Big) \le \frac{\phi(\frac{x}{a^n},\, \frac{y}{a^n})}{t + \phi(\frac{x}{a^n},\, \frac{y}{a^n})}$$

for all $x,\, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Replacing t by $\frac{t}{|a|^{6\beta n}}$,

$$N\Big(a^{6n} D_a f\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big),\, t\Big) \le \frac{\phi(\frac{x}{a^n},\, \frac{y}{a^n})}{\frac{t}{|a|^{6\beta n}} + \phi(\frac{x}{a^n},\, \frac{y}{a^n})} \le \frac{L^n\, \phi(x,\, y)}{t + L^n\, \phi(x,\, y)}$$

for all $x,\, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{L^n\, \phi(x,\, y)}{t + L^n\, \phi(x,\, y)} = 0$ for all $x,\, y \in X$ and all $t > 0$, we may conclude that

$$N\Big(D_a S(x,\, y),\, t\Big) = 0$$

for all $x,\, y \in X$ and all $t > 0$. Thus the mapping $S : X \to Y$ is the generalized sextic mapping. □

Corollary 4.2. *Let $\theta \ge 0$, $p > 6$ be a real number and X be a normed linear space with norm $||\cdot||$. Suppose $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and*

$$N(D_a f(x, y),\, t) \le \frac{\theta(||x||^p + ||y||^p)}{t + \theta(||x||^p + ||y||^p)} \tag{4.7}$$

for all $x,\, y \in X$ and all $t > 0$. Then $S(x) := N\text{-}\lim_{n\to\infty} a^{6n} f\Big(\frac{x}{a^n}\Big)$ exists for each $x \in X$ and defines a generalized sextic mapping $S : X \to Y$ such that

$$N(f(x) - S(x),\, t) \le \frac{\theta\, ||x||^p}{2^{\beta}(|a|^{p\beta} - |a|^{6\beta})\, t + \theta\, ||x||^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 4.1 by taking $\phi(x,\, y) = \theta(||x||^p + ||y||^p)$ for all $x\,, y \in X$ and $L = |a|^{(6-p)\beta}$. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.2, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Existence and uniqueness of solutions to SFDEs driven by G-Brownian motion with non-Lipschitz conditions

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January 1, 2016

Abstract

The main aim of this paper is to study the existence, uniqueness and stability of solution for stochastic functional differential equations driven by G-Brownian motion (in short G-SFDEs). The existence-and-uniqueness theorem is established for G-SFDEs under non-Lipschitz condition and weakened linear growth condition. We have used the Picard approximation scheme, Gronwall's inequality, Bihari's inequality and Burkholder-Davis-Gundy (in short BDG) inequalities to develop the existence theory for the above mentioned stochastic dynamical systems. In addition, the mean square stability of solutions for these systems has been obtained.

Key words: Existence, uniqueness, stability, G-Brownian motion, stochastic functional differential equations.

1 Introduction

Responding to the contemporary developments in the fields of physics, control engineering, economics, and social sciences, a growing concern has recently been witnessed in both stochastic differential and deterministic models. The applications of functional differential equations have been applied in a number of cases in physical phenomena, such as in the relocation of soil moisture, where the fluid flows through the crack of rocks, and the problem of conduction of heat as well as its share in order fluids is investigated. The idea of G-Brownian motion as well as the associated stochastic differential equations were introduced by Peng [8, 10]. These equations were extended to stochastic functional differential equations, which are driven by G-Brownian motion (in short G-SFDEs) by Ren, Bi and Sakthivel [12]. While Faizullah, developed the existence-and-uniqueness theorem for G-SFDEs with Cauchy-Maruyama approximation scheme [3], they used the strong Lipschitz and linear growth conditions to develop the mentioned theory. In this article, we have generalized the existence theory for functional stochastic dynamical systems, driven by G-Brownian motion. We have used non-Lipschitz condition and weak linear growth condition to study the existence, uniqueness and stability theory for G-SFDEs. We have considered the following stochastic dynamical system that

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is driven by G-Brownian motion. Let $0 \leq t \leq T < \infty$. Suppose $g : [0,T] \times BC([-\theta,0];\mathbb{R}^n) \to \mathbb{R}^n$, $h : [0,T] \times BC([-\theta,0];\mathbb{R}^n) \to \mathbb{R}^n$ and $w : [0,T] \times BC([-\theta,0];\mathbb{R}^n) \to \mathbb{R}^n$ are Borel measurable. Consider stochastic functional differential equation driven by G-Brownian motion of the type

$$dX(t) = g(t,X_t)dt + h(t,X_t)d\langle B,B\rangle(t) + w(t,X_t)dB(t), \tag{1.1}$$

where $X(t)$ is the value of stochastic process at time t and $X_t = \{X(t+\delta) : -\theta \leq \delta \leq 0, \theta > 0\}$ is a $BC([-\theta,0];\mathbb{R}^n)$-valued stochastic process, which presents the family of bounded continuous $\mathbb{R}^n$-valued functions φ defined on $[-\theta,0]$ having norm $\|\varphi\| = \sup_{-\theta\leq\delta\leq 0} \mid \varphi(\delta) \mid$. $\{\langle B,B\rangle(t), t \geq 0\}$ is the quadratic variation process of G-Brownian motion $\{B(t), t \geq 0\}$ and $g,h,w \in M_G^2([-\tau,T];\mathbb{R}^n)$. Denote the space of all $\mathcal{F}_t$-adapted process $X(t), 0 \leq t \leq T$, such that $\| X \|_{L^2} = \sup_{-\theta\leq t\leq T} |X(t)| < \infty$ by L^2. The initial data of equation (1.1) is given as follows

$$\begin{aligned} X_{t_0} =& \zeta = \{\zeta(\delta) : -\theta < \delta \leq 0\}\ \textit{is}\ \mathcal{F}_0 - \textit{measurable},\ BC([-\theta,0];\mathbb{R}^n) -\ \textit{valued} \\ &\textit{random variable such that}\ \zeta \in M_G^2\left([-\theta,0];\mathbb{R}^n\right). \end{aligned} \tag{1.2}$$

The integral form of G-SFDE (1.1) with initial data (1.2) is given by

$$X(t) = \zeta(0) + \int_0^t g(s,X_s)ds + \int_0^t h(s,X_s)d\langle B,B\rangle(s) + \int_0^t w(s,X_s)dB(s).$$

The solution of G-SFDE (1.1) with initial data (1.2) is an $\mathbb{R}^n$ valued stochastic processes $X(t)$, $t \in [-\theta,T]$ such that

- **(i)** $X(t)$ is $\mathcal{F}_t$-adapted and continuous for all $t \in [0,T]$;
- **(ii)** $g(t,X_t) \in \mathcal{L}^1([o,T];\mathbb{R}^n)$ and $h(t,X_t), w(t,X_t) \in \mathcal{L}^2([0,T];\mathbb{R}^n)$;
- **(iii)** $X_0 = \zeta$ and for each $t \in [0,T]$, $dX(t) = g(t,X_t)dt + h(t,X_t)d\langle B,B\rangle(t) + w(t,X_t)dB(t)$ q.s.

$X(t)$ is called a unique solution if it is indistinguishable from any other solution $Y(t)$, that is,

$$E[\sup_{-\theta\leq q\leq t} |X(q) - Y(q)|^2] = 0.$$

Throughout this paper we assume the following two conditions, known as non-uniform Lipschitz condition and weakened linear growth condition respectively.

($\mathbf{A_i}$) For all $\varphi,\psi \in BC([-\theta,0];\mathbb{R}^d)$ and $t \in [0,T]$,

$$|g(t,\varphi) - g(t,\psi)|^2 + |h(t,\varphi) - |h(t,\psi)|^2 + |w(t,\varphi) - w(t,\psi)|^2 \leq \lambda(|\varphi - \psi|^2), \tag{1.3}$$

where $\lambda(.) : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing and concave function such that $\lambda(0) = 0$, $\lambda(v) > 0$ for $v > 0$ and

$$\int_{0+} \frac{dv}{\lambda(v)} = \infty. \tag{1.4}$$

As λ is concave and $\lambda(0) = 0$, there exists two positive constants c and d such that

$$\lambda(v) \leq c + dv, \tag{1.5}$$

for all $v \geq 0$.

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($\mathbf{A_{ii}}$) For all $t \in [0,T]$, $g(t,0), h(t,0), w(t,0) \in L^2$ and

$$|g(t,0)|^2 + |h(t,0)|^2 + |w(t,0)|^2 \leq K, \tag{1.6}$$

where K is a positive constant.

We have organized the rest of the paper as follows. In section 2, some well-known basic notions and results are included. In section 3, several important lemmas are developed. In section 4, the existence-and-uniqueness theorem is proved. In section 5, the mean square stability for the solution of G-SFDEs is given.

2 Preliminaries

The main purpose of this section is to give some basic concepts and results, which are used in the subsequent sections of this paper. For more detailed literature of G-expectation, we refer the readers to book [9] and papers [1, 2, 4, 5, 13].

Definition 2.1. Let $\mathcal{H}$ be a linear space of real valued functions defined on a nonempty basic space Ω. Then a sub-linear expectation E is a real valued functional on $\mathcal{H}$ with the following properties:

(**i**) For all $X, Y \in \mathcal{H}$, if $X \leq Y$ then $E[X] \leq E[Y]$.

(**ii**) For any real constant α, $E[\alpha] = \alpha$.

(**iii**) For all $X, Y \in \mathcal{H}$, $E[X+Y] \leq E[X] + E[Y]$.

(**iv**) For any $\theta > 0$ $E[\theta X] = \theta E[X]$.

Let $C_{b.Lip}(\mathbb{R}^{l\times d})$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^{l\times d}$ and

$$L^p_G(\Omega_T) = \{\phi(B_{t_1}, B_{t_2}, ..., B_{t_l}/l \geq 1, t_1, t_2, ..., t_l \in [0,T], \phi \in C_{b.Lip}(\mathbb{R}^{l\times d}))\}.$$

Let $\xi_i \in L^p_G(\Omega_{t_i})$, $i = 0, 1, ..., N-1$ then $M^0_G(0,T)$ denotes the collection of processes of the following type: For a given partition $\pi_T = \{t_0, t_1, ..., t_N\}$ of $[0,T]$,

$$\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1}]}(t).$$

Under the norm $\|\eta\| = \{\int_0^T E[|\eta_u|^p] du\}^{1/p}$, $M^p_G(0,T)$, $p \geq 1$, is the completion of $M^0_G(0,T)$. For every $\eta_t \in M^{2,0}_G(0,T)$, the G-Itô's integral $I(\eta)$ and G-quadratic variation process $\{\langle B\rangle_t\}_{t\geq 0}$ are respectively given by

$$I(\eta) = \int_0^T \eta_u dB_u = \sum_{i=0}^{N-1} \xi_i (B_{t_{i+1}} - B_{t_i}),$$

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_u dB_u.$$

The following definition and lemmas are borrowed from [7, 11].

Definition 2.2. A solution $X(t)$ of dynamical system (1.1) with initial data (1.2) is said to be stable in mean square if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $E|\zeta - \xi|^2 \leq \delta(\epsilon)$ follows that $E|X(t) - Y(t)|^2 < \epsilon$ for all $t \geq 0$, where $Y(t)$ is an other solution of system (1.1) having initial data $\xi \in M^2([-\theta, 0] : \mathbb{R}^l)$.

Lemma 2.3. *(Hölder's inequality) If* $\frac{1}{q} + \frac{1}{r} = 1$ *for any* $q, r > 1$, $g \in L^2$ *and* $h \in L^2$ *then* $gh \in L^1$ *and*

$$\int_c^d gh \leq (\int_c^d |g|^q)^{\frac{1}{q}} (\int_c^d |h|^r)^{\frac{1}{r}}.$$

Lemma 2.4. *(Gronwall's inequality) Let* $C \geq 0$, $h(t) \geq 0$ *and* $w(t)$ *be a real valued continuous function on* $[c, d]$. *If for all* $c \leq t \leq d$, $w(t) \leq C + \int_c^d h(s)w(s)ds$, *then*

$$w(t) \leq Ce^{\int_c^t h(s)ds},$$

for all $c \leq t \leq d$.

Lemma 2.5. *(Bihari's inequality) Suppose* $T \geq 0$ *and* $h_0 \geq 0$. *Assume* $h(t)$ *and* $w(t)$ *be continuous functions on* $[0, T]$. *Let* $\lambda(.) : \mathbb{R}^+ \to \mathbb{R}^+$ *be non-decreasing and concave continuous function such that* $\lambda(v) > 0$ *for* $v > 0$. *If for all* $0 \leq t \leq T$, $h(t) \leq h(0) + \int_0^T w(s)\lambda(h(s))ds$, *then for all* $0 \leq t \leq T$,

$$h(t) \leq H^{-1}(H(h_0) + \int_t^T w(s)ds),$$

such that $H(h_0) + \int_t^T w(s)ds \in Dom(H^{-1})$ *where* $H(q) = \int_t^q \frac{1}{\lambda(s)}ds$, $q \geq 0$ *and* H^{-1} *is the inverse function of* H.

Lemma 2.6. *Assume the assumptions of lemma 2.5 are satisfied and for* $0 \leq t \leq T$, $w(t) \geq 0$. *If for all* $\epsilon > 0$, *there exists* $t_1 \geq 0$ *such that for* $0 \leq h_0 \leq \epsilon$, $\int_{t_1}^T w(s)ds \leq \int_{h_0}^T \frac{1}{\lambda(s)}ds$ *holds, then for each* $t_1 \leq t \leq T$

$$h(t) \leq \epsilon,$$

holds.

3 Important results

In this section, we show some important lemmas. They will be used in the forth coming existence-and-uniqueness theorem. Let $X^0(t) = \zeta(0)$ for $t \in [0, T]$. Set $X^l(0) = \zeta$ for each $l = 1, 2, ...,$ and define the following Picard iterations sequence,

$$\begin{aligned} X^l(t) = \zeta(0) &+ \int_0^t g(s, X_s^{l-1})ds + \int_0^t h(s, X_s^{l-1})d\langle B, B\rangle(s) \\ &+ \int_0^t w(s, X_s^{l-1})dB(s), \quad t \in [0, T]. \end{aligned} \tag{3.1}$$

First, we show that $X^l(.) \in M_G^2([-\theta, T]; \mathbb{R}^n)$.

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Lemma 3.1. *Let assumptions A_i and A_{ii} hold. Then for all $l \geq 1$,*

$$\sup_{-\theta \leq t \leq T} E|X^l(t)|^2 \leq C,$$

where C is a positive constant.

Proof. Obviously, $X^0(.) \in M_G^2([-\theta, T]; \mathbb{R}^n)$. Using the basic inequality $|a+b+c+d|^2 \leq 4|a|^2 + 4|b|^2 + 4|c|^2 + 4|d|^2$, equation (3.1) yields

$$\begin{aligned}|X^l(t)|^2 \leq\ & 4|\zeta(0)|^2 + 4|\int_0^t g(s, X_s^{l-1})ds|^2 + 4|\int_0^t h(s, X_s^{l-1})d\langle B, B\rangle(s)|^2 \\ & + 4|\int_0^t w(s, X_s^{l-1})dB(s)|^2.\end{aligned}$$

Taking G-expectation on both sides, using the Burkholder-Davis-Gundy (BDG) inequalities [6] and Hölder inequality (lemma 2.3) we have

$$\begin{aligned}E|X^l(t)|^2 &\leq 4E|\zeta(0)|^2 + 4C_1 E\int_0^t |g(s, X_s^{l-1})|^2 ds \\ &+ 4C_2 E\int_0^t |h(s, X_s^{l-1})|^2 ds + 4C_3 \int_0^t |w(s, X_s^{l-1})|^2 ds \\ &\leq 4E|\zeta(0)|^2 + 8C_1 E\int_0^t (|g(s, X_s^{l-1}) - g(s,0)|^2 + |g(s,0)|^2) ds \\ &+ 8C_2 E\int_0^t (|h(s, X_s^{l-1}) - h(s,0)|^2 + |h(s,0)|^2) ds \\ &+ 8C_3 \int_0^t (|w(s, X_s^{l-1}) - w(s,0)|^2 + |w(s,0)|^2) d(s) \\ &\leq 4E|\zeta(0)|^2 + 8C_1 E\int_0^t |g(s,0)|^2 ds + 8C_1 E\int_0^t |g(s, X_s^{l-1}) - g(s,0)|^2 ds \\ &+ 8C_2 E\int_0^t |h(s,0)|^2 d(s) + 8C_2 E\int_0^t |h(s, X_s^{l-1}) - h(s,0)|^2 ds \\ &+ 8C_3 \int_0^t |w(s,0)|^2 ds + 8C_3 \int_0^t |w(s, X_s^{l-1}) - w(s,0)|^2 ds\end{aligned}$$

By assumptions A_i and A_{ii}, the above inequality yields

$$\begin{aligned}E|X^l(t)|^2 &\leq 4E|\zeta(0)|^2 + 8C_1 KT + 8C_2 KT + 8C_3 KT \\ &+ 8C_1 E\int_0^t \lambda(|X_s^{l-1}|^2) ds + 8C_2 E\int_0^t \lambda(|X_s^{l-1}|^2) d(s) + 8C_3 \int_0^t \lambda(|X_s^{l-1}|^2) d(s) \\ &= 4E|\zeta(0)|^2 + 8KT(C_1 + C_2 + C_3) + 8(C_1 + C_2 + C_3) E\int_0^t \lambda(|X_s^{l-1}|^2) ds \\ &\leq 4E|\zeta(0)|^2 + 8KT(C_1 + C_2 + C_3) + 8a(C_1 + C_2 + C_3)T \\ &+ 8b(C_1 + C_2 + C_3) E\int_0^t |X_s^{l-1})|^2 ds \\ &= K_1 + 8b(C_1 + C_2 + C_3) E\int_0^t |X_s^{l-1})|^2 ds,\end{aligned}$$

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where $K_1 = 4E|\zeta(0)|^2 + 8C_0KT + 8aC_0T$. and $C_0 = C_1 + C_2 + C_3$. Noting that

$$\sup_{0\le s\le t}|X_s^l|^2 \le \sup_{0\le s\le t}\sup_{-\theta\le u\le 0}|X^l(s+u)|^2 \le \sup_{-\theta\le q\le t}|X^l(q)|^2 \le |\zeta|^2 + \sup_{0\le q\le t}|X^l(q)|^2,$$

we have

$$\sup_{-\theta\le q\le t} E|X^l(q)|^2 \le E|\zeta|^2 + K_1 + 8b(C_1+C_2+C_3)E\int_0^t \sup_{-\theta\le q\le t}|X^{l-1}(q)|^2 ds.$$

Again noting that for any $j \ge 1$

$$\max_{1\le l\le j} E|X_s^{l-1}|^2 \le E|\zeta|^2 + \max_{1\le l\le j} E|X^l(q)|^2,$$

we obtain

$$\begin{aligned}\max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2 &\le E|\zeta|^2 + K_1 + 8b(C_1+C_2+C_3)\int_0^t [E|\zeta|^2 + \max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2]ds \\ &\le E|\zeta|^2 + K_1 + 8b(C_1+C_2+C_3)TE|\zeta|^2 + \int_0^t \max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2 ds \\ &= K_2 + 8b(C_1+C_2+C_3)\int_0^t \max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(q)|^2 ds,\end{aligned}$$

where $K_2 = K_1 + (1 + 8bC_0T)E|\zeta|^2$. Now the Gronwall inequality (lemma 2.4) yields

$$\max_{1\le l\le j}\sup_{-\theta\le q\le t} E|X^l(t)|^2 \le C,$$

where $C = K_2 e^{8bC_0T}$, but j is arbitrary, so

$$\sup_{-\theta\le t\le T} E|X^l(t)|^2 \le C.$$

The proof is complete. □

Lemma 3.2. *Under the assumptions A_i and A_{ii} there exists a positive constant C^* such that for all $l, d \ge 1$,*

$$\begin{aligned}E\sup_{-\theta\le s\le t}|X^{l+d}(s) - X^l(s)|^2 &\le \hat{C}\int_0^t \lambda(E\sup_{-\theta\le q\le s}|X^{l+d-1}(q) - X^{l-1}(q)|^2)ds \\ &\le C^* t.\end{aligned}$$

Proof. Using the basic inequality $|a+b+c|^2 \le 3|a|^2 + 3|b|^2 + 3|c|^2$, equation (3.1) yields

$$\begin{aligned}|X^{l+d}(t) - X^l(t)|^2 \le 3|\int_0^t [g(s, X_s^{l+d-1}) - g(s, X_s^{l-1})]ds|^2 + 3|\int_0^t [h(s, X_s^{l+d-1}) - h(s, X_s^{l-1})]d\langle B, B\rangle(s)|^2 \\ + 3|\int_{t_0}^t [w(s, X_s^{l+d-1}) - w(s, X_s^{l-1})]dB(s)|^2\end{aligned}$$

Taking G-expectation on both sides, using the BDG inequalities [6], Jensen inequality $E(\lambda(x)) \leq \lambda(E(x))$, Holder inequality and assumptions A_i, A_i it gives

$$\begin{aligned}
E[\sup_{-\theta\leq s\leq t}|X^{l+d}(s)-X^l(s)|^2] &\leq 3C_1\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds\\
&+3C_2\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds\\
&+3C_3\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds\\
&\leq 3(C_1+C_2+C_3)\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds.
\end{aligned}$$

$$E[\sup_{-\theta\leq s\leq t}|X^{l+d}(s)-X^l(s)|^2] \leq \hat{C}\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s}|X^{l+d-1}(q)-X^{l-1}(q)|^2])ds,$$

where $\hat{C}=3C_0$. Finally, using lemma 3.1 it yields

$$E[\sup_{-\theta\leq s\leq t}|X^{l+d}(s)-X^l(s)|^2] \leq \hat{C}\lambda(4C)t = C^*t,$$

where $C^*=\hat{C}\lambda(4C)$. The proof is complete. □

4 Existence and uniqueness results for G-SFDEs

We introduce the following new notations to prepare a key lemma. Choose $T_1 \in [0,T]$ such that for all $t\in[0,T_1]$

$$\hat{C}\lambda(C^*t) \leq C^*. \tag{4.1}$$

For all $l,d\geq 1$, define the following recursive function

$$\phi_1(t)=C^*t. \tag{4.2}$$

$$\begin{aligned}
\phi_{l+1}(t) &= \hat{C}\int_0^t \lambda(\phi_l(s))ds,\\
\phi_{l,d}(t) &= E[\sup_{-\theta\leq q\leq t}|X^{l+d}(q)-X^l(q)|^2].
\end{aligned} \tag{4.3}$$

Lemma 4.1. *Under the hypothesis A_i and A_{ii} for any $d\geq 1$ and all $l\geq 1$ there exists a positive $T_1\in[0,T]$ such that*

$$0\leq \phi_{l,d}(t)\leq \phi_l(t)\leq \phi_{l-1}(t)\leq ... \leq \phi_1(t), \tag{4.4}$$

for all $t\in[0,T_1]$.

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Proof. We use mathematical induction to prove the inequality (4.4). Using the definition of function $\phi(.)$ and lemma 3.2, we have

$$\phi_{1,d}(t) = E[\sup_{-\theta\leq q\leq t} |X^{1+d}(q) - X^1(q)|^2] \leq C^* t = \phi_1(t).$$

$$\begin{aligned}\phi_{2,d}(t) &= E[\sup_{-\theta\leq q\leq t} |X^{2+d}(q) - X^2(q)|^2] \\ &\leq \hat{C}\int_0^t \lambda(E[\sup_{-\theta\leq q\leq t} |X^{1+d}(q) - X^1(q)|^2])ds \\ &\leq \hat{C}\int_0^t \lambda(\phi_1(s))ds = \phi_2(t).\end{aligned}$$

Using (4.1), we have

$$\phi_2(t) = \hat{C}\int_0^t \lambda(\phi_1(s))ds = \int_0^t \hat{C}\lambda(C^* t)ds \leq C^* t = \phi_1(t).$$

Hence for all $t \in [0, T_1]$, we derive that $\phi_{2,d}(t) \leq \phi_2(t) \leq \phi_1(t)$. Next, suppose that the inequality (4.4) holds for some $l \geq 1$. We now show that lemma 4.1 is valid for $l + 1$, as follows

$$\begin{aligned}\phi_{l+1,d}(t) &= E[\sup_{-\theta\leq q\leq t} |X^{l+d+1}(q) - X^{l+1}(q)|^2] \\ &\leq \hat{C}\int_0^t \lambda(E[\sup_{-\theta\leq q\leq s} |X^{l+d}(q) - X^l(q)|^2])ds \\ &= \hat{C}\int_0^t \lambda(\phi_{l,d}(s))ds \\ &\leq \hat{C}\int_0^t \lambda(\phi_l(s))ds \\ &= \phi_{l+1}(t).\end{aligned}$$

Also

$$\phi_{l+1}(t) = \hat{C}\int_0^t \lambda(\phi_l(s))ds \leq \hat{C}\int_0^t \lambda(\phi_{l-1}(s))ds = \phi_l(s).$$

Hence for all $t \in [0, T_1]$, we derive that $\phi_{l+1,d}(t) \leq \phi_{l+1}(t) \leq \phi_l(s)$, that is, lemma 4.1 holds for $l + 1$. The proof is complete. ☐

Theorem 4.2. *Let assumptions A_i and A_{ii} hold. Then the stochastic system* (1.1) *with initial data* (1.2) *has a unique solution.*

Proof. We split the whole proof in two steps. First, we show uniqueness and then existence. Let system (1.1) with initial data (1.2) has two solutions $X(t)$ and $Y(t)$. Then we have

$$\begin{aligned}|X(t) - Y(t)| \leq &\int_0^t |g(s, X_s) - g(s, Y_s)|ds + \int_0^t |h(s, X_s) - h(s, Y_s)|d\langle B, B\rangle(s) \\ &+ \int_0^t |w(s, X_s) - w(s, Y_s)|dB(s).\end{aligned}$$

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Taking G-expectation on both sides and using the basic inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, Hölder inequality and BDG inequalities [6], it follows

$$E|X(t)-Y(t)|^2 \leq 3C_1 \int_0^t E|g(s,X_s)-g(s,Y_s)|^2 ds + 3C_2 \int_0^t E|h(s,X_s)-h(s,Y_s)|^2 ds$$
$$+ 3C_3 \int_0^t E|w(s,X_s)-w(s,Y_s)|ds.$$

Using assumptions A_i and A_{ii} we have

$$E[\sup_{-\theta\leq q\leq t} |X(q)-Y(q)|^2] \leq 3(C_1+C_2+C_3) \int_0^t \lambda(E[\sup_{-\theta\leq q\leq s} |X(q)-Y(q)|^2])ds,$$

Then lemma 2.5 and lemma 2.6 gives $E[\sup_{-\theta\leq q\leq t}|X(q)-Y(q)|^2] = 0$, $t \in [0,T]$. The proof of uniqueness is complete.

Next we show existence. We note that on $t \in [0,T_1]$, $\phi_l(t)$ is continuous. For $l \geq 1$, it is decreasing on $t \in [0,T_1]$. By dominated convergence theorem, we define the function $\phi(t)$ as follows

$$\phi(t) = \lim_{l\to\infty} \phi_l(t) = \lim_{l\to\infty} \hat{C} \int_0^t \lambda(\phi_{l-1}(s))ds = \hat{C} \int_0^t \lambda(\phi(s))ds, \quad 0 \leq t \leq T_1.$$

So,

$$\phi(t) \leq \phi(0) + \hat{C} \int_0^t \lambda(\phi(s))ds.$$

Thus for all $0 \leq t \leq T_1$, lemma 2.5 and lemma 2.6 follow that $\phi(t) = 0$. From lemma 4.1 for all $t \in [0,T_1]$ we get $\phi_{l,d}(s) \leq \phi_l(s) \to 0$ as $l \to \infty$, which yields $E|X^{l+d}(t) - X^l(t)|^2 \to 0$ as $l \to \infty$. By the property of function $\lambda(.)$, assumptions A_i, A_{ii} and completeness of L^2, it follows that for all $t \in [0,T_1]$,

$$g(t,X_t^l) \to g(t,X_t), h(t,X_t^l) \to h(t,X_t), w(t,X_t^l) \to w(t,X_t) \;\; in\ L^2 \;\; as\ \ l \to \infty.$$

Hence for all $t \in [0,T_1]$,

$$\lim_{l\to\infty} X^l(t) = \zeta(0) + \lim_{l\to\infty} \int_0^t g(s,X_s^{l-1})ds$$
$$+ \lim_{l\to\infty} \int_0^t h(s,X_s^{l-1})d\langle B,B\rangle(s) + \lim_{l\to\infty} \int_0^t w(s,X_s^{l-1})dB(s),$$

that is,

$$X(t) = \zeta(0) + \int_0^t g(s,X_s)ds + \int_0^t h(s,X_s)d\langle B,B\rangle(s) + \int_0^t w(s,X_s)dB(s).$$

Thus $X(t)$ is a unique solution of stochastic system (1.1) with initial data (1.2) on $t \in [0,T_1]$. Thus by iteration, one can obtain that the system (1.1) has a unique solution on $t \in [0,T]$. The proof is complete. □

5 Dependence of solutions

In this section, we use lemma 2.5 and lemma 2.6 to give continuous dependence of solutions for stochastic system (1.1) with initial data (1.2).

Theorem 5.1. *Let assumptions A_i and A_{ii} hold. Assume $X(t)$ and $Y(t)$ be two solutions of dynamical system (1.1) with initial data ζ and ξ respectively. If for all $\epsilon > 0$ and $t \in [0,T]$ there exists $\delta(\epsilon) > 0$ such that $E|\zeta - \xi|^2 < \delta(\epsilon)$, then*

$$E|X(t) - Y(t)|^2 \leq \epsilon.$$

Proof. Since $X(t)$ and $Y(t)$ are any two solutions of system (1.1). It follows that for any $t \in [0,T]$,

$$X(t) = \zeta(0) + \int_0^t g(s, X_s)ds + \int_0^t h(s, X_s)d\langle B, B\rangle(s) + \int_0^t w(s, X_s)dB(s) \;\; q.s.$$

$$Y(t) = \xi(0) + \int_0^t g(s, Y_s)ds + \int_0^t h(s, Y_s)d\langle B, B\rangle(s) + \int_0^t w(s, Y_s)dB(s) \;\; q.s.$$

Then

$$\begin{aligned} X(t) - Y(t) = \zeta(0) - \xi(0) + \int_0^t [g(s, X_s) - g(s, Y_s)]ds + \int_0^t [h(s, X_s) - h(s, Y_s)]d\langle B, B\rangle(s) \\ + \int_0^t [w(s, X_s) - w(s, Y_s)]dB(s) \;\; q.s. \end{aligned}$$

Taking G-expectation on both sides, using the fundamental inequality $(a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)$, BDG inequalities [6] and Hölder inequality, it follows

$$E[\sup_{-\theta \leq r \leq t} |X(r) - Y(r)|^2] \leq 4E|\zeta(0) - \xi(0)|^2 + 4(C_1 + C_2 + C_3)\int_0^t \lambda(E[\sup_{-\theta \leq r \leq t} |X(r) - Y(r)|^2])ds.$$

Thus from lemma 2.5 and 2.6 we have

$$E[|X(t) - Y(t)|^2] \leq \epsilon,$$

for $t \in [0,T]$. The proof is complete. □

6 Acknowledgement

The financial support of NUST research directorate for this research work is acknowledged and deeply appreciated.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.2, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Approximation of a kind of new Bernstein-Bézier type operators

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Abstract. In this paper, a kind of new Bernstein-Bézier type operators is introduced. The Korovkin type approximation theorem of these operators is investigated. The rates of convergence of these operators are studied by means of modulus of continuity. Then, by using the Ditzian-Totik modulus of smoothness, a direct theorem concerned with an approximation for these operators is also obtained.

Keywords: Bernstein-Bézier type operators; Korovich type approximation theorem; rate of convergence; direct theorem; modulus of smoothness

Mathematical subject classification: 41A10, 41A25, 41A36

1. Introduction

In view of the Bézier basis function, which was introduced by Bézier [1], in 1983, Chang [2] defined the generalized Bernstein-Bézier polynomials for any $\alpha > 0$, and a function f defined on $[0,1]$ as follows:

$$B_{n,\alpha}(f;x) = \sum_{k=0}^{n} f(\frac{k}{n})[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)], \tag{1}$$

where $J_{n,n+1}(x) = 0$, and $J_{n,k}(x) = \sum\limits_{i=k}^{n} P_{n,i}(x)$, $k = 0,1,...,n$, $P_{n,i}(x) = \left(\begin{array}{c} n \\ i \end{array} \right) x^i(1-x)^{n-i}$. $J_{n,k}(x)$ is the Bézier basis function of degree n.

Obviously, when $\alpha = 1$, $B_{n,\alpha}(f;x)$ become the well-known Bernstein polynomials $B_n(f;x)$, and for any $x \in [0,1]$, we have $1 = J_{n,0}(x) > J_{n,1}(x) > ... > J_{n,n}(x) = x^n, J_{n,k}(x) - J_{n,k+1}(x) = P_{n,k}(x)$.

During the last ten years, the Bézier basis function was extensively used for constructing various generalizations of many classical approximation processes. Some Bézier type operators, which are based on the Bézier basis function, have been introduced and studied (e.g., see [3-9]).

In 2013, Ren [10] introduced generalized Bernstein operators as follows:

$$E_{n,\beta}(f;x) = f(0)P_{n,0}(x) + \sum_{k=1}^{n-1} P_{n,k}(x)F_{n,k}^{(\beta)}(f) + f(1)P_{n,n}(x), \tag{2}$$

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where $f \in C[0,1]$, $x \in [0,1]$, $P_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k(1-x)^{n-k}$, $k = 0,1,...,n$, and

$$F_{n,k}^{(\beta)}(f) = \frac{1}{B(nk, n(n-k))} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} f(\beta t + (1-\beta)\frac{k}{n})dt, \quad (3)$$

where $k = 1,...,n-1$, $\beta \in [0,1]$, $B(.,.)$ is the beta function.

The moments of the operators $E_{n,\beta}(f;x)$ were obtained as follows (see [10]).

Remark For $E_{n,\beta}(t^j;x)$, $j = 0,1,2$, we have

(i) $E_{n,\beta}(1;x) = 1$;
(ii) $E_{n,\beta}(t;x) = x$;
(iii) $E_{n,\beta}(t^2;x) = x^2 + \left[\frac{1}{n} + \frac{(n-1)\beta^2}{(n^2+1)n}\right] x(1-x)$.

In the present paper, we will study the Bézier variant of the generalized Bernstein operators $E_{n,\beta}(f;x)$ given by (2). We introduce Bernstein-Bézier type operators as follows:

$$E_{n,\beta}^{(\alpha)}(f;x) = f(0)Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(f) + f(1)Q_{n,n}^{(\alpha)}(x), \quad (4)$$

where $f \in C[0,1]$, $x \in [0,1]$, $\beta \in [0,1]$, $\alpha > 0$, $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{n,n+1}(x) = 0$, $J_{n,k}(x) = \sum\limits_{i=k}^{n} P_{n,i}(x)$, $k = 0,1,...,n$, $P_{n,i}(x) = \left(\begin{array}{c} n \\ i \end{array}\right) x^i(1-x)^{n-i}$, and $F_{n,k}^{(\beta)}(f)$ is defined as above (3).

It is clear that $E_{n,\beta}^{(\alpha)}(f;x)$ are bounded and positive on C[0,1]. When $\alpha = 1$, $E_{n,\beta}^{(\alpha)}(f;x)$ become the operators $E_{n,\beta}(f;x)$. When $\beta = 0$, $E_{n,\beta}^{(\alpha)}(f;x)$ become the generalized Bernstein-Bézier operators $B_{n,\alpha}(f;x)$.

The goal of this paper is to study the approximation properties of these operators with the help of the Korovkin type approximation theorem. We also estimate the rates of convergence of these operators by using a modulus of continuity. Then we obtain the direct theorem concerned with an approximation for these operators by means of the Ditzian-Totik modulus of smoothness.

In the paper, for $f \in C[0,1]$, we denote $\|f\| = max\{|f(x)| : x \in [0,1]\}$. $\omega(f,\delta)$ $(\delta > 0)$ denotes the usual modulus of continuity of $f \in C[0,1]$.

2. Auxiliary results

In the sequel, we shall need the following auxiliary results.

Lemma 1 (see [2]) *Let $\alpha > 0$. We have*

(i) $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} J_{n,k}^{\alpha}(x) = x$ *uniformly on* $[0,1]$;

(ii) $\lim\limits_{n\to\infty} \frac{1}{n^2} \sum\limits_{k=1}^{n} kJ_{n,k}^{\alpha}(x) = \frac{x^2}{2}$ *uniformly on* $[0,1]$.

Lemma 2 *Let $\alpha > 0$. We have*

$$\text{(i) } E_{n,\beta}^{(\alpha)}(1;x) = 1;$$
$$\text{(ii) } \lim_{n\to\infty} E_{n,\beta}^{(\alpha)}(t;x) = x \ \textit{uniformly on } [0,1];$$
$$\text{(iii) } \lim_{n\to\infty} E_{n,\beta}^{(\alpha)}(t^2;x) = x^2 \ \textit{uniformly on } [0,1].$$

Proof By simple calculation, we obtain $F_{n,k}^{(\beta)}(1) = 1,\ F_{n,k}^{(\beta)}(t) = \frac{k}{n},\ F_{n,k}^{(\beta)}(t^2) = \frac{\beta^2}{n^2+1}\cdot\frac{k}{n} + (1-\frac{\beta^2}{n^2+1})\frac{k^2}{n^2}$.

(i) Since $\sum\limits_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) = 1$, by (4) we can get $E_{n,\beta}^{(\alpha)}(1;x) = 1$.

(ii) By (4), we have

$$\begin{aligned}
&E_{n,\beta}^{(\alpha)}(t;x)\\
&= \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)\frac{k}{n} + Q_{n,n}^{(\alpha)}(x)\\
&= [J_{n,1}^{\alpha}(x) - J_{n,2}^{\alpha}(x)]\frac{1}{n} + ... + [J_{n,n-1}^{\alpha}(x) - J_{n,n}^{\alpha}(x)]\frac{n-1}{n} + J_{n,n}^{\alpha}(x)\frac{n}{n}\\
&= \frac{1}{n}\sum_{k=1}^{n} J_{n,k}^{\alpha}(x),
\end{aligned}$$

thus, by Lemma 1 (i), we have $\lim\limits_{n\to\infty} E_{n,\beta}^{(\alpha)}(t;x) = x$ uniformly on $[0,1]$.

(iii) By (4), we have

$$\begin{aligned}
&E_{n,\beta}^{(\alpha)}(t^2;x)\\
&= \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x)\left[\frac{\beta^2}{n^2+1}\cdot\frac{k}{n} + (1-\frac{\beta^2}{n^2+1})\frac{k^2}{n^2}\right] + Q_{n,n}^{(\alpha)}(x)\\
&= \frac{\beta^2}{n^2+1}\cdot\frac{1}{n}\sum_{k=1}^{n} kQ_{n,k}^{(\alpha)}(x) + (1-\frac{\beta^2}{n^2+1})\cdot\frac{1}{n^2}\sum_{k=1}^{n} k^2 Q_{n,k}^{(\alpha)}(x)\\
&= \frac{\beta^2}{n^2+1}\cdot\frac{1}{n}\sum_{k=1}^{n} J_{n,k}^{\alpha}(x) + (1-\frac{\beta^2}{n^2+1})\cdot\frac{1}{n^2}\sum_{k=1}^{n}(2k-1)J_{n,k}^{\alpha}(x),
\end{aligned}$$

thus, by Lemma 1, we have $\lim\limits_{n\to\infty} E_{n,\beta}^{(\alpha)}(t^2;x) = x^2$ uniformly on $[0,1]$.

Lemma 3 (see [11]) *For $x \in [0,1]$, $k = 0,1,...,n$, we have*

$$0 \le Q_{n,k}^{(\alpha)}(x) \le \begin{cases} \alpha P_{n,k}(x), & \alpha \ge 1;\\ P_{n,k}^{\alpha}(x), & 0 < \alpha < 1. \end{cases}$$

Lemma 4 (see [12]) *For $0 < \alpha < 1$, $\gamma > 0$, we have*

$$\sum_{k=0}^{n} |k - nx|^{\gamma} P_{n,k}^{\alpha}(x) \le (n+1)^{1-\alpha}(A_{\frac{\gamma}{\alpha}})^{\alpha} n^{\frac{\gamma}{2}},$$

where the constant A_s only depends on s.

Lemma 5 *For $\alpha \geq 1$, we have*

$$\text{(i) } E_{n,\beta}^{(\alpha)}((t-x)^2;x) \leq \frac{\alpha}{4}(1+\frac{\beta^2}{5}) \cdot \frac{1}{n};$$
$$\text{(ii) } E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})} \cdot \sqrt{\frac{1}{n}}.$$

Proof Let $\alpha \geq 1$.

(i) By (4), Lemma 3 and Remark 1, we obtain

$$\begin{aligned}
&E_{n,\beta}^{(\alpha)}((t-x)^2;x) \\
&= x^2 Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2 Q_{n,n}^{(\alpha)}(x) \\
&\leq \alpha[x^2 P_{n,0}(x) + \sum_{k=1}^{n-1} P_{n,k}(x) F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2 P_{n,n}(x)] \\
&= \alpha E_{n,\beta}((t-x)^2;x) \\
&= \frac{\alpha}{n}\left(1+\frac{n-1}{n^2+1}\beta^2\right) x(1-x). \qquad (5)
\end{aligned}$$

Since $\max\limits_{0\leq x\leq 1} x(1-x) = \frac{1}{4}$, and for any $n \in N$, one can get $\frac{n-1}{n^2+1} \leq \frac{1}{5}$, so we have

$$E_{n,\beta}^{(\alpha)}((t-x)^2;x) \leq \frac{\alpha}{4}(1+\frac{\beta^2}{5}) \cdot \frac{1}{n}.$$

(ii) In view of $E_{n,\beta}^{(\alpha)}(1;x) = 1$, by the Cauchy-Schwarz inequality, we have

$$E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)},$$

thus, we get $E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})} \cdot \sqrt{\frac{1}{n}}$.

Lemma 6 *For $0 < \alpha < 1$, we have*

$$\text{(i) } E_{n,\beta}^{(\alpha)}((t-x)^2;x) \leq M_\alpha^{(\beta)} n^{-\alpha};$$
$$\text{(ii) } E_{n,\beta}^{(\alpha)}(|t-x|;x) \leq \sqrt{M_\alpha^{(\beta)}} \cdot n^{-\frac{\alpha}{2}}.$$

Where the constant $M_\alpha^{(\beta)}$ only depends on α, β.

Proof Let $0 < \alpha < 1$.

(i) In view of (4), Lemma 3 and $F_{n,k}^{(\beta)}((t-x)^2) = \frac{(k-nx)^2}{n^2} + \frac{\beta^2}{n^2+1}(\frac{k}{n} - \frac{k^2}{n^2})$,

we obtain

$$\begin{aligned}
&E_{n,\beta}^{(\alpha)}((t-x)^2;x)\\
&= x^2 Q_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{n-1} Q_{n,k}^{(\alpha)}(x) F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2 Q_{n,n}^{(\alpha)}(x)\\
&\le x^2 P_{n,0}^{\alpha}(x) + \sum_{k=1}^{n-1} P_{n,k}^{\alpha}(x) F_{n,k}^{(\beta)}((t-x)^2) + (1-x)^2 P_{n,n}^{\alpha}(x)\\
&= \sum_{k=0}^{n} P_{n,k}^{\alpha}(x) \left[\frac{(k-nx)^2}{n^2} + \frac{\beta^2}{n^2+1}\left(\frac{k}{n} - \frac{k^2}{n^2}\right)\right]\\
&= \frac{1}{n^2}\sum_{k=0}^{n}(k-nx)^2 P_{n,k}^{\alpha}(x) + \frac{\beta^2}{n^2+1}\sum_{k=0}^{n} P_{n,k}^{\alpha}(x)\left(\frac{k}{n} - \frac{k^2}{n^2}\right)\\
&:= I_1 + I_2.
\end{aligned}$$

By Lemma 4, we have $I_1 \le \frac{n+1}{n}(n+1)^{-\alpha}(A_{\frac{2}{\alpha}})^{\alpha} \le 2(A_{\frac{2}{\alpha}})^{\alpha} n^{-\alpha}$, where the constant $A_{\frac{2}{\alpha}}$ only depends on α.

Using the Hölder inequality, we have $\sum\limits_{k=0}^{n} P_{n,k}^{\alpha}(x) \le (n+1)^{1-\alpha}[\sum\limits_{k=0}^{n} P_{n,k}(x)]^{\alpha}$, and $(\frac{k}{n} - \frac{k^2}{n^2}) \le 1$, so we have

$$I_2 \le \frac{\beta^2}{n^2+1}(n+1)^{1-\alpha}[\sum_{k=0}^{n} P_{n,k}(x)]^{\alpha} = \frac{\beta^2}{n^2+1}(n+1)^{1-\alpha} \le \beta^2 n^{-\alpha}.$$

Denote $M_{\alpha}^{(\beta)} = 2(A_{\frac{2}{\alpha}})^{\alpha} + \beta^2$, then we can get $E_{n,\beta}^{(\alpha)}((t-x)^2;x) \le M_{\alpha}^{(\beta)} n^{-\alpha}$.

(ii) Since

$$E_{n,\beta}^{(\alpha)}(|t-x|;x) \le \sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)},$$

thus, we get

$$E_{n,\beta}^{(\alpha)}(|t-x|;x) \le \sqrt{M_{\alpha}^{(\beta)}} \cdot n^{-\frac{\alpha}{2}}.$$

Lemma 7 *For $f \in C[0,1],\ x \in [0,1]$ and $\alpha > 0,$ we have*

$$|\ E_{n,\beta}^{(\alpha)}(f;x)\ | \le \|\ f\ \|\ .$$

Proof By (4) and Lemma 2 (i), we have

$$|\ E_{n,\beta}^{(\alpha)}(f;x)\ | \le \|f\| E_{n,\beta}^{(\alpha)}(1;x) = \|f\|.$$

3. Main results

First of all we give the following convergence theorem for the sequence $\{E_{n,\beta}^{(\alpha)}(f;x)\}$.

Theorem 1 *Let $\alpha > 0$. Then the sequence $\{E_{n,\beta}^{(\alpha)}(f;x)\}$ converges to f uniformly on $[0,1]$ for any $f \in C[0,1]$.*

Proof Since $E_{n,\beta}^{(\alpha)}(f;x)$ is bounded and positive on $C[0,1]$, and by Lemma 2, we have $\lim\limits_{n\to\infty}\|E_{n,\beta}^{(\alpha)}(e_j;\cdot)-e_j\|=0$ for $e_j(t)=t^j$, $j=0,1,2$. So, according to the well-known Bohman-korovkin theorem ([13, P.40, Theorem 1.9]), we see that the sequence $\{E_{n,\beta}^{(\alpha)}(f;x)\}$ converges to f uniformly on [0,1] for any $f\in C[0,1]$.

Next we estimate the rates of convergence of the sequence $\{E_{n,\beta}^{(\alpha)}\}$ by means of the modulus of continuity.

Theorem 2 *Let $f\in C[0,1]$, $x\in[0,1]$. Then*

(i) *when* $\alpha\geq 1,$ *we have* $\|E_{n,\beta}^{(\alpha)}(f;\cdot)-f\|\leq\left[1+\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\right]\omega(f,\frac{1}{\sqrt{n}});$

(ii) *when* $0<\alpha<1,$ *we have* $\|E_{n,\beta}^{(\alpha)}(f;\cdot)-f\|\leq(1+\sqrt{M_\alpha^{(\beta)}})\omega(f,n^{-\frac{\alpha}{2}}).$

Where the constant $M_\alpha^{(\beta)}$ only depends on α,β.

Proof (i) When $\alpha\geq 1$, by Lemma 2 (i), we have

$$
\begin{aligned}
&|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\\
&\leq|f(0)-f(x)|Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(|f(t)-f(x)|)+|f(1)-f(x)|Q_{n,n}^{(\alpha)}(x)\\
&\leq\omega(f,|0-x|)Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(\omega(f,|t-x|))+\omega(f,|1-x|)Q_{n,n}^{(\alpha)}(x)\\
&\leq(1+\sqrt{n}|0-x|)\omega(f,\frac{1}{\sqrt{n}})Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}((1+\sqrt{n}|t-x|)\omega(f,\frac{1}{\sqrt{n}}))\\
&\quad+(1+\sqrt{n}|1-x|)\omega(f,\frac{1}{\sqrt{n}})Q_{n,n}^{(\alpha)}(x)\\
&\leq\omega(f,\frac{1}{\sqrt{n}})+\sqrt{n}\omega(f,\frac{1}{\sqrt{n}})E_{n,\beta}^{(\alpha)}(|t-x|;x),
\end{aligned}
$$

so, by Lemma 5 (ii), we obtain

$$
|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\leq\left[1+\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\right]\omega(f,\frac{1}{\sqrt{n}}).
$$

The desired result follows immediately.

(ii) When $0<\alpha<1$, by Lemma 2 (i), we have

$$
\begin{aligned}
&|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\\
&\leq\omega(f,|0-x|)Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(\omega(f,|t-x|))+\omega(f,|1-x|)Q_{n,n}^{(\alpha)}(x)\\
&\leq(1+n^{\frac{\alpha}{2}}|0-x|)\omega(f,n^{-\frac{\alpha}{2}})Q_{n,0}^{(\alpha)}(x)+\sum_{k=1}^{n-1}Q_{n,k}^{(\alpha)}(x)F_{n,k}^{(\beta)}(1+n^{\frac{\alpha}{2}}|t-x|)\omega(f,n^{-\frac{\alpha}{2}})\\
&\quad+(1+n^{\frac{\alpha}{2}}|1-x|)\omega(f,n^{-\frac{\alpha}{2}})Q_{n,n}^{(\alpha)}(x)\\
&=\omega(f,n^{-\frac{\alpha}{2}})+n^{\frac{\alpha}{2}}\omega(f,n^{-\frac{\alpha}{2}})E_{n,\beta}^{(\alpha)}(|t-x|;x),
\end{aligned}
$$

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so, by Lemma 6 (ii), we obtain

$$|E_{n,\beta}^{(\alpha)}(f;x)-f(x)| \le (1+\sqrt{M_\alpha^{(\beta)}})\omega(f,n^{-\frac{\alpha}{2}}).$$

The desired result follows immediately.

Theorem 3 *Let $f\in C^1[0,1]$, $x\in[0,1]$. Then*
(i) *when $\alpha\ge 1$, we have*

$$\begin{aligned}|E_{n,\beta}^{(\alpha)}(f;x)-f(x)| &\le \|f'\|\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\cdot\sqrt{\frac{1}{n}}\\ &+ \omega(f',\frac{1}{\sqrt{n}})\left[1+\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\right]\cdot\sqrt{\frac{\alpha}{4}(1+\frac{\beta^2}{5})}\cdot\sqrt{\frac{1}{n}};\end{aligned}$$

(ii) *when $0<\alpha<1$, we have*

$$|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\le\|f'\|\sqrt{M_\alpha^{(\beta)}n^{-\alpha}}+\omega(f',n^{-\frac{\alpha}{2}})(1+\sqrt{M_\alpha^{(\beta)}})\sqrt{M_\alpha^{(\beta)}n^{-\alpha}}.$$

Where the constant $M_\alpha^{(\beta)}$ only depends on α,β.

Proof Let $f\in C^1[0,1]$. For any $t,x\in[0,1]$, $\delta>0$, we have

$$\begin{aligned}|f(t)-f(x)-f'(x)(t-x)| &\le |\int_x^t|f'(u)-f'(x)|du|\\ &\le \omega(f',|t-x|)|t-x|\\ &\le \omega(f',\delta)(|t-x|+\delta^{-1}(t-x)^2),\end{aligned}$$

hence, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&|E_{n,\beta}^{(\alpha)}(f(t)-f(x)-f'(x)(t-x);x)|\\ &\le\omega(f',\delta)\left(E_{n,\beta}^{(\alpha)}(|t-x|;x)+\delta^{-1}E_{n,\beta}^{(\alpha)}((t-x)^2;x)\right)\\ &\le\omega(f',\delta)\left[\sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\right.\\ &\left.\quad+\delta^{-1}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}\right]\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}.\end{aligned}$$

So, we get

$$\begin{aligned}&|E_{n,\beta}^{(\alpha)}(f;x)-f(x)|\\ &\le\|f'\|E_{n,\beta}^{(\alpha)}(|t-x|;x)\\ &+\omega(f',\delta)\left[1+\delta^{-1}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}\right]\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)}.\end{aligned}\tag{6}$$

(i) When $\alpha\ge1$, taking $\delta=\frac{1}{\sqrt{n}}$ in (6), by Lemma 5 and inequality (6), we obtain the desired result.
(ii)When $0<\alpha<1$, taking $\delta=n^{-\frac{\alpha}{2}}$ in (6), by Lemma 6 and inequality (6), we obtain the desired result.

Finally we study the direct theorem concerned with an approximation for the sequence $\{E_{n,\beta}^{(\alpha)}\}$ by means of the Ditzian-Totik modulus of smoothness. For the following theorem we shall use some notations.

For $f \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $0 \le \lambda \le 1$, $x \in [0,1]$, let

$$\omega_{\varphi^\lambda}(f,t) = \sup_{0<h\le t} \sup_{x\pm\frac{h\varphi^\lambda(x)}{2}\in[0,1]} |f(x+\frac{h\varphi^\lambda(x)}{2}) - f(x-\frac{h\varphi^\lambda(x)}{2})|$$

be the Ditzian-Totik modulus of first order, and let

$$K_{\varphi^\lambda}(f,t) = \inf_{g\in W_\lambda}\{\| f-g \| + t\|\varphi^\lambda g'\|\} \tag{7}$$

be the corresponding K-functional, where $W_\lambda = \{f | f \in AC_{loc}[0,1], \|\varphi^\lambda f'\| < \infty, \|f'\| < \infty\}$.

It is well known that (see [14])

$$K_{\varphi^\lambda}(f,t) \le C\omega_{\varphi^\lambda}(f,t), \tag{8}$$

for some absolute constant $C > 0$.

Now we state our following main result.

Theorem 4 *Let $f \in C[0,1]$, $\alpha \ge 1$, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$, $0 \le \beta, \lambda \le 1$. Then there exists an absolute constant $C > 0$ such that*

$$|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \le C\omega_{\varphi^\lambda}(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}).$$

Proof Let $g \in W_\lambda$, by Lemma 2 (i) and Lemma 7, we have

$$\begin{aligned} &|E_{n,\beta}^{(\alpha)}(f;x) - f(x)| \\ &\le |E_{n,\beta}^{(\alpha)}(f-g;x)| + |f(x)-g(x)| + |E_{n,\beta}^{(\alpha)}(g;x) - g(x)| \\ &\le 2\|f-g\| + |E_{n,\beta}^{(\alpha)}(g;x) - g(x)|. \end{aligned} \tag{9}$$

Since $g(t) = \int_x^t g'(u)du + g(x)$, $E_{n,\beta}^{(\alpha)}(1;x) = 1$, so, we have

$$\begin{aligned} |E_{n,\beta}^{(\alpha)}(g;x) - g(x)| &\le |E_{n,\beta}^{(\alpha)}(\int_x^t |g'(u)|du;x)| \\ &\le \|\varphi^\lambda g'\| E_{n,\beta}^{(\alpha)}(|\int_x^t \varphi^{-\lambda}(u)du|;x). \end{aligned} \tag{10}$$

By the Hölder inequality, we get

$$|\int_x^t \varphi^{-\lambda}(u)du| \le |\int_x^t \frac{1}{\sqrt{u(1-u)}}du|^\lambda |t-x|^{1-\lambda}, \tag{11}$$

also, in view of $1 \le \sqrt{u} + \sqrt{1-u} < 2$, $0 \le u \le 1$, we have

$$\begin{aligned} |\int_x^t \frac{1}{\sqrt{u(1-u)}}du| &\le |\int_x^t (\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}})du| \\ &\le 2(|\sqrt{t}-\sqrt{x}| + |\sqrt{1-x}-\sqrt{1-t}|) \\ &\le 2(\frac{|t-x|}{\sqrt{t}+\sqrt{x}} + \frac{|t-x|}{\sqrt{1-t}+\sqrt{1-x}}) \\ &\le 2|t-x|(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}) \\ &\le 4|t-x|\varphi^{-1}(x), \end{aligned} \tag{12}$$

thus, by (11) and (12), we obtain

$$|\int_x^t \varphi^{-\lambda}(u)du| \le C\varphi^{-\lambda}(x)|t-x|. \tag{13}$$

Also, by (10) and (13), we have

$$\begin{aligned} |E_{n,\beta}^{(\alpha)}(g;x)-g(x)| &\le C\|\varphi^\lambda g'\|E_{n,\beta}^{(\alpha)}(\varphi^{-\lambda}(x)|t-x|;x) \\ &= C\|\varphi^\lambda g'\|\varphi^{-\lambda}(x)E_{n,\beta}^{(\alpha)}(|t-x|;x). \end{aligned} \tag{14}$$

In view of (5) and Lemma 2 (i), by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} E_{n,\beta}^{(\alpha)}(|t-x|;x) &\le \sqrt{E_{n,\beta}^{(\alpha)}(1;x)}\sqrt{E_{n,\beta}^{(\alpha)}((t-x)^2;x)} \\ &\le \sqrt{\frac{\alpha}{n}\left(1+\frac{n-1}{n^2+1}\beta^2\right)x(1-x)} \\ &\le C\frac{\varphi(x)}{\sqrt{n}}, \end{aligned} \tag{15}$$

so, by (14) and (15), we obtain

$$|E_{n,\beta}^{(\alpha)}(g;x)-g(x)| \le C\|\varphi^\lambda g'\|\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}, \tag{16}$$

thus, by (9) and (16), we have

$$|E_{n,\beta}^{(\alpha)}(f;x)-f(x)| \le 2\|f-g\| + C\|\varphi^\lambda g'\|\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}.$$

Then, in view of (17), (7) and (8), we obtain

$$|E_{n,\beta}^{(\alpha)}(f;x)-f(x)| \le CK_{\varphi^\lambda}(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}) \le C\omega_{\varphi^\lambda}(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}),$$

where C is a positive constant, in different places, the value of C may be different.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No. 61572020), the Class A Science and Technology Project of Education Department of Fujian Province of China (Grant No. JA12324), and the Natural Science Foundation of Fujian Province of China (Grant No. 2014J01021 and 2013J01017).

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Approximation by complex Stancu type summation-integral operators in compact disks

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Abstract. In this paper we introduce a class of complex Stancu type summation-integral operators and study the approximation properties of these operators. We obtain a Voronovskaja-type result with quantitative estimate for these operators attached to analytic functions on compact disks. We also study the exact order of approximation. More important, our results show the overconvergence phenomenon for these complex operators.

Keywords: complex Stancu type summation-integral operators; Voronovskaja-type result; Exact order of approximation; Simultaneous approximation; Overconvergence

Mathematical subject classification: 30E10, 41A25 , 41A36

1. Introduction

In 1986, some approximation properties of complex Bernstein polynomials in compact disks were initially studied by Lorentz [11]. Very recently, the problem of the approximation of complex operators has been causing great concern, which is becoming a hot topic of research. A Voronovskaja-type result with quantitative estimate for complex Bernstein polynomials in compact disks was obtained by Gal [3]. Also, in [1-2, 4-10, 12-15] similar results for complex Bernstein-Kantorovich polynomials, Bernstein-Stancu polynomials, Kantorovich-Schurer polynomials, Kantorovich-Stancu polynomials, complex Favard-Szász-Mirakjan operators, complex Beta operators of first kind, complex Baskajov-Stancu operators, complex Bernstein-Durrmeyer operators based on Jacobi weights, complex genuine Durrmeyer Stancu polynomials, complex Schurer-Stancu operators, complex q-Szász-Mirakjan operators, complex q-Gamma operators, and complex q-Durrmeyer type operators were obtained.

The aim of the present article is to obtain approximation results for complex Stancu type summation-integral operators which are defined for $f : [0, 1] \to \mathbf{C}$ continuous on $[0, 1]$ by

$$M_n^{(\alpha,\beta)}(f; z) := p_{n,0}(z)f(\frac{\alpha}{n+\beta}) + \sum_{k=1}^{n-1} p_{n,k}(z)L_{n,k}^{(\alpha,\beta)}(f) + p_{n,n}(z)f(\frac{n+\alpha}{n+\beta}), \quad (1)$$

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where α, β are two given real parameters satisfying the condition $0 \le \alpha \le \beta$, $z \in \mathbf{C}, n \in \mathbf{N}$, $L_{n,k}^{(\alpha,\beta)}(f) = \frac{1}{B(n(n-k),nk)} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} f(\frac{nt+\alpha}{n+\beta})dt$ with $B(x,y)$ is Beta function, and $p_{n,k}(z) = \left(\begin{array}{c} n \\ k \end{array} \right) z^k(1-z)^{n-k}$.

Note that, for $\alpha = \beta = 0$, these operators become the complex summation-integral type operators $M_n(f;z) = M_n^{(0,0)}(f;z)$, this case has been investigated in [16].

2. Auxiliary results

In the sequel, we shall need the following auxiliary results.

Lemma 1 *Let $e_m(z) = z^m$, $m \in \mathbf{N} \cup \{0\}$, $z \in \mathbf{C}$, $n \in \mathbf{N}$, $0 \le \alpha \le \beta$, we have $M_n^{(\alpha,\beta)}(e_m;z)$ is a polynomial of degree less than or equal to min (m,n) and*

$$M_n^{(\alpha,\beta)}(e_m;z) = \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array} \right) \frac{n^j \alpha^{m-j}}{(n+\beta)^m} M_n(e_j;z).$$

Proof By the definition given by (1) , the proof is easy, here the proof is omitted.

Let $m = 0,1,2$, according to [16, Lemma 1] , by simple computation, we have

$$\begin{aligned}
M_n^{(\alpha,\beta)}(e_0;z) &= 1; \\
M_n^{(\alpha,\beta)}(e_1;z) &= \frac{nz}{n+\beta} + \frac{\alpha}{n+\beta}; \\
M_n^{(\alpha,\beta)}(e_2;z) &= \frac{n^2}{(n+\beta)^2}\left[\frac{n(n-1)}{n^2+1}z^2 + \frac{n+1}{n^2+1}z\right] \\
&\quad + \frac{2n\alpha z}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.
\end{aligned}$$

Lemma 2 *Let $e_m(z) = z^m$, $m \in \mathbf{N} \cup \{0\}$, $z \in \mathbf{C}$, $n \in \mathbf{N}$, $0 \le \alpha \le \beta$, for all $|z| \le r$, $r \ge 1$, we have $|M_n^{(\alpha,\beta)}(e_m;z)| \le r^m$.*

Proof The proof follows directly Lemma 1 and [16, Corollary 1].

Lemma 3 *Let $e_m(z) = z^m$, $m, n \in$ N, $z \in \mathbf{C}$ and $0 \le \alpha \le \beta$, we have*

$$\begin{aligned}
M_n^{(\alpha,\beta)}(e_{m+1};z) &= \frac{z(1-z)n^2}{(n+\beta)(n^2+m)}(M_n^{(\alpha,\beta)}(e_m;z))' \\
&\quad + \frac{(m+n^2z)n + \alpha(n^2+2m)}{(n+\beta)(n^2+m)} M_n^{(\alpha,\beta)}(e_m;z) \\
&\quad - \frac{\alpha m(n+\alpha)}{(n+\beta)^2(n^2+m)} M_n^{(\alpha,\beta)}(e_{m-1};z). \qquad (2)
\end{aligned}$$

Proof Let

$$\widetilde{L}_{n,k}^{(\alpha,\beta)}(f) := \frac{1}{B(n(n-k),nk)} \int_0^1 t^{nk-1}(1-t)^{n(n-k)-1} t f(\frac{nt+\alpha}{n+\beta})dt,$$

$$\widehat{L}_{n,k}^{(\alpha,\beta)}(f) := \frac{1}{B(n(n-k),nk)}\int_0^1 t^{nk-1}(1-t)^{n(n-k)-1}t^2 f(\frac{nt+\alpha}{n+\beta})dt,$$
$$E_n^{(\alpha,\beta)}(f;z) := \sum_{k=1}^{n-1} p_{n,k}(z)L_{n,k}^{(\alpha,\beta)}(f),$$

we have

$$M_n^{(\alpha,\beta)}(f;z) = p_{n,0}(z)f(\frac{\alpha}{n+\beta}) + E_n^{(\alpha,\beta)}(f;z) + p_{n,n}(z)f(\frac{n+\alpha}{n+\beta}),$$
$$\widetilde{L}_{n,k}^{(\alpha,\beta)}(e_m) = \frac{n+\beta}{n}L_{n,k}^{(\alpha,\beta)}(e_{m+1}) - \frac{\alpha}{n}L_{n,k}^{(\alpha,\beta)}(e_m),$$
$$\widehat{L}_{n,k}^{(\alpha,\beta)}(e_m) = (\frac{n+\beta}{n})^2 L_{n,k}^{(\alpha,\beta)}(e_{m+2}) - \frac{2\alpha(n+\beta)}{n^2}L_{n,k}^{(\alpha,\beta)}(e_{m+1}) + (\frac{\alpha}{n})^2 L_{n,k}^{(\alpha,\beta)}(e_m).$$

By simple calculation, we obtain

$$z(1-z)p'_{n,k}(z) = (k-nz)p_{n,k}(z),$$
$$t(1-t)[t^{nk-1}(1-t)^{n(n-k)-1}]' = [nk-1-(n^2-2)t]t^{nk-1}(1-t)^{n(n-k)-1},$$

it follows that

$$\begin{aligned}
&z(1-z)(E_n^{(\alpha,\beta)}(e_m;z))' \\
&= \sum_{k=1}^{n-1}(k-nz)\,p_{n,k}(z)L_{n,k}^{(\alpha,\beta)}(e_m) \\
&= \sum_{k=1}^{n-1} kp_{n,k}(z)\frac{1}{B(n(n-k),nk)}\int_0^1 t^{nk-1}(1-t)^{n(n-k)-1}(\frac{nt+\alpha}{n+\beta})^m dt - nzE_n^{(\alpha,\beta)}(e_m;z) \\
&= \frac{1}{n}\sum_{k=1}^{n-1} p_{n,k}(z)\frac{1}{B(n(n-k),nk)}\int_0^1 [nk-1-(n^2-2)t]t^{nk-1}(1-t)^{n(n-k)-1}(\frac{nt+\alpha}{n+\beta})^m dt \\
&\quad + \frac{1}{n}E_n^{(\alpha,\beta)}(e_m;z) + \frac{n^2-2}{n}\sum_{k=1}^{n-1} p_{n,k}(z)\widetilde{L}_{n,k}^{(\alpha,\beta)}(e_m) - nzE_n^{(\alpha,\beta)}(e_m;z),
\end{aligned}$$

where

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n-1} p_{n,k}(z)\frac{1}{B(n(n-k),nk)}\int_0^1 [nk-1-(n^2-2)t]t^{nk-1}(1-t)^{n(n-k)-1}(\frac{nt+\alpha}{n+\beta})^m dt \\
&= \frac{1}{n}\sum_{k=1}^{n-1} p_{n,k}(z)\frac{1}{B(n(n-k),nk)}\int_0^1 (t-t^2)[t^{nk-1}(1-t)^{n(n-k)-1}]'(\frac{nt+\alpha}{n+\beta})^m dt \\
&= -\frac{1}{n}E_n^{(\alpha,\beta)}(e_m;z) + \frac{2}{n}\sum_{k=1}^{n-1} p_{n,k}(z)\widetilde{L}_{n,k}^{(\alpha,\beta)}(e_m) - \frac{m}{n+\beta}\sum_{k=1}^{n-1} p_{n,k}(z)\widetilde{L}_{n,k}^{(\alpha,\beta)}(e_{m-1}) \\
&\quad + \frac{m}{n+\beta}\sum_{k=1}^{n-1} p_{n,k}(z)\widehat{L}_{n,k}^{(\alpha,\beta)}(e_{m-1}).
\end{aligned}$$

So, in conclusion, we have

$$\begin{aligned}
z(1-z)(E_n^{(\alpha,\beta)}(e_m;z))' = &\frac{(n+\beta)(n^2+m)}{n^2}E_n^{(\alpha,\beta)}(e_{m+1};z) \\
&- (\frac{\alpha n^2+mn+2\alpha m}{n^2} + nz)E_n^{(\alpha,\beta)}(e_m;z) \\
&+ \frac{\alpha mn+\alpha^2 m}{n^2(n+\beta)}E_n^{(\alpha,\beta)}(e_{m-1};z),
\end{aligned}$$

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which implies the recurrence in the statement.

Lemma 4 *Let $n \in \mathbf{N}$, $m = 2, 3, \cdots$, $e_m(z) = z^m$, $S_{n,m}^{(\alpha,\beta)}(z) := M_n^{(\alpha,\beta)}(e_m; z) - z^m$, $z \in \mathbf{C}$ and $0 \le \alpha \le \beta$, we have*

$$\begin{aligned}
S_{n,m}^{(\alpha,\beta)}(z) = & \frac{z(1-z)n^2}{(n+\beta)(n^2+m-1)}(M_n^{(\alpha,\beta)}(e_{m-1}; z))' \\
& + \frac{(m-1+n^2z)n + \alpha(n^2+m-1)}{(n+\beta)(n^2+m-1)} S_{n,m-1}^{(\alpha,\beta)}(z) \\
& + \frac{\alpha(m-1)}{(n+\beta)(n^2+m-1)} M_n^{(\alpha,\beta)}(e_{m-1}; z) \\
& - \frac{\alpha(m-1)(n+\alpha)}{(n+\beta)^2(n^2+m-1)} M_n^{(\alpha,\beta)}(e_{m-2}; z) \\
& + \frac{(m-1+n^2z)n + \alpha(n^2+m-1)}{(n+\beta)(n^2+m-1)} z^{m-1} - z^m. \qquad (3)
\end{aligned}$$

Proof Using the recurrence formula (2), by simple calculation, we can easily get the recurrence (3), the proof is omitted.

3. Main results

The first main result is expressed by the following upper estimates.

Theorem 1 *Let $0 \le \alpha \le \beta$, $1 \le r \le R$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R, i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$.*

(i) for *all $|z| \le r$ and $n \in \mathbf{N}$, we have*

$$|M_n^{(\alpha,\beta)}(f; z) - f(z)| \le \frac{K_r^{(\alpha,\beta)}(f)}{n+\beta},$$

where $K_r^{(\alpha,\beta)}(f) = (1+r)\sum_{m=1}^{\infty} |c_m| m(m+1+\alpha+\beta) r^{m-1} < +\infty$.

(ii) (Simultaneous approximation) *If $1 \le r < r_1 < R$ are arbitrary fired, then for all $|z| \le r$ and $n, p \in \mathbf{N}$ we have*

$$|(M_n^{(\alpha,\beta)}(f; z))^{(p)} - f^{(p)}(z)| \le \frac{K_{r_1}^{(\alpha,\beta)}(f) p! r_1}{(n+\beta)(r_1 - r)^{p+1}},$$

where $K_{r_1}^{(\alpha,\beta)}(f)$ is defined as at the above point (i).

Proof Taking $e_m(z) = z^m$, by hypothesis that $f(z)$ is analytic in D_R, i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$, it is easy for us to obtain

$$M_n^{(\alpha,\beta)}(f; z) = \sum_{m=0}^{\infty} c_m M_n^{(\alpha,\beta)}(e_m; z),$$

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therefore, we get

$$\begin{aligned}|M_n^{(\alpha,\beta)}(f;z)-f(z)| &\le \sum_{m=0}^{\infty}|c_m|\cdot|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)|\\ &= \sum_{m=1}^{\infty}|c_m|\cdot|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)|,\end{aligned}$$

as $M_n^{(\alpha,\beta)}(e_0;z)=e_0(z)=1$.

(i) For $m\in\mathbf{N}$, taking into account that $M_n^{(\alpha,\beta)}(e_{m-1};z)$ is a polynomial degree $\le min(m-1,n)$, by the well-known Bernstein inequality and Lemma 2 we get

$$|(M_n^{(\alpha,\beta)}(e_{m-1};z))'|\le\frac{m-1}{r}max\{|M_n^{(\alpha,\beta)}(e_{m-1};z)|:|z|\le r\}\le(m-1)r^{m-2}.$$

On the one hand, when $m=1$, for $|z|\le r$, by Lemma 1, we have

$$|M_n^{(\alpha,\beta)}(e_1;z)-e_1(z)|=|\frac{nz}{n+\beta}+\frac{\alpha}{n+\beta}-z|\le\frac{1+r}{n+\beta}(2+\alpha+\beta).$$

When $m\ge 2$, for $n\in\mathbf{N}$, $|z|\le r$, $0\le\alpha\le\beta$, in view of $|(m-1+n^2z)n+\alpha(n^2+m-1)|\le(n+\beta)(n^2+m-1)r$, using the recurrence formula (3) and the above inequality, we have

$$\begin{aligned}|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)| &= |S_{n,m}^{(\alpha,\beta)}(z)|\\ &\le \frac{r(1+r)}{n+\beta}\cdot(m-1)r^{m-2}+r|S_{n,m-1}^{(\alpha,\beta)}(z)|\\ &\quad+\frac{\alpha}{n+\beta}r^{m-1}+\frac{\alpha}{n+\beta}r^{m-2}+\frac{m+1+\beta}{n+\beta}(1+r)r^{m-1}\\ &\le \frac{m-1}{n+\beta}(1+r)r^{m-1}+r|S_{n,m-1}^{(\alpha,\beta)}(z)|\\ &\quad+\frac{\alpha}{n+\beta}(1+r)r^{m-1}+\frac{m+1+\beta}{n+\beta}(1+r)r^{m-1}\\ &= r|S_{n,m-1}^{(\alpha,\beta)}(z)|+\frac{2m+\alpha+\beta}{n+\beta}(1+r)r^{m-1}.\end{aligned}$$

By writing the last inequality, for $m=2,\cdots$, we easily obtain step by step the following

$$\begin{aligned}|M_n^{(\alpha,\beta)}(e_m;z)-e_m(z)| &\le r\left(r|S_{n,m-2}^{(\alpha,\beta)}(z)|+\frac{2(m-1)+\alpha+\beta}{n+\beta}(1+r)r^{m-2}\right)\\ &\quad+\frac{2m+\alpha+\beta}{n+\beta}(1+r)r^{m-1}\\ &= r^2|S_{n,m-2}^{(\alpha,\beta)}(z))|+\frac{2(m-1+m)+2(\alpha+\beta)}{n+\beta}(1+r)r^{m-1}\\ &\le\ldots\le\frac{1+r}{n+\beta}m(m+1+\alpha+\beta)r^{m-1}.\end{aligned}$$

In conclusion, for any $m,n\in\mathbf{N}$, $|z|\le r$, $0\le\alpha\le\beta$, we have

$$|M_{n+\beta}^{(\alpha,\beta)}(e_m;z)-e_m(z)|\le\frac{1+r}{n+\beta}m(m+1+\alpha+\beta)r^{m-1},$$

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it follows that

$$|M_n^{(\alpha,\beta)}(f;z)-f(z)| \le \frac{1+r}{n+\beta}\sum_{m=1}^{\infty}|c_m|m(m+1+\alpha+\beta)r^{m-1}.$$

By assuming that $f(z)$ is analytic in D_R, we have $f^{(2)}(z)=\sum\limits_{m=2}^{\infty}c_m m(m-1)z^{m-2}$ and the series is absolutely convergent in $|z|\le r$, so we get $\sum\limits_{m=2}^{\infty}|c_m|m(m-1)r^{m-2}<+\infty$, which implies $K_r^{(\alpha,\beta)}(f)=(1+r)\sum\limits_{m=1}^{\infty}|c_m|m(m+1+\alpha+\beta)r^{m-1}<+\infty$.

(ii) For the simultaneous approximation, denoting by Γ the circle of radius $r_1>r$ and center 0, since for any $|z|\le r$ and $\upsilon\in\Gamma$, we have $|\upsilon-z|\ge r_1-r$, by Cauchy's formulas it follows that for all $|z|\le r$ and $n\in\mathbf{N}$, we have

$$\begin{aligned}
|(M_n^{(\alpha,\beta)}(f;z))^{(p)}-f^{(p)}(z)| &= \frac{p!}{2\pi}\left|\int_\Gamma \frac{M_n^{(\alpha,\beta)}(f;\upsilon)-f(\upsilon)}{(\upsilon-z)^{p+1}}d\upsilon\right| \\
&\le \frac{K_{r_1}^{(\alpha,\beta)}(f)}{n+\beta}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}} \\
&= \frac{K_{r_1}^{(\alpha,\beta)}(f)}{n+\beta}\cdot\frac{p!r_1}{(r_1-r)^{p+1}},
\end{aligned}$$

which proves the theorem.

Theorem 2 *Let $0\le\alpha\le\beta$, $R>1$, $D_R=\{z\in\mathbf{C}:|z|<R\}$. Suppose that $f:D_R\to\mathbf{C}$ is analytic in D_R, i.e. $f(z)=\sum\limits_{k=0}^{\infty}c_kz^k$ for all $z\in D_R$. For any fixed $r\in[1,R]$ and all $n\in\mathbf{N}$, $|z|\le r$, we have*

$$\begin{aligned}
&\left|M_n^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-\beta z}{n+\beta}f'(z)-\frac{z(1-z)}{2(n+\beta)}f''(z)\right| \\
&\le \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)}+\frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2}+\frac{M_{r,2}(f)}{n^2}, \qquad (4)
\end{aligned}$$

where $M_{r,2}(f)=M_r(f)+M_{r,1}(f)$, $M_r(f)=\sum\limits_{k=2}^{\infty}|c_k|(k-1)F_{k,r}r^k$ with $F_{k,r}=10k^3-30k^2+39k-16+4(k-2)(k-1)^2(1+r)$, $M_{r,1}(f)=\sum\limits_{k=2}^{\infty}|c_k|(\beta+1)k(k-1)(1+r)r^{k-1}$, $M_{r,1}^{(\alpha,\beta)}(f)=\sum\limits_{k=2}^{\infty}|c_k|[2k(k-1)^2\alpha+2k^3\beta r]r^{k-1}$, $M_{r,2}^{(\alpha,\beta)}(f)=\sum\limits_{k=2}^{\infty}|c_k|[\frac{k(k-1)(\alpha^2+\beta^2r^2)}{2}+k^2\alpha\beta r+k^2\beta^2r^2]r^{k-2}$.

Proof For all $z\in D_R$, we have

$$\begin{aligned}
&M_n^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-\beta z}{n+\beta}f'(z)-\frac{z(1-z)}{2(n+\beta)}f''(z) \\
&=\left[M_n(f;z)-f(z)-\frac{(n+1)z(1-z)}{2(n^2+1)}f''(z)\right] \\
&\quad+\left[M_n^{(\alpha,\beta)}(f;z)-M_n(f;z)-\frac{\alpha-\beta z}{n+\beta}f'(z)+\frac{(\beta+1)n+(\beta-1)}{2(n+\beta)(n^2+1)}z(1-z)f''(z)\right] \\
&:=I_1+I_2.
\end{aligned}$$

6

By [16, Theorem 2], we have $|I_1| \le \frac{M_r(f)}{n^2}$, where $M_r(f) = \sum\limits_{k=2}^{\infty} |c_k|(k-1)F_{k,r}r^k$ and $F_{k,r} = 10k^3 - 30k^2 + 39k - 16 + 4(k-2)(k-1)^2(1+r)$.

Next let us to estimate $|I_2|$.

Denote $Q_{n,k}^{(\beta)}(z) = \frac{k(k-1)((\beta+1)n+(\beta-1))}{2(n+\beta)(n^2+1)} z^{k-1}(1-z)$. By f is analytic in D_R, i.e. $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$, and $M_n^{(\alpha,\beta)}(e_1; z) = M_n(e_1; z) + \frac{\alpha-\beta z}{n+\beta}$, we have

$$\begin{aligned} |I_2| &= \left| \sum_{k=2}^{\infty} c_k \left[M_n^{(\alpha,\beta)}(e_k; z) - M_n(e_k; z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \right] \right| \\ &\le \sum_{k=2}^{\infty} |c_k| \left| M_n^{(\alpha,\beta)}(e_k; z) - M_n(e_k; z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \right|. \end{aligned}$$

When $k \ge 2$, since $\frac{n^k}{(n+\beta)^k} - 1 = -\sum\limits_{j=0}^{k-1} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k}$, by Lemma 1, we obtain

$$\begin{aligned} & M_n^{(\alpha,\beta)}(e_k; z) - M_n(e_k; z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j; z) + \left[\frac{n^k}{(n+\beta)^k} - 1 \right] M_n(e_k; z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} \\ &\quad + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j; z) + \frac{k n^{k-1} \alpha}{(n+\beta)^k} M_n(e_{k-1}; z) \\ &\quad - \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} M_n(e_k; z) - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j; z) + \frac{k n^{k-1} \alpha}{(n+\beta)^k} \left[M_n(e_{k-1}; z) - e_{k-1}(z) \right] \\ &\quad + \frac{k n^{k-1} \alpha}{(n+\beta)^k} z^{k-1} - \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} M_n(e_k; z) \\ &\quad - \frac{k n^{k-1} \beta}{(n+\beta)^k} \left[M_n(e_k; z) - e_k(z) \right] - \frac{k n^{k-1} \beta}{(n+\beta)^k} z^k - \frac{\alpha-\beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z) \\ &= \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j; z) + \frac{k n^{k-1} \alpha}{(n+\beta)^k} \left[M_n(e_{k-1}; z) - e_{k-1}(z) \right] \\ &\quad - \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n+\beta)^k} M_n(e_k; z) - \frac{k n^{k-1} \beta}{(n+\beta)^k} \left[M_n(e_k; z) - e_k(z) \right] \\ &\quad - \left[\frac{1}{n+\beta} - \frac{n^{k-1}}{(n+\beta)^k} \right] k \alpha z^{k-1} + \left[\frac{1}{n+\beta} - \frac{n^{k-1}}{(n+\beta)^k} \right] k \beta z^k + Q_{n,k}^{(\beta)}(z). \end{aligned}$$

By the proof of the [16, Theorem 1], for any $k \in \mathbf{N}$, $|z| \le r$, $r \ge 1$, we have

$$|M_n(e_k; z)| \le r^k, \ |M_n(e_k; z) - e_k| \le \frac{2k^2}{n} r^k,$$

hence, for any $k \geq 2$, $|z| \leq r$, $r \geq 1$, we can get

$$\left|\sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} M_n(e_j;z)\right|$$
$$\leq \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} r^{k-2}$$
$$= \sum_{j=0}^{k-2} \frac{k(k-1)}{(k-j)(k-j-1)} \binom{k-2}{j} \frac{n^j \alpha^{k-2-j}}{(n+\beta)^{k-2}} \cdot \frac{\alpha^2}{(n+\beta)^2} r^{k-2}$$
$$\leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{n^j \alpha^{k-2-j}}{(n+\beta)^{k-2}} r^{k-2}$$
$$\leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} r^{k-2}$$

and

$$\left|\frac{kn^{k-1}\alpha}{(n+\beta)^k}\left[M_n(e_{k-1};z) - e_{k-1}(z)\right]\right| \leq \frac{2k(k-1)^2\alpha}{n(n+\beta)} r^{k-1}.$$

Also, using

$$\frac{1}{n+\beta} - \frac{n^{k-1}}{(n+\beta)^k} = \frac{\sum_{j=0}^{k-2} \binom{k-1}{j} n^j \beta^{k-1-j}}{(n+\beta)^k} \leq \frac{(k-1)\beta}{(n+\beta)^2},$$

thus, for any $k \geq 2$, $|z| \leq r$, $r \geq 1$, we get

$$|M_n^{(\alpha,\beta)}(e_k;z) - M_n(e_k;z) - \frac{\alpha - \beta z}{n+\beta} k z^{k-1} + Q_{n,k}^{(\beta)}(z)|$$
$$\leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n+\beta)^2} r^{k-2} + \frac{2k(k-1)^2\alpha}{n(n+\beta)} r^{k-1} + \frac{k(k-1)}{2} \cdot \frac{\beta^2}{(n+\beta)^2} r^k$$
$$+ \frac{2k^3\beta}{n(n+\beta)} r^k + \frac{k^2\alpha\beta}{(n+\beta)^2} r^{k-1} + \frac{k^2\beta^2}{(n+\beta)^2} r^k + \frac{(\beta+1)k(k-1)(1+r)r^{k-1}}{n^2}$$
$$= \frac{r^{k-1}}{n(n+\beta)}\left[2k(k-1)^2\alpha + 2k^3\beta r\right] + \frac{(\beta+1)k(k-1)(1+r)r^{k-1}}{n^2}$$
$$+ \frac{r^{k-2}}{(n+\beta)^2}\left[\frac{k(k-1)(\alpha^2+\beta^2r^2)}{2} + k^2\alpha\beta r + k^2\beta^2r^2\right].$$

Hence, we have

$$|I_2| \leq \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)} + \frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2} + \frac{M_{r,1}(f)}{n^2},$$

where

$$M_{r,1}(f) = \sum_{k=2}^{\infty} |c_k|(\beta+1)k(k-1)(1+r)r^{k-1},$$
$$M_{r,1}^{(\alpha,\beta)}(f) = \sum_{k=2}^{\infty} |c_k|[2k(k-1)^2\alpha + 2k^3\beta r]r^{k-1},$$
$$M_{r,2}^{(\alpha,\beta)}(f) = \sum_{k=2}^{\infty} |c_k|\left[\frac{k(k-1)(\alpha^2+\beta^2r^2)}{2} + k^2\alpha\beta r + k^2\beta^2r^2\right]r^{k-2}.$$

In conclusion, we obtain

$$\left|M_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z) - \frac{z(1-z)}{2(n+\beta)} f''(z)\right|$$

$$\leq |I_1| + |I_2| \leq \frac{M_{r,1}^{(\alpha,\beta)}(f)}{n(n+\beta)} + \frac{M_{r,2}^{(\alpha,\beta)}(f)}{(n+\beta)^2} + \frac{M_{r,2}(f)}{n^2},$$

where $M_{r,2}(f) = M_r(f) + M_{r,1}(f)$.

In the following theorem, we will obtain the exact order in approximation.

Theorem 3 *Let $0 < \alpha \leq \beta$, $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. If f is not a polynomial of degree 0 , then for any $r \in [1, R)$ we have*

$$\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r \geq \frac{C_r^{(\alpha,\beta)}(f)}{n+\beta},\ n \in \mathbf{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \leq r\}$ and the constant $C_r^{(\alpha,\beta)}(f) > 0$ depends on f, r and α, β but it is independent of n.

Proof Denote $e_1(z) = z$ and

$$H_n^{(\alpha,\beta)}(f;z) = M_n^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z) - \frac{z(1-z)}{2(n+\beta)} f''(z).$$

For all $z \in D_R$ and $n \in \mathbf{N}$ we have

$$M_n^{(\alpha,\beta)}(f;z) - f(z)$$

$$= \frac{1}{n+\beta}\left\{(\alpha - \beta z) f'(z) + \frac{z(1-z)}{2} f''(z) + (n+\beta) H_n^{(\alpha,\beta)}(f;z)\right\}.$$

In view of the property: $\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r$, it follows

$$\|M_n^{(\alpha,\beta)}(f;\cdot) - f\|_r$$

$$\geq \frac{1}{n+\beta}\left\{\|(\alpha - \beta e_1) f' + \frac{e_1(1-e_1)}{2} f''\|_r - (n+\beta)||H_n^{(\alpha,\beta)}(f;\cdot)||_r\right\}.$$

Considering the hypothesis that f is not a polynomial of degree 0 in D_R, we have $\|(\alpha - \beta e_1) f' + \frac{e_1(1-e_1)}{2} f''\|_r > 0$.

Indeed, supposing the contrary, it follows that

$$(\alpha - \beta z) f'(z) + \frac{z(1-z)}{2} f''(z) = 0, \text{ for all } z \in \overline{D_r}.$$

Denoting $y(z) = f'(z)$ and looking for the analytic function $y(z)$ under the form $y(z) = \sum_{k=0}^{\infty} a_k z^k$, after replacement in the differential equation, the identification of the coefficients method immediately leads to $a_k = 0$, for all $k \in \mathbb{N} \bigcup \{0\}$. This implies that $y(z) = 0$ for all $z \in \overline{D_r}$ and therefore f is constant on $\overline{D_r}$, a contradiction with the hypothesis.

Using the inequality (4), we get

$$\lim_{n\to\infty} (n+\beta)\|H_n^{(\alpha,\beta)}(f;\cdot)\|_r = 0, \tag{5}$$

therefore, there exists an index n_0 depending only on f, r and α, β, such that for all $n \geq n_0$, we have

$$\|(\alpha-\beta e_1)f'+\frac{e_1(1-e_1)}{2}f''\|_r-(n+\beta)||H_n^{(\alpha,\beta)}(f;\cdot)||_r$$
$$\geq \frac{1}{2}\left\|(\alpha-\beta e_1)f'+\frac{e_1(1-e_1)}{2}f''\right\|_r,$$

which implies

$$\|M_n^{(\alpha,\beta)}(f;\cdot)-f\|_r \geq \frac{1}{2n}\left\|(\alpha-\beta e_1)f'+\frac{e_1(1-e_1)}{2}f''\right\|_r, \text{ for all } n \geq n_0.$$

For $n \in \{1,2,\cdots,n_0-1\}$, we have $\|M_n^{(\alpha,\beta)}(f;\cdot)-f\|_r \geq \frac{W_{r,n}^{(\alpha,\beta)}(f)}{n+\beta}$, where $W_{r,n}^{(\alpha,\beta)}(f)=(n+\beta)\|M_n^{(\alpha,\beta)}(f;\cdot)-f\|_r>0$.

As a conclusion, we have $\|M_n^{(\alpha,\beta)}(f;\cdot)-f\|_r \geq \frac{C_r^{(\alpha,\beta)}(f)}{n+\beta}$, for all $n \in \mathbf{N}$, where

$$C_r^{(\alpha,\beta)}(f)=min\Big\{W_{r,1}^{(\alpha,\beta)}(f),W_{r,2}^{(\alpha,\beta)}(f),\ldots,W_{r,n_0-1}^{(\alpha,\beta)}(f),$$
$$\frac{1}{2}\|(\alpha-\beta e_1)f'+\frac{e_1(1-e_1)}{2}f''\|_r\Big\},$$

this complete the proof.

Combining Theorem 3 with Theorem 1, we get the following result.

Corollary *Let $0 \leq \alpha \leq \beta$, $R>1$, $D_R=\{z \in \mathbf{C}:|z|<R\}$. Suppose that $f:D_R \to \mathbf{C}$ is analytic in D_R. If f is not a polynomial of degree 0 , then for any $r \in [1,R)$ we have*

$$\|M_n^{(\alpha,\beta)}(f;\cdot)-f\|_r \asymp \frac{1}{n+\beta}, \quad n \in \mathbf{N},$$

where $\|f\|_r=max\{|f(z)|;|z| \leq r\}$ and the constants in the equivalence depend on f, r and α, β but it is independent of n.

Theorem 4 *Let $0 \leq \alpha \leq \beta$, $R>1$, $D_R=\{z \in \mathbf{C}:|z|<R\}$. Suppose that $f:D_R \to \mathbf{C}$ is analytic in D_R. Also, let $1 \leq r<r_1<R$ and $p \in \mathbf{N}$ be fixed. If f is not a polynomial of degree $\leq p-1$, then we have*

$$\|(M_n^{(\alpha,\beta)}(f;\cdot))^{(p)}-f^{(p)}\|_r \asymp \frac{1}{n+\beta}, \ n \in \mathbf{N},$$

where $\|f\|_r=max\{|f(z)|;|z| \leq r\}$ and the constants in the equivalence depend on f, r, r_1, p, α and β, but it is independent of n.

Proof Taking into account that the upper estimate in Theorem 1 , it remains to prove the lower estimate only. Denoting by Γ the circle of radius $r_1>r$ and center 0 , by the Cauchy's formula, it follows that for all $|z| \leq r$ and $n \in \mathbf{N}$, we have

$$(M_n^{(\alpha,\beta)}(f;z))^{(p)}-f^{(p)}(z)=\frac{p!}{2\pi i}\int_\Gamma \frac{M_n^{(\alpha,\beta)}(f;v)-f(v)}{(v-z)^{p+1}}dv.$$

Keeping the notation there for $H_n^{(\alpha,\beta)}(f;z)$, for all $n \in \mathbf{N}$, we have

$$M_n^{(\alpha,\beta)}(f;z) - f(z)$$
$$= \frac{1}{n+\beta}\left\{(\alpha-\beta z)f'(z) + \frac{z(1-z)}{2}f''(z) + (n+\beta)H_n^{(\alpha,\beta)}(f;z)\right\}.$$

by using Cauchy's formula, for all $v \in \Gamma$ we get

$$(M_n^{(\alpha,\beta)}(f;z))^{(p)} - f^{(p)}(z) = \frac{1}{n+\beta}\left\{\left[(\alpha-\beta z)f'(z) + \frac{z(1-z)}{2}f''(z)\right]^{(p)}\right.$$
$$\left. + \frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-z)^{p+1}}dv\right\},$$

passing now to $\|\cdot\|_r$ and denoting $e_1(z) = z$, it follows

$$\left\|(M_n^{(\alpha,\beta)}(f;\cdot))^{(p)} - f^{(p)}\right\|_r \geq \frac{1}{n+\beta}\left[\left\|\left[(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f''\right]^{(p)}\right\|_r\right.$$
$$\left. - \left\|\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r\right].$$

Since for any $|z| \leq r$ and $v \in \Gamma$, we have $|v-z| \geq r_1 - r$, so,

$$\left\|\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r \leq \frac{p!}{2\pi}\cdot\frac{2\pi r_1(n+\beta)\|H_n^{(\alpha,\beta)}(f;\cdot)\|_{r_1}}{(r_1-r)^{p+1}},$$

thus, by the inequality (5), we can get $\lim_{n\to\infty}\left\|\frac{p!}{2\pi i}\int_\Gamma \frac{(n+\beta)H_n^{(\alpha,\beta)}(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r = 0$.

Taking into account the function f is analytic in D_R, by following exactly the lines in Gal [5], seeing also the book Gal [6, pp. 77-78], we have $\left\|[(\alpha-\beta e_1)f' + \frac{e_1(1-e_1)}{2}f'']^{(p)}\right\|_r > 0$,

In continuation, reasoning exactly as in the proof of Theorem 3, we can get the desired conclusion.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No. 61572020) and the Natural Science Foundation of Fujian Province of China (Grant No. 2014J01021 and 2013J01017).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.2, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

On right multidimensional Riemann-Liouville fractional integral

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Abstract

Here we study some important properties of right multidimensional Riemann-Liouville fractional integral operator, such as of continuity and boundedness.

2010 AMS Subject Classification: 26A33.
Key Words and Phrases: Riemann-Liouville fractional integral, continuity, boundedness.

1 Motivation

From [1] we have

Theorem 1 *Let $r > 0$, $F \in L_{\infty}(a,b)$, and*

$$G(s) = \int_s^b (t-s)^{r-1} F(t)\, dt,$$

all $s \in [a,b]$. Then $G \in AC([a,b])$ (absolutely continuous functions) for $r \geq 1$, and $G \in C([a,b])$, only for $r \in (0,1)$.

2 Main Results

We give

Theorem 2 *Let $f \in L_{\infty}([a,b] \times [c,d])$, $\alpha_1, \alpha_2 > 0$. Consider the function*

$$F(x_1,x_2) = \int_{x_1}^{b_1} \int_{x_2}^{b_2} (t_1 - x_1)^{\alpha_1 - 1} (t_2 - x_2)^{\alpha_2 - 1} f(t_1,t_2)\, dt_1 dt_2, \qquad (1)$$

where $x_1, b_1 \in [a,b]$, $x_2, b_2 \in [c,d] : x_1 \le b_1$, $x_2 \le b_2$.

Then F *is continuous on* $[a, b_1] \times [c, b_2]$.

Proof. (I) Let $a_1, a_1^*, b_1 \in [a,b]$ with $a_1 < a_1^* < b_1$, and $a_2, a_2^*, b_2 \in [c,d]$ with $a_2 < a_2^* < b_2$.

We observe that

$$F(a_1, a_2) - F(a_1^*, a_2^*) =$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 -$$

$$\int_{a_1^*}^{b_1}\int_{a_2^*}^{b_2} (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 = \tag{2}$$

$$\int_{a_1^*}^{b_1}\int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$

$$\int_{a_1^*}^{b_1}\int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$

$$\int_{a_1}^{a_1^*}\int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$

$$\int_{a_1}^{a_1^*}\int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 - \tag{3}$$

$$\int_{a_1^*}^{b_1}\int_{a_2^*}^{b_2} (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 =$$

$$\int_{a_1^*}^{b_1}\int_{a_2^*}^{b_2} \left[(t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} \right] f(t_1, t_2)\, dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1}\int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 + \tag{4}$$

$$\int_{a_1}^{a_1^*}\int_{a_2}^{a_2^*} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2 +$$

$$\int_{a_1}^{a_1^*}\int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2.$$

Call

$$I(a_1^*, a_2^*) =$$

$$\int_{a_1^*}^{b_1}\int_{a_2^*}^{b_2} \left| (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} \right| dt_1 dt_2. \tag{5}$$

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Thus

$$|F(a_1,a_2) - F(a_1^*,a_2^*)| \leq$$

$$\left\{ I(a_1^*,a_2^*) + \left(\frac{(b_1-a_1)^{\alpha_1} - (a_1^*-a_1)^{\alpha_1}}{\alpha_1} \right) \frac{(a_2^*-a_2)^{\alpha_2}}{\alpha_2} + \right.$$

$$\left. \frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1} \frac{(a_2^*-a_2)^{\alpha_2}}{\alpha_2} + \frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1} \left(\frac{(b_2-a_2)^{\alpha_2} - (a_2^*-a_2)^{\alpha_2}}{\alpha_2} \right) \right\} \|f\|_\infty .$$

Hence, by the last inequality, it holds

$$\delta := \lim_{\substack{(a_1^*,a_2^*)\to(a_1,a_2) \\ \text{or} \\ (a_1,a_2)\to(a_1^*,a_2^*)}} |F(a_1,a_2) - F(a_1^*,a_2^*)| \leq$$

$$\left(\lim_{\substack{(a_1^*,a_2^*)\to(a_1,a_2) \\ \text{or} \\ (a_1,a_2)\to(a_1^*,a_2^*)}} I(a_1^*,a_2^*) \right) \|f\|_\infty =: \rho. \tag{6}$$

If $\alpha_1 = \alpha_2 = 1$, then $\rho = 0$, proving $\delta = 0$.

If $\alpha_1 = 1$, $\alpha_2 > 0$ we get

$$I(a_1^*,a_2^*) = (b_1 - a_1^*) \int_{a_2^*}^{b_2} \left| (t_2-a_2)^{\alpha_2-1} - (t_2-a_2^*)^{\alpha_2-1} \right| dt_2. \tag{7}$$

Assume $\alpha_2 > 1$, then $\alpha_2 - 1 > 0$. Hence

$$I(a_1^*,a_2^*) = (b_1 - a_1^*) \int_{a_2^*}^{b_2} \left((t_2-a_2)^{\alpha_2-1} - (t_2-a_2^*)^{\alpha_2-1} \right) dt_2$$

$$= (b_1 - a_1^*) \left\{ \left(\frac{(b_2-a_2)^{\alpha_2} - (a_2^*-a_2)^{\alpha_2}}{\alpha_2} \right) - \frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2} \right\}. \tag{8}$$

Clearly, then

$$\lim_{\substack{(a_1,a_2)\to(a_1^*,a_2^*) \\ \text{or} \\ (a_1^*,a_2^*)\to(a_1,a_2)}} I(a_1^*,a_2^*) = 0. \tag{9}$$

Similarly and symmetrically, we obtain that

$$\lim_{\substack{(a_1,a_2)\to(a_1^*,a_2^*) \\ \text{or} \\ (a_1^*,a_2^*)\to(a_1,a_2)}} I(a_1^*,a_2^*) = 0 \tag{10}$$

for the case of $\alpha_2 = 1$, $\alpha_1 > 1$.

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If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I\left(a_1^*, a_2^*\right) = \left(b_1 - a_1^*\right) \int_{a_2^*}^{b_2} \left[\left(t_2 - a_2^*\right)^{\alpha_2 - 1} - \left(t_2 - a_2\right)^{\alpha_2 - 1}\right] dt_2 =$$

$$\left(b_1 - a_1^*\right) \left\{ \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} - \left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(a_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \right) \right\}. \tag{11}$$

Clearly, then

$$\lim_{\substack{a_2^* \to a_2 \\ \text{or} \\ a_2 \to a_2^*}} I\left(a_1^*, a_2^*\right) = 0. \tag{12}$$

Similarly and symmetrically, we derive that

$$\lim_{\substack{a_1^* \to a_1 \\ \text{or} \\ a_1 \to a_1^*}} I\left(a_1^*, a_2^*\right) = 0, \tag{13}$$

for the case of $\alpha_2 = 1$, $0 < \alpha_1 < 1$.

Case now of $\alpha_1, \alpha_2 > 1$, then

$$I\left(a_1^*, a_2^*\right) =$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left(\left(t_1 - a_1\right)^{\alpha_1 - 1} \left(t_2 - a_2\right)^{\alpha_2 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \right) dt_1 dt_2 =$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right) \left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(a_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \right)$$

$$- \frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2}. \tag{14}$$

That is

$$\lim_{\substack{\left(a_1^*, a_2^*\right) \to \left(a_1, a_2\right) \\ \text{or} \\ \left(a_1, a_2\right) \to \left(a_1^*, a_2^*\right)}} I\left(a_1^*, a_2^*\right) = 0. \tag{15}$$

Case now of $0 < \alpha_1, \alpha_2 < 1$, then

$$I\left(a_1^*, a_2^*\right) =$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} \left(\left(t_1 - a_1^*\right)^{\alpha_1 - 1} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} - \left(t_1 - a_1\right)^{\alpha_1 - 1} \left(t_2 - a_2\right)^{\alpha_2 - 1} \right) dt_1 dt_2 =$$

$$\frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} -$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1} - \left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right) \left(\frac{\left(b_2 - a_2\right)^{\alpha_2} - \left(a_2^* - a_2\right)^{\alpha_2}}{\alpha_2} \right). \tag{16}$$

Hence, when $0 < \alpha_1, \alpha_2 < 1$, we get

$$\lim_{\substack{(a_1^*, a_2^*) \to (a_1, a_2) \\ (a_1, a_2) \to (a_1^*, a_2^*)}} I(a_1^*, a_2^*) = 0. \tag{17}$$

We observe that

$$I(a_1^*, a_2^*) \leq I^*(a_1^*, a_2^*) :=$$

$$\int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left| (t_2 - a_2)^{\alpha_2 - 1} - (t_2 - a_2^*)^{\alpha_2 - 1} \right| dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_2 - a_2^*)^{\alpha_2 - 1} \left| (t_1 - a_1)^{\alpha_1 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} \right| dt_1 dt_2. \tag{18}$$

Next we treat the case of $\alpha_1 > 1$, $0 < \alpha_2 < 1$.

Therefore it holds

$$I^*(a_1^*, a_2^*) = \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left((t_2 - a_2^*)^{\alpha_2 - 1} - (t_2 - a_2)^{\alpha_2 - 1} \right) dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_2 - a_2^*)^{\alpha_2 - 1} \left((t_1 - a_1)^{\alpha_1 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} \right) dt_1 dt_2 = \tag{19}$$

$$\left(\frac{(b_1 - a_1)^{\alpha_1} - (a_1^* - a_1)^{\alpha_1}}{\alpha_1} \right) \left(\frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} - \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} + \frac{(a_2^* - a_2)^{\alpha_2}}{\alpha_2} \right) +$$

$$\frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} \left(\frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} - \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} - \frac{(b_1 - a_1^*)^{\alpha_1}}{\alpha_1} \right).$$

Clearly then ($\alpha_1 > 1$, $0 < \alpha_2 < 1$)

$$\lim_{\substack{(a_1, a_2) \to (a_1^*, a_2^*) \\ \text{or} \\ (a_1^*, a_2^*) \to (a_1, a_2)}} I(a_1^*, a_2^*) = 0. \tag{20}$$

Finally, we prove the case of $\alpha_2 > 1$ and $0 < \alpha_1 < 1$. We have that

$$I^*(a_1^*, a_2^*) = \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_1 - a_1)^{\alpha_1 - 1} \left((t_2 - a_2)^{\alpha_2 - 1} - (t_2 - a_2^*)^{\alpha_2 - 1} \right) dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2^*}^{b_2} (t_2 - a_2^*)^{\alpha_2 - 1} \left((t_1 - a_1^*)^{\alpha_1 - 1} - (t_1 - a_1)^{\alpha_1 - 1} \right) dt_1 dt_2 = \tag{21}$$

$$\left(\frac{(b_1 - a_1)^{\alpha_1} - (a_1^* - a_1)^{\alpha_1}}{\alpha_1} \right) \left(-\frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} + \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} - \frac{(a_2^* - a_2)^{\alpha_2}}{\alpha_2} \right) +$$

$$\frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} \left(-\frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} + \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} + \frac{(b_1 - a_1^*)^{\alpha_1}}{\alpha_1} \right).$$

Clearly then ($\alpha_2 > 1$, $0 < \alpha_1 < 1$)

$$\lim_{\substack{(a_1,a_2)\to(a_1^*,a_2^*)\\ \text{or}\\ (a_1^*,a_2^*)\to(a_1,a_2)}} I\left(a_1^*, a_2^*\right) = 0. \tag{22}$$

We proved $\rho = 0$, and $\delta = 0$ in all cases of this section.

The case of $a_1 > a_1^*$ and $a_2 > a_2^*$, as symmetric to the already treated one of $a_1 < a_1^*$ and $a_2 < a_2^*$, is omitted.

(II) The remaining cases are: let $a_1, a_1^*, b_1 \in [a,b]$; $a_2, a_2^*, b_2 \in [c,d]$, we can have

$$\begin{array}{l} (\mathrm{II}_1)\ a_1 > a_1^* \ \text{ and } \ a_2 < a_2^*, \\ \text{or} \\ (\mathrm{II}_2)\ a_1 < a_1^* \ \text{ and } \ a_2 > a_2^*. \end{array} \tag{23}$$

Notice that the subcases (II_1) and (II_2) are symmetric, and treated the same way. As such we treat only the case (II_2).

We observe again that

$$F\left(a_1, a_2\right) - F\left(a_1^*, a_2^*\right) = \tag{24}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \left(t_1 - a_1\right)^{\alpha_1 - 1}\left(t_2 - a_2\right)^{\alpha_2 - 1} f\left(t_1, t_2\right) dt_1 dt_2 -$$

$$\int_{a_1^*}^{b_1}\int_{a_2^*}^{b_2} \left(t_1 - a_1^*\right)^{\alpha_1 - 1}\left(t_2 - a_2^*\right)^{\alpha_2 - 1} f\left(t_1, t_2\right) dt_1 dt_2 =$$

$$\int_{a_1}^{a_1^*}\int_{a_2}^{b_2} \left(t_1 - a_1\right)^{\alpha_1 - 1}\left(t_2 - a_2\right)^{\alpha_2 - 1} f\left(t_1, t_2\right) dt_1 dt_2 +$$

$$\int_{a_1^*}^{b_1}\int_{a_2}^{b_2} \left(t_1 - a_1\right)^{\alpha_1 - 1}\left(t_2 - a_2\right)^{\alpha_2 - 1} f\left(t_1, t_2\right) dt_1 dt_2 -$$

$$\int_{a_1^*}^{b_1}\int_{a_2^*}^{a_2} \left(t_1 - a_1^*\right)^{\alpha_1 - 1}\left(t_2 - a_2^*\right)^{\alpha_2 - 1} f\left(t_1, t_2\right) dt_1 dt_2 - \tag{25}$$

$$\int_{a_1^*}^{b_1}\int_{a_2}^{b_2} \left(t_1 - a_1^*\right)^{\alpha_1 - 1}\left(t_2 - a_2^*\right)^{\alpha_2 - 1} f\left(t_1, t_2\right) dt_1 dt_2 =$$

$$\int_{a_1^*}^{b_1}\int_{a_2}^{b_2} \left(\left(t_1 - a_1\right)^{\alpha_1 - 1}\left(t_2 - a_2\right)^{\alpha_2 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1}\left(t_2 - a_2^*\right)^{\alpha_2 - 1}\right) f\left(t_1, t_2\right) dt_1 dt_2$$

$$+\int_{a_1}^{a_1^*}\int_{a_2}^{b_2} \left(t_1 - a_1\right)^{\alpha_1 - 1}\left(t_2 - a_2\right)^{\alpha_2 - 1} f\left(t_1, t_2\right) dt_1 dt_2 - \tag{26}$$

$$\int_{a_1^*}^{b_1}\int_{a_2^*}^{a_2} \left(t_1 - a_1^*\right)^{\alpha_1 - 1}\left(t_2 - a_2^*\right)^{\alpha_2 - 1} f\left(t_1, t_2\right) dt_1 dt_2.$$

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Call

$$I\left(a_1^*, a_2\right) :=$$

$$\int_{a_1^*}^{b_1} \int_{a_2}^{b_2} \left| (t_1 - a_1)^{\alpha_1 - 1} (t_2 - a_2)^{\alpha_2 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} (t_2 - a_2^*)^{\alpha_2 - 1} \right| dt_1 dt_2. \tag{27}$$

Hence, we have

$$\left|F\left(a_1, a_2\right) - F\left(a_1^*, a_2^*\right)\right| \leq$$

$$\left\{ I\left(a_1^*, a_2\right) + \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} + \frac{(b_1 - a_1^*)^{\alpha_1}}{\alpha_1} \frac{(a_2 - a_2^*)^{\alpha_2}}{\alpha_2} \right\} \|f\|_\infty . \tag{28}$$

Therefore it holds

$$\delta := \lim_{\substack{|a_1 - a_1^*| \to 0, \\ |a_2 - a_2^*| \to 0}} \left|F\left(a_1, a_2\right) - F\left(a_1^*, a_2^*\right)\right| \leq \left(\lim_{\substack{|a_1 - a_1^*| \to 0, \\ |a_2 - a_2^*| \to 0}} I\left(a_1^*, a_2\right) \right) \|f\|_\infty =: \theta. \tag{29}$$

We will prove that $\theta = 0$, hence $\delta = 0$, in all possible cases.

If $\alpha_1 = \alpha_2 = 1$, then $I\left(a_1^*, a_2\right) = 0$, hence $\theta = 0$.

If $\alpha_1 = 1$, $\alpha_2 > 0$ we get

$$I\left(a_1^*, a_2\right) = \left(b_1 - a_1^*\right) \int_{a_2}^{b_2} \left| (t_2 - a_2)^{\alpha_2 - 1} - (t_2 - a_2^*)^{\alpha_2 - 1} \right| dt_2. \tag{30}$$

Assume $\alpha_2 > 1$, then $\alpha_2 - 1 > 0$. Hence

$$I\left(a_1^*, a_2\right) = \left(b_1 - a_1^*\right) \int_{a_2}^{b_2} \left((t_2 - a_2^*)^{\alpha_2 - 1} - (t_2 - a_2)^{\alpha_2 - 1} \right) dt_2$$

$$= \left(b_1 - a_1^*\right) \left\{ \frac{(b_2 - a_2^*)^{\alpha_2}}{\alpha_2} - \frac{(a_2 - a_2^*)^{\alpha_2}}{\alpha_2} - \frac{(b_2 - a_2)^{\alpha_2}}{\alpha_2} \right\}. \tag{31}$$

Clearly, then

$$\lim_{|a_2 - a_2^*| \to 0,} I\left(a_1^*, a_2\right) = 0, \tag{32}$$

hence $\theta = 0$.

Let the case now of $\alpha_2 = 1$, $\alpha_1 > 1$. Then

$$I\left(a_1^*, a_2\right) = \left(b_2 - a_2\right) \int_{a_1^*}^{b_1} \left| (t_1 - a_1)^{\alpha_1 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} \right| dt_1$$

$$= \left(b_2 - a_2\right) \int_{a_1^*}^{b_1} \left((t_1 - a_1)^{\alpha_1 - 1} - (t_1 - a_1^*)^{\alpha_1 - 1} \right) dt_1 \tag{33}$$

$$= \left(b_2 - a_2\right) \left(\frac{(b_1 - a_1)^{\alpha_1}}{\alpha_1} - \frac{(a_1^* - a_1)^{\alpha_1}}{\alpha_1} - \frac{(b_1 - a_1^*)^{\alpha_1}}{\alpha_1} \right).$$

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Then $\theta = 0$.

If $\alpha_1 = 1$, and $0 < \alpha_2 < 1$, then $\alpha_2 - 1 < 0$. Hence

$$I\left(a_1^*, a_2\right) = \left(b_1 - a_1^*\right) \int_{a_2}^{b_2} \left(\left(t_2 - a_2\right)^{\alpha_2 - 1} - \left(t_2 - a_2^*\right)^{\alpha_2 - 1}\right) dt_2 =$$

$$\left(b_1 - a_1^*\right) \left\{ \frac{\left(b_2 - a_2\right)^{\alpha_2}}{\alpha_2} - \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} + \frac{\left(a_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} \right\}, \tag{34}$$

hence $\theta = 0$.

Let now $\alpha_2 = 1$, $0 < \alpha_1 < 1$. Then

$$I\left(a_1^*, a_2\right) = \left(b_2 - a_2\right) \int_{a_1^*}^{b_1} \left(\left(t_1 - a_1^*\right)^{\alpha_1 - 1} - \left(t_1 - a_1\right)^{\alpha_1 - 1}\right) dt_1$$

$$= \left(b_2 - a_2\right) \left\{ \frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} - \frac{\left(b_1 - a_1\right)^{\alpha_1}}{\alpha_1} + \frac{\left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right\}, \tag{35}$$

hence $\theta = 0$.

We observe that:

$$I\left(a_1^*, a_2\right) \leq \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} \left(t_1 - a_1\right)^{\alpha_1 - 1} \left|\left(t_2 - a_2\right)^{\alpha_2 - 1} - \left(t_2 - a_2^*\right)^{\alpha_2 - 1}\right| dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \left|\left(t_1 - a_1\right)^{\alpha_1 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1}\right| dt_1 dt_2 =: J\left(a_1^*, a_2\right). \tag{36}$$

I.e.

$$I\left(a_1^*, a_2\right) \leq J\left(a_1^*, a_2\right). \tag{37}$$

Case of $\alpha_1, \alpha_2 > 1$. Then

$$J\left(a_1^*, a_2\right) = \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} \left(t_1 - a_1\right)^{\alpha_1 - 1} \left(\left(t_2 - a_2^*\right)^{\alpha_2 - 1} - \left(t_2 - a_2\right)^{\alpha_2 - 1}\right) dt_1 dt_2$$

$$+ \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} \left(t_2 - a_2^*\right)^{\alpha_2 - 1} \left(\left(t_1 - a_1\right)^{\alpha_1 - 1} - \left(t_1 - a_1^*\right)^{\alpha_1 - 1}\right) dt_1 dt_2 = \tag{38}$$

$$\left(\frac{\left(b_1 - a_1\right)^{\alpha_1}}{\alpha_1} - \frac{\left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} \right) \left\{ \frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} - \frac{\left(a_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} - \frac{\left(b_2 - a_2\right)^{\alpha_2}}{\alpha_2} \right\}$$

$$+ \left(\frac{\left(b_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} - \frac{\left(a_2 - a_2^*\right)^{\alpha_2}}{\alpha_2} \right) \left\{ \frac{\left(b_1 - a_1\right)^{\alpha_1}}{\alpha_1} - \frac{\left(a_1^* - a_1\right)^{\alpha_1}}{\alpha_1} - \frac{\left(b_1 - a_1^*\right)^{\alpha_1}}{\alpha_1} \right\}, \tag{39}$$

hence $\theta = 0$.

Case of $0 < \alpha_1, \alpha_2 < 1$, then

$$J\left(a_1^*, a_2\right) = \int_{a_1^*}^{b_1} \int_{a_2}^{b_2} \left(t_1 - a_1\right)^{\alpha_1 - 1} \left(\left(t_2 - a_2\right)^{\alpha_2 - 1} - \left(t_2 - a_2^*\right)^{\alpha_2 - 1}\right) dt_1 dt_2$$

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$$+\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_2-a_2^*)^{\alpha_2-1}\left((t_1-a_1^*)^{\alpha_1-1}-(t_1-a_1)^{\alpha_1-1}\right)dt_1dt_2= \tag{40}$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}-\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right)\left\{\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}-\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}+\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right\}$$
$$+\left(\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right)\left\{\frac{(b_1-a_1^*)^{\alpha_1}}{\alpha_1}-\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}+\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right\}, \tag{41}$$

hence $\theta=0$.

Next case of $\alpha_1>1$, $0<\alpha_2<1$. We observe that

$$J(a_1^*,a_2)=\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}\left((t_2-a_2)^{\alpha_2-1}-(t_2-a_2^*)^{\alpha_2-1}\right)dt_1dt_2 \tag{42}$$

$$+\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_2-a_2^*)^{\alpha_2-1}\left((t_1-a_1)^{\alpha_1-1}-(t_1-a_1^*)^{\alpha_1-1}\right)dt_1dt_2=$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}-\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right)\left\{\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}-\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}+\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right\} \tag{43}$$
$$+\left(\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right)\left\{\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}-\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}-\frac{(b_1-a_1^*)^{\alpha_1}}{\alpha_1}\right\},$$

hence $\theta=0$.

Finally, we prove the case of $\alpha_2>1$ and $0<\alpha_1<1$. In that case it holds

$$J(a_1^*,a_2)=\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}\left((t_2-a_2^*)^{\alpha_2-1}-(t_2-a_2)^{\alpha_2-1}\right)dt_1dt_2$$

$$+\int_{a_1^*}^{b_1}\int_{a_2}^{b_2}(t_2-a_2^*)^{\alpha_2-1}\left((t_1-a_1^*)^{\alpha_1-1}-(t_1-a_1)^{\alpha_1-1}\right)dt_1dt_2= \tag{44}$$

$$\left(\frac{(b_1-a_1)^{\alpha_1}-(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right)\left\{\frac{(b_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(a_2-a_2^*)^{\alpha_2}}{\alpha_2}-\frac{(b_2-a_2)^{\alpha_2}}{\alpha_2}\right\}$$
$$+\left(\frac{(b_2-a_2^*)^{\alpha_2}-(a_2-a_2^*)^{\alpha_2}}{\alpha_2}\right)\left\{\frac{(b_1-a_1^*)^{\alpha_1}}{\alpha_1}-\frac{(b_1-a_1)^{\alpha_1}}{\alpha_1}+\frac{(a_1^*-a_1)^{\alpha_1}}{\alpha_1}\right\}, \tag{45}$$

hence $\theta=0$.

We have proved that $\delta=0$, in all possible subcases of (II_2).

We have proved that F is a continuous function over $[a,b_1]\times[c,b_2]$. ■

Now we can state:

9

Theorem 3 *Let $f \in L_{\infty}\left(\prod_{i=1}^{k}[a_i, b_i]\right)$, $\alpha_i > 0$, $i = 1, ..., k \in \mathbb{N}$. Consider the function*

$$F(x_1, ..., x_k) = \int_{x_1}^{b_1^*} ... \int_{x_k}^{b_k^*} \prod_{i=1}^{k} (t_i - x_i)^{\alpha_i - 1} f(t_1, ..., t_k) \, dt_1 ... dt_k, \tag{46}$$

where $a_i \leq x_i \leq b_i^* \leq b_i$, $i = 1, ..., k$.

Then F is continuous on $\prod_{i=1}^{k}[a_i, b_i^*]$.

Remark 4 *In the setting of Theorem 3: Consider the right multidimensional Riemann-Liouville fractional integral of order $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_i > 0$, $i = 1, ..., k$:*

$$\left(I_{b_-^*}^{\alpha} f\right)(x) = \frac{1}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \int_{x_1}^{b_1^*} ... \int_{x_k}^{b_k^*} \prod_{i=1}^{k} (t_i - x_i)^{\alpha_i - 1} f(t_1, ..., t_k) \, dt_1 ... dt_k, \tag{47}$$

where $a_i \leq x_i \leq b_i^* \leq b_i$, $i = 1, ..., k$, where $b^* = (b_1^*, ..., b_k^*)$, $x = (x_1, ..., x_k)$, Γ is the gamma function.

By Theorem 3 we get that $\left(I_{b_-^*}^{\alpha} f\right)$ is a continuous function for every $x \in \prod_{i=1}^{k}[a_i, b_i^*]$.

We notice that

$$\left|\left(I_{b_-^*}^{\alpha} f\right)(x)\right| \leq \frac{1}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \left(\int_{x_1}^{b_1^*} ... \int_{x_k}^{b_k^*} \prod_{i=1}^{k} (t_i - x_i)^{\alpha_i - 1} dt_1 ... dt_k\right) \|f\|_{\infty} \tag{48}$$

$$= \frac{1}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} \left(\int_{x_i}^{b_i^*} (t_i - x_i)^{\alpha_i - 1} dt_i\right) \|f\|_{\infty} =$$

$$\frac{1}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} \frac{(b_i^* - x_i)^{\alpha_i}}{\alpha_i} \|f\|_{\infty} = \left(\prod_{i=1}^{k} \frac{(b_i^* - x_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right) \|f\|_{\infty}. \tag{49}$$

That is

$$\left|\left(I_{b_-^*}^{\alpha} f\right)(x)\right| \leq \left(\prod_{i=1}^{k} \frac{(b_i^* - x_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right) \|f\|_{\infty}. \tag{50}$$

In particular we get

$$\left(I_{b_-^*}^{\alpha} f\right)(b^*) = 0, \tag{51}$$

and

$$\left\|I_{b_-^*}^{\alpha} f\right\|_{\infty, \prod_{i=1}^{k}[a_i, b_i^*]} \leq \left(\prod_{i=1}^{k} \frac{(b_i^* - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right) \|f\|_{\infty}. \tag{52}$$

That is $I_{b_-^*}^{\alpha} f$ is a bounded linear operator, which here is also a positive operator.

10

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Volume 23, Number 3 September 2017
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of

Computational

Analysis and

Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fourteen times annually)

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Dynamics of a difference equation with maximum

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Abstract The purpose of this work is to investigate the convergence of the solutions of the following max-type difference equation

$$z_n = \max\{\frac{1}{z_{n-s}}, \frac{P_n}{z_{n-t}^{\alpha_n}}\}, \quad n = 0, 1, 2, \cdots,$$

where $s, t \in \{1, 2, 3, \cdots\}$ with $s \neq t$, $\alpha_n \in (0, 1)$ is an s-periodic sequence, $\{P_n\}_{n=0}^{+\infty}$ is a constant sequence satisfying $P_n \in (0, 1]$ for every $n \geq 0$. We show that if $\{z_n\}_{n=-r}^{+\infty}$ ($r = \max\{s, t\}$) is a positive solution of the above equation with the initial conditions $z_{-r}, z_{-r+1}, \cdots, z_{-1} \in (0, +\infty)$, then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_{2sn+k}\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s - 1$. Further, we show that if P_n is a periodic sequence, $s = 1$ and t is even, then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_n\}_{n=-t}^{+\infty}$ is eventually periodic with period 2.

AMS Subject Classification: 39A10; 39A11.
Keywords: max-type equation, positive solution, eventual periodicity, monotonicity, periodic sequence.

1. Introduction

The max operator arises naturally in certain models in automatic control theory (see [6,7]). In the recent years, there has been a lot of interest in studying the convergence and boundedness of max-type difference equations (see [1,3,5,8-11]). In [2], Chen studied the second order max-type difference equation

$$z_{n+1} = \max\{\frac{1}{z_n}, \frac{A_n}{z_{n-1}}\}, \quad n = 0, 1, 2, \cdots, \tag{1.1}$$

and showed that every positive solution of (1.1) is eventually periodic with period 2 when $\{A_n\}_{n=0}^{+\infty}$ is a periodic sequence with period $k \geq 2$ and $A_n \in (0, 1)$ for all $n \geq 0$.

In [4], the authors studied the following non-autonomous max-type difference equation with two delays

$$z_n = \max\{\frac{f_n}{z_{n-m}^{\alpha}}, \frac{A}{z_{n-r}^{\beta}}\}, \quad n = 0, 1, 2, \cdots,$$

⋆ Project Supported by NNSF of China (11461003)
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where $\alpha, \beta \in \mathbb{R}$, $\{A_n\}_{n=0}^{+\infty}$ is a sequence of positive real numbers with a finite limit and $m, r \in \mathbb{N} \equiv \{1, 2, 3, \cdots\}$ with $m \neq r$.

In this paper, we study the periodicity, the boundedness and the convergence of the following max-type difference equation

$$z_n = \max\{\frac{1}{z_{n-s}}, \frac{P_n}{z_{n-t}^{\alpha_n}}\}, \quad n = 0, 1, 2, \cdots, \tag{1.2}$$

where $s, t \in \mathbb{N}$ with $s \neq t$, $\alpha_n \in (0, 1)$ is an s-periodic sequence, $\{P_n\}_{n=0}^{+\infty}$ is a constant sequence satisfying $P_n \in (0, 1]$ for every $n \geq 0$.

2. Some Propositions

In the following, suppose that $\{z_n\}_{n=-r}^{+\infty}$ is a positive solution of (1.2). To obtain the main results of this paper, we need the following propositions.

Proposition 2.1 (i) $z_n z_{n-s} \geq 1$ for all $n \geq 0$.

(ii) For any $n \geq r$, $z_n \leq \max\{z_{n-2s}, P_n z_{n-s-t}^{\alpha_{n-s}}\}$.

(iii) If $z_n = P_n/z_{n-t}^{\alpha_n} > 1/z_{n-s}$ for some $n \geq r$, then $z_n > z_{n-2s}$. If $z_n = 1/z_{n-s}$ for some $n \geq s$, then $z_n \leq z_{n-2s}$.

Proof (i) Since $z_n \geq 1/z_{n-s}$ for any $n \geq 0$, we have $z_n z_{n-s} \geq 1$.

(ii) According to (i), we get that for every $n \geq r$,

$$z_n = \max\Big\{\frac{z_{n-2s}}{z_{n-s}z_{n-2s}}, \frac{P_n z_{n-s-t}^{\alpha_n}}{z_{n-s-t}^{\alpha_n} z_{n-t}^{\alpha_n}}\Big\} \leq \max\{z_{n-2s}, P_n z_{n-s-t}^{\alpha_n}\}.$$

(iii) If $z_n = P_n/z_{n-t}^{\alpha_n} > 1/z_{n-s}$ for some $n \geq r$, then by (i) we obtain that

$$\begin{aligned} 1 \quad < \quad z_n z_{n-s} &= \max\{\frac{z_n}{z_{n-2s}}, \frac{z_n z_{n-t}^{\alpha_n} P_{n-s}}{z_{n-t-s}^{\alpha_{n-s}} z_{n-t}^{\alpha_n}}\} \\ \leq \quad &\max\{\frac{z_n}{z_{n-2s}}, P_n P_{n-s}\} = \frac{z_n}{z_{n-2s}}. \end{aligned}$$

Which implies $z_n > z_{n-2s}$. If $z_n = 1/z_{n-s}$ for some $n \geq s$, then by (i) we obtain that

$$z_n = \frac{z_{n-2s}}{z_{n-s}z_{n-2s}} \leq z_{n-2s}.$$

The proof is complete.

Define

$$U_n = \max\{z_{n-1}, z_{n-2}, \cdots, z_{n-s-r}\} \quad (n \geq r). \tag{2.1}$$

According to Proposition 2.1 (i), we get $\max\{z_{n-1}, z_{n-s-1}\} \geq 1$, from which it follows $U_n \geq 1$ for any $n \geq r$.

Proposition 2.2 (i) Let U_n be as in (2.1). Then $z_n \leq U_n$ for any $n \geq r$ and $\{U_n\}_{n=r}^{+\infty}$ is a decreasing sequence.

(ii) There exist constants $R \geq R' > 0$ such that $R' \leq z_n \leq R$ for any $n \geq -r$.

Proof (i) If $z_{n-s-t} \leq 1$, then $z_{n-s-t}^{\alpha_n} \leq 1$. If $z_{n-s-t} \geq 1$, then $z_{n-s-t}^{\alpha_n} \leq z_{n-s-t}$. According to Proposition 2.1 (ii), we have that for any $n \geq r$,

$$z_n \leq \max\{z_{n-2s}, z_{n-s-t}^{\alpha_n}\} \leq \max\{z_{n-1}, z_{n-2}, \cdots, z_{n-s-r}\} = U_n.$$

Further, we get

$$U_{n+1} = \max\{z_n, z_{n-1}, \cdots, z_{n-s-r+1}\} \le U_n.$$

(ii) Let $R = \max\{U_r, z_{r-1}, \cdots, z_{-r}\}$ and $R' = \min\{1/U_r, z_{r-1}, \cdots, z_{-r}\}$. Then $R' \le z_n \le R$ for any $n \ge -r$. The proof is complete.

Now we assume $\lim_{n\longrightarrow\infty} U_n = U$ and $\liminf_{n\longrightarrow\infty} U_n = u$. According to Proposition 2.2 (i), we obtain the following corollary.

Corollary 2.3 There exists a sequence $1 < n_1 < n_2 < \cdots < n_k < \cdots$ such that $z_{n_k} \ge U$ and $n_{k+1} - n_k \le s + r$.

Proposition 2.4 The following statements hold:

(i) $U = \limsup_{n\longrightarrow\infty} z_n$.

(ii) Assume that $U > 1$. Then $\{n : U \le z_n = P_n/z_{n-t}^{\alpha_n}\}$ is a finite set. Further, there exists $N \in \mathbb{N}$ such that:

i) $z_{N+2ks} \ge U$ and $z_{N+2ks} = 1/z_{N+(2k-1)s}$ for any $k \ge 0$, and z_{N+2ks} is decreasing.

ii) $\lim_{k\longrightarrow\infty} z_{N+(2k-1)s} = u = 1/U$.

Proof (i) According to (2.1), we see that U_n is a subsequence of z_n. Thus $U \le \limsup_{n\longrightarrow\infty} z_n$. Further, since $z_n \le U_n$ for all $n \ge r$, we obtain

$$\limsup_{n\longrightarrow\infty} z_n \le \limsup_{n\longrightarrow\infty} U_n = U.$$

(ii) If $\{n : U \le z_n = P_n/z_{n-t}^{\alpha_n}\}$ is an infinite set, then there exists a sequence $t < n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$U \le z_{n_k} = \frac{P_{n_k}}{z_{n_k-t}^{\alpha_{n_k}}} \le P_{n_k} z_{n_k-t-s}^{\alpha_{n_k}} \le z_{n_k-t-s}^{\alpha_{n_k}}.$$

Without loss of generality, suppose that $\lim_{k\longrightarrow\infty} z_{n_k-t-s} = u_1$ and $\lim_{k\longrightarrow\infty} \alpha_{n_k} = \alpha < 1$. Thus we obtain $U = \lim_{k\longrightarrow\infty} z_{n_k} \le u_1^\alpha \le U^\alpha < U$ since $U > 1$. A contradiction.

It follows from the above that there exists $M \in \mathbb{N}$ such that if $n \ge M$ and $z_n \ge U$, then $z_n = 1/z_{n-s}$. By Corollary 2.3 we see that there exists a sequence $1 < n_1 < n_2 < \cdots < n_k < \cdots$ such that $z_{n_k} \ge U$ and $\lim_{k\longrightarrow\infty} z_{n_k} = U$. Without loss of generality, suppose that $n_k = 2sr_k + \tau > M$ with $0 \le \tau < 2s$ for all $k \in \mathbb{N}$. Then $z_{n_k} = 1/z_{n_k-s}$. Write $N = 2sr_1 + \tau$. By Proposition 2.1 (iii), we see that for any $k \ge 0$,

$$z_{N+2ks} \ge U \quad \text{and} \quad \frac{1}{z_{N+2ks-s}} = z_{N+2ks} \ge z_{N+2(k+1)s} = \frac{1}{z_{N+2(k+1)s-s}}.$$

Let $i_k \longrightarrow +\infty$ such that $z_{i_k} \longrightarrow u$ and $z_{i_k-s} \longrightarrow u_1$. Then

$$\frac{1}{U} = \lim_{k\longrightarrow\infty} \frac{1}{z_{N+2ks}} = \lim_{k\longrightarrow\infty} z_{N+(2k-1)s} \ge u = \lim_{k\longrightarrow\infty} z_{i_k} \ge \lim_{k\longrightarrow\infty} \frac{1}{z_{i_k-s}} = \frac{1}{u_1} \ge \frac{1}{U},$$

this implies $\lim_{k\longrightarrow\infty} z_{N+(2k-1)s} = u = 1/U$. The proof is complete.

Proposition 2.5 Let $N, p, q \in \mathbb{N}$ with $q \ge 2$ such that

(i) $\{z_{N+2ks}\}_{k=0}^{+\infty}$ is monotone.

(ii) $z_{N+2s(p+\lambda)+t} = P_{N+2s(p+\lambda)+t}/z_{N+2s(p+\lambda)}^{\alpha_{N+2s(p+\lambda)+t}} > 1/z_{N+2s(p+\lambda)+t-s}$ for every $\lambda \in \{0, q\}$.

(iii) $z_{N+2s(p+\lambda)+t} = 1/z_{N+2s(p+\lambda)+t-s}$ for every $1 \le \lambda \le q-1$.

Then $z_{N+2s(p+\lambda)+t} = z_{N+2s(p+\lambda+1)+t}$ for every $0 \le \lambda \le q-2$.

Proof There are two cases to be considered.

Case 1 $\{z_{N+2sk}\}_{k=0}^{+\infty}$ is decreasing. In this case, we claim that $z_{N+2s(p+\lambda)+t-s} = 1/z_{N+2s(p+\lambda-1)+t}$ for any $1 \le \lambda \le q-1$. Since, otherwise, if for some $1 \le \lambda \le q-1$,

$$z_{N+2s(p+\lambda)+t-s} = \frac{P_{N+2s(p+\lambda)+t-s}}{z_{N+2s(p+\lambda)-s}^{\alpha_{N+2s(p+\lambda)+t-s}}} > 1/z_{N+2s(p+\lambda-1)+t},$$

then by Proposition 2.1 (iii) it follows that

$$\begin{aligned}\frac{P_{N+2sp+t}}{z_{N+2s(p+\lambda)}^{\alpha_{N+2s(p+\lambda)+t}}} &\ge \frac{P_{N+2sp+t}}{z_{N+2sp}^{\alpha_{N+2sp+t}}} = z_{N+2sp+t} \ge z_{N+2s(p+\lambda-1)+t}\\ &> \frac{1}{z_{N+2s(p+\lambda)+t-s}} = \frac{z_{N+2s(p+\lambda)-s}^{\alpha_{N+2s(p+\lambda)+t-s}}}{P_{N+2s(p+\lambda)+t-s}}.\end{aligned}$$

This implies

$$1 \ge P_{N+2sp+t}P_{N+2s(p+\lambda)+t-s} > z_{N+2s(p+\lambda)}^{\alpha_{N+2s(p+\lambda)+t}} z_{N+2s(p+\lambda)-s}^{\alpha_{N+2s(p+\lambda)+t-s}} \ge 1.$$

A contradiction. From the above claim it follows that

$$z_{N+2s(p+\lambda)+t} = \frac{1}{z_{N+2s(p+\lambda)+t-s}} = z_{N+2s(p+\lambda-1)+t} \ge z_{N+2s(p+\lambda)+t}.$$

Thus $z_{N+2s(p+\lambda-1)+t} = z_{N+2s(p+\lambda)+t}$ for every $1 \le \lambda \le q-1$.

Case 2 $\{z_{N+2ks}\}_{k=0}^{+\infty}$ is increasing. In this case, it follows from Proposition 2.1 (iii) that

$$\begin{aligned}\frac{P_{N+2s(p+q)+t}}{z_{N+2s(p+q-1)}^{\alpha_{N+2s(p+q-1)+t}}} &\ge \frac{P_{N+2s(p+q)+t}}{z_{N+2s(p+q)}^{\alpha_{N+2s(p+q)+t}}} = z_{N+2s(p+q)+t} > z_{N+2s(p+q-1)+t}\\ &= \frac{1}{z_{N+2s(p+q-1)+t-s}} = \min\{z_{N+2s(p+q-2)+t}, \frac{z_{N+2s(p+q-1)-s}^{\alpha_{N+2s(p+q-1)+t-s}}}{P_{N+2s(p+q-1)+t-s}}\}\\ &= z_{N+2s(p+q-2)+t} \ge z_{N+2s(p+q-1)+t}\end{aligned}$$

since

$$P_{N+2s(p+q)+t}P_{N+2s(p+q-1)+r-s} \le 1 \text{ and } z_{N+2s(p+q-1)}^{\alpha_{N+2s(p+q-1)+t}} z_{N+2s(p+q-1)-s}^{\alpha_{N+2s(p+q-1)+t-s}} \ge 1,$$

we have

$$z_{N+2s(p+q-1)+t} = z_{N+2s(p+q-2)+t}.$$

In a similar fashion, we may obtain that $z_{N+2s(p+q-1)+t} = z_{N+2s(p+\lambda)+t}$ for any $0 \le \lambda \le q-2$. The proof is complete.

Proposition 2.6 If there exists $N \in \mathbb{N}$ such that $\{z_{N+2ks}\}_{k=0}^{+\infty}$ is monotone, then $\{z_{N+t+2ks}\}_{k=0}^{+\infty}$ is eventually monotone.

Proof If there exists $K \in \mathbb{N}$ such that

$$z_{N+2ks+t} = 1/z_{N+2sk+t-s} \text{ for all } k \ge K$$

or

$$z_{N+2ks+t} = P_{N+2ks+t}/z_{N+2ks}^{\alpha_{N+2ks+t}} > 1/z_{N+2ks+t-s} \quad \text{for all} \quad k \geq K,$$

then by Proposition 2.1 (iii) we obtain that $z_{N+2ks+t} \leq z_{N+2(k-1)s+t}$ for all $k \geq K$ (or $z_{N+2ks+t} > z_{N+2(k-1)s+t}$ for all $k \geq K$). Thus $\{z_{N+t+2ks}\}_{k=K}^{+\infty}$ is monotone.

If there exists a sequence $1 < p_1 < q_1 < p_2 < q_2 < \cdots < p_k < q_k < \cdots$ such that

$$z_{N+2rs+t} = \frac{P_{N+2rs+t}}{z_{N+2rs}^{\alpha_{N+2rs+t}}} > \frac{1}{z_{N+2rs+t-s}} \quad \text{for every} \quad p_i \leq r < q_i$$

and

$$z_{N+2rs+t} = \frac{1}{z_{N+2rs+t-s}} \quad \text{for every} \quad q_i \leq r < p_{i+1},$$

then by Proposition 2.1 (iii) and Proposition 2.5 it follows that $z_{N+2(r-1)s+t} < z_{N+2rs+t}$ for every $p_i \leq r < q_i$ and $z_{N+2(r-1)s+t} = z_{N+2rs+t}$ for every $q_i \leq r < p_{i+1}$, this follows that $\{z_{N+t+2rs}\}_{r=p_1}^{+\infty}$ is increasing. The proof is complete.

3. Main Results

In section, we state the main results of this paper.

Theorem 3.1 Let $\{z_n\}_{n=-r}^{+\infty}$ be a positive solution of (1.2). Then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_{2ns+k}\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s-1$.

Proof If $U = \limsup_{n\longrightarrow\infty} z_n = 1$, then let $i_k \longrightarrow +\infty$ such that $z_{i_k} \longrightarrow u = \liminf_{n\longrightarrow\infty} z_n$ and $z_{i_k-s} \longrightarrow u_1$. Thus

$$1 \geq u = \lim_{k\longrightarrow\infty} z_{i_k} \geq \lim_{k\longrightarrow\infty} \frac{1}{z_{i_k-s}} = \frac{1}{u_1} \geq \frac{1}{U} = 1.$$

Which implies $\lim_{n\longrightarrow\infty} z_n = 1$. Now assume that $U = \limsup_{n\longrightarrow\infty} z_n > 1$.

First we suppose that $\gcd(s,t) = 1$. Then by Proposition 2.4 (iii) we see that there exists $N \in \mathbb{N}$ such that the following statements hold:

(1) $z_{N+2ns}z_{N+(2n-1)s} = 1$ for any $n \geq 0$.

(2) z_{N+2ns} is decreasing $(n \geq 0)$ and $\lim_{n\longrightarrow\infty} z_{N+2ns} = U$. $x_{N+(2n-1)s}$ is increasing $(n \geq 0)$ and $\lim_{n\longrightarrow\infty} z_{N+(2n-1)s} = u = 1/U$.

Using Proposition 2.6 repeatedly, it follows that for every $1 \leq i \leq s-1$, $\{z_{N+2ns+it}\}_{n=0}^{+\infty}$ and $\{z_{N+(2n-1)s+it}\}_{n=0}^{+\infty}$ are eventually monotone. Since $\gcd(s,t) = 1$, it follows that for every $j \in \{0, 1, 2, \cdots, s-1\}$ there exist some $0 \leq i_j \leq s-1$ and integer λ_j such that $i_jt = \lambda_js + j$ and $i_jt - s = (\lambda_j - 1)s + j$. Thus $\{z_{N+2ns+\lambda_js+j}\}_{n=0}^{+\infty}$ and $\{z_{N+2ns+(\lambda_j-1)s+j}\}_{n=0}^{+\infty}$ are eventually monotone for every $j \in \{0, 1, 2, \cdots, s-1\}$, which implies that $\{z_{2ns+k}\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s-1$.

If $\gcd(s,t) = d > 1$, then we consider the max-type equation

$$z_n = \max\{\frac{1}{z_{n-ds_1}}, \frac{P_n}{z_{n-dt_1}^{\alpha_n}}\}, \quad n = 0, 1, 2, \cdots\cdots, \tag{3.1}$$

where $s = ds_1$ and $t = dt_1$ with $\gcd(s_1, t_1) = 1$. Write $y_n^i = z_{nd+i}$ for every $0 \leq i \leq d-1$ and $n = 0, 1, 2, \cdots$. Then (3.1) reduces to the equations

$$y_n^i = \max\{\frac{1}{y_{n-s_1}^i}, \frac{P_{nd+i}}{(y_{n-t_1}^i)^{\alpha_{nd+i}}}\}, \quad 0 \leq i \leq d-1, \quad n = 0, 1, 2, \cdots. \tag{3.2}$$

By an analogous way as in the above, we obtain that for every $0 \leq i \leq d-1$, y_n^i is a solution of equation

$$y_n^i = \max\{1/y_{n-s_1}^i, \frac{P_{nd+i}}{(y_{n-t_1}^i)^{\alpha_{nd+i}}}\}.$$

Then $\{y_{2s_1n+k}^i\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s_1 - 1$. Thus for every $0 \leq k \leq 2s-1$, $\{z_{2ns+k}\}_{n=0}^{+\infty}$ is eventually monotone. The proof is complete.

Theorem 3.2 Assume that $s = 1$, and t is even, and P_n is a periodic sequence. Let $\{z_n\}_{n=-t}^{+\infty}$ be a positive solution of (1.2). Then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_n\}_{n=-t}^{+\infty}$ is eventually periodic with period 2.

Proof If $U = \limsup_{n\longrightarrow\infty} z_n = 1$, then using arguments similar to ones developed in the proof of Theorem 3.1 we can obtain $\lim_{n\longrightarrow\infty} z_n = 1$. Now assume that $U = \limsup_{n\longrightarrow\infty} z_n > 1$.

According to Proposition 2.4 (iii) and Theorem 3.1, we see that there exists $N \in \mathbb{N}$ such that the following statements hold:

(1) $z_{N+2n}z_{N+2n-1} = 1$ for any $n \geq 0$.

(2) z_{N+2n} is decreasing $(n \geq 0)$ and $\lim_{n\longrightarrow\infty} z_{N+2n} = U$. z_{N+2n-1} is increasing $(n \geq 0)$ and $\lim_{n\longrightarrow\infty} z_{N+2n-1} = u = 1/U$.

We claim that $z_{N+2n+1} = 1/z_{N+2n}$ eventually. In fact, if there exist $1 \leq k_1 < k_2 < \cdots < k_i < \cdots$ such that

$$z_{N+2k_i+1} = \frac{P_{N+2k_i+1}}{z_{N+2k_i+1-t}^{\alpha_{N+2k_i+1}}},$$

then by taking a subsequence we may assume that P_{N+2k_i+1} and α_{N+2k_i+1} are constant sequences since P_n and α_n are periodic sequences. Thus z_{N+2k_i+1} is decreasing since $z_{N+2k_i+1-t}^{\alpha_{N+2k_i+1}}$ is increasing. A contradiction. Which implies that $\{z_n\}_{n=-t}^{+\infty}$ is eventually periodic with period 2. The proof is complete.

Example 3.3 Assume that $s = 1$ and t is odd. Let $P_n = P \in (0, 1)$ and $\alpha_n = \alpha \in (0, 1)$ for any $n \geq 0$. Then there exists a positive solution $\{z_n\}_{n=-t}^{\infty}$ of (1.2) which is not eventually periodic such that $\lim_{n\longrightarrow\infty} z_n \neq 1$.

Proof Choose the initial values $z_{-t}, z_{1-t}, \cdots, z_{-1} \in (0, +\infty)$ satisfying

$$z_{-t} < z_{2-t} < \cdots < z_{-1} < z_{-t}/P, \quad z_{-t} < P^{2/(1-\alpha)}, \quad z_{k-t} = 1/z_{k-t-1} \quad k \in \{1, 3, \cdots, t-2\}.$$

Now we show that $z_{2k-1} < z_{2k+1}$ and $z_{2k} < z_{2k-2}$ for any $k \in \mathbb{N}$.

By $z_{-1} < z_{-t}/P$ and $z_{-t} < P^{2/(1-\alpha)}$, we have $z_{-1} < z_{-t}/P < Pz_{-t}^{\alpha}$. Which implies

$$\begin{aligned}
z_0 &= \max\{\frac{1}{z_{-1}}, \frac{P}{z_{-t}^{\alpha}}\} = \frac{1}{z_{-1}} < \frac{1}{z_{-3}} = z_{-2}.\\
z_1 &= \max\{\frac{1}{z_0}, \frac{P}{z_{1-t}^{\alpha}}\} = \max\{z_{-1}, Pz_{-t}^{\alpha}\} = Pz_{-t}^{\alpha} > z_{-1}.\\
z_2 &= \max\{\frac{1}{z_1}, \frac{P}{z_{2-t}^{\alpha}}\} = \max\{\frac{1}{Pz_{-t}^{\alpha}}, \frac{P}{z_{2-t}^{\alpha}}\} = \frac{1}{Pz_{-t}^{\alpha}} = \frac{1}{z_1} < \frac{1}{z_{-1}} = z_0.\\
z_3 &= \max\{\frac{1}{z_2}, \frac{P}{z_{3-t}^{\alpha}}\} = \max\{z_1, Pz_{2-t}^{\alpha}\} = \max\{Pz_{-t}^{\alpha}, Pz_{2-t}^{\alpha}\} = Pz_{2-t}^{\alpha} > \frac{1}{z_2} = z_1.\\
z_4 &= \max\{\frac{1}{z_3}, \frac{P}{z_{4-t}^{\alpha}}\} = \max\{\frac{1}{Pz_{2-t}^{\alpha}}, \frac{P}{z_{4-t}^{\alpha}}\} = \frac{1}{Pz_{2-t}^{\alpha}} = \frac{1}{z_3} < \frac{1}{z_1} = z_2.
\end{aligned}$$

Assume that there exists some $m \in \mathbb{N}$ such that

(1) $z_{2k-1} < z_{2k+1}$ and $z_{2k+2} < z_{2k}$ for any $(-t+1)/2 \leq k \leq m$.

(2) $z_{2k+1} = Pz^{\alpha}_{2k-t}$ for any $0 \leq k \leq m$ and $z_{2k+2}z_{2k+1} = 1$ for any $(-t+1)/2 \leq k \leq m$.

Then

$$\begin{aligned} z_{2m+3} &= \max\{\frac{1}{z_{2m+2}}, \frac{P}{z^{\alpha}_{2m+3-t}}\} = \max\{z_{2m+1}, Pz^{\alpha}_{2m+2-t}\} \\ &= \max\{Pz^{\alpha}_{2m-t}, Pz^{\alpha}_{2m+2-t}\} = Pz^{\alpha}_{2m+2-t} > Pz^{\alpha}_{2m-t} = z_{2m+1}. \\ z_{2m+4} &= \max\{\frac{1}{z_{2m+3}}, \frac{P}{z^{\alpha}_{2m+4-t}}\} = \max\{\frac{1}{Pz^{\alpha}_{2m+2-t}}, \frac{P}{z^{\alpha}_{2m+4-t}}\} \\ &= \frac{1}{Pz^{\alpha}_{2m+2-t}} = \frac{1}{z_{2m+3}} < \frac{1}{z_{2m+1}} = z_{2m+2}. \end{aligned}$$

Therefore $z_{2k-1} < z_{2k+1}$ and $z_{2k+2} < z_{2k}$ for any $k \geq (-t+1)/2$, which implies that $\{z_n\}_{n=-t}^{\infty}$ is not eventually periodic. Since $z_{2n+1} = Pz^{\alpha}_{2n-t}$ $(n \in \mathbb{N})$, we obtain $\lim_{n\longrightarrow\infty} z_n \neq 1$. The proof is complete.

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General properties of concave functions defined by the generalized Srivastava-Attiya operator

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January 27, 2016

Abstract

In this paper we introduce a class $\Im_{\mu,b}^{m,k} C_0(\alpha)$ of concave functions by using the generalized Srivastava-Attiya operator. Also, we get distortion bounds for this class.

Keywords: Hadamard product, concave functions, linear operator, distortion theorem, Hurwitz-Lerch Zeta functions, Srivastava-Attiya operator.

2010, Mathematics Subject Classification: 30C45, 30C50.

1 Introduction

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The study of operators plays an important role in Geometric Function Theory in Complex Analysis and its related fields. Many derivative and integral operators can be written in terms of convolution of certain analytic functions. For functions

$$f_j(z) = \sum_{n=0}^{\infty} a_{n,j} z^n \quad (j = 1, 2)$$

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analytic in U, we define the Hadamard product of f_1 and f_2 as

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{n,1}, a_{n,2} z^n = (f_2 * f_1)(z) \quad (z \in U). \tag{2}$$

In terms of the Hadamard product (or convolution), the Dziok-Srivastava linear convolution operator involving the generalized hypergeometric function was introduced and studied systematically by Dziok and Srivastava [9], [10] and (subsequently) by many other authors (see, for details, [11] and [20]).

We recall here a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined in [19] by

$$\Phi(z, s, a) := \sum_{n=2}^{\infty} \frac{z^n}{(n+a)^s}$$

$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C},\ \text{when } |z| < 1;\ Re(s) > 1 \text{ when } |z| = 1\big)$ where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \backslash \mathbb{N},$ and $\mathbb{N} := \{1, 2, 3, \ldots\}$). Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in [8], and the references stated there in (see also [16], [21], [22]). Srivastava and Attiya [21] (also see [4], [12]) introduced and investigated the linear operator.

$$\Im_b^{\mu} : A \to A$$

defined in terms of the Hadamard product by

$$\Im_b^{\mu} f\ (z) = (G_b^{\mu} * f)\ (z), \qquad \big(z \in U;\ b \in \mathbb{C} \backslash \mathbb{Z}_0^-;\ \mu \in \mathbb{C};\ f \in A\big) \tag{3}$$

where, for convenience,

$$G_b^{\mu}(z) := (1+b)^{\mu}[\Phi(z, \mu, b) - b^{-\mu}] \quad (z \in U). \tag{4}$$

We recall here the following relationships which follow easily by using (1), (3) and (4)

$$\Im_b^{\mu} f\ (z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} a_n z^n. \tag{5}$$

Motivated essentially by the Srivastava-Attiya operator, Murugusundaramoorthy [17] introduced the generalized integral operator $\Im_{\mu,b}^{m,k}$ given by

$$\Im_{\mu,b}^{m,k}\ f(z) = z + \sum_{n=2}^{\infty} C_n^m(b, \mu, k) a_n z^n \tag{6}$$

where

$$\Psi_n = C_n^m(b, \mu, k) = \left| \left(\frac{1+b}{n+b}\right)^{\mu} \right| \frac{m!(n+k-2)!}{(k-2)!(n+m-1)!} \tag{7}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \backslash \mathbb{Z}_0^-; \mu \in \mathbb{C}, k \geq 2$ and $m > -1$. It is of interest to note that $\Im_{\mu,b}^{1,2}$ is the Srivastava-Attiya operator and $\Im_{0,b}^{m,k}$ is the well-known Choi-Saigo-Srivastava operator (see [15]). Suitably specializing the parameters m, k, μ and b in $\Im_{\mu,b}^{m,k}\ f(z)$ we can get various integral operators introduced by Alexander [1] and Bernardi [5], Libera and Livingston [13], [14].

2

2 Preliminaries

Conformal maps of the unit disk onto convex domains are a classical topic. Recently Avkhadiev and Wirths [2] discovered that conformal maps onto concave domains (the complements of convex closed sets) have some novel properties.

A function $f : U \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

- f is analytic in U with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.
- f maps U conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
- The opening angle of $f(U)$ at ∞ is less than or equal to $\pi\alpha,\ \alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev et al. [3], Cruz and Pommerenke [7] and references there in.

In particular, the inequality

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0 \qquad (z \in U)$$

is used - sometimes also as a definition - for concave functions $f \in C_{0_O}$ (see e.g. [18] and others).

Bhowmik et al. [6] showed that an analytic function f maps U onto a concave domain of angle $\pi\alpha$, if and only if $ReP_f(z) > 0$, where

$$P_f(z) = \frac{2}{\alpha - 1}\left[\frac{\alpha + 1}{2}\frac{1 + z}{1 - z} - 1 - z\frac{f''(z)}{f'(z)}\right].$$

Definition 1 *Let $f(z) \in A$ and $\alpha \in (1, 2]$. Then $f(z) \in \Im_{\mu,b}^{m,k} C_0(\alpha)$ if and only if*

$$Re\frac{2}{\alpha - 1}\left[\frac{\alpha + 1}{2}\frac{1 + z}{1 - z} - 1 - z\frac{\left[\Im_{\mu,b}^{m,k} f(z)\right]''}{\left[\Im_{\mu,b}^{m,k} f(z)\right]'}\right] > 0 \qquad (z \in U).$$

3 Main results

Theorem 2 *If $f(z) \in A$ satisfies the inequality*

$$\sum_{n=2}^{\infty}\left[(\alpha - 1)n + 2n^2\right]|C_n^m(b, \mu, k)||a_n| < 3 - \alpha,$$

for some $\alpha \in (1, 2],\ n \in \mathbb{N}$, then $f(z) \in \Im_{\mu,b}^{m,k} C_0(\alpha)$.

3

Proof. We want to prove that

$$Re\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-z\frac{\left[\Im_{\mu,b}^{m,k}f(z)\right]''}{\left[\Im_{\mu,b}^{m,k}f(z)\right]'}\right]>0.$$

By using the fact that

$$Re\frac{1}{w}>\frac{1}{2}\Leftrightarrow|w-1|<1,$$

it is enough to show that $|w|<1$.

$$\frac{1}{w}=\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-z\frac{g'(z)}{g(z)}\right] \tag{8}$$

where

$$g(z)=z\left(\Im_{\mu,b}^{m,k}f(z)\right)'=z\left\{1+\sum_{n=2}^{\infty}C_n^m(b,\mu,k)na_nz^{n-1}\right\} \tag{9}$$

and

$$g'(z)=1+\sum_{n=2}^{\infty}C_n^m(b,\mu,k)n^2a_nz^{n-1}. \tag{10}$$

Using (9) and (10), in (8) we obtain

$$|w|\leq\frac{\alpha-1}{2}\left|\frac{2(1-z)z\left[1+\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)na_nz^{n-1}\right]}{(\alpha+1)(1+z)z\left(1+\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)na_nz^{n-1}\right)-2(1-z)z\left(1+\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)n^2a_nz^{n-1}\right)}\right|.$$

Using triangle inequality and letting $z\to-1$, then

$$|w|<\frac{\alpha-1}{2}\left(\frac{1+\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n}{1-\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n^2}\right).$$

The last expression is bounded by 1, if

$$\frac{1+\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n}{1-\sum\limits_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n^2}<\frac{2}{\alpha-1}.$$

Finally, we can easily see that

$$\sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n|<3-\alpha. \tag{11}$$

■

4

4 Distortion Bounds

Theorem 3 *If* $f(z) \in \Im_{\mu,b}^{m,k} C_0(\alpha)$, *then*

$$|z| - \frac{3-\alpha}{2(3+\alpha)}|z|^2 \leq \left|\Im_{\mu,b}^{m,k} f(z)\right| \leq |z| + \frac{3-\alpha}{2(3+\alpha)}|z|^2.$$

Proof. From the Theorem 2, we have

$$2(3+\alpha)\sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right] C_n^m(b,\mu,k)|a_n| < 3-\alpha,$$

That is

$$\sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n| \leq \frac{3-\alpha}{2(3+\alpha)}.$$

According to (11) we obtain

$$\begin{aligned} |\Im_{\mu,b}^{m,k} f(z)| &\leq |z| + \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^2 \\ &\leq |z| + \frac{3-\alpha}{2(3+\alpha)}|z|^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |\Im_{\mu,b}^{m,k} f(z)| &\geq |z| - \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^n \\ &\geq |z| - \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^2 \\ &\geq |z| - \frac{3-\alpha}{2(3+\alpha)}|z|^2. \end{aligned}$$

This completes the proof. ∎

Theorem 4 *If* $f(z) \in \Im_{\mu,b}^{m,k} C_0(\alpha)$, *then*

$$|z| - \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2 \leq |f(z)| \leq |z| + \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2.$$

Proof. According to the Theorem 2 we get that

$$2(3+\alpha)\left|\left(\frac{1+b}{2+b}\right)^{\mu}\right|\frac{k(k-1)}{m+1}\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right] C_n^m(b,\mu,k)|a_n| < 3-\alpha.$$

Thus we get

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right|.$$

Next from (1), we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^2 \\ &\leq |z| + \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2. \end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^n \\ &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^2 \\ &\geq |z| - \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2 \end{aligned}$$

This completes the proof. ■

Theorem 5 *If $f(z) \in \Im_{\mu,b}^{1,2}C_0(\alpha)$, then*

$$|z| - \frac{(3-\alpha)}{2(3+\alpha)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2 \leq |f(z)| \leq |z| + \frac{(3-\alpha)}{2(3+\alpha)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2.$$

Proof. According to the Theorem 2 we get that

$$2(3+\alpha)\sum_{n=2}^{\infty}C_n^1(b,\mu,2)|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n| < 3-\alpha,$$

or, equivalently

$$2(3+\alpha)\left|\left(\frac{1+b}{2+b}\right)^{\mu}\right|\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n| < 3-\alpha.$$

Thus we get

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(3-\alpha)}{2(3+\alpha)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right|.$$

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Next from (1), we have

$$\begin{aligned}|f(z)| &\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^2 \\ &\leq |z| + \frac{(3-\alpha)}{2(3+\alpha)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2.\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}|f(z)| &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^n \\ &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^2 \\ &\geq |z| - \frac{(3-\alpha)}{2(3+\alpha)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2.\end{aligned}$$

∎

Theorem 6 *If $f(z) \in \Im_{0,b}^{m,k}C_0(\alpha)$, then*

$$|z|-\frac{(3-\alpha)}{2(3+\alpha)}\left|\frac{(k-2)!(n+m-1)!}{m!(n+k-2)!}\right||z|^2 \leq |f(z)| \leq |z|+\frac{(3-\alpha)}{2(3+\alpha)}\left|\frac{(k-2)!(n+m-1)!}{m!(n+k-2)!}\right||z|^2.$$

Proof. According to the Theorem 2 we get that

$$2(3+\alpha)\left|\frac{m!(n+k-2)!}{(k-2)!(n+m-1)!}\right|\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n| \leq 3-\alpha.$$

Thus we get

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(3-\alpha)}{2(3+\alpha)}\left|\frac{(k-2)!(n+m-1)!}{m!(n+k-2)!}\right|.$$

Next from (1), we have

$$\begin{aligned}|f(z)| &\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^2 \\ &\leq |z| + \frac{(3-\alpha)}{2(3+\alpha)}\left|\frac{(k-2)!(n+m-1)!}{m!(n+k-2)!}\right||z|^2.\end{aligned}$$

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The other assertion can be proved as follows

$$\begin{aligned}
|f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n||z|^n \\
&\geq |z| - \sum_{n=2}^{\infty} |a_n||z|^2 \\
&\geq |z| - \frac{(3-\alpha)}{2(3+\alpha)} \left| \frac{(k-2)!(n+m-1)!}{m!(n+k-2)!} \right| |z|^2.
\end{aligned}$$

■

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.3, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

On the zeros of eigenfunctions of discontinuous Sturm-Liouville problems

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Abstract : In this paper, we prove analogues of the classical Sturm comparison and oscillation theorems for Sturm-Liouville problem together with boundary -transmission conditions on two disjoint intervals. We present a new version for Sturm's comparison and oscillation theorems. The obtained results generalizes the recently obtained oscillation and comparison theorems for regular Sturm-Liouville problem which contained transmission conditions.

Keywords : Sturm-Liouville problems, transmission conditions, Sturm comparison and oscillation theorems.

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1 Introduction

The oscillation theory for the solutions of differential equations is one of the traditional trends in the qualitative theory of differential equations. Its essence is to establish conditions for the existence of oscillating (nonoscillating) solutions, to study the laws of distribution of the zeros, to obtain estimates of the distance between the consecutive zeros and of the number of zeros in a given interval. The relationship between the oscillatory and other fundamental properties of the solutions of Sturm-Liouville type differential equations are of central importance in the theory of boundary value problems There are substantial literature on this subject. Many authors have expounded on various aspects of this theory, see [1, 9, 10] and the references cited therein. A considerable number of studies have been made on the oscillation and nonoscillation for a long time. Those results can be found in [14, 15] and the references contained therein. While the extensions and generalizations have much intrinsic interest, we believe their continued relevance is due in no small part to their important connection with problems of physical origin. Particularly the connections with the minimization problems of the calculus of variations and optimal control as well as the spectral theory of differential operators are important. Since the second order equations have applications in various problems in physics, biology, and economics (see for example [1, 5, 13], and the references cited therein) there is a permanent interest in obtaining new sufficient conditions for the oscillation or nonoscillation of solutions of various types of second order equations. In this study we investigated same aspects of comparison and oscillation properties for one discontinuous eigenvalue problem which consists of Sturm-Liouville

equation,

$$Ly := -y''(x) + q(x)y(x) = \lambda y(x) \tag{1.1}$$

to hold on two disjoint intervals $(-1, 0)$ and $(0, 1)$, where discontinuity in y and y' at the interior singular point $x = 0$ are prescribed by transmission conditions

$$y(0-) = \delta y(0+), \quad y'(0-) = \frac{1}{\delta} y'(0+), \tag{1.2}$$

together with the boundary conditions

$$y(-1) = y(1) = 0 \tag{1.3}$$

where the potential $q(x)$ is real-valued, continuous on $[-1, 0) \cup \in (0, 1]$ and has a finite limits $q(c\mp) = \lim_{x\to 0\mp} q(x)$; λ is a complex eigenparameter; $\delta \neq 0$ any real number. Since various type transmission problems appear frequently in various fields of physics and technics, Sturm-Liouville problems with transmission conditions have been an important research topic in mathematical physics [2, 8, 11]. For the earlier developments about Sturm comparison and oscillation theory, we refer to [4, 5, 6, 9, 14, 15] and for recent developments, we refer to [1, 3, 7, 13, 16, 17].

2 Comparison Theorem for discontinuous Sturm-Liouville problems

At first we shall extend and generalize the classical Sturm-liouville comparison theorem.

Theorem 2.1. *Let $y = y_1(x)$ be solution of the equation*

$$L_1 y := -y'' + q_1(x)y = 0 \tag{2.1}$$

satisfying transmission conditions at the point of interaction $x = 0$ given by

$$y(0-) = \delta y(0+), \; y'(0-) = \frac{1}{\delta} y'(0+) \tag{2.2}$$

and let $y = y_2(x)$ be the solution of the equation

$$L_2 y := -y'' + q_2(x)y = 0 \tag{2.3}$$

satisfying the same transmission conditions (2.2) where $\delta \neq 0$ any real number if $q_1(x) > q_2(x)$ on $[-1, 0) \cup \in (0, 1]$, then between any two consecutive zeros of $y_1(x)$ there is at least one zero of $y_2(x)$.

Proof. Let x_1 and x_2 with $x_1 < x_2$ be consecutive zeroes of y_1. Suppose, it possible,that y_2 does not have a zero on (x_1, x_2). Lagrange's identity (see, [12]) gives

$$y_2 L_1 y_1 - y_1 L_2 y_2 = \frac{d}{dx}\{y_2' y_1 - y_1' y_2\} + \{q_1(x) - q_2(x)\} y_1 y_2 \tag{2.4}$$

Hence

$$\frac{d}{dx}\{y_1'y_2 - y_2'y_1\} = \{q_1(x) - q_2(x)\}y_1y_2 \tag{2.5}$$

Case 1. Let $x_1 \in [-1, 0)$, $x_2 \in (0, 1]$ and $\delta > 0$. Integrating on both sides of the equation (2.9) over $[x_1, 0)$ and $(0, x_2]$ and then adding we get

$$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} (y_1'y_2 - y_2'y_1)|_{x_1}^{0-\epsilon_1} + \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} (y_1'y_2 - y_2'y_1)|_{0+\epsilon_2}^{x_2}$$

$$= \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} \int_{x_1}^{0-\epsilon_1} \{q_1(x) - q_2(x)\}y_1y_2dx + \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} \int_{0+\epsilon_2}^{x_2} \{q_1(x) - q_2(x)\}y_1y_2dx \tag{2.6}$$

Since $y_1(x_1) = y_1(x_2) = 0$ we get

$$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} W(y_1, y_2; 0 - \epsilon_1) - \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} W(y_1, y_2; 0 + \epsilon_2) - y_1'(x_1)y_2(x_1) + y_1'(x_2)y_2(x_2)$$

$$= \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} \int_{x_1}^{0-\epsilon_1} \{q_1(x) - q_2(x)\}y_1y_2dx + \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} \int_{0+\epsilon_2}^{x_2} \{q_1(x) - q_2(x)\}y_1y_2dx \tag{2.7}$$

Using the transmission conditions we obtain

$$-y_1'(x_1)y_2(x_1) + y_1'(x_2)y_2(x_2) = \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} \int_{x_1}^{0-\epsilon_1} \{q_1(x) - q_2(x)\}y_1y_2dx$$

$$+ \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} \int_{0+\epsilon_2}^{x_2} \{q_1(x) - q_2(x)\}y_1y_2dx \tag{2.8}$$

In this case with no restriction we can assume that $y_1(x) > 0$ and $y_2(x) > 0$ over $(x_1, 0) \cup (0, x_2)$. These conditions ensure that the integral on the right in (2.8) is positive. On the left, since $y_1(x) > 0$ by assumption, the function is increasing at the point x_1. Hence $y_1'(x_1) > 0$(it cannot vanish, because then it would follow from the uniqueness theorem for the solutions of (2.1) that $y_1(x) \equiv 0$, which is impossible). Similarly, $y_1'(x_2) < 0$. Thus, the left-hand side of the equation (2.8) is less or equal to zero, which is a contradiction.
Case 2. Let $x_1 \in [-1, 0)$, $x_2 \in (0, 1]$ and $\delta < 0$. In this case with no restriction it can be assumed that, $y_1(x) > 0$ over $(x_1, 0)$, $y_1(x) < 0$ over $(0, x_2)$, $y_2(x) > 0$ over $(x_1, 0)$ and $y_2(x) < 0$ over $(0, x_2)$. Since $y_1(x_1) = 0$ and $y_1(x_1) > 0$ over $(x_1, 0)$ $y_1'(x_1) > 0$. Further, since $y_2(x_2) = 0$ and $y_2(x_2) < 0$ immediately to left of x_2, $y_2'(x) < 0$. Hence, the left-hand side of (2.8) is is less

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or equal zero, but the right-hand side is positive which shows that (2.8) is impossible.
Case 3. Let $(x_1, x_2) \subset [-1, 0)$. Integrating on both sides of the equation (2.5) from x_1 to x_2, we get

$$(y_1' y_2 - y_2' y_1)|_{x_1}^{x_2} = \int\limits_{x_1}^{x_2} \{q_1(x) - q_2(x)\} y_1 y_2 dx \tag{2.9}$$

Then with no restriction it can be assumed that $y_1(x) > 0$ and $y_2(x) > 0$ over (x_1, x_2). These conditions ensure that the integral on the right in (2.9) is positive. However, on the left, we have $y_1(x_1) = y_1(x_2) = 0$ with $y_1'(x_1) > 0$ and $y_1'(x_2) < 0$. The left-hand side therefore becomes

$$y_1'(x_2) y_2(x_2) - y_1'(x_1) y_2(x_1) \leq 0$$

which presents us with a contradiction: right-hand side > 0 and left-hand side < 0. Thus $y_2(x) = 0$ (at least once) between the zeros of $y_1(x)$. Since the conditions describing $y_1(x)$ are given, we conclude that $y_2(x)$ must change sign between $x = x_1$ and $x = x_2$.
Case 4. Let $(x_1, x_2) \subset (0, 1]$. This case is totaly similar to the previous case. □

3 On the zeros of eigenfunctions

In this section we examine the number of zeros of eigenfunctions.

Lemma 3.1. *There is an unique solution $y(x, \lambda)$ of the equation (1.1) satisfying the initial conditions*

$$y(x_0, \lambda) = \alpha(\lambda), \;\; y'(x_0, \lambda) = \beta(\lambda) \tag{3.1}$$

and the transmission conditions (1.2) where $\alpha(\lambda), \beta(\lambda)$ are given entire functions of $\lambda \in \mathbb{C}$ and $x_0 \in [-1, 0) \cup (0, 1]$. Moreover, $y(x, \lambda)$ is entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [-1, 0) \cup (0, 1]$.

Proof. The proof is totally similar to [**?**] and therefore is omitted. □

Theorem 3.2. *Let $\phi(x, \lambda_1) = \begin{cases} \phi_1(x, \lambda_1), & x \in [-1, 0) \\ \phi_2(x, \lambda_1), & x \in (0, 1] \end{cases}$ be solution of the equation (1.1), for $\lambda = \lambda_1$ satisfying the initial conditions*

$$\phi_1(-1, \lambda_1) = \alpha, \quad \phi_1'(-1, \lambda_1) = \beta \tag{3.2}$$

and the transmission conditions

$$\phi_2(0^+, \lambda_1) = \frac{1}{\delta} \phi_1(0^-, \lambda_1), \phi_2'(0^+, \lambda_1) = \delta \phi_1'(0^-, \lambda_1) \tag{3.3}$$

and $\varphi(x, \lambda_2) = \begin{cases} \varphi_1(x, \lambda_2), & x \in [-1, 0) \\ \varphi_2(x, \lambda_2), & x \in (0, 1] \end{cases}$ be solution of the equation (1.1), for $\lambda = \lambda_2$ satisfying the initial conditions

$$\varphi_1(-1, \lambda_2) = \alpha, \quad \varphi_1'(-1, \lambda_2) = \beta \tag{3.4}$$

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and the transmission conditions

$$\varphi_2(0^+,\lambda_2)=\frac{1}{\delta}\phi_2(0^-,\lambda_2),\varphi_2'(0^+,\lambda_2 1)=\delta\phi_1'(0^-,\lambda_1). \tag{3.5}$$

where δ,β,δ any real numbers with $\alpha^2+\beta^2\neq 0,\delta\neq 0$. Suppose that $\phi(x,\lambda_1)$ has a zeros in $[-1,0)\cup(0,1]$ and let $x_1(x_1\neq -1)$ be zero of the function $\phi(x,\lambda_1)$, nearest to $x=-1$. If $\lambda_2>\lambda_1$ then $\varphi(x_2,\lambda_2)$ has at least one zero in $[-1,x_1)$.

Proof. From the well-known Lagrange's identity (see,for example, [12]) we have

$$\frac{d}{dx}\{\phi_1'\varphi_1-\varphi_1'\phi_1\}=\{\lambda_2-\lambda_1\}\phi_1\varphi_1 \tag{3.6}$$

in the interval $(0,1)$.

$$\frac{d}{dx}\{\phi_2'\varphi_2-\varphi_2'\phi_2\}=\{\lambda_2-\lambda_1\}\phi_2\varphi_2 \tag{3.7}$$

Case 1. Let $x_1>0$ and $\delta>0$. Integrating on both sides of the equation (3.11) from -1 to x_1, we get

$$\begin{aligned}&\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}(\phi_1'\varphi_1-\varphi_1'\phi_1)|_{-1}^{0-\epsilon_1}+\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}(\phi_2'\varphi_2-\varphi_2'\phi_2)|_{0+\epsilon_2}^{x_1}\\ &=\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}\{\lambda_2-\lambda_1\}\int_{-1}^{0-\epsilon_1}\phi_1\varphi_1dx+\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}\{\lambda_2-\lambda_1\}\int_{0+\epsilon_2}^{x_1}\phi_2\varphi_2dx\end{aligned} \tag{3.8}$$

Since $W(\phi_1,\varphi_1;-1)=0$ by (3.2) and (3.4) we get

$$\begin{aligned}&\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}W(\phi_1,\varphi_1;0-\epsilon_1)-\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}W(\phi_2,\varphi_2;0+\epsilon_2)+\phi_2'(x_1,\lambda_1)\varphi_2(x_1,\lambda_2)\\ &=\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}\{\lambda_2-\lambda_1\}\int_{-1}^{0-\epsilon_1}\phi_1\varphi_1dx+\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}\{\lambda_2-\lambda_1\}\int_{0+\epsilon_2}^{x_1}\phi_2\varphi_2dx\end{aligned} \tag{3.9}$$

Using the transmission conditions we obtain

$$\begin{aligned}\phi_2'(x_1,\lambda_1)\varphi_2(x_1,\lambda_2)&=\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}\{\lambda_2-\lambda_1\}\int_{-1}^{0-\epsilon_1}\phi_1\varphi_1dx\\ &+\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}\{\lambda_2-\lambda_1\}\int_{0+\epsilon_2}^{x_1}\phi_2\varphi_2dx\end{aligned} \tag{3.10}$$

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With no restriction it can be assumed that $\phi(x,\lambda_1) < 0$ and $\varphi(x,\lambda_2) < 0$ in $[-1,x_1)$. These conditions ensure that the integral on the right in (3.10) is positive. Since $\phi_2(x_1,\lambda_1) = 0$ and $\phi_2(x,\lambda_1) > 0$ immediately to the left of x_1 by assumption, the function is increasing at the point x_1. Hence $\phi_2'(x_1,\lambda_1) > 0$(it cannot vanish, because then it would follow from the uniqueness theorem for the solutions of (2.1) that $\phi_2(x,\lambda_1) \equiv 0$, which is impossible). Thus, the left-hand side of the equation (3.10) is less or equal to zero, but the right-hand side is positive, which is a contradiction.

Case 2. Let $x_1 > 0$ and $\delta < 0$. In this case with no restriction it can be assumed that $\phi(x,\lambda_1) > 0$ and $\varphi(x,\lambda_2) < 0$ in $[-1,0)$ but $\phi(x,\lambda_1) < 0$ and $\varphi(x,\lambda_2) > 0$ in $(0,x_1]$. As in the previous case, these conditions ensure that the integral on the right of (3.10) is negative, but left hand side of (3.10) is positive or is equal to zero, i.e. the equality (3.10)is impossible.

Case 3. Let $x_1 \in [-1,0)$. Integrating on both sides of the equation (2.5) from a to x_1, we get

$$(\phi_1'\varphi_1 - \varphi_1'\phi_1)|_{-1}^{x_1} = \int_{-1}^{x_1} \{\lambda_2 - \lambda_1\}\phi_1\varphi_1 dx \tag{3.11}$$

Since $\phi_1(x,\lambda_1) = 0$ by using the initial conditions $\phi_1(-1,\lambda_1) = 0, \phi_1'(-1,\lambda_1) = 0$ we get

$$\phi_1'(x_1)\varphi_1(x_1) = \int_{-1}^{x_1} \{\lambda_2 - \lambda_1\}\phi_1\varphi_1 dx \tag{3.12}$$

Let $x_1 < 0$. Without loss of generality, we can put $\phi(x,\lambda_1) > 0$ and $\varphi(x,\lambda_2) > 0$ in $[-1,x_1)$. Since, by assumption, $\phi_1(x,\lambda_1) > 0$ and $\varphi_1(x,\lambda_2) > 0$ in $[-1,x_1)$ and $\lambda_2 > \lambda_1$, the right-hand side of the equality (3.12) is positive. However, on the left-hand side, since $\phi_1(x_1,\lambda_1) = 0$ and $\phi_1(x,\lambda_1) > 0$ immediately to the left of x_1, the function $\phi_1(x,\lambda_1)$ is decreasing in the vicinity of the point x_1. Therefore, $\phi_1'(x_1,\lambda_1) \le 0$(it cannot vanish, because then it would follow from the uniqueness theorem for the solutions of (1.1) that $\phi_1(x,\lambda_1) \equiv 0$, which is impossible). The left-hand side therefore becomes

$$\phi_1'(x_1,\lambda_1)\varphi_1(x_1,\lambda_1) \le 0$$

which presents us with a contradiction: right-hand side > 0 and left-hand side ≤ 0. The proof is complete. □

Now we are ready to establish the main result.

Theorem 3.3. *Let $\psi_1(x)$ and $\psi_2(x)$ be two eigenfunction corresponding to the eigenvalues λ_1 and λ_2 of the problem (1.1)-(1.3) and let $\lambda_2 > \lambda_1$. Then if $\psi_1(x)$ has m zeros in $[-1,0) \cup (0,1]$, $\psi_2(x)$ has not fewer than m zeros in the same two-interval $[-1,0) \cup (0,1]$. Moreover, $n-th$ zero of $\psi_2(x)$ is less than the $n-th$ zero of $\psi_1(x)$.*

Proof. Let $x_1', x_2', ..., x_m'$ with $x_1' < x_2 <, ... < x_m'$ be zeros of the eigenfunctions $\psi_1(x)$. By virtue of the Theorem 3.2 $\psi_2(x)$ has at least one zero in $[-1,x_1')$. Moreover, by applying the Theorem 2.1 to the solutions ψ_1 and ψ_2 we see that $\psi_2(x)$ has at least one zero in each of the intervals $(x_1',x_2'), (x_2',x_3'), ..., (x_{m-1}',x_m')$. Consequently the number of zeros of $\psi_2(x)$ is not fewer than the number of zeros $\psi_1(x)$ and $n-th$ zero of $\psi_2(x)$ is less than $n-th$ zero of $\psi_1(x)$. The proof is complete. □

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Fuzzy stability of an additive-quadratic functional equation in matrix fuzzy normed spaces

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Abstract. A mapping $f : X \times X \to Y$ is called additive-quadratic if f satisfies the system of equations

$$\begin{cases} f(x+y,z) = f(x,z) + f(y,z), \\ f(x,y+z) + f(x,y-z) = 2f(x,y) + 2f(x,z). \end{cases}$$

In this paper, using the fixed point method, we prove the Hyers-Ulam stability in matrix fuzzy normed spaces associated to the following additive-quadratic functional equation

$$f(x+y,z+w) + f(x+y,z-w) = 2f(x,z) + 2f(x,w) + 2f(y,z) + 2f(y,w)$$

for all $x, y, z, w \in X$.

1. Introduction and preliminaries

A definition of fuzzy norm on a vector space, to construct a fuzzy vector topological structure, introduced by Katsaras [15]. During the last four decades some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 16, 32]. In particular, Bag and Samanta [1], following Cheng and Mordeson [6], presented an idea of a fuzzy norm in such a manner the corresponding fuzzy metric is of Kramosil and Michalek type [6]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [2].

We use the definition of fuzzy normed spaces given in [1, 19, 21] to investigate a fuzzy version of the Hyers-Ulam stability of an additive-quadratic additive functional equation in the fuzzy normed vector space setting.

Definition 1.1. *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a* fuzzy norm *on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

(N_1) $N(x,t) = 0$ *for* $t \leqslant 0$;

(N_2) $x = 0$ *if and only if* $N(x,t) = 1$ *for all* $t > 0$;

(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ *if* $c \neq 0$

(N_4) $N(x+y, s+t \geqslant \min\{N(x,s), N(y,t)\}$;

[0]**2010 Mathematics Subject Classification**: 47H10; 47L25; 46S40; 39B52; 39B72.

[0]**Keywords**: additive-quadratic functional equation; matrix fuzzy normed space; fixed point; Hyers-Ulam stability.

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(N_5) *$N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;*
(N_6) *for $x \neq 0, N(x,.)$ is continuous on $\mathbb{R}$.*

The pair (X, N) is called a *fuzzy normed vector space*. To see more properties and examples of fuzzy normed vector spaces, we refer to [19, 20].

Definition 1.2. *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be* convergent *or* converge *if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the* limit *of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 1.3. *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called a* Cauchy sequence *if for each $\varepsilon > 0$ and each $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

It is well known that every convergent sequence in a fuzzy normed vector space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges $f(x_0)$. If $f : X \to Y$ continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [2]).

We will use the following notations:
$M_n(X)$ is the set of all $n \times n$-matrices in X;
$e_j \in M_{1,n}(\mathbb{C})$ is that the jth component is 1 and the other components are zero;
$E_{ij} \in M_n(\mathbb{C})$ is that the (i, j)-component is 1 and the other components are zero;
$E_{ij} \otimes x \in M_n(X)$ is that the (i, j)-component is x and the other components are zero.
For $x \in M_n(X), y \in M_k(X)$,

$$x \otimes y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Let $(X, \|\cdot\|)$ be a normed space. Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leqslant \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C}), x = (x_{ij}) \in M_n(X)$ and $B \in M_{n,k}(\mathbb{C})$, and that $(X, \{\|.\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

A matrix normed space $(X, \{\|\cdot\|_n\})$ is called an *L^∞-matrix normed space* if $\|x \oplus y\|_{n+k} = \max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

Throughout this paper, let $(X, \{\|\cdot\|_n\})$ be a matrix normed space and $(Y, \{\|\cdot\|_n\})$ be a matrix Banach space.

We introduce the concept of a matrix fuzzy normed space.

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Definition 1.4. *Let (X, N) be a fuzzy normed space.*

(1) (X, N) *is called a* matrix fuzzy normed space *if for each positive integer* n, $(M_n(X), N_n)$ *is a fuzzy normed space and* $N_k(AxB, t) \geqslant N_n\left(x, \frac{t}{\|A\|\cdot\|B\|}\right)$ *for all* $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ *and* $B \in M_{n,k}(\mathbb{R})$ *with* $\|A\| \neq 0.\|B\| \neq 0$.

(2) $(X, \{N_n\})$ *is called a* matrix fuzzy Banach space *if* (X, N) *is a fuzzy Banach space and* $(X, \{N_n\})$ *is a matrix fuzzy normed space.*

Example 1.5. *Let $(X, \{\|\cdot\|_n\})$ be a matrix normed space. Let $N_n(x, t) := \frac{t}{t+\|x\|_n}$ for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Then*

$$N_k(AxB, t) = \frac{t}{t + \|AxB\|_k} \geqslant \frac{t}{t + \|A\|.\|x\|_n.\|B\|} = \frac{\frac{t}{\|A\|.\|B\|}}{\frac{t}{\|A\|.\|B\|} + \|x\|_n}$$

for all $t > 0, A \in M_{k,n}(\mathbb{R}), x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\|.\|B\| \neq 0$. So, $(X, \{N_n\})$ is a matrix fuzzy normed space.

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [29] implies that quotients, mapping spaces, and various tensor product of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces have an increasingly significant effect on operator algebra theory(see [10]).

The proof given in [29] appealed to the theory of ordered operator spaces [7]. Effros and Ruan [11] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [26] and Effors [9].

The study of stability problems have been formulated by Ulam [31] in 1940: Under what condition does there exist a homomorphism near an approximate homomorphism? In the following year, Hyers [14] answered affirmatively the question of Ulam for Banach spaces, which was stated that if $\varepsilon > 0$ and $f : X \to Y$ is a mapping with X a normed space and Y is a Banach space such that

$$\|f(x + y) - f(x) - f(y)\| \leqslant \varepsilon \tag{1.1}$$

for all $x, y \in X$, then there exists a unique additive map $T : X \to Y$ such that

$$\|f(x + y) - f(x) - f(y)\| \leqslant \varepsilon$$

for all $x \in X$. A generalized version of the theorem of Hyers for approximately linear mappings presented by Rassias [27] in 1978 by considering the case when (1.1) is unbounded.

In 2003, Cădariu and Radu applied the fixed point method to the investigation of the Jensen functional equation [3]. They could present a short and a simple proof (different of the "direct method", initiated by Hyers in 1941) for the Hyers-Ulam stability of the Jensen functional equation [3] and forthe quadratic functional equation [4]. See [12, 22, 23, 24, 28, 30] for more information on functional equations.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,y)\leqslant d(x,z)+d(z,y)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.6. [8] *Let (Ω,d) be a complete generalized metric space and $J:\Omega\to\Omega$ be a strictly contractive mapping with Lipschitz constant $0<L<1$. Then for each given $x\in\Omega$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x)<\infty, \quad \forall n\geqslant n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $\Lambda=\{y\in\Omega:\ d(J^{n_0}x,y)<\infty\}$;*
(4) *$d(y,y^*)\leqslant\frac{1}{1-L}d(y,Jy)$ for all $y\in\Lambda$.*

Definition 1.7. *A mapping $f:X\times X\to Y$ is called additive-quadratic if f satisfies the system of equations*

$$\begin{cases} f(x+y,z)=f(x,z)+f(y,z), \\ f(x,y+z)+f(x,y-z)=2f(x,y)+2f(x,z). \end{cases} \tag{1.2}$$

When $X=Y=\mathbb{R}$, the function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by $f(x,y):=cxy^2$ is a solution of (1.2). In particular, letting $x=y$, we get a cubic function $g:\mathbb{R}\to\mathbb{R}$ given by $g(x):=f(x,x)=cx^3$.

For a mapping $f:X\times X\to Y$, consider the functional equation:

$$f(x+y,z+w)+f(x+y,z-w)=2f(x,z)+2f(x,w)+2f(y,z)+2f(y,w). \tag{1.3}$$

for all $x,y,z,w\in X$. The solution of (1.3) was discussed in [25].

In this paper, by using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.3) in matrix fuzzy normed spaces.

2. Fuzzy stability of the additive-quadratic functional equation (1.3)

In this section, using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.3) in matrix fuzzy normed space.

We need the following lemma.

Lemma 2.1. [17, Lemma 2.1] *Let $(X,\{N_n\})$ be a matrix fuzzy normed space.*

(1) *$N_n(E_{kl}\otimes x,t)=N(x,t)$ for all $t>0$ and $x\in X$.*
(2) *for all $[x_{ij}]\in M_n(X)$ and $t=\sum_{i,j=1}^n t_{ij}$,*

$$N(x_{kl},t)\geqslant N([x_{ij}],t)\geqslant\min\{N(x_{ij},t_{ij}):i,j=1,2,\cdots,n\},$$

$$N(x_{kl},t)\geqslant N([x_{ij}],t)\geqslant\min\left\{N\Big(x_{ij},\frac{t}{n^2}\Big):i,j=1,2,\cdots,n\right\}$$

(3) $\lim_{n\to\infty}x_n=x$ *if and only if* $\lim_{n\to\infty}x_{ijn}=x_{ij}$ *for* $x_n=[x_{ijn}],x=[x_{ij}]\in M_k(X)$

Proof. (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1, N_n(E_{kl} \otimes x, t) \geqslant N(x,t)$. Since $e_k(E_{kl} \otimes x)e_l^* = x, N_n(E_{kl} \otimes x, t) \leqslant N(x,t)$. So $N(E_{kl} \otimes x, t)$=N(x,t).

(2) $N(x_{kl}, t) = N(e_k[x_{ij}]e_l^*, t) \geqslant N_n\left([x_{ij}], \frac{t}{\|e_k\| \cdot \|e_l\|}\right) = N_n([x_{ij}], t)$.

$$\begin{aligned}N_n([x_{ij}], t) &= N_n\Big(\sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}, t\Big) \geqslant \min\{N_n(E_{ij} \otimes x_{ij}, t_{ij}) : i,j = 1, 2, \cdots, n\}\\ &= \min\{N(x_{ij}, t_{ij}) : i,j = 1, 2, \cdots, n\},\end{aligned}$$

where $t = \sum_{i,j=1}^{n} t_{ij}$. So, $N_n([x_{ij}], t) \geqslant \min\{N(x_{ij}, \frac{t}{n^2}) : i,j = 1, 2, \cdots, n\}$.

(3) By $N(x_{kl}, t) \geqslant N_n([x_{ij}], t) \geqslant \min\{N(x_{ij}, \frac{t}{n^2}) : i,j = 1, 2, \cdots, n\}$, we obtain the result. This completes the proof. □

For a mapping $f : X \to Y$, define $Df : X^m \to Y$ and $Df_n : M_n(X^4) \to M_n(Y)$ by

$$\begin{aligned}Df(a,b,c,d) &:= f(a+b, c+d) + f(a+b, c-d)\\ &\quad - 2f(a,c) - 2f(a,d) - 2f(b,c) - 2f(b,d),\end{aligned}$$

$$\begin{aligned}Df_n\Big([x_{ij}], [y_{ij}], [z_{ij}],[w_{ij}]\Big) &:= f_n\Big([x_{ij}] + [y_{ij}], [z_{ij}] + [w_{ij}]\Big) + f_n\Big([x_{ij}] + [y_{ij}], [z_{ij}] - [w_{ij}]\Big)\\ &\quad - 2f_n\Big([x_{ij}], [z_{ij}]\Big) - 2f_n\Big([x_{ij}], [w_{ij}]\Big) - 2f_n\Big([y_{ij}], [z_{ij}]\Big) - 2f_n\Big([y_{ij}], [w_{ij}]\Big)\end{aligned}$$

for all $a, b, c, d \in X$ and all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}], w = [w_{ij}] \in M_n(X)$.

Theorem 2.2. *Let $f : X \to Y$, with $f(x, 0) = 0$, be a mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$ such that*

$$N_n\Big(f_n([x_{ij}], [y_{ij}], [z_{ij}], [w_{ij}]), t\Big) \geqslant \frac{t}{t + \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}, z_{ij}, w_{ij})} \tag{2.1}$$

for all $t > 0$ and all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}], w = [w_{ij}] \in M_n(X)$. If there exists an $\alpha < 1$ such that

$$\varphi(a,b,c,d) \leqslant 8\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \frac{d}{2}\right) \tag{2.2}$$

for all $a, b, c, d \in X$, then there exists a unique additive-quadratic mapping $T : X \times X \to Y$ such that

$$N_n\left(f_n\left([x_{ij}], [y_{ij}]\right) - T_n\left([x_{ij}], [y_{ij}]\right), t\right) \geqslant \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2 \sum_{i,j=1}^{n} \varphi(x_{ij}, x_{ij}, y_{ij}, y_{ij})} \tag{2.3}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. Putting $n = 1$ in (2.1), we have

$$N(Df(x,y,z,w), t) \geqslant \frac{t}{t + \varphi(x,y,z,w)} \tag{2.4}$$

for all $t > 0$ and $x, y, z, w \in X$.

Letting $x = y$ and $z = w$ in (2.4), we obtain

$$N(f(2x,2z) - 8f(x,z),t) \geqslant \frac{t}{t+\varphi(x,x,z,z)} \tag{2.5}$$

and also

$$N\left(\frac{1}{8}f(2x,2z) - f(x,z), \frac{t}{8}\right) \geqslant \frac{t}{t+\varphi(x,x,z,z)}$$

for all $t > 0$ and $x, z \in X$. Also it can be written as

$$N\left(\frac{1}{8}f(2x,2y) - f(x,y), \frac{t}{8}\right) \geqslant \frac{t}{t+\varphi(x,x,y,y)} \tag{2.6}$$

for all $t > 0$ and $x, y \in X$.

By considering the set of

$$\Omega := \{g : X \to Y\},$$

we introduce the generalized metric on Ω as following:

$$d(g,h) = \inf\left\{k \in \mathbb{R}^+ : N(g(x,y) - h(x,y), kt) \geqslant \frac{t}{t+\varphi(x,x,y,y)}, \forall x,y \in X, \forall t > 0\right\}$$

where, as usual $\inf \emptyset = +\infty$. It is easy to show that (Ω, d) is complete (see [5, 18]).

Now we define $J : \Omega \to \Omega$ by

$$Jg(x,y) := \frac{1}{8}h(2x,2y)$$

for all $x, y \in X$.

Let $g, h \in \Omega$ be given such that $d(g,h) = c$. Then

$$\begin{aligned}
&N(g(x,y) - h(x,y), ct) \geqslant \frac{t}{t+\varphi(2x,2x,2y,2y)} \\
&\Rightarrow N\left(\frac{1}{8}g(2x,2y) - \frac{1}{8}h(2x,2y), \frac{c}{8}t\right) \geqslant \frac{t}{t+\varphi(2x,2x,2y,2y)} \\
&\Rightarrow N\left(\frac{1}{8}g(2x,2y) - \frac{1}{8}h(2x,2y), \frac{c}{8}t\right) \geqslant \frac{t}{t+8\alpha\varphi(x,x,y,y)} \\
&\Rightarrow N\left(\frac{1}{8}g(2x,2y) - \frac{1}{8}h(2x,2y), \alpha ct\right) \geqslant \frac{t}{t+\varphi(x,x,y,y)} \\
&\Rightarrow d(Jg, Jh) \leqslant \alpha c
\end{aligned}$$

for all $x, y \in X$. Hence we get that

$$d(Jg, Jh) \leqslant \alpha d(g,h)$$

for all $g, h \in \Omega$. It follows from (2.6) that $d(f, Jf) \leqslant \frac{1}{8}$.

By Theorem 1.6, there exists a mapping $T : X \to Y$ satisfying the following:

(1) T is a fixed point of J, i.e., $T(2x,2y) = 8T(x,y)$ for all $x \in X$. The mapping T is a unique fixed point of J in the set $X = \{g \in \Omega : d(f,g) < \infty\}$.

(2) $d(J^k f, T) \to 0$ as $k \to \infty$. This implies the inequality $N - \lim_{k\to\infty} \frac{1}{8^k} f(2^k x, 2^k y) = T(x,y)$ for all $x, y \in X$.

(3) $d(f,T) \leqslant \frac{1}{1-\alpha} d(f,Jf)$, which implies the inequality

$$(f,T) \leqslant \frac{1}{8(1-\alpha)}. \tag{2.7}$$

By (2.2) and (2.4),

$$\begin{aligned} N\left(\frac{1}{8^k} Df(2^k x, 2^k y, 2^k z, 2^k w)\right) &\geqslant \frac{t}{t+\varphi(2^k x, 2^k y, 2^k z, 2^k w)} \\ &\geqslant \frac{8^k t}{8^k t + 8^k \alpha^k \varphi(x,y,z,w)} \end{aligned}$$

for all $x,y,z,w \in X$ and $t>0$. Since $\lim_{k\to\infty} \frac{8^k t}{8^k t+8^k\alpha^k\varphi(x,y,z,w)} = 1$ for all $x,y,z,w \in X$ and $t>0$,

$$N(DT(x,y,z,w),t) = 1$$

for all $x,y,z,w \in X$ and $t>0$. Therefore

$$T(x+y,z+w) + T(x+y,z-w) = 2T(x,z) + 2T(x,w) + 2T(y,z) + 2T(y,w).$$

for all $x,y,z,w \in X$. Then, the mapping $T : X \times X \to Y$ is additive-quadratic.

It follows from Lemma 2.1 and (2.7) that

$$\begin{aligned} N_n\Big(f_n([x_{ij}],[y_{ij}]) - T_n([x_{ij}],[y_{ij}]),t\Big) &\geqslant \left\{ N\left(f(x_{ij},y_{ij}) - T(x_{ij},y_{ij}), \frac{t}{n^2}\right) : i,j = 1,2,\cdots,n \right\} \\ &\geqslant \min\left\{ \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2\varphi(x_{ij},x_{ij},y_{ij},y_{ij})} : i,j=1,2,\cdots,n \right\} \\ &\geqslant \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2\sum_{i,j=1}^n \varphi(x_{ij},x_{ij},y_{ij},y_{ij})} \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Therefore, we conclude that $T : X \times X \to Y$ is the unique mapping satisfying (2.3). $\square$

Corollary 2.3. *Let p, θ be positive real numbers $p<1$. Let $f : X \times X \to Y$, with $f(x,0)=0$, be a mapping satisfying*

$$N_n(Df_n([x_{ij}],[y_{ij}],[z_{ij}],[w_{ij}]),t) \geqslant \frac{t}{t+\sum_{i,j=1}^n \theta(\|x_{ij}\|^p + \|y_{ij}\|^p + \|z_{ij}\|^p + \|w_{ij}\|^p)} \tag{2.8}$$

for all $x=[x_{ij}], y=[y_{ij}], z=[z_{ij}], w=[w_{ij}] \in M_n(X)$ and $t>0$. Then $T(x,y) := N-\lim_{k\to\infty} \frac{1}{8^k} f(2^k x, 2^k y)$ exists for each $x,y \in X$ and defines an additive-quadratic mapping $T : X \times X \to Y$ such that

$$N_n\Big(f_n\left([x_{ij}],[y_{ij}]\right) - T_n\left([x_{ij}],[y_{ij}]\right),t\Big) \geqslant \frac{2(2-2^p)t}{2(2-2^p)t + n^2\sum_{i,j=1}^n \theta\left(\|x_{ij}\|^p + \|y_{ij}\|^p\right)}$$

for all $x=[x_{ij}], y=[y_{ij}] \in M_n(X)$ and $t>0$.

Proof. Putting $\varphi(a,b,c,z) := \theta\sum_{i=1}^m (\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$ for all $a,b,c,d \in X$ and letting $\alpha = 2^{p-1}$ in Theorem 2.2, we obtain the desired result. $\square$

Theorem 2.4. *Let $f : X \times X \to Y$, with $f(x, 0) = 0$, be a mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$ satisfying (2.1). If there exists an $\alpha < 1$ such that*

$$\varphi\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \frac{d}{2}\right) \leqslant \frac{\alpha}{8}\varphi(a, b, c, d)$$

for all $a, b, c, d \in X$, then there exists a unique additive-quadratic mapping $T : X \times X \to Y$ such that

$$N\Big(f_n([x_{ij}], [y_{ij}]) - T_n([x_{ij}], [y_{ij}]), t\Big) \geqslant \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2\alpha \sum_{i,j=1}^{n} \varphi(x_{ij}, x_{ij}, y_{ij}, y_{ij})}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. Let (Ω, d) be the generalized metric space defined in the proof of Theorem 2.2. Here, we define the linear mapping $J : \Omega \to \Omega$ such that

$$Jg(x, y) := 8g(\frac{x}{2}, \frac{y}{2})$$

for all $x, y \in X$.

It follows from (2.5) that $d(f, Jf) \leqslant \frac{\alpha}{8}$. Thus

$$d(f, T) \leqslant \frac{\alpha}{8(1-\alpha)}.$$

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let p, θ be positive real numbers with $p > 1$. Let $f : X \times X \to Y$, with $f(x, 0) = 0$, be a mapping satisfying (2.8). Then $T(x, y) := N - \lim_{k\to\infty} 8^k f(\frac{x}{2^k}, \frac{y}{2^k})$ exists for all $x \in X$ and defines an additive-quadratic mapping $T : X \times X \to Y$ such that*

$$N_n\Big(f_n([x_{ij}], [y_{ij}]) - T_n([x_{ij}], [y_{ij}]), t\Big) \geqslant \frac{4(2^p-2)t}{4(2^p-2)t + n^2 \cdot 2^p \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^p + \|y_{ij}\|^p)}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and $t > 0$.

Proof. Putting $\varphi(a, b, c, d) := \theta(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$ for all $a, b, c, d \in X$ and letting $\alpha = 2^{1-p}$ in Theorem 2.4, we get the desired result. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.3, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Closed Form Expressions of some systems of Nonlinear Partial Difference Equations

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January 13, 2016

Abstract

In this paper we give the closed form expressions of some two dimensional systems of nonlinear rational partial difference equations of second order.We shall use a new method to prove the results by using (odd-even) double mathematical induction. As a direct consequences , we investigate and drive the explicit solutions of some partial difference equations and some (systems of) ordinary difference equations .

AMS Subject Classification: 39A10, 39A14.
Key Words and Phrases: (partial)difference equations, solutions , double mathematical induction.

1 Introduction

While the study of (ordinary)difference equations has been widely treated in the past , partial difference equations (PΔEs) have not received the same full attention .Both of ordinary and partial difference equations may be found in the study of probability ,dynamics and other branches of mathematical physics .Moreover,partial difference equations arise in applications involving population dynamics with spatial migrations , chemical reactions and finite difference schemes . Indeed Laplace and Lagrange considered the solution of partial difference equations in their studies of dynamics and probability.
An example of a partial difference equation is the following well known relation

$$C_m^{(n)} = C_{m-1}^{(n-1)} + C_m^{(n-1)} \quad ,1 \leq m < n.$$

The solution of this equation is the celebrated binomial coefficient function $C_m^{(n)}$ defined by

$$C_m^{(n)} = \frac{n!}{m!(n-m)!} \quad ,0 \leq m < n.$$

An another example , the following PΔEs :

$$s_k^{(n+1)} = s_{k-1}^{(n)} - ns_k^{(n)} \quad ,1 \leq k < n.$$

$$S_k^{(n+1)} = S_{k-1}^{(n)} + kS_k^{(n)} \quad ,1 \leq k < n.$$

The solutions of these PΔEs are the stirling numbers of the first kind $s_k^{(n)}$ and the stirling numbers of the second kind $S_k^{(n)}$ respectively .

Some authors investigate the closed form solutions for certain Partial difference equations .
For instance , Heins [[9]] considered the solution of the partial difference equation

$$X_{n+1,m} + X_{n-1,m} = 2X_{n,m+1}$$

under some conditions .
In [[3]] Carlitz has studied a solution of the partial difference equation

$$X_{n,m} - X_{n,m-1} - X_{n-1,m} - X_{n,m-2} + 3X_{n-1,m-1} - X_{n-2,m} = 0$$

He used a power series expansion related to the Fibonacci numbers .

For more results about partial difference equations we refer to ([1],[2], [4]-[8],[10],[11]-[15]).

In this paper , we studied the closed form solutions of the following systems of partial difference equations

$$\alpha X_{n,m} + \beta X_{n,m}X_{n-2,m-2}Y_{n-1,m-1} - X_{n-2,m-2} = 0 \tag{1}$$

$$\gamma Y_{n,m} + \delta Y_{n,m}Y_{n-2,m-2}X_{n-1,m-1} - Y_{n-2,m-2} = 0 \tag{2}$$

where $n, m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$,$\alpha, \beta, \gamma, \delta \in \{1, -1\}$ and the initial values $X_{n,0}$,$X_{n,-1}$,$X_{0,m}$,$X_{-1,m}$, $Y_{n,0}$,$Y_{n,-1}$,$Y_{0,m}$,and $Y_{-1,m}$ are real numbers .

As a direct consequence , we can drive the explicit solutions of a family of partial difference equations in the following form

$$\alpha X_{n,m} + \beta X_{n,m}X_{n-2,m-2}X_{n-1,m-1} - X_{n-2,m-2} = 0$$

where $n, m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$,$\alpha, \beta \in \{1, -1\}$ and the initial values $X_{n,0}$,$X_{n,-1}$,$X_{0,m}$, ,and $X_{-1,m}$ are real numbers .
Moreover , we can derive the exact solution for the following systems of ordinary difference equations

$$\alpha X_n + \beta X_n X_{n-2} Y_{n-1} - X_{n-2} = 0$$

$$\gamma Y_n + \delta Y_n Y_{n-2} X_{n-1} - Y_{n-2} = 0$$

where $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$,$\alpha, \beta, \gamma, \delta \in \{1, -1\}$ and the initial values X_0,X_{-1},Y_0,and Y_{-1} are real numbers .

2 Forms of Solutions

In this section we shall give explicit forms of solutions of the system (1)-(2) for particular values of $\alpha, \beta, \gamma, \delta \in \{1, -1\}$. We can rewrite system (1)-(2) in the following form

$$X_{n,m} = \frac{X_{n-2,m-2}}{\alpha + \beta X_{n-2,m-2}Y_{n-1,m-1}} \quad , \quad Y_{n,m} = \frac{Y_{n-2,m-2}}{\gamma + \delta Y_{n-2,m-2}X_{n-1,m-1}} \tag{3}$$

2.1 Form of Solutions when $(\alpha, \beta) = (\gamma, \delta) = (1, -1)$

In this case we have the system

$$X_{n,m} = \frac{X_{n-2,m-2}}{1 - X_{n-2,m-2}Y_{n-1,m-1}} \quad , \quad Y_{n,m} = \frac{Y_{n-2,m-2}}{1 - Y_{n-2,m-2}X_{n-1,m-1}} \tag{4}$$

Theorem 1. *Let* $\{X_{n,m}, Y_{n,m}\}_{n,m=-k}^{\infty}$ *be a solution of system* (4) *with initial conditions*

$$X_{n,0}, X_{n,-1}, X_{0,m}, X_{-1,m}, Y_{n,0}, Y_{n,-1}, Y_{0,m}, Y_{-1,m}$$

where $n, m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. *Suppose* $X_{-1,m-2}Y_{0,m-1} \neq 1$,$X_{n-2,-1}Y_{n-1,0} \neq 1$,$Y_{-1,m-2}X_{0,m-1} \neq 1$,$Y_{n-2,-1}X_{n-1,0} \neq 1$. *Then, the form of solutions of system* (4) *,for* $n, m \geq 1$ *and* $n \geq m$ *, are as follows:*

$$X_{n,m} = \begin{cases} X_{n-m,0} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)X_{n-m,0}Y_{n-m-1,-1}}{-1+(2k+2)X_{n-m,0}Y_{n-m-1,-1}}, & m \quad even; \\ X_{n-m-1,-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{n-m-1,-1}Y_{n-m,0}}{1-(2k+1)X_{n-m-1,-1}Y_{n-m,0}}, & m \quad odd; \end{cases} \tag{5}$$

$$Y_{n,m} = \begin{cases} Y_{n-m,0} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)Y_{n-m,0}X_{n-m-1,-1}}{-1+(2k+2)Y_{n-m,0}X_{n-m-1,-1}}, & m \quad even; \\ Y_{n-m-1,-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)Y_{n-m-1,-1}X_{n-m,0}}{1-(2k+1)Y_{n-m-1,-1}X_{n-m,0}}, & m \quad odd; \end{cases} \tag{6}$$

$$X_{m,n} = \begin{cases} X_{-1,n-m-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{-1,n-m-1}Y_{0,n-m}}{1-(2k+1)X_{-1,n-m-1}Y_{0,n-m}}, & m \quad odd; \\ X_{0,n-m} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)X_{0,n-m}Y_{-1,n-m-1}}{-1+(2k+2)X_{0,n-m}Y_{-1,n-m-1}}, & m \quad even; \end{cases} \tag{7}$$

$$Y_{m,n} = \begin{cases} Y_{-1,n-m-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)Y_{-1,n-m-1}X_{0,n-m}}{1-(2k+1)Y_{-1,n-m-1}X_{0,n-m}}, & m \quad odd; \\ Y_{0,n-m} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)Y_{0,n-m}X_{-1,n-m-1}}{-1+(2k+2)Y_{0,n-m}X_{-1,n-m-1}}, & m \quad even; \end{cases} \tag{8}$$

Proof. We shall use the principle of (odd-even)double mathematical induction . Firstly , we shall prove that the relations (5)-(8) hold for $(n, m) = (1, 1)$. From equations in system (4)we can see

$$X_{1,1} = \frac{X_{-1,-1}}{1 - X_{-1,-1}Y_{0,0}} = X_{-1,-1} \prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)X_{-1,-1}Y_{0,0}}{1 - (2k+1)X_{-1,-1}Y_{0,0}}$$

$$Y_{1,1} = \frac{Y_{-1,-1}}{1 - Y_{-1,-1}X_{0,0}} = Y_{-1,-1} \prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)Y_{-1,-1}X_{0,0}}{1 - (2k+1)Y_{-1,-1}X_{0,0}}$$

Now , we shall prove that the relations (5)-(8)hold for $(n, m) = (2, 2)$.

$$X_{2,2} = \frac{X_{0,0}}{1 - X_{0,0}Y_{1,1}} = \frac{X_{0,0}}{1 - X_{0,0}(\frac{Y_{-1,-1}}{1-Y_{-1,-1}X_{0,0}})} = X_{0,0}(\frac{1 - X_{0,0}Y_{-1,-1}}{1 - 2X_{0,0}Y_{-1,-1}})$$

$$= X_{0,0}\prod_{k=0}^{\frac{2-2}{2}} \frac{-1 + (2k+1)X_{0,0}Y_{-1,-1}}{-1 + (2k+2)X_{0,0}Y_{-1,-1}}$$

$$Y_{2,2} = \frac{Y_{0,0}}{1 - Y_{0,0}X_{1,1}} = \frac{Y_{0,0}}{1 - Y_{0,0}(\frac{X_{-1,-1}}{1-X_{-1,-1}Y_{0,0}})} = Y_{0,0}(\frac{1 - Y_{0,0}X_{-1,-1}}{1 - 2Y_{0,0}X_{-1,-1}})$$

$$= Y_{0,0}\prod_{k=0}^{\frac{2-2}{2}} \frac{-1 + (2k+1)Y_{0,0}X_{-1,-1}}{-1 + (2k+2)Y_{0,0}X_{-1,-1}}$$

Moreover ,We shall prove that the relations (5)-(8) hold for $(n, m) = (1, 2)$ and $(n, m) = (2, 1)$.

$$X_{1,2} = \frac{X_{-1,0}}{1 - X_{-1,0}Y_{0,1}} = X_{-1,0}\prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)X_{-1,0}Y_{0,1}}{1 - (2k+1)X_{-1,0}Y_{0,1}}$$

$$Y_{1,2} = \frac{Y_{-1,0}}{1 - Y_{-1,0}X_{0,1}} = Y_{-1,0}\prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)Y_{-1,0}X_{0,1}}{1 - (2k+1)Y_{-1,0}X_{0,1}}$$

$$X_{2,1} = \frac{X_{0,-1}}{1 - X_{0,-1}Y_{1,0}} = X_{0,-1}\prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)X_{0,-1}Y_{1,0}}{1 - (2k+1)X_{0,-1}Y_{1,0}}$$

$$Y_{2,1} = \frac{Y_{0,-1}}{1 - Y_{0,-1}X_{1,0}}$$

Now suppose that the relations (5)-(8) hold for $m = 1$ and $m = 2$ with $n \in \mathbb{N}$. So we have ,

$$X_{n,1} = X_{n-2,-1}\prod_{k=0}^{0} \frac{1 - (2k)X_{n-2,-1}Y_{n-1,0}}{1 - (2k+1)X_{n-2,-1}Y_{n-1,0}} = \frac{X_{n-2,-1}}{1 - X_{n-2,-1}Y_{n-1,0}}$$

$$Y_{n,1} = \frac{Y_{n-2,-1}}{1 - Y_{n-2,-1}X_{n-1,0}}$$

$$X_{n,2} = X_{n-2,0}(\frac{1 - X_{n-2,0}Y_{n-3,-1}}{1 - 2X_{n-2,0}Y_{n-3,-1}})$$

$$Y_{n,2} = Y_{n-2,0}(\frac{1 - Y_{n-2,0}X_{n-3,-1}}{1 - 2Y_{n-2,0}X_{n-3,-1}})$$

Now we try to prove that relations (5)-(8) hold for $m=1$ with $n+2$.

$$X_{n+2,1}=\frac{X_{n,-1}}{1-X_{n,-1}Y_{n+1,0}}=X_{n,-1}\prod_{k=0}^{\frac{0}{2}}\frac{1-(2k)X_{n,-1}Y_{n+1,0}}{1-(2k+1)X_{n,-1}Y_{n+1,0}}$$

$$Y_{n+2,1}=\frac{Y_{n,-1}}{1-Y_{n,-1}X_{n+1,0}}=Y_{n,-1}\prod_{k=0}^{\frac{1-1}{2}}\frac{1-(2k)Y_{n,-1}X_{n+1,0}}{1-(2k+1)Y_{n,-1}X_{n+1,0}}$$

Now we try to prove that relations (5)-(8) hold for $m=2$ with $n+2$.

$$X_{n+2,2}=\frac{X_{n,0}}{1-X_{n,0}Y_{n+1,1}}=\frac{X_{n,0}}{1-X_{n,0}(\frac{Y_{n-1,-1}}{1-Y_{n-1,-1}X_{n,0}})}$$

$$=\frac{X_{n,0}(1-Y_{n-1,-1}X_{n,0})}{1-2Y_{n-1,-1}X_{n,0}}=X_{n,0}\prod_{k=0}^{\frac{2-2}{2}}\frac{1-(2k+1)X_{n,0}Y_{n-1,-1}}{1-(2k+2)X_{n,0}Y_{n-1,-1}}$$

$$Y_{n+2,2}=\frac{Y_{n,0}}{1-Y_{n,0}X_{n+1,1}}=Y_{n,0}\prod_{k=0}^{\frac{2-2}{2}}\frac{1-(2k+1)Y_{n,0}X_{n-1,-1}}{1-(2k+2)Y_{n,0}X_{n-1,-1}}$$

Finally , we suppose that relations (5)-(8) hold for $n,m\in\mathbb{N}$. We shall prove that relations (5)-(8) hold for $n,m+2\in\mathbb{N}$.
From (4)we have

$$X_{n,m+2}=\frac{X_{n-2,m}}{1-X_{n-2,m}Y_{n-1,m+1}} \tag{9}$$

There are four cases :

(1) If $n>m+2$ and m even .

$$X_{n,m+2}=\frac{X_{n-2,m}}{1-X_{n-2,m}Y_{n-1,m+1}}$$

$$=\frac{X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}}}{1-(X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}})(Y_{n-m-3,-1}\prod_{k=0}^{\frac{m}{2}}\frac{1-(2k)Y_{n-m-3,-1}X_{n-m-2,0}}{1-(2k+1)Y_{n-m-3,-1}X_{n-m-2,0}})}$$

$$=\frac{X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}}}{1-\frac{X_{n-m-2,0}Y_{n-m-3,-1}}{1-(m+1)X_{n-m-2,0}Y_{n-m-3,-1}}}$$

$$=X_{n-m-2,0}\prod_{k=0}^{\frac{m}{2}}\frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}}$$

(2) If $n > m+2$ and m odd

$$X_{n,m+2} = \frac{X_{n-2,m}}{1 - X_{n-2,m}Y_{n-1,m+1}}$$

$$= \frac{X_{n-m-3,-1}\prod_{k=0}^{\frac{m-1}{2}}\frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}}}{1-(X_{n-m-3,-1}\prod_{k=0}^{\frac{m-1}{2}}\frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}})(Y_{n-m-2,0}\prod_{k=0}^{\frac{m-1}{2}}\frac{1-(2k+1)Y_{n-m-2,0}X_{n-m-3,-1}}{1-(2k+2)Y_{n-m-2,0}X_{n-m-3,-1}})}$$

$$= \frac{X_{n-m-3,-1}\prod_{k=0}^{\frac{m-1}{2}}\frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}}}{1-\frac{X_{n-m-3,-1}Y_{n-m-2,0}}{1-(m+1)X_{n-m-3,-1}Y_{n-m-2,0}}}$$

$$= X_{n-m-3,-1}\prod_{k=0}^{\frac{m+1}{2}}\frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}}$$

(3) If $n < m+2$ and m even
By symmetry ,using (7) and (8), we can prove it like part (1) .

(4) If $n < m+2$ and m odd
By symmetry ,using (7) and (8), we can prove it like part (2) ..

$$Y_{n,m+2} = \frac{Y_{n-2,m}}{1 - Y_{n-2,m}X_{n-1,m+1}}$$

We can do that by the same way in proving equation (9)

□

Proposition 1. *We have the following properties for the solutions of system (4) :*

(1) If m even and $X_{n-m,0} = 0$, then $X_{n,m} = 0$.

(2) If m odd and $X_{n-m,0} = 0$, then $Y_{n,m} = Y_{n-m-1,-1}$.

(3) If m even and $Y_{n-m,0} = 0$, then $Y_{n,m} = 0$.

(4) If m odd and $Y_{n-m,0} = 0$, then $X_{n,m} = X_{n-m-1,-1}$.

(5) If m even and $X_{n-m-1,-1} = 0$, then $Y_{n,m} = Y_{n-m,0}$.

(6) If m odd and $X_{n-m-1,-1} = 0$, then $X_{n,m} = 0$.

(7) If m even and $Y_{n-m-1,-1} = 0$, then $X_{n,m} = X_{n-m,0}$.

(8) If m odd and $Y_{n-m-1,-1} = 0$, then $Y_{n,m} = 0$.

Proposition 2. *We have the following properties for the solutions of system (4) :*

(1) *If m even and $X_{0,n-m}=0$, then $X_{m,n}=0$.*

(2) *If m odd and $X_{0,n-m}=0$, then $Y_{m,n}=Y_{-1,n-m-1}$.*

(3) *If m even and $Y_{0,n-m}=0$, then $Y_{m,n}=0$.*

(4) *If m odd and $Y_{0,n-m}=0$, then $X_{m,n}=X_{-1,n-m-1}$.*

(5) *If m even and $X_{-1,n-m-1}=0$, then $Y_{m,n}=Y_{0,n-m}$.*

(6) *If m odd and $X_{-1,n-m-1}=0$, then $X_{m,n}=0$.*

(7) *If m even and $Y_{-1,n-m-1}=0$, then $X_{m,n}=X_{0,n-m}$.*

(8) *If m odd and $Y_{-1,n-m-1}=0$, then $Y_{m,n}=0$.*

Remark 1. *If we take into account the one dimensional case of system (4) we have a partial difference equation in the form*

$$X_{n,m}=\frac{X_{n-2,m-2}}{1-X_{n-2,m-2}X_{n-1,m-1}} \tag{10}$$

We can see that the closed form solution of equation(10) is given ,from theorem(1) , by the following corollary .

Corollary 2. *Let $\{X_{n,m}\}_{n,m=-k}^{\infty}$ be a solution of equation (10) with initial conditions $X_{n,0},X_{n,-1},X_{0,m},X_{-1,m}$, where $n,m\in\mathbb{N}_0$, $\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$. Suppose $X_{-1,m-2}X_{0,m-1}\neq 1$,$X_{n-2,-1}X_{n-1,0}\neq 1$. Then, the form of solutions of equation (10) ,for $n,m\geq 1$ and $n\geq m$, are as follows:*

$$X_{n,m}=\begin{cases} X_{n-m,0}\prod\limits_{k=0}^{\frac{m-2}{2}}\frac{-1+(2k+1)X_{n-m,0}X_{n-m-1,-1}}{-1+(2k+2)X_{n-m,0}X_{n-m-1,-1}}, & m \quad even;\\ X_{n-m-1,-1}\prod\limits_{k=0}^{\frac{m-1}{2}}\frac{1-(2k)X_{n-m-1,-1}X_{n-m,0}}{1-(2k+1)X_{n-m-1,-1}X_{n-m,0}}, & m \quad odd;\end{cases}$$

$$X_{m,n}=\begin{cases} X_{-1,n-m-1}\prod\limits_{k=0}^{\frac{m-1}{2}}\frac{1-(2k)X_{-1,n-m-1}X_{0,n-m}}{1-(2k+1)X_{-1,n-m-1}X_{0,n-m}}, & m \quad odd;\\ X_{0,n-m}\prod\limits_{k=0}^{\frac{m-2}{2}}\frac{-1+(2k+1)X_{0,n-m}X_{-1,n-m-1}}{-1+(2k+2)X_{0,n-m}X_{-1,n-m-1}}, & m \quad even;\end{cases}$$

Proposition 3. *We have the following properties for the solutions of equation (4):*

(1) *If m even and $X_{n-m,0} = 0$, then $X_{n,m} = 0$.*

(2) *If m odd and $X_{n-m,0} = 0$, then $X_{n,m} = X_{n-m-1,-1}$.*

(3) *If m even and $X_{n-m-1,-1} = 0$, then $X_{n,m} = X_{n-m,0}$.*

(4) *If m odd and $X_{n-m-1,-1} = 0$, then $X_{n,m} = 0$.*

(5) *If m even and $X_{0,n-m} = 0$, then $X_{m,n} = 0$.*

(6) *If m odd and $X_{0,n-m} = 0$, then $X_{m,n} = X_{-1,n-m-1}$.*

(7) *If m even and $X_{-1,n-m-1} = 0$, then $X_{m,n} = X_{0,n-m}$.*

(8) *If m odd and $X_{-1,n-m-1} = 0$, then $X_{m,n} = 0$.*

Remark 2. *If we put $n = m$ in system (4) we have a system of ordinary difference equations in the following form*

$$X_n = \frac{X_{n-2}}{1 - X_{n-2}Y_{n-1}}, \qquad Y_n = \frac{Y_{n-2}}{1 - Y_{n-2}X_{n-1}} \tag{11}$$

Corollary 3. *Let $\{X_n, Y_n\}_{n=-k}^{\infty}$ be a solution of system (11) with initial conditions X_0, X_{-1}, Y_0, Y_{-1} . Suppose $X_{-1}Y_0 \neq 1$,and $Y_{-1}X_0 \neq 1$,. Then, the form of solutions of system (11) ,for $n \geq 1$ are as follows:*

$$X_n = \begin{cases} X_0 \prod\limits_{k=0}^{\frac{n-2}{2}} \frac{-1+(2k+1)X_0Y_{-1}}{-1+(2k+2)X_0Y_{-1}}, n, even \\ X_{-1} \prod\limits_{k=0}^{\frac{n-1}{2}} \frac{1-(2k)X_{-1}Y_0}{1-(2k+1)X_{-1}Y_0}, n, odd \end{cases} \qquad Y_n = \begin{cases} Y_0 \prod\limits_{k=0}^{\frac{n-2}{2}} \frac{-1+(2k+1)Y_0X_{-1}}{-1+(2k+2)Y_0X_{-1}}, n, even \\ Y_{-1} \prod\limits_{k=0}^{\frac{n-1}{2}} \frac{1-(2k)Y_{-1}X_0}{1-(2k+1)Y_{-1}X_0}, n, odd \end{cases}$$

Remark 3. *If we put $X = Y$ in system(11) we get an ordinary difference equation in the form*

$$X_n = \frac{X_{n-2}}{1 - X_{n-2}X_{n-1}} \tag{12}$$

We can see that the closed form solution of equation(12) is given ,from corollary(3) , by the following

$$X_n = \begin{cases} X_0 \prod\limits_{k=0}^{\frac{n-2}{2}} \frac{-1+(2k+1)X_0X_{-1}}{-1+(2k+2)X_0X_{-1}}, & n \quad even; \\ X_{-1} \prod\limits_{k=0}^{\frac{n-1}{2}} \frac{1-(2k)X_{-1}X_0}{1-(2k+1)X_{-1}X_0}, & n \quad odd; \end{cases}$$

where $n \in \mathbb{N}$, and $X_{-1}X_0 \neq -1$ .We can easy see that if n even (or odd) and $X_0 = 0$ then $X_n = 0(X_n = X_{-1})$. Also if n even (or odd) and $X_{-1} = 0$ then $X_n = X_0(X_n = 0)$.

2.2 Form of Solutions when $(\alpha,\beta)=(1,1)\&(\gamma,\delta)=(1,-1)$

In this case we have the system

$$X_{n,m}=\frac{X_{n-2,m-2}}{1+X_{n-2,m-2}Y_{n-1,m-1}} \quad , \quad Y_{n,m}=\frac{Y_{n-2,m-2}}{1-Y_{n-2,m-2}X_{n-1,m-1}} \tag{13}$$

Theorem 4. *Let $\{X_{n,m},Y_{n,m}\}_{n,m=-k}^{\infty}$ be a solution of system (13) with initial conditions $X_{n,0},X_{n,-1},X_{0,m},X_{-1,m},Y_{n,0},Y_{n,-1},Y_{0,m},Y_{-1,m}$ where $n,m\in\mathbb{N}_0$, $\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$. Suppose $X_{-1,m-2}Y_{0,m-1}\neq-1$,$X_{n-2,-1}Y_{n-1,0}\neq-1$, $Y_{-1,m-2}X_{0,m-1}\neq1$,$Y_{n-2,-1}X_{n-1,0}\neq1$. Then, the form of solutions of system (13) ,for $n,m\geq1$ and $n\geq m$, are as follows:*

$$X_{n,m}=\begin{cases}\frac{X_{n-m-1,-1}}{(1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+1}{2}}}, & m \quad odd;\\ (-1)^{\frac{m}{2}}X_{n-m,0}(-1+X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

$$Y_{n,m}=\begin{cases}\frac{(-1)^{\frac{m+1}{2}}Y_{n-m-1,-1}}{(-1+Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{2}}}, & m \quad odd;\\ Y_{n-m,0}(1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

$$X_{m,n}=\begin{cases}\frac{X_{-1,n-m-1}}{(1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+1}{2}}}, & m \quad odd;\\ (-1)^{\frac{m}{2}}X_{0,n-m}(-1+X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

$$Y_{m,n}=\begin{cases}\frac{(-1)^{\frac{m+1}{2}}Y_{-1,n-m-1}}{(-1+Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{2}}}, & m \quad odd;\\ Y_{0,n-m}(1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

Proof. We can prove the theorem by odd-even double mathematical induction as in theorem (1). □

Remark 4. *We can see that both of proposition (1) and proposition (2) hold for the solutions of system (13) included in theorem(4) .*

Remark 5. *If we put $n=m$ in system (13) we have a system of ordinary difference equations in the following form*

$$X_n=\frac{X_{n-2}}{1+X_{n-2}Y_{n-1}}, \qquad Y_n=\frac{Y_{n-2}}{1-Y_{n-2}X_{n-1}} \tag{14}$$

We can drive the formulas for solutions from theorem(4) in the following corollary .

Corollary 5. *Let $\{X_n,Y_n\}_{n=-k}^{\infty}$ be a solution of system (14) with initial conditions X_0,X_{-1},Y_0,Y_{-1} . Suppose $X_{-1}Y_0\neq-1$,and $Y_{-1}X_0\neq1$,. Then, the form of solutions of system (14) ,for $n\geq1$ are as follows:*

$$X_n=\begin{cases}\frac{X_{-1}}{(1+X_{-1}Y_0)^{\frac{n+1}{2}}}; n, odd\\ (-1)^{\frac{n}{2}}X_0(-1+X_0Y_{-1})^{\frac{n}{2}}; n, even\end{cases} \qquad Y_n=\begin{cases}\frac{(-1)^{\frac{n+1}{2}}Y_{-1}}{(-1+Y_{-1}X_0)^{\frac{n+1}{2}}}; n, odd\\ Y_0(1+Y_0X_{-1})^{\frac{n}{2}}; n, even\end{cases}$$

.

2.3 Form of Solutions when $(\alpha,\beta)=(1,1)\&(\gamma,\delta)=(-1,1)$

In this case we have the system

$$X_{n,m}=\frac{X_{n-2,m-2}}{1+X_{n-2,m-2}Y_{n-1,m-1}} \quad , \quad Y_{n,m}=\frac{Y_{n-2,m-2}}{-1+Y_{n-2,m-2}X_{n-1,m-1}} \qquad (15)$$

Theorem 6. *Let $\{X_{n,m},Y_{n,m}\}_{n,m=-k}^{\infty}$ be a solution of system (15) with initial conditions $X_{n,0},X_{n,-1},X_{0,m},X_{-1,m},Y_{n,0},Y_{n,-1},Y_{0,m},Y_{-1,m}$ where $n,m\in\mathbb{N}_0$, $\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$. Suppose $X_{-1,m-2}Y_{0,m-1}\neq-1$,$X_{n-2,-1}Y_{n-1,0}\neq-1$, $Y_{-1,m-2}X_{0,m-1}\neq 1$,$Y_{n-2,-1}X_{n-1,0}\neq 1$. Then, the form of solutions of system (15) ,for $n,m\geq 1$ and $n\geq m$, are as follows:*

$$X_{n,m}=\begin{cases}\dfrac{(-1)^{\frac{m-1}{4}}X_{n-m-1,-1}}{(1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+3}{4}}(-1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m-1}{4}}}, & m=4K+1;\\ \dfrac{(-1)^{\frac{m-2}{4}}X_{n-m,0}(-1+X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{2}}}{(-1+2X_{n-m,0}Y_{n-m-1,-1})^{\frac{m+2}{4}}}, & m=4K+2;\\ \dfrac{(-1)^{\frac{m+1}{4}}X_{n-m-1,-1}}{(-1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+1}{4}}(1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+1}{4}}}, & m=4K+3;\\ \dfrac{(-1)^{\frac{m}{4}}X_{n-m,0}(-1+X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{2}}}{(-1+2X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{4}}}, & m=4K+4;\end{cases}$$

$$Y_{n,m}=\begin{cases}\dfrac{(-1)^{\frac{m-1}{4}}Y_{n-m-1,-1}(-1+2Y_{n-m-1,-1}X_{n-m,0})^{\frac{m-1}{4}}}{(-1+Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{2}}}, & m=4K+1;\\ (-1)^{\frac{m+2}{4}}Y_{n-m,0}(-1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m-2}{4}} \\ \quad .(1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m+2}{4}}, & m=4K+2;\\ \dfrac{(-1)^{\frac{m+1}{4}}Y_{n-m-1,-1}(-1+2Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{4}}}{(-1+Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{2}}}, & m=4K+3;\\ (-1)^{\frac{m}{4}}Y_{n-m,0}(-1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m}{4}} \\ \quad .(1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m}{4}}, & m=4K+4;\end{cases}$$

$$X_{m,n}=\begin{cases}\dfrac{(-1)^{\frac{m-1}{4}}X_{-1,n-m-1}}{(1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+3}{4}}(-1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m-1}{4}}}, & m=4K+1;\\ \dfrac{(-1)^{\frac{m-2}{4}}X_{0,n-m}(-1+X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{2}}}{(-1+2X_{0,n-m}Y_{-1,n-m-1})^{\frac{m+2}{4}}}, & m=4K+2;\\ \dfrac{(-1)^{\frac{m+1}{4}}X_{-1,n-m-1}}{(-1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+1}{4}}(1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+1}{4}}}, & m=4K+3;\\ \dfrac{(-1)^{\frac{m}{4}}X_{0,n-m}(-1+X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{2}}}{(-1+2X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{4}}}, & m=4K+4;\end{cases}$$

$$Y_{m,n}=\begin{cases}\dfrac{(-1)^{\frac{m-1}{4}}Y_{-1,n-m-1}(-1+2Y_{-1,n-m-1}X_{0,n-m})^{\frac{m-1}{4}}}{(-1+Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{2}}}, & m=4K+1;\\ (-1)^{\frac{m+2}{4}}Y_{0,n-m}(-1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m-2}{4}} \\ \quad .(1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m+2}{4}}, & m=4K+2;\\ \dfrac{(-1)^{\frac{m+1}{4}}Y_{-1,n-m-1}(-1+2Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{4}}}{(-1+Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{2}}}, & m=4K+3;\\ (-1)^{\frac{m}{4}}Y_{0,n-m}(-1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m}{4}} \\ \quad .(1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m}{4}}, & m=4K+4;\end{cases}$$

where $k = 0, 1, 2, 3......$.

Proof. We can prove the theorem by piecewise double mathematical induction as in theorem (1). □

Proposition 4. *We have the following properties for the solutions of system* (15) *:*

(1) *If* m *even and* $X_{n-m,0} = 0$, *then* $X_{n,m} = 0$.

(2) *If* m *odd and* $X_{n-m,0} = 0$, *then* $Y_{n,m} = \pm Y_{n-m-1,-1}$.

(3) *If* m *even and* $Y_{n-m,0} = 0$, *then* $Y_{n,m} = 0$.

(4) *If* m *odd and* $Y_{n-m,0} = 0$, *then* $X_{n,m} = X_{n-m-1,-1}$.

(5) *If* m *even and* $X_{n-m-1,-1} = 0$, *then* $Y_{n,m} = \pm Y_{n-m,0}$.

(6) *If* m *odd and* $X_{n-m-1,-1} = 0$, *then* $X_{n,m} = 0$.

(7) *If* m *even and* $Y_{n-m-1,-1} = 0$, *then* $X_{n,m} = \pm X_{n-m,0}$.

(8) *If* m *odd and* $Y_{n-m-1,-1} = 0$, *then* $Y_{n,m} = 0$.

Proposition 5. *We have the following properties for the solutions of system* (15) *:*

(1) *If* m *even and* $X_{0,n-m} = 0$, *then* $X_{m,n} = 0$.

(2) *If* m *odd and* $X_{0,n-m} = 0$, *then* $Y_{m,n} = \pm Y_{-1,n-m-1}$.

(3) *If* m *even and* $Y_{0,n-m} = 0$, *then* $Y_{m,n} = 0$.

(4) *If* m *odd and* $Y_{0,n-m} = 0$, *then* $X_{m,n} = X_{-1,n-m-1}$.

(5) *If* m *even and* $X_{-1,n-m-1} = 0$, *then* $Y_{m,n} = \pm Y_{0,n-m}$.

(6) *If* m *odd and* $X_{-1,n-m-1} = 0$, *then* $X_{m,n} = 0$.

(7) *If* m *even and* $Y_{-1,n-m-1} = 0$, *then* $X_{m,n} = \pm X_{0,n-m}$.

(8) *If* m *odd and* $Y_{-1,n-m-1} = 0$, *then* $Y_{m,n} = 0$.

Remark 6. *If we put* $n = m$ *in system* (15) *we have a system of ordinary difference equations in the following form*

$$X_n = \frac{X_{n-2}}{1 + X_{n-2}Y_{n-1}}, \qquad Y_n = \frac{Y_{n-2}}{-1 + Y_{n-2}X_{n-1}} \tag{16}$$

We can drive the formulas for solutions from theorem(6) *in the following corollary .*

Corollary 7. *Let* $\{X_n, Y_n\}_{n=-k}^{\infty}$ *be a solution of system* (16) *with initial conditions* X_0, X_{-1}, Y_0, Y_{-1} . *Suppose* $X_{-1}Y_0 \neq -1$ *,and* $Y_{-1}X_0 \neq 1$ *,. Then, the form of solutions of system* (16) *,for* $n \geq 1$ *are as follows:*

$$X_n = \begin{cases} \frac{(-1)^{\frac{n-1}{4}} X_{-1}}{(1+X_{-1}Y_0)^{\frac{n+3}{4}}(-1+X_{-1}Y_0)^{\frac{n-1}{4}}}, & n = 4K+1; \\ \frac{(-1)^{\frac{n-2}{4}} X_0(-1+X_0Y_{-1})^{\frac{n}{2}}}{(-1+2X_0Y_{-1})^{\frac{n+2}{4}}}, & n = 4K+2; \\ \frac{(-1)^{\frac{n+1}{4}} X_{-1}}{(-1+X_{-1}Y_0)^{\frac{n+1}{4}}(1+X_{-1}Y_0)^{\frac{n+1}{4}}}, & n = 4K+3; \\ \frac{(-1)^{\frac{n}{4}} X_0(-1+X_0Y_{-1})^{\frac{n}{2}}}{(-1+2X_0Y_{-1})^{\frac{n}{4}}}, & n = 4K+4; \end{cases}$$

$$Y_n = \begin{cases} \frac{(-1)^{\frac{n-1}{4}} Y_{-1}(-1+2Y_{-1}X_0)^{\frac{n-1}{4}}}{(-1+Y_{-1}X_0)^{\frac{n+1}{2}}}, & n = 4K+1; \\ (-1)^{\frac{n+2}{4}} Y_0(-1+Y_0X_{-1})^{\frac{n-2}{4}}(1+Y_0X_{-1})^{\frac{n+2}{4}}, & n = 4K+2; \\ \frac{(-1)^{\frac{n+1}{4}} Y_{-1}(-1+2Y_{-1}X_0)^{\frac{n+1}{4}}}{(-1+Y_{-1}X_0)^{\frac{n+1}{2}}}, & n = 4K+3; \\ (-1)^{\frac{n}{4}} Y_0(-1+Y_0X_{-1})^{\frac{n}{4}}(1+Y_0X_{-1})^{\frac{n}{4}}, & n = 4K+4; \end{cases}$$

where $k = 0, 1, 2, 3......$.

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TWO-DIMENSIONAL CHLODOWSKY VARIANT OF q-BERNSTEIN-SCHURER-STANCU OPERATORS

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Abstract. In this paper, two-dimensional Chlodowsky variant q-based Bernstein-Schurer-Stancu operators are introduced. Korovkin-type approximation theorems in different function spaces are studied. The error of approximation by using full modulus of continuity and partial modulus of continuities are given. Moreover, we introduce a generalization of our operators and investigate its approximation in more general weighted space.

1. Introduction

It was Chlodowsky [3] who introduced the classical Bernstein-Chlodowsky operators as

$$C_n(f;x)=\sum_{r=0}^{n} f\left(\frac{r}{n}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}\right)^r\left(1-\frac{x}{b_n}\right)^{n-r},$$

where the function f is defined on $[0,\infty)$ and $\{b_n\}$ is a positive increasing sequence with $b_n\to\infty$ and $\dfrac{b_n}{n}\to 0$ as $n\to\infty$.
In 2008, the q-analogue of Chlodowsky operators were introduced and investigated by Karslı and Gupta [8] as

$$C_n(f;q;x)=\sum_{k=0}^{n+p} f\left(\frac{[k]}{[n]}b_n\right)\begin{bmatrix}n+p\\k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right),\qquad 0\le x\le b_n$$

where $\{b_n\}$ has the same property of Bernstein-Chlodowsky operators.
On the other hand, the q-Bernstein-Schurer operators were defined by Muraru [9], for fixed $p\in\mathbb{N}_0$ and for all $x\in[0,1]$, by

$$B_n^p(f;q;x)=\sum_{k=0}^{n+p} f\left(\frac{[k]}{[n]}\right)\begin{bmatrix}n+p\\k\end{bmatrix}x^k\prod_{s=0}^{n+p-k-1}(1-q^sx).\tag{1.1}$$

Note that the case $q\to1^-$ in (1.1) reduces to the operators considered by Schurer [12]. Then, some properties of the q-Bernstein-Schurer operators were given in [13].
In 2013, the q-analogue of Bernstein-Schurer-Stancu operators $S_{n,p}^{\alpha,\beta}:C[0,1+p]\to C[0,1]$ were introduced by Agrawal, et al in [4] by

$$S_{n,p}^{(\alpha,\beta)}(f;q;x)=\sum_{k=0}^{n+p} f\left(\frac{[k]+\alpha}{[n]+\beta}\right)\begin{bmatrix}n+p\\k\end{bmatrix}x^k\prod_{s=0}^{n+p-k-1}(1-q^sx),\tag{1.2}$$

Key words and phrases. q-Bernstein operators, Chlodowsky operators, Chlodowsky variant of q-Bernstein-Schurer-Stancu operators, weighted space, modulus of continuity.
2010 *AMS Math. Subject Classification.* Primary 41A10, 41A25; Secondary 41A36.

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where α and β are non-negative numbers which satisfy $0 \leq \alpha \leq \beta$ and also p is a non-negative integer. Notice that, if we choose $\alpha = \beta = 0$ in (1.2), $S_{n,p}^{(\alpha,\beta)}(f;q;x)$ reduces to the classical q-Bernstein operator [10].

Recently, Chlodowsky variant of q-Bernstein-Schurer-Stancu operators were introduced by the authors in [14] as

$$(1.3) \quad C_{n,p}^{(\alpha,\beta)}(f;q;x) := \sum_{k=0}^{n+p} f\left(\frac{[k]+\alpha}{[n]+\beta} b_n\right) \begin{bmatrix} n+p \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1} \left(1 - q^s \frac{x}{b_n}\right),$$

where $n \in \mathbb{N}$, $p \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $0 \leq \alpha \leq \beta$, $0 \leq x \leq b_n$ and $0 < q < 1$. If $\alpha = \beta = p = 0$ in (1.3), we get the operators $C_n(f;q;x)$ and if $q \to 1^-$ and $\alpha = \beta = p = 0$ in (1.3), we get the operators $C_n(f;x)$.

In 2009, Büyükyazıcı [1] defined the two-dimensional q-Bernstein-Chlodowsky polynomials as

$$\widetilde{B}_{n,m}^{q_n,q_m}(f;x,y) = \sum_{k=0}^{n} \sum_{j=0}^{m} f\left(\frac{[k]_{q_n}}{[n]_{q_n}} \alpha_n, \frac{[j]_{q_m}}{[m]_{q_m}} \beta_m\right) \Omega_{k,n,q_n}\left(\frac{x}{\alpha_n}\right) \Omega_{k,m,q_m}\left(\frac{y}{\beta_m}\right)$$

where $\Omega_{k,n,q_n}(u) = \begin{bmatrix} n \\ k \end{bmatrix} u^k \prod_{s=0}^{n-k-1} (1 - q_n^s)$ and investigated its approximation properties on the rectangular unbounded domain.

On the other hand, Büyükyazıcı and Sharma [2] defined the two-dimensional q-Bernstein-Chlodowsky-Durrmeyer operators on the rectangular unbounded domain and derived the Korovkin type approximation properties. They also computed the order of convergence by means of the modulus of continuity and then examined the weighted approximation properties for these operators.

In the present paper we consider the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. Some of the results about the operators $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ defined in (1.3) will be useful in our investigations. For instance, the first three moments first three moments of the operator $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ are as follows [14]:

Lemma 1.1. *Let $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ defined. Then the first few moments of the operators are,*

(i) $C_{n,p}^{(\alpha,\beta)}(1;q;x) = 1,$

(ii) $C_{n,p}^{(\alpha,\beta)}(t;q;x) = \dfrac{[n+p]\,x + \alpha b_n}{[n]+\beta},$

(iii) $C_{n,p}^{(\alpha,\beta)}(\mathrm{t}^2;q;x) = \dfrac{1}{([n]+\beta)^2}\left\{[n+p-1]\,[n+p]\,qx^2\right.$

$\left. + (2\alpha+1)\,[n+p]\,b_n x + \alpha^2 b_n^2\right\}.$

Before proceeding further let us recall that the some basic definitions of q-calculus. The q-integer of $k \in \mathbb{R}$ is [7]

$$[k]_q = \begin{cases} \left(1-q^k\right)/(1-q), & q \neq 1 \\ k, & q = 1, \end{cases}$$

the q-factorial is defined by

$$[k]_q! = \begin{cases} [k]_q [k-1]_q \ldots [1]_q, & k = 1, 2, 3, \ldots, \\ 1, & k = 0 \end{cases}$$

and q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}$$

The organization of the paper as follows:
In section two, the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators is established and the first few moments of the operator is given. In section three, some Korovkin-type theorems in different function spaces are studied. In section four, we obtain the order of convergence of the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators by means of the first modulus of continuity and partial modulus of continuity. In section five, we study the generalization of the two-dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators and seek its approximation properties in more general weighted space.

2. Construction of the operators

Let $\{a_n\}$ and $\{b_m\}$ be increasing sequences of real numbers satisfying

$$\lim_{n\to\infty} a_n = \lim_{m\to\infty} b_m = \infty.$$

Let, D_{a_n,b_m} denotes

$$D_{a_n,b_m} = \{(x,y) : 0 \le x \le a_n,\, 0 \le y \le b_m\}. \tag{2.1}$$

For $(x,y) \in D_{a_n,b_m}$, we construct the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators as

$$\begin{aligned} & C_{n,m,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y) \\ & := \sum_{k=0}^{n+p} \sum_{j=0}^{m+p} f\left(\frac{[k]_{q_n} + \alpha}{[n]_{q_n} + \beta} a_n, \frac{[j]_{q_m} + \alpha}{[m]_{q_m} + \beta} b_m \right) \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right) \Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \end{aligned} \tag{2.2}$$

where $n \in \mathbb{N}$, $p \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $0 \le \alpha \le \beta$. $\Phi_{k,n,q_n}(z) = \begin{bmatrix} n+p \\ k \end{bmatrix}_{q_n} z^k \prod_{s=0}^{n+p-k-1} (1 - q_n^s z)$.
We also let $0 < q_n < 1$ $(n \in \mathbb{N})$ for the positivity of the operators. It is easy to show that $C_{n,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y)$ is a linear and positive operator.
Now, we start by giving the following lemma which will be used throughout the paper.

Lemma 2.1. *Let $C_{n,m,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y)$ be given in (2.2). Then the first few moments of the operators are,*

(i) $C_{n,m,p}^{(\alpha,\beta)}(1; q_n, q_m; x, y) = 1,$

(ii) $C_{n,m,p}^{(\alpha,\beta)}(t_1; q_n, q_m; x, y) = \dfrac{[n+p]_{q_n} x + \alpha a_n}{[n]_{q_n} + \beta},$

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$$(iii)\ C_{n,m,p}^{(\alpha,\beta)}\left(t_2;q_n,q_m;x,y\right)=\frac{[m+p]_{q_m}y+\alpha b_m}{[m]_{q_m}+\beta}$$

$$(iv)\ C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;x,y\right)$$

$$=\frac{1}{\left([n]_{q_n}+\beta\right)^2}\left\{[n+p-1]_{q_n}[n+p]_{q_n}q_nx^2+(2\alpha+1)[n+p]_{q_n}a_nx+\alpha^2a_n^2\right\}$$

$$+\frac{1}{\left([m]_{q_m}+\beta\right)^2}\left\{[m+p-1]_{q_m}[m+p]_{q_m}q_my^2+(2\alpha+1)[m+p]_{q_m}b_my+\alpha^2b_m^2\right\}.$$

Proof. Using Lemma 1.1 and the linearity of the operators, the proof is easily obtained. □

3. Korovkin-type approximation theorems

In this section, Korovkin-type approximation theorems are given for the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. For fixed $\nu\geq 0$ consider the space C_{ρ^ν} which consists of all continuous functions f, satisfying the condition

$$|f(x,y)|\leq M_f\rho^\nu(x,y),\quad (x,y)\in[0,\infty)\times[0,\infty):=\mathbb{R}_+^2 \text{ and } \rho(x,y)=1+x^2+y^2.$$

Clearly, C_{ρ^ν} is a linear normed space with the following norm

$$\|f\|_{\rho^\nu}=\sup_{0\leq x,y<\infty}\frac{|f(x,y)|}{\rho^\nu(x,y)}.$$

The following theorem will be used in the investigation of approximation properties of $C_{n,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)$ in the weighted spaces.

Theorem 3.1. *Let the numbers A and B be any fixed positive real numbers. Let $D_{A,B}=\{(x,y):0\leq x\leq A,\ 0\leq y\leq B\}$, $q:=\{q_n\}$ with $0<q_n<1$, $\lim_{n\to\infty}q_n=1$ and $\{a_n\}$ and $\{b_m\}$ be increasing sequences of positive real numbers that satisfy the following properties:*

$$\lim_{n\to\infty}a_n=\lim_{m\to\infty}b_m=\infty \text{ and } \lim_{n\to\infty}\frac{a_n}{[n]_{q_n}}=\lim_{m\to\infty}\frac{b_m}{[m]_{q_m}}=0.$$

For all $f\in C(D_{A,B})$, we have

$$\lim_{n,m\to\infty}\max_{(x,y)\in D_{A,B}}\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|=0.$$

Proof. Using Lemma 2.1, we get

$$\left\| C_{n,m,p}^{(\alpha,\beta)}\left(1;q_n,q_m;\cdot,\cdot\right)-1\right\|_{C(D_{A,B})}=0$$

$$\left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_1;q_n,q_m;\cdot,\cdot\right)-x\right\|_{C(D_{A,B})}\leq A\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|+\frac{\alpha a_n}{[n]_{q_n}+\beta}$$

$$\left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_2;q_n,q_m;\cdot,\cdot\right)-y\right\|_{C(D_{A,B})}\leq B\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|+\frac{\alpha b_m}{[m]_{q_m}+\beta}.$$

And again using Lemma 2.1 we have

$$C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)=\frac{1}{\left([n]_{q_n}+\beta\right)^2}$$

$$\times\left\{\left([n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right)x^2+(2\alpha+1)[n+p]_{q_n}a_nx+\alpha^2a_n^2\right\}$$

$$+\frac{1}{\left([m]_{q_m}+\beta\right)^2}$$

$$\times\left\{\left([m+p+1]_{q_m}[m+p]_{q_m}q_m-\left([m]_{q_m}+\beta\right)^2\right)y^2+(2\alpha+1)[m+p]_{q_m}b_my+\alpha^2b_m^2\right\}.$$

Finally, from the above equality we obtain

$$\left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_{C(D_{A,B})}$$

$$\leq\frac{1}{\left([n]_{q_n}+\beta\right)^2}$$

$$\times\left\{\left|[n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right|A^2+(2\alpha+1)[n+p]_{q_n}a_nA+\alpha^2a_n^2\right\}$$

$$+\frac{1}{\left([m]_{q_m}+\beta\right)^2}$$

$$\times\left\{\left|[m+p+1]_{q_m}[m+p]_{q_m}q_m-\left([m]_{q_m}+\beta\right)^2\right|B^2+(2\alpha+1)[m+p]_{q_m}b_mB+\alpha^2b_m^2\right\}.$$

Therefore, from the hypothesis of the theorem, we have

$$\begin{aligned}\left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_1;q_n,q_m;\cdot,\cdot\right)-x\right\|_{C(D_{A,B})}&\rightarrow 0\\ \left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_2;q_n,q_m;\cdot,\cdot\right)-y\right\|_{C(D_{A,B})}&\rightarrow 0\\ \left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_{C(D_{A,B})}&\rightarrow 0\end{aligned}$$

when n and $m\rightarrow\infty$.

Hence, the proof is completed by the two dimensional Korovkin theorem. □

In studying Korovkin-type weighted approximation, the following theorem plays an important role.

Theorem 3.2. *(See [6]) There exists a sequence of positive operators $T_{n,m}$, acting from $C_\rho\left(\mathbb{R}_+^2\right)$ to $C_\rho\left(\mathbb{R}_+^2\right)$, satisfying the conditions*

$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(1;\cdot,\cdot\right)-1\right\|_\rho=0$$

$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(t_1;\cdot,\cdot\right)-x\right\|_\rho=0$$

$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(t_2;\cdot,\cdot\right)-y\right\|_\rho=0$$

$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(t_1^2+t_2^2;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_\rho=0$$

and there exists a function $f^*\in C_\rho$ for which

$$\lim_{n,m\to\infty}\left\|T_{n,m}f^*-f^*\right\|_\rho\geq\frac{1}{4}$$

where $\rho=1+x^2+y^2$.

Now, consider the following operator

$$T_{n,m}\left(f;q_n,q_m;x,y\right)=\left\{\begin{array}{cc}C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right), & (x,y)\in D_{a_n,b_n}\\ f\left(x,y\right), & \mathbb{R}_+^2\backslash D_{a_n,b_n}\end{array}\right..$$

Theorem 3.3. *Let $f\in C_\rho\left(\mathbb{R}_+^2\right)$. Then for any $\gamma>0$*

$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(f;q_n,q_m;\cdot,\cdot\right)-f\left(\cdot\right)\right\|_{C_{\rho^{1+\gamma}}}=0$$

where $\{a_n\}$, $\{b_m\}$, $\{q_n\}$ and $\{q_m\}$have the same conditions as in Theorem 3.1.

Proof. For all $\varepsilon>0$, there exist sufficiently large positive real numbers A and B such that

$$\left(1+x^2+y^2\right)^{-\gamma}<\varepsilon \tag{3.1}$$

when $x>A$ and $y>B$.
Let n, m be sufficiently large so that $D_{A,B}\subset D_{a_n,b_m}$

$$\begin{aligned}&\left\|T_{n,m}\left(f;q_n,q_m;\cdot,\cdot\right)-f\left(\cdot\right)\right\|_{C_{\rho^{1+\gamma}}}\\ &\leq\sup_{(x,y)\in D_{A,B}}\frac{\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\gamma}}\\ &+\sup_{(x,y)\in D_{a_n,b_n}\backslash D_{A,B}}\frac{\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\gamma}}\\ &=y_{n,m}^{'}+y_{n,m}^{''}.\end{aligned}$$

By Theorem 3.1, $\lim\limits_{n,m\to\infty}y_{n,m}^{'}=0$ and for the proof of the second term we have

$$y_{n,m}^{''}\leq\left(1+x^2+y^2\right)^{-\gamma}\left(\frac{\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)\right|}{1+x^2+y^2}+\frac{|f(x,y)|}{1+x^2+y^2}\right).$$

Finally, since $f\in C_\rho\left(\mathbb{R}_+^2\right)$, the term $\frac{|f(x,y)|}{1+x^2+y^2}$ is bounded. Furthermore, because of the fact that

$$\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)\right|\leq\left|C_{n,m,p}^{(\alpha,\beta)}\left(1+t_1^2+t_2^2;q_n,q_m;x,y\right)\right|,$$

using Lemma 2.1, the term $\frac{\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)\right|}{1+x^2+y^2}$ is bounded for sufficiently large n and m. Hence, we get by (3.1) that

$$y_{n,m}^{''} \leq \varepsilon(1+M)$$

Since $\varepsilon > 0$ is arbitrary, then $\lim\limits_{n,m\to\infty} y_{n,m}^{''} = 0$. This completes the proof. □

Now, consider the subspace C_ρ^0 of C_ρ which is defined by

$$C_\rho^0 := \left\{ f \in C_\rho : \lim_{x,y\to 0} \frac{|f(x,y)|}{1+x^2+y^2} = 0 \right\}.$$

Theorem 3.4. *Let the sequences $\{q_n\}$, $\{a_n\}$ and $\{b_m\}$ satisfy the same properties as in Theorem 3.1. Then for all $f \in C_\rho^0\left(\mathbb{R}_+^2\right)$, we obtain*

$$\lim_{n,m\to\infty} \left\| T_{n,m}(f;q_n,q_m;\cdot,\cdot) - f(\cdot) \right\|_{C_\rho} = 0.$$

Proof. For all $f \in C_\rho^0\left(\mathbb{R}_+^2\right)$, observe that

$$\lim_{x,y\to\infty} \frac{|f(x,y)|}{1+x^2+y^2} = 0, \qquad \lim_{n,m\to\infty} \frac{\left| f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta} b_m \right) \right|}{1 + \left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} a_n \right)^2 + \left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta} b_m \right)^2} = 0.$$

Therefore, for all $\varepsilon > 0$, we can find sufficiently large numbers A and B such that

$$|f(x,y)| < \varepsilon\left(1+x^2+y^2\right) \tag{3.2}$$

for $x > A$ and $y > B$ and there exists natural numbers n_0 and m_0 such that

$$\left| f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta} b_m \right) \right| < \varepsilon \left(1 + \left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} a_n \right)^2 + \left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta} b_m \right)^2 \right) \tag{3.3}$$

for all $n > n_0$ and $m > m_0$.
Hence, for large n and m, we have

$$\begin{aligned}
&\left\| T_{n,m}(f;q_n,q_m;\cdot,\cdot) - f(\cdot) \right\|_{C_\rho} \\
&\leq \sup_{(x,y)\in D_{A,B}} \frac{\left| C_{n,m}^{\alpha,\beta}(f;q_n,q_m;x,y) - f(x,y) \right|}{1+x^2+y^2} \\
&+ \sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} \frac{\left| C_{n,m}^{\alpha,\beta}(f;q_n,q_m;x,y) - f(x,y) \right|}{1+x^2+y^2} = z_{n,m}^{'} + z_{n,m}^{''}.
\end{aligned}$$

By Theorem 3.1 it is sufficient to show that $z_{n,m}^{''} \to 0$ as $n \to \infty$.
Using (3.2) and (3.3), we get

$$\begin{aligned}
z_{n,m}^{''} &\leq \varepsilon + \sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} \frac{\left| C_{n,m}^{\alpha,\beta}(f;q_n,q_m;x,y)) \right|}{1+x^2+y^2} \\
&\leq \varepsilon + \varepsilon \sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} t_{n,m}(q_n,q_m;x;y) \\
&= \varepsilon\left(1 + \sup_{(x,y)\in D_{a_n,b_m}/D_{A,B}} t_{n,m}(q_n,q_m;x;y) \right)
\end{aligned}$$

where $t_{n,m}(q_n,q_m;x;y) := \frac{C^{\alpha,\beta}_{n,m}(1;q_n,q_m;x,y))+C^{\alpha,\beta}_{n,m}\left(t_1^2;q_n,q_m;x,y)\right)+C^{\alpha,\beta}_{n,m}\left(t_2^2;q_n,q_m;x,y)\right)}{1+x^2+y^2}$. By Lemma 2.1, it is clear that there exist K independent of n and m such that

$$\sup_{(x,y)\in D_{a_n,b_m}/D_{A,B}} t_{n,m}(q_n,q_m;x;y) \le K.$$

Therefore, for $n > n_0$ and $m > m_0$ we have

$$z''_{n,m} < (1+K)\varepsilon.$$

This completes the proof. □

4. Order of convergence

In this section, we compute the rate of convergence of the operators in terms of the the full modulus of continuity and partial modulus of continuities.

Let $f \in D_{A,B}$ and $x \ge 0$. Then the definition of the modulus of continuity of f is given by

$$\omega(f;\delta) = \max_{\substack{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\le\delta \\ x,y\in C(D_{A,B})}} |f(x_1,y_1)-f(x_2,y_2)|. \tag{4.1}$$

It is known that for any $\delta > 0$ we know that

$$|f(x_1,y_1)-f(x_2,y_2)| \le \omega(f,\delta)\left(\frac{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{\delta}+1\right) \tag{4.2}$$

and its partial modulus of continuies are defined by

$$\begin{aligned}
\omega^{(1)}(f;\delta) &= \max_{0\le y\le A}\max_{|x_1-x_2|\le\delta}|f(x_1,y)-f(x_2,y)| \\
\omega^{(2)}(f;\delta) &= \max_{0\le x\le B}\max_{|y_1-y_2|\le\delta}|f(x,y_1)-f(x,y_2)|.
\end{aligned}$$

Also, for any $\delta > 0$ we have

$$\begin{aligned}
|f(x_1,y_1)-f(x_2,y_2)| &\le \omega^{(1)}(f,\delta)\left(\frac{|x_1-x_2|}{\delta}+1\right), \\
|f(x_1,y_1)-f(x_2,y_2)| &\le \omega^{(2)}(f,\delta)\left(\frac{|y_1-y_2|}{\delta}+1\right).
\end{aligned}$$

Theorem 4.1. *For any $f \in C(D_{A,B})$, the following inequalities*

$$\left|C^{(\alpha,\beta)}_{n,m,p}(f;q_n,q_m;x,y)-f(x,y)\right| \le 2\left[\omega^{(1)}(f;\delta_m)+\omega^{(2)}(f;\delta_n)\right] \tag{4.3}$$

$$\left|C^{(\alpha,\beta)}_{n,m,p}(f;q_n,q_m;x,y)-f(x,y)\right| \le 2\omega\left(f;\sqrt{\delta_m^2+\delta_n^2}\right) \tag{4.4}$$

are satisfied where

(4.5)

$$\delta_n^2 := \frac{1}{\left([n]_{q_n}+\beta\right)^2}$$
$$\times\left\{\left|[n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right|A^2+(2\alpha+1)[n+p]_{q_n}a_nA+\alpha^2a_n^2\right\}$$

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and

(4.6)
$$\delta_m^2 := \frac{1}{\left([m]_{q_m}+\beta\right)^2} \times \left\{ \left| [m+p+1]_{q_m} [m+p]_{q_m} q_m - \left([m]_{q_m}+\beta\right)^2 \right| B^2 + (2\alpha+1)[m+p]_{q_m} b_m B + \alpha^2 b_m^2 \right\}.$$

Proof. We directly have,

$$\begin{aligned}
& C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y) - f(x,y) \\
&= \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left[f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) - f(x,y)\right] \\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&= \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left[f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) - f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y)\right. \\
&\left. + f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y) - f(x,y)\right]\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}$$

By linearity and positivity of the operators, we get

$$\begin{aligned}
& \left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y) - f(x,y)\right| \\
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) - f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y)\right| \\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&+ \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y) - f(x,y)\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m - y\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&+ \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(1)}\left(f;\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n - x\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&= \Omega_1(x,y) + \Omega_2(x,y).
\end{aligned}$$

Using Lemma 1.1 and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\Omega_1(x,y)\\
=&\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
=&\sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
\leq&\ \omega^{(2)}(f;\delta_m)\left\{1+\frac{1}{\delta_m}\left[\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\right]^{1/2}\right\}.
\end{aligned}$$

Finally, using Lemma 2.1, we get

$$\Omega_1(x,y)\leq 2\omega^{(2)}(f;\delta_m) \tag{4.7}$$

where we choose δ_m as in (4.6).
In the same way, we obtain

$$\Omega_2(x,y)\leq 2\omega^{(1)}(f;\delta_n) \tag{4.8}$$

where δ_n is given in (4.5). Combining (4.7) and (4.8), we obtain (4.3) .
Now, by using linearity and the monotonicity of the operators, and taking into account (4.1), we have

$$\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|\\
&\leq\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega\left(f;\sqrt{\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&\leq\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m)-f(x,y)\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&\leq 1+\frac{1}{\delta}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega(f;\sqrt{\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2})
\end{aligned}$$

(4.9)

$$\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)$$

Using (4.2) and the Cauchy-Schwartz inequality, we get (4.4). □

Theorem 4.2. *Let $f(x,y)$ have continuous partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$, let $\omega^1(f_x;.)$ and $\omega^2(f_y;.)$ denote the partial moduli of $\partial f/\partial x$ and $\partial f/\partial y$, respectively*

on $D_{A,B}$. Then the inequality

$$\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right| \\
&\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)+2\left[\delta_n\omega^{(1)}\left(\frac{\partial f}{\partial x};\delta_n\right)\right] \\
&+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)+2\left[\delta_m\omega^{(2)}\left(\frac{\partial f}{\partial y};\delta_m\right)\right].
\end{aligned}$$

where δ_n and δ_m are the same as in Theorem 4.1 and $\left|\frac{\partial f}{\partial x}\right|\leq N$, $\left|\frac{\partial f}{\partial y}\right|\leq M$ on $D_{A,B}$.

Proof. By the mean value theorem, we can write

$$\begin{aligned}
&f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)-f(x,y) \\
&=f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y\right)-f(x,y)+f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]+\alpha}{[m]+\beta}b_m\right) \\
&-f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y\right)
\end{aligned}$$

$$\begin{aligned}
&=\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\frac{\partial f(x,y)}{\partial x}+\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\left[\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right] \\
&+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\frac{\partial f(x,y)}{\partial y}+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right) \\
&\times\left[\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right]
\end{aligned} \tag{4.10}$$

for any fixed $y\in[0,B]$ and $x\in[0,A]$, where

$$x<\psi_1<\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n$$

and

$$y<\psi_2<\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m.$$

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Applying the operator $C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)$ to (4.10)

$$
\begin{aligned}
&C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\\
&=\frac{\partial f(x,y)}{\partial x}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\left[\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right]\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\frac{\partial f(x,y)}{\partial y}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\left[\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right]\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}
$$

Hence, taking $\left|\frac{\partial f}{\partial x}\right|\leq N$ and $\left|\frac{\partial f}{\partial y}\right|\leq M$, we get

$$
\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|\\
&\leq\left|\frac{\partial f(x,y)}{\partial x}\right|\left|C_{n,m,p}^{(\alpha,\beta)}(t_1-x;q_n,q_m;x,y)\right|\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|\left|\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right|
\end{aligned}
$$

$$
\begin{aligned}
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\left|\frac{\partial f(x,y)}{\partial y}\right|\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\left|\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right|\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq N\left|C_{n,m,p}^{(\alpha,\beta)}\left(t_1-x;q_n,q_m;x,y\right)\right| \\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|\left|\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right| \\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&+M\left|C_{n,m,p}^{(\alpha,\beta)}\left(t_2-x;q_n,q_m;x,y\right)\right| \\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\left|\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right| \\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}
$$

Then using the properties of partial modulus of continuities, we have

$$
\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right| \\
&\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right) \\
&+\omega^{(1)}\left(f_x;\delta_n\right)\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|\left(\frac{\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|}{\delta_n}+1\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right) \\
&+\omega^{(2)}\left(f_y;\delta_m\right)\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\left(\frac{\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|}{\delta_n}+1\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}
$$

since

$$
|\psi_1-x|\leq\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|,\quad|\psi_2-y|\leq\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|.
$$

Applying the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right| \\
&\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right) \\
&+\omega^{(1)}\left(f_x;\delta_n\right)\left(\sum_{k=0}^{n+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\right)^{1/2} \\
&+\frac{\omega^{(1)}(f_x;\delta_n)}{\delta_n}\sum_{k=0}^{n+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)
\end{aligned}
$$

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$$+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)$$
$$+\omega^{(2)}(f_y;\delta_m)\left(\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\right)^{1/2}$$
$$+\frac{\omega^{(2)}(f_y;\delta_m)}{\delta_m}\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).$$

Therefore

$$\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)$$
$$+\omega^{(1)}(f_x;\delta_n)\left(\left(\sqrt{C_{n,m,p}^{(\alpha,\beta)}\left((t_1-x)^2;q_n,q_m;x,y\right)}\right)\right.$$
$$\left.+\frac{\omega^{(1)}(f_x;\delta_n)}{\delta_n}\left(C_{n,m,p}^{(\alpha,\beta)}\left((t_1-x)^2;q_n,q_m;x,y\right)\right)\right)$$
$$+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)$$
$$+\omega^{(2)}(f_y;\delta_m)\sqrt{C_{n,m,p}^{(\alpha,\beta)}\left((t_2-y)^2;q_n,q_m;x,y\right)}$$
$$+\frac{\omega^{(2)}(f_y;\delta_m)}{\delta_m}C_{n,m,p}^{(\alpha,\beta)}\left(C_{n,m,p}^{(\alpha,\beta)}\left((t_2-y)^2;q_n,q_m;x,y\right)\right).$$

Now using Lemma 2.1 and choosing δ_n and δ_m as in (4.5) and (4.6), respectively, we get

$$\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|$$
$$\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)+2\left[\delta_n\omega^{(1)}\left(\frac{\partial f}{\partial x};\delta_n\right)\right]$$
$$+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)+2\left[\delta_m\omega^{(2)}\left(\frac{\partial f}{\partial y};\delta_m\right)\right].$$

Whence the result. □

5. Generalization of the Two Dimensional of Chlodowsky Variant of q-Bernstein-Schurer-Stancu Operators

In this section, we introduce generalization of Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. The generalized operators help us to approximate continuous functions defined on more general weighted spaces. Note that this kind of generalization was considered earlier for the Chlodowsky-Bernstein polynomials [5]. For $x\geq 0$, consider any continuous function $\omega(x,y)\geq 1$ and define

$$G_f(t,s)=f(t,s)\frac{1+t^2+s^2}{w(t,s)}.$$

Let us consider the generalization of the $C_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)$ as follows

(5.1)
$$L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)=\begin{cases}\begin{array}{c}\frac{w(x,y)}{1+x^2+y^2}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}G_f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)\\ \times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\end{array}, & (x,y)\in D_{a_n,b_n}\\ f(x,y), & \mathbb{R}_+^2\backslash D_{a_n,b_n}\end{cases}$$

where $(x,y)\in D_{a_nb_m}$ and $\{a_n\}$ and $\{b_m\}$ have the same properties of two dimensional of Chlodowsky variant of q-Bernstein-Schurer-Stancu operators.

Theorem 5.1. *For all continuous functions f satisfying $|f(x,y)|\leq M_fw(x,y)$, $x,y\geq 0$, and $\lim\limits_{x,y\to\infty}\frac{f(x,y)}{w(x,y)}=0$, we have*

$$\lim_{n,m\to\infty}\left\|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;\cdot,\cdot)-f(\cdot,\cdot)\right\|_w=0$$

where $\rho(x,y)=1+x^2+y^2$.

Proof. Clearly,

$$\begin{aligned}&\left|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)-f(x,y)\right|\\ =\ &\frac{w(x,y)}{1+x^2+y^2}\left|\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}G_f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)\right.\\ &\left.\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)-G_f(x,y)\right|,\end{aligned}$$

thus

$$\begin{aligned}&\left\|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;\cdot,\cdot)-f(\cdot,\cdot)\right\|_w\\ =\ &\sup_{x,y\in\mathbb{R}_+^2}\frac{\left|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)-f(x,y)\right|}{w(x,y)}=\sup_{x,y\in\mathbb{R}_+^2}\frac{\left|T_{n,p}(G_f;q_n,q_m;x,y)-G_f(x,y)\right|}{1+x^2+y^2}.\end{aligned}$$

Since $|f(x,y)|\leq M_fw(x,y)$, then $|G_f(x,y)|\leq M_f\rho(x,y)$ for $x,y\geq 0$ and $G_f(x,y)$ is continuous function on R_+^2. Furthermore, from $\lim\limits_{x,y\to\infty}\frac{f(x,y)}{w(x,y)}=0$, we have

$$\lim_{x,y\to\infty}\frac{G_f(x,y)}{\rho(x,y)}=0.$$

Thus, from Theorem 3.4 we get the result. □

Finally, note that, taking $w(x,y)=1+x^2+y^2$, then the operators $L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)$ reduces $T_{n,p}^{\alpha,\beta}(G_f;q_n,q_m;x,y)$.

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Global stability in stochastic difference equations for predator-prey models

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Abstract

There are many publications on theoretical analysis of deterministic difference equations and stochastic differential equations. However, relatively few theoretical papers are published to consider the positivity of solutions of discrete-time stochastic difference equations (DSDEs), and no theoretical papers investigate the global stability of nontrivial solutions of DSDEs with nonlinear terms. In this paper, we consider a DSDE model that is a generalization of two-dimensional nonlinear models of stochastic predator-prey interactions, and show the positivity and global stability of the nontrivial solutions by using our new discretized version of the Itô formula. In addition, our results are compared with those of continuous-time stochastic differential equations and discrete-time deterministic difference equations. Numerical simulations are introduced to support the results.

Key words: Discrete-time stochastic difference equations, Positivity, Global stability.

1. Introduction

Many predator-prey models have been studied to describe the dynamics of biological systems in which two species interact, one as a predator and the other as a prey. A classic predator-prey model is given by

$$\frac{dx}{dt} = x(r_1 - a_{11}x - a_{12}y), \ \frac{dy}{dt} = y(r_2 + a_{21}x - a_{22}y), \tag{1}$$

where $x(t)$ and $y(t)$ denote the population density of the prey and predator at time t, respectively. In the model (1), r_1 is the intrinsic growth rate of the prey in the absence of the predator, $-r_2$ is the death rate of the predator in the absence of the prey, the coefficients $a_{ij}(i \neq j)$ give the strength of the interaction between the two species, and $a_{ii}(i = 1, 2)$ measure the inhibiting effect of environment on the two species.

In the model (1), the predator consumes the prey with functional response of type $a_{12}x(t)y(t)$. However the rate of prey capture is saturated when the population of the prey is relatively large. Such phenomena are described by nonlinear functions including Holling types [1–5], Beddington-DeAngelis type [6–8], Crowley-Martin type [9–11], and

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Ivlev-type of functional responses [12–14]. Other types of nonlinear functions have been applied to express the Allee effect [15–19], which describes a positive relation between the population density and the per capita growth rate of a species. There have been also models to take into account of diffusion of species ([15] and [20–22]).

On the other hand, the population is inevitably affected by environmental noise in nature, so that the reproduction rates can change randomly. In order to be more realistic, stochastic models should be considered. Stochastic differential equation (SDE) models have been increasingly used in a range of application areas, including biology, chemistry, mechanics, economics, and finance. The SDE models have been studied to understand extinction, stochastic permanence and stationary distributions of the stochastic systems. In particular, many authors have taken stochastic perturbation into deterministic predator prey models with Beddington-DeAngelis and Holling types of functional responses [23–33]. For example, putting noise into the deterministic model (1) gives the SDE model

$$\begin{aligned} dx(t) &= x(t)\{r_1 - a_{11}x(t) - a_{12}y(t)\}dt + \sigma_1 x(t)dW_1(t), \\ dy(t) &= y(t)\{r_2 + a_{21}x(t) - a_{22}y(t)\}dt + \sigma_2 y(t)dW_2(t), \end{aligned} \tag{2}$$

which is a special model studied in [25] with zero-time delays. Here the positive coefficients σ_1 and σ_2 measure the intensity of environmental perturbations on the underlying growth rate of the prey and the death rate of the predator, respectively. The processes W_i are independent and real valued Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In general, the exact solutions of SDEs are not known, so one has to numerically solve these SDEs. This leads us to consider and analyze discrete-time stochastic difference equations (DSDEs), which can be also viewed as stochastically perturbed versions of deterministic difference equations (DDEs) (see [34], [35] and references therein). There are many publications on estimations of the difference between solutions of SDEs and DSDEs. The global asymptotic stability of the trivial solution of DSDEs has been also widely addressed (see [36], [37], [38] and references therein). However, relatively few theoretical studies consider the positivity of solutions of DSDEs that are scalar equations on a finite time interval (see [39] references therein). In particular, to the best of our knowledge, there is no paper that theoretically deals with the global stability of nontrivial solutions of DSDEs. Therefore, to investigate the positivity and global stability, we consider the DSDE model for (2)

$$x_{k+1}^i = x_k^i \left\{ 1 + h\left(r_i + \sum\nolimits_{j=1}^{i-1} a_{ij}x_k^j - \sum\nolimits_{j=i}^{2} a_{ij}x_k^j \right) + h^{0.5}\sigma_i \xi_{k+1}^i \right\}, \tag{3}$$

where $1 \le i \le 2$, $k \ge 0$, $x_0^i > 0$ and $0 < h < 1$. Although $r_1 > 0$, $r_2 < 0$ and $a_{ij} > 0$ in the SDE model (2) and the DDE model (3) with $\sigma_i = 0$ (see [34] and [35]), we weaken the conditions on the parameters and use the following conditions in the DSDE model (3): for $1 \le i, j \le 2$ and $i \ne j$

$$r_i \in \mathbb{R}, a_{ii} > 0, a_{ij} \ge 0, \sigma_i > 0. \tag{4}$$

The discrete Wiener processes $W_i(t_{k+1}) - W_i(t_k)$ are $h^{0.5}\xi_{k+1}^i$ with a mutually independent and identically distributed sequence $(\xi_k^1, \xi_k^2)_{k=1}^\infty$ of the standard normal random variables. The solutions of (3) are defined with respect to a complete, filtered probability space $(\Omega_h, \mathcal{F}_h, \{\mathcal{F}_k\}_{k=1}^\infty, \mathbb{P}_h)$, where $\{\mathcal{F}_k\}_{k=1}^\infty$ is the natural filtration generated by the stochastic sequence $(\xi_k^1, \xi_k^2)_{k=1}^\infty$, i.e., $\mathcal{F}_k = \sigma(\xi_1^1, \xi_1^2, \cdots, \xi_k^1, \xi_k^2)$ for $k \ge 1$. Therefore $(x_k^1, x_k^2)_{k=1}^\infty$ is

adapted to the filtration for any initial vector (x_0^1, x_0^2), which is supposed to be non-random.

The positivity of solutions of the SDEs (2) is obtained in the infinite time interval $[0, \infty)$ without boundedness of the noises $W_i(t)$ by using the concept of explosion time (see [25] and [40]). However, to the best of our knowledge, there is no method for applying the concept of explosion time to DSDEs. Then for obtaining the positivity of solutions of the DSDE model (3) in the infinite time interval, we restrict the noises to bounded noises, which means that $\xi_k^i (1 \le i \le 2, k \ge 1)$ are assumed to be doubly truncated standard normal random variables with support $[-\varsigma, \varsigma]$ for a positive constant ς

$$-\varsigma \le \xi_k^i \le \varsigma \tag{5}$$

and the probability density function

$$\psi(x) = \begin{cases} q(x) \left\{\Phi(\varsigma) - \Phi(-\varsigma)\right\}^{-1} & \text{if } x \in [-\varsigma, \varsigma], \\ 0 & \text{if } x \notin [-\varsigma, \varsigma], \end{cases} \tag{6}$$

where q and Φ are the probability density and cumulative distribution functions of the standard normal random variable, respectively. Denoting $\eta_\varsigma = 2\varsigma q(\varsigma) \left\{\Phi(\varsigma) - \Phi(-\varsigma)\right\}^{-1}$ gives that for $1 \le i \le 2$ and $k \ge 1$

$$E(\xi_k^i) = 0, \; E\left((\xi_k^i)^2\right) = 1 - \eta_\varsigma, \tag{7}$$

in which the positive value η_ς can be assumed to be sufficiently close to 0. For example, when $\varsigma = 20$, we have $0 < \eta_\varsigma < 10^{-85}$. The truncation constant ς will be first used in (12) for the positivity of the solutions x_k^i of the DSDE model (3).

The paper is organized as follows. Section 2 gives the positivity and boundedness of solutions of the model (3). In Section 3, we develop a new discrete Itô formula for (3) by using a known discrete Itô formula for DSDEs (see [41], [42] and [43]). The new discrete Itô formula is the main tool for finding conditions for the global stability of solutions of (3). Section 4 introduces auxiliary equations, the solutions of which are used for the upper bounds of solutions of (3). In Section 5, we present sufficient conditions for extinction and non-extinction of solutions of (3). Our results are compared with those for the DDEs in [35] and the SDEs in [25]. Section 6 gives simulation results to confirm the theoretical analysis obtained in this paper.

2. Positivity and boundedness of solutions of DSDEs

In this section, we show the positivity and boundedness of solutions of the DSDE model (3) by applying the approach used in the DDE model (3) with $\sigma_1 = \sigma_2 = 0$ (see [34] and [35]).

Notation 1. For simplicity, we use the symbols $\tilde{a}$ and $\hat{a}$ for every constant a to denote

$$\tilde{a} = a \cdot h^{0.5}, \; \hat{a} = a \cdot h$$

and the symbols $\varkappa_k^1$ and $\varkappa_k^2$ for a vector $\varkappa_k = (x_k^1, x_k^2)$ to denote

$$\varkappa_k^1 = x_k^2, \; \varkappa_k^2 = x_k^1.$$

3

Write the model (3) as

$$x_{k+1}^i = F_{k,\varkappa_k^i}^i(x_k^i),$$

where

$$\begin{aligned} F_{k,y}^1(x) = &\ x\left(1+\hat{r}_1-\hat{a}_{11}x-\hat{a}_{12}y+\tilde{\sigma}_1\xi_{k+1}^1\right), \\ F_{k,x}^2(y) = &\ y\left(1+\hat{r}_2+\hat{a}_{21}x-\hat{a}_{22}y+\tilde{\sigma}_2\xi_{k+1}^2\right). \end{aligned} \tag{8}$$

Note that for a vector $\boldsymbol{\zeta}_k = (\zeta_k^1, \zeta_k^2)$ of real numbers ζ_k^1 and ζ_k^2,

$$F_{k,\zeta_k^i}^i(\tau) \text{ is strictly increasing on } 0 \le \tau < V_k^i(\boldsymbol{\zeta}_k), \tag{9}$$

in which

$$V_k^i(\boldsymbol{\zeta}_k) = (2\hat{a}_{ii})^{-1}\left(1+\hat{r}_i+\sum\nolimits_{j=1}^{i-1}\hat{a}_{ij}\zeta_k^j-\sum\nolimits_{j=i+1}^{2}\hat{a}_{ij}\zeta_k^j+\tilde{\sigma}_i\xi_{k+1}^i\right). \tag{10}$$

Denote that for $1 \le i \le 2$

$$\chi_i = \hat{a}_{ii}^{-1}\left(\hat{r}_i+\sum\nolimits_{j=1}^{i-1}\hat{a}_{ij}\chi_j+\tilde{\sigma}_i\varsigma_*\right), \tag{11}$$

where ς_* is a constant satisfying

$$\varsigma_* > \varsigma, \tag{12}$$

$$\chi_i \le (2\hat{a}_{ii})^{-1}\left(1+\hat{r}_i-\sum\nolimits_{j=i+1}^{2}\hat{a}_{ij}\chi_j-\tilde{\sigma}_i\varsigma_*\right), \tag{13}$$

$$\hat{r}_i+\sum\nolimits_{j=1}^{i-1}\hat{a}_{ij}\chi_j+\tilde{\sigma}_i\varsigma_* < 1. \tag{14}$$

The relation (12) will be first used in (69) to find upper solutions of the model (3). The initial condition of the model (3) is assumed to satisfy

$$(x_0^1, x_0^2) \in (0,\chi_1)\times(0,\chi_2). \tag{15}$$

Remark 1. The definition (11) gives that $\chi_1 = \frac{\hat{r}_1+\tilde{\sigma}_1\varsigma_*}{\hat{a}_{11}}$ and $\chi_2 = \hat{a}_{22}^{-1}\left(\hat{r}_2+\hat{a}_{21}\chi_1+\tilde{\sigma}_2\varsigma_*\right)$. Letting h in (3) be small, we can choose ς_* satisfying the two conditions (13) and (14). For example, let $h = 0.0001$, $\varsigma_* = 20$, $r_1 = 2$, $r_2 = a_{ij} = 1$ and $\sigma_i = 0.1$ $(1 \le i, j \le 2)$. Denoting by R_i and L_i the right and left-hand sides of (13) and (14), respectively, gives

$$(\chi_1, R_1, L_1) = (202, 4699.5, 0.3848), (\chi_2, R_2, L_2) = (403, 4900.5, 0.3518),$$

which show that the conditions (13) and (14) are satisfied.

Theorem 1. *Let x_k^i be the solutions of (3) and χ_i be defined in (11). Assume that (5), (12), (13), (14) and (15) hold. Then*

$$(x_k^1, x_k^2) \in (0,\chi_1)\times(0,\chi_2),\ \ k \ge 0.$$

Proof. The proof is divided into the following three steps.
Step 1. We prove the positivity: $x_1^i > 0$ for $1 \le i \le 2$.
Note that for $\varkappa_0 = (x_0^1, x_0^2)$

$$0 < x_0^i < \chi_i \le (2\hat{a}_{ii})^{-1}\left(1 + \hat{r}_i - \sum_{j=i+1}^{2} \hat{a}_{ij}\chi_j - \tilde{\sigma}_i\varsigma_*\right) < V_0^i(\varkappa_0),$$

where the first two inequalities are obtained from (15), the third from (13) and the last from (10), (15), (5) and (12). Then using (9) with $\boldsymbol{\zeta}_0 = \varkappa_0$ and (15), we have the positivity

$$x_1^i = F^i_{0,\varkappa_0^i}(x_0^i) > F^i_{0,\varkappa_0^i}(0) = 0.$$

Step 2. We prove the upper-bound property: $x_1^i < \chi_i$ for $1 \le i \le 2$.
Let $\omega \in \Omega_h$. If $\hat{r}_i + \sum_{j=1}^{i-1} \hat{a}_{ij}x_0^j - \sum_{j=i+1}^{2} \hat{a}_{ij}x_0^j + \tilde{\sigma}_i\xi_1^i(\omega) \le 0$, then

$$x_1^i(\omega) = F^i_{0,\varkappa_0^i}(x_0^i)(\omega) \le x_0^i < \chi_i.$$

Otherwise, we have $0 < x_0^i < f_{0,i}(\varkappa_0^i)(\omega)$ with

$$f_{0,i}(\varkappa_0^i) = \hat{a}_{ii}^{-1}\left(\hat{r}_i + \sum_{j=1}^{i-1} \hat{a}_{ij}x_0^j - \sum_{j=i+1}^{2} \hat{a}_{ij}x_0^j + \tilde{\sigma}_i\xi_1^i\right).$$

Since $0 < f_{0,i}(\varkappa_0^i) < V_0^i(\varkappa_0)$ by (14), we get

$$0 < x_0^i < f_{0,i}(\varkappa_0^i)(\omega) < V_0^i(\varkappa_0)(\omega)$$

and further

$$x_1^i(\omega) = F^i_{0,\varkappa_0^i}(x_0^i)(\omega) < F^i_{0,\varkappa_0^i}(f_{0,i}(\varkappa_0^i))(\omega) = f_{0,i}(\varkappa_0^i)(\omega) < \chi_i,$$

where the first inequality is obtained from (9) with $\boldsymbol{\zeta}_0 = \varkappa_0$ and the last inequality from (11) and (15).
Step 3. We prove the boundedness: $(x_k^1, x_k^2) \in (0,\chi_1) \times (0,\chi_2)$ for $k \ge 0$.
Since Step1 and 2 give that

$$\text{if } (x_0^1, x_0^2) \in (0,\chi_1) \times (0,\chi_2), \text{ then } (x_1^1, x_1^2) \in (0,\chi_1) \times (0,\chi_2),$$

we can obtain the desired result by both applying mathematical induction and replacing $\left(x_0, \xi_1^i, \varkappa_0, \boldsymbol{\zeta}_0, V_0^i, F^i_{0,\varkappa_0^i}, f_{0,i}\right)$ with $\left(x_k, \xi_{k+1}^i, \varkappa_k, \boldsymbol{\zeta}_k, V_k^i, F^i_{k,\varkappa_k^i}, f_{k,i}\right)$ in Step 1 and 2. Here the function $f_{k,i}$ is defined as $f_{k,i}(\varkappa_k^i) = \hat{a}_{ii}^{-1}\left(\hat{r}_i + \sum_{j=1}^{i-1} \hat{a}_{ij}x_k^j - \sum_{j=i+1}^{2} \hat{a}_{ij}x_k^j + \tilde{\sigma}_i\xi_{k+1}^i\right)$. □

Remark 2. For simplicity, from now on we assume that the conditions (5), (12), (13), (14) and (15) used in Theorem 1 hold. Then we will not write the conditions explicitly in later sections when we need the positivity and boundedness of the solutions x_k^i.

3. A new discretized version of the Itô formula

In order to find conditions for the stability of (3), we need a discretized form of the Itô formula. Although there are discretized versions of the Itô formula (see [41], [42] and [43]), we need to formulate a variant which is suitable for our model (3). The proof of our new discrete Itô formula is almost the same as that of the discrete Itô formula in [42] and [43]. For the completeness of this paper, we reproduce the proof in the Appendix.

We write $q_1(h) = O(q_2(h))$ (or $q_1(h) = O(q_2(h))$ for $h \to 0$ to be more precise) if there exist positive constants C and h_0 such that $|q_1(h)| \le C|q_2(h)|$ for all h with $0 < h \le h_0$.

We make the two assumptions about the noise ξ: First, the noise ξ satisfies that for some constants M_1 and μ with $0 < \mu < 1$

$$E(\xi) = 0, \; E\left(\xi^2\right) = 1 - \mu, \; E\left(|\xi|^\ell\right) \leq M_1 \; (\ell = 1, 3). \tag{16}$$

Second, the probability density function p of the noise ξ exists with the property that for some constant M_2 and all sufficiently large $|x|$

$$|x|^3 p(x) \leq M_2 |x|^{-1}. \tag{17}$$

Using $\mu = \eta_\varsigma$ in (7) and the probability density function $p(x) = \psi(x)$ in (6), one can obtain that the truncated standard normal random variables ξ_k^i satisfy the two assumptions (16) and (17). Let the symbol $\mathbb{R}$ denote the set of all real numbers and $C^3(\mathbb{R})$ denote the set of all functions defined on $\mathbb{R}$ that are continuously differentiable up to the order 3.

Lemma 1. *Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}_h$. Consider functions $\phi, \; \varphi : \mathbb{R} \to \mathbb{R}$ satisfying that for some $\delta > 0$,*

(i) *$\varphi = \phi$ on $[1 - \delta, 1 + \delta]$*
(ii) *$\varphi \in C^3(\mathbb{R})$ and $|\varphi'''(x)| \leq M_3$ for some constant M_3 and all $x \in \mathbb{R}$*
(iii) *$\int_{\mathbb{R}} |\varphi(x) - \phi(x)| dx < M_4$ for some constant M_4*

and ϕ is almost everywhere continuous. Let f and g be $\mathcal{G}$-measurable random variables satisfying that for some positive constants ε and M_5,

$$\max\{h|f|, h^{0.5}|g|\} \leq M_5 h^\varepsilon. \tag{18}$$

Let ξ be a $\mathcal{G}$-independent random variable satisfying (16) and (17). Then the conditional expectation of the random variable $\phi\left(1 + hf + h^{0.5} g\xi\right)$ with respect to the σ-algebra $\mathcal{G}$ becomes

$$\begin{aligned} &E\left[\phi\left(1 + hf + h^{0.5} g\xi\right) | \mathcal{G}\right] \\ &\quad = \phi(1) + \phi'(1) hf + 2^{-1} \phi''(1) hg^2 \cdot (1 - \mu) + hfO\left(h^\varepsilon\right) + hg^2 O\left(h^\varepsilon\right), \end{aligned}$$

where the first big O denotes

$$2^{-1} \varphi''(1) M_5 h^\varepsilon + 6^{-1} M_3 \left(M_5 h^\varepsilon\right)^2 \{1 + 3(1 - \mu)\}$$

and the last denotes

$$\left(M_1 M_5 + M_4 M_2 M_5 \delta_1\right) h^\varepsilon$$

for some positive constant δ_1 less than δ. Here M_1 and M_2 are defined in (16) and (17).

Proof. See the Appendix. □

Remark 3. Differently from the discretized Itô formulas in [43], [41] and [42], our discretized Itô formula in Lemma 1 does not require that the upper bounds of f and g are independent of h. Let $\mathcal{G} = \mathcal{F}_k$ and

$$f = r_i + \sum_{j=1}^{i-1} a_{ij} x_k^j - \sum_{j=i}^{2} a_{ij} x_k^j, \; g = \sigma_i, \; \xi = \xi_{k+1}^i \tag{19}$$

for the solutions x_k^i of (3) with $1 \leq i \leq 2$. Then f and g are $\mathcal{F}_k$-measurable and satisfy (18) with $\varepsilon = 0.5$ by applying the upper bound $\chi_i = O(h^{-0.5})$ of x_k^i to the definition of f. In addition, $\xi = \xi_{k+1}^i$ is an $\mathcal{F}_k$-independent random variable satisfying (16) and (17).

Remark 4. In order to construct φ in Lemma 1 corresponding to the function

$$\phi(x) = \begin{cases} \ln|x| & (|x| > 0), \\ 0 & (x = 0), \end{cases}$$

we modify the function φ used in [37]. Define the function φ as follows.

$$\varphi(x) = \begin{cases} \ln|x| & (|x| \geq e^{-1}), \\ -4^{-1}e^4x^4 + e^2x^2 - 4^{-1}7 + 6^{-1}e^6(x - e^{-1})^3(x + e^{-1})^3 & (|x| \leq e^{-1}). \end{cases}$$

Then ϕ and φ satisfy all the conditions in Lemma 1 with $\delta = 1 - e^{-1}$.

Notation 2. For simplicity, we use the notations

$$\overline{E}(x_k^i) = k^{-1}\sum_{s=0}^{k-1} E\left(x_s^i\right) \tag{20}$$

and

$$\mathring{a} = a \cdot \left\{1 + O(h^{0.5})\right\}, \ a_\eta = a \cdot (1 - \eta_\varsigma), \ r_{i\sigma} = r_i - 0.5\sigma_{i\eta}^2$$

for $k > 0$, $1 \leq i \leq 2$, constants a and η_ς in (7). Here $\sigma_{i\eta}^2$ is equal to $\{\sigma_i \cdot (1 - \eta_\varsigma)\}^2$.

Remark 5. Since the solutions x_k^i of (3) are positive by Theorem 1, we can take logarithm of (3), which gives

$$E\left[\ln x_{k+1}^i \middle| \mathcal{F}_k\right] = E\left[\ln x_k^i \middle| \mathcal{F}_k\right] + E\left[\ln\left(1 + hf + h^{0.5}g\xi_{k+1}^i\right) \middle| \mathcal{F}_k\right], \tag{21}$$

where f and g are defined in (19). In order to simplify the equation (21), applying $\mathcal{F}_k$-independence of ξ_{k+1}, $\mathcal{F}_k$-measurability of x_k^i and Lemma 1 with Remarks 3 and 4 to the three expectation terms in (21), respectively, we have

$$\begin{aligned} E(\ln x_{k+1}^i) &= \ln x_k^i + hf - \frac{1}{2}hg^2 \cdot (1 - \eta_\varsigma) + hfO\left(h^{0.5}\right) + hg^2O\left(h^{0.5}\right) \\ &= \ln x_k^i + \mathring{h}\left(r_i - \frac{1}{2}\sigma_{i\eta}^2 + \sum_{j=1}^{i-1} a_{ij}x_k^j - \sum_{j=i}^{2} a_{ij}x_k^j\right). \end{aligned} \tag{22}$$

Taking expectation of (22) and adding the result, we obtain

$$E(\ln x_k^i) = E(\ln x_0^i) + k\mathring{h}\left\{r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\overline{E}(x_k^j) - \sum_{j=i}^{2} a_{ij}\overline{E}(x_k^j)\right\}. \tag{23}$$

4. Auxiliary equations

In order to find upper bounds of x_k^i, we consider the auxiliary equations

$$z_{k+1}^i = z_k^i\left(1 + \hat{r}_i + \sum_{j=1}^{i-1} \hat{a}_{ij}z_k^j - \hat{a}_{ii}z_k^i + \tilde{\sigma}_i\xi_{k+1}^i\right), \ z_0^i = x_0^i \tag{24}$$

for $1 \leq i \leq 2$ and $k \geq 0$. Since (24) is the system (3) with $a_{12} = 0$, Theorem 1 with (4) gives that for $k \geq 0$

$$(z_k^1, z_k^2) \in (0, \chi_1) \times (0, \chi_2). \tag{25}$$

Let β_i be the solutions of the equations

$$r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j - a_{ii}\beta_i = 0 \tag{26}$$

for $1 \le i \le 2$. Note that (22) and (23) with $a_{12} = 0$ become

$$\begin{aligned} E\left(\ln z_{k+1}^1\right) &= \ln z_k^1 + \mathring{h}\left(r_{1\sigma} - a_{11} z_k^1\right), & (27)\\ E\left(\ln z_k^1\right) &= E\left(\ln z_0^1\right) + k\mathring{h}\left\{r_{1\sigma} - a_{11}\overline{E}\left(z_k^1\right)\right\} \\ &= E\left(\ln z_0^1\right) + k\mathring{h}a_{11}\left\{\beta_1 - k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right)\right\} & (28) \end{aligned}$$

due to (20) and $\beta_1 = a_{11}^{-1} r_{1\sigma}$ in (26). Similarly, we have

$$\begin{aligned} E\left(\ln z_{k+1}^2\right) &= \ln z_k^2 + \mathring{h}\left(r_{2\sigma} + a_{21} z_k^1 - a_{22} z_k^2\right), & (29)\\ E\left(\ln z_k^2\right) &= E\left(\ln z_0^2\right) + k\mathring{h}\left\{r_{2\sigma} + a_{21}\overline{E}(z_k^1) - a_{22}\overline{E}(z_k^2)\right\} \\ &= E\left(\ln z_0^2\right) + k\mathring{h}a_{22}\left\{\frac{r_{2\sigma}}{a_{22}} + \frac{a_{21}}{a_{22}}\overline{E}(z_k^1) - k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right)\right\}. & (30) \end{aligned}$$

Lemma 2. *Let z_k^1 and β_1 be the solutions of (24) and (26), respectively. If $\beta_1 \ge 0$, then for $\epsilon > 0$ and all sufficiently large k*

$$k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) \le \beta_1 + \epsilon.$$

Proof. Suppose, on the contrary, that the theorem is false, which means that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ satisfying both for all k_m

$$k_m^{-1}\sum_{s=0}^{k_m-1} E\left(z_s^1\right) > \beta_1 + \varepsilon_0 \tag{31}$$

and for all k with $k \ne k_m$

$$k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) \le \beta_1 + \varepsilon_0. \tag{32}$$

Combining (31) and (28), we have

$$\lim_{m\to\infty} E\left(\ln z_{k_m}^1\right) = -\infty. \tag{33}$$

Substituting (33) and the boundedness of z_k^1 into (27) gives

$$\lim_{m\to\infty} \ln z_{k_m-1}^1 = -\infty \quad a.s.$$

and then

$$\lim_{m\to\infty} z_{k_m-1}^1 = 0 \quad a.s. \tag{34}$$

Thus the dominated convergence theorem with (25) leads to

$$\lim_{m\to\infty} E(z_{k_m-1}^1) = 0. \tag{35}$$

In order to obtain a contraction we follow the two steps:

Step 1. If there exists $k = k_m - 1$ satisfying (32), then the system of (31) and (32) becomes

$$\begin{aligned} \sum\nolimits_{s=0}^{k_m-1} E\left(z_s^1\right) &> k_m\left(\beta_1+\varepsilon_0\right), \\ \sum\nolimits_{s=0}^{k_m-2} E\left(z_s^1\right) &\leq \left(k_m-1\right)\left(\beta_1+\varepsilon_0\right), \end{aligned}$$

which gives

$$E(z_{k_m-1}^1) > \beta_1 + \varepsilon_0, \tag{36}$$

and hence there exist finitely many k satisfying (32) due to (35) and (36). Therefore for all sufficiently large k

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^1\right) > \beta_1 + \varepsilon_0. \tag{37}$$

Step 2. As (31) implies (35), the equation (37) implies

$$\lim_{k\to\infty} E(z_k^1) = 0,$$

which is contradictory to (37) due to $\beta_1 + \varepsilon_0 > 0$ and so the proof is completed. □

Lemma 3. *Let (z_k^1, z_k^2) and (β_1, β_2) be the solutions of (24) and (26), respectively.*

(a) *Assume $r_{1\sigma} < 0$. Then $\lim_{k\to\infty} z_k^1 = 0$ a.s.*
  (i) *If $r_{1\sigma} < 0$ and $r_{2\sigma} < 0$, then $\lim_{k\to\infty} z_k^2 = 0$ a.s.*
  (ii) *If $r_{1\sigma} < 0$ and $r_{2\sigma} \geq 0$, then $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) = a_{22}^{-1} r_{2\sigma}$.*

(b) *Assume $r_{1\sigma} \geq 0$. Then $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) = \beta_1$.*
  (i) *If $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$, then $\lim_{k\to\infty} z_k^2 = 0$ a.s.*
  (ii) *If $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}\beta_1 \geq 0$, then $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) = \beta_2$.*

Proof. (a) Since $r_{1\sigma} < 0$ is equivalent to $\beta_1 = a_{11}^{-1} r_{1\sigma} < 0$, it follows from (28) and the positivity of z_k^1 in (25) that if $r_{1\sigma} < 0$, then $\lim_{k\to\infty} E\left(\ln z_k^1\right) = -\infty$, and further

$$\lim_{k\to\infty} z_k^1 = 0 \ \ a.s. \tag{38}$$

as (33) implies (34).
(a)-(i) Assume that $r_{1\sigma} < 0$ and $r_{2\sigma} < 0$.
As (34) implies (35), the equation (38) yields $\lim_{m\to\infty} E(z_k^1) = 0$ and then

$$\lim_{k\to\infty} \overline{E}(z_k^1) = 0. \tag{39}$$

Combining (39) and (30) with $r_{2\sigma} < 0$ and using $z_k^2 > 0$, we have from (30) that

$$\lim_{k\to\infty} E\left(\ln z_k^2\right) = -\infty. \tag{40}$$

Therefore, as (33) implies (34), the equation (40) gives

$$\lim_{k\to\infty} z_k^2 = 0 \ \ a.s.$$

(a)-(ii) Assume that $r_{1\sigma} < 0$ and $r_{2\sigma} \geq 0$.
Using $(z_k^2, a_{22}^{-1} r_{2\sigma})$, (29) and (30) instead of (z_k^1, β_1), (27) and (28) in the proof of Lemma 2, respectively, and applying (39) to (30), we can obtain that for $\epsilon > 0$ and all sufficiently large k

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^2\right) \leq a_{22}^{-1} r_{2\sigma} + \epsilon. \tag{41}$$

In order to show $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) = a_{22}^{-1} r_{2\sigma}$, it is enough to prove that for $\epsilon > 0$ and all sufficiently large k

$$a_{22}^{-1} r_{2\sigma} - \epsilon \leq k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right). \tag{42}$$

Suppose that (42) is false, which means that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ satisfying

$$a_{22}^{-1} r_{2\sigma} - \varepsilon_0 > k_m^{-1}\sum_{s=0}^{k_m-1} E\left(z_s^2\right). \tag{43}$$

Then the boundedness of z_k^2 and (30) imply that for all k_m

$$\infty > E\left(\ln z_{k_m}^2\right) > E\left(\ln z_0^2\right) + k_m \mathring{h} a_{22}\varepsilon_0, \tag{44}$$

which is a contradiction. Therefore (42) is true and so the proof is completed due to (41) and (42).
(b) Assume $r_{1\sigma} \geq 0$, which means $\beta_1 = a_{11}^{-1} r_{1\sigma} \geq 0$.
In order to show $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) = \beta_1$, it is enough to prove that for $\epsilon > 0$ and all sufficiently large k

$$\beta_1 - \epsilon \leq k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) \tag{45}$$

due to Lemma 2. Suppose that (45) is false, so that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ such that

$$\beta_1 - \varepsilon_0 > k_m^{-1}\sum_{s=0}^{k_m-1} E\left(z_s^1\right). \tag{46}$$

Then the boundedness of z_k^1 and (28) imply that for all k_m

$$\infty > E\left(\ln z_{k_m}^1\right) > E\left(\ln z_0^1\right) + k_m \mathring{h} a_{11}\varepsilon_0, \tag{47}$$

which is a contradiction. Hence (45) is true and, therefore, Lemma 2 with (45) gives

$$\lim_{k\to\infty} \overline{E}(z_k^1) = \beta_1. \tag{48}$$

(b)-(i) Assume that $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$.
Applying (48) to (30) with both $r_{2\sigma} + a_{21}\beta_1 < 0$ and $z_k^2 > 0$, we have

$$\lim_{k\to\infty} E(\ln z_k^2) = -\infty.$$

Therefore, as (33) implies (34), we can obtain $\lim_{k\to\infty} z_k^2 = 0 \;\; a.s.$
(b)-(ii) Assume that $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}\beta_1 \geq 0$.
Following the proof of Lemma 2, we can obtain that

$$k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) \leq \beta_2 + \epsilon \tag{49}$$

for $\epsilon > 0$ and all sufficiently large k by using (z_k^2, β_2), (29) and (30) instead of (z_k^1, β_1), (27) and (28), respectively, and applying (48) and $\beta_2 = a_{22}^{-1}\left(r_{2\sigma} + a_{21}\beta_1\right) \geq 0$ to (30).
Similarly, following the proof of (45), we can obtain that

$$\beta_2 - \epsilon \leq k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) \tag{50}$$

for $\epsilon > 0$ and all sufficiently large k by replacing (z_k^1, β_1) and (28) with (z_k^2, β_2) and (30), respectively, and applying (48) to (30). Therefore (49) and (50) give the desired result. □

Remark 6. The equations (28) and (30) can be written as

$$E(\ln z_k^i) = E(\ln z_0^i) + k\mathring{h}\left\{r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\overline{E}(z_k^j) - a_{ii}\overline{E}(z_k^i)\right\}. \tag{51}$$

Substituting (26) to (51) yields

$$E(\ln z_k^i) = E(\ln z_0^i) + k\mathring{h}\left[\sum_{j=1}^{i-1} a_{ij}\left\{\overline{E}(z_k^j) - \beta_j\right\} - a_{ii}\left\{\overline{E}(z_k^i) - \beta_i\right\}\right]. \tag{52}$$

Applying Lemma 3-(b) and (b)-(ii) to (52) with the notation (20), we have

$$\lim_{k\to\infty} k^{-1}E(\ln z_k^i) = 0 \tag{53}$$

under the condition that $\min\{r_{1\sigma}, r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j\} \geq 0$ for $1 \leq i \leq 2$.

Lemma 4. *Let x_k^i and z_k^i be the solutions of (3) and (24), respectively for $i = 1, 2$. Then for $k \geq 0$*

$$0 < x_k^i \leq z_k^i.$$

Proof. Theorem 1 with Remark 2 gives

$$0 < x_k^i. \tag{54}$$

Note that

$$F_{k,y}^1(x) \text{ is nonincreasing in } y \text{ for } x \geq 0 \text{ and } k \geq 0 \tag{55}$$

and

$$F_{k,x}^2(y) \text{ is nondecreasing in } x \text{ for } y \geq 0 \text{ and } k \geq 0 \tag{56}$$

by the definition (8). The proof of this lemma is divided into the following two cases.
Case 1. Let $i = 1$.
Using $\varkappa_0^1 = x_0^2 > 0$ and (55), we have

$$x_1^1 = F_{0,\varkappa_0^1}^1(x_0^1) \leq F_{0,0}^1(x_0^1). \tag{57}$$

It follows from Remark 2, (24), (25), (10) and (13) that

$$0 < x_0^1 \leq z_0^1 < \chi_1 < V_0^1(0,0),$$

with which (9) yields

$$F_{0,0}^1(x_0^1) \leq F_{0,0}^1(z_0^1) = z_1^1. \tag{58}$$

Hence combining (54), (57) and (58) gives

$$0 < x_1^1 \leq z_1^1. \tag{59}$$

Assume that for some positive integer k

$$0 < x_k^1 \leq z_k^1. \tag{60}$$

Using (54), (60), (25), (10) and (13), we have

$$\varkappa_k^1 > 0,\ 0 < x_k^1 \leq z_k^1 < \chi_1 < V_k^1(0,0)$$

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and so

$$x^1_{k+1} = F^1_{k,\not{x}^1_k}(x^1_k) \le F^1_{k,0}(x^1_k) \le F^1_{k,0}(z^1_k) = z^1_{k+1},$$

where the first inequality is obtained from (55) and the second inequality from (9).
Case 2. Let $i = 2$.
Using $\not{x}^2_0 = x^1_0 \le z^1_0$ and $0 < x^2_0 \le z^2_0 < \chi_2 < V^2_0(0,0)$, we have

$$x^2_1 = F^2_{0,\not{x}^2_0}(x^2_0) \le F^2_{0,z^1_0}(x^2_0) \le F^2_{0,z^1_0}(z^2_0) = z^2_1 \tag{61}$$

due to (56) and (9). Similarly as in Case 1, using mathematical induction and $z^2_k \le \chi_2 < V^2_k(0,0)$ instead of $z^1_k < \chi_1 < V^1_k(0,0)$ in Case 1, we can obtain the desired result. □

Remark 7. If $\min\{r_{1\sigma}, r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j\} \ge 0$ for $1 \le i \le 2$, then Lemma 4 and (53) imply that for $\epsilon > 0$ and all sufficiently large k

$$k^{-1}E(\ln x^i_k) \le \epsilon, \tag{62}$$

which will be first used in Theorem 4.

5. Extinction and persistence of the discrete solutions

In this section, we present several conditions sufficient for the extinction and persistence (non-extinction) of the solutions x^i_k of (3).

Theorem 2. *Let x^i_k and β_i be the solutions of (3) and (26), respectively for $i = 1, 2$.*

(a) *If $r_{1\sigma} < 0$, then $\lim_{k\to\infty} x^1_k = 0$ a.s.*
(b) *If $r_{1\sigma} < 0$ and $r_{2\sigma} < 0$, then $\lim_{k\to\infty} x^2_k = 0$ a.s.*

Proof. The proof is followed by combining Lemma 3-(a) and (a)-(i) with Lemma 4. □

Remark 8. Since $r_{1\sigma} = 0$ gives $\beta_1 = a^{-1}_{11} r_{1\sigma} = 0$, we obtain that

$$\text{if } r_{1\sigma} = 0, \text{ then } \lim_{k\to\infty} k^{-1} \textstyle\sum_{s=0}^{k-1} E\left(x^1_s\right) = 0$$

by combining Lemma 3-(b) with Lemma 4. Similarly, Lemma 3-(b)-(ii) gives

$$\text{if } r_{1\sigma} = r_{2\sigma} = 0, \text{ then } \lim_{k\to\infty} k^{-1} \textstyle\sum_{s=0}^{k-1} E\left(x^2_s\right) = 0$$

since $\beta_2 = a^{-1}_{22}\left(r_{2\sigma} + a_{21}\beta_1\right) = 0$.

Remark 9. By Theorem 2-(a), we find that if $r_1 < \frac{1}{2}\sigma^2_{1\eta}$, then the prey population will be extinct in the future, no matter whether the predator exists. It implies that environmental noise plays a very important role in the biological system.

In order to establish the sufficient condition for the extinction of the predator and the persistence of the prey, we will use the following Lemma 5 as well as Lemma 3-(b).
Using Lemmas 4 and 3-(b) with $\beta_1 = a^{-1}_{11} r_{1\sigma}$ we obtain that

$$\text{if } r_{1\sigma} > 0, \text{ then } \lim_{k\to\infty} k^{-1} \sum_{s=0}^{k-1} E\left(x^1_s\right) \le a^{-1}_{11} r_{1\sigma}. \tag{63}$$

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For finding a lower function of x_k^1, we consider the solution $u_{k,\epsilon}$ of the equation

$$u_{k+1,\epsilon} = u_{k,\epsilon}(1 + \hat{r}_1 - \hat{a}_{11}u_{k,\epsilon} - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}),\ u_{0,\epsilon} = x^1_{N_\epsilon}, \tag{64}$$

in which ϵ satisfies that for some positive integer N_ϵ and all $k \geq N_\epsilon$

$$0 < x_k^2 \leq \epsilon, \tag{65}$$

$$\hat{r}_1 - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\varsigma_* < 1, \tag{66}$$

$$\hat{a}_{12}\epsilon + \tilde{\sigma}_1\varsigma < \tilde{\sigma}_1\varsigma_*, \tag{67}$$

where (65) is possible under the conditions $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$ due to Lemmas 4 and 3-(b)-(i). The inequalities (66) and (67) are possible by (14) and (12), respectively.

Lemma 5. *Assume that $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$. Let ϵ and N_ϵ satisfy (65)–(67). Let x_k^1 and $u_{k,\epsilon}$ be the solutions of (3) and (64), respectively. Then*

(a) $0 < u_{k,\epsilon} < \chi_1$ *for* $k \geq 0$.

(b) $u_{k,\epsilon} \leq x^1_{N_\epsilon+k}$ *for* $k \geq 0$.

(c) *If* $r_{1\sigma} - a_{12}\epsilon > 0$, *then* $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(u_{s,\epsilon}\right) = a_{11}^{-1}\left(r_{1\sigma} - a_{12}\epsilon\right)$.

Proof. (a)We proceed by induction on k.
Since (64) and Theorem 1 with Remark 2 give

$$u_{0,\epsilon} = x^1_{N_\epsilon},\ 0 < x^1_{N_\epsilon} < \chi_1,$$

the statement (a) is true for $k = 0$.
Assume that for a nonnegative integer k

$$0 < u_{k,\epsilon} < \chi_1. \tag{68}$$

Now, in the case of $k + 1$, the proof of (a) is divided into the following two steps.
Step 1. We prove the positivity of $u_{k+1,\epsilon}$.
Denoting

$$\mathcal{U}_k = (2\hat{a}_{11})^{-1}\left(1 + \hat{r}_1 - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}\right)$$

gives that for $k \geq 0$

$$0 < \chi_1 < (2\hat{a}_{11})^{-1}\left(1 + \hat{r}_1 - \tilde{\sigma}_1\varsigma_*\right) < \mathcal{U}_k, \tag{69}$$

where the second inequality is obtained from (13) and the last from (67), (12) and (5).
Letting

$$G_k(x) = x\left(1 + \hat{r}_1 - \hat{a}_{11}x - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}\right),$$

we have

$$G_k(x) \text{ is strictly increasing on } 0 \leq x < \mathcal{U}_k. \tag{70}$$

Applying (68) and (69) to (70), we have the desired positivity.
Step 2. We prove that χ_1 is an upper bound of $u_{k+1,\epsilon}$.
Let $\omega \in \Omega_h$. If $\hat{r}_1 - \hat{a}_{11}u_{k,\epsilon}(\omega) - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}(\omega) \leq 0$, then

$$u_{k+1,\epsilon}(\omega) = G_k(u_{k,\epsilon})(\omega) \leq u_{k,\epsilon}(\omega) < \chi_1,$$

in which (68) gives the last inequality. Otherwise, we have $0 < u_{k,\epsilon}(\omega) < \Delta_k(\omega)$ with

$$\Delta_k = \hat{a}_{11}^{-1}\left(\hat{r}_1 - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}\right).$$

Since $\Delta_k < \mathcal{U}_k$ by (66), we have $0 < u_{k,\epsilon}(\omega) < \Delta_k(\omega) < \mathcal{U}_k(\omega)$ and then (70) gives

$$u_{k+1,\epsilon}(\omega) = G_k(u_{k,\epsilon})(\omega) < G_k(\Delta_k)(\omega) = \Delta_k(\omega) < \chi_1,$$

where the last inequality is obtained from (11), (12) and (5).
(b)We proceed by induction on k.
The statement (b) is true for $k = 0$ due to (64).
Assume that for a nonnegative integer k

$$u_{k,\epsilon} \le x^1_{N_\epsilon+k}. \tag{71}$$

It follows from (a) in this theorem, (71), Theorem 1, Remark 2 and (69) that

$$0 < u_{k,\epsilon} \le x^1_{N_\epsilon+k} < \chi_1 < \mathcal{U}_k$$

and then

$$u_{k+1,\epsilon} = G_k(u_{k,\epsilon}) \le G_k(x^1_{N_\epsilon+k}) = F^1_{N_\epsilon+k,\epsilon}(x^1_{N_\epsilon+k}) \tag{72}$$

due to (70). Combining (55) and (65) also gives

$$F^1_{N_\epsilon+k,\epsilon}(x^1_{N_\epsilon+k}) \le F^1_{N_\epsilon+k,x^2_{N_\epsilon+k}}(x^1_{N_\epsilon+k}) = x^1_{N_\epsilon+k+1}. \tag{73}$$

Therefore, (72) and (73) give the desired result.
(c) Let $\gamma_1 = a_{11}^{-1}(r_{1\sigma} - a_{12}\epsilon)$. Note that

$$\begin{aligned} E(\ln u_{k+1,\epsilon}) &= \ln u_{k,\epsilon} + \mathring{h}\,(r_{1\sigma} - a_{11}u_{k,\epsilon} - a_{12}\epsilon), &(74)\\ E(\ln u_{k,\epsilon}) &= E(\ln u_{0,\epsilon}) + k\mathring{h}\left\{r_{1\sigma} - a_{12}\epsilon - a_{11}\overline{E}\,(u_{k,\epsilon})\right\} \\ &= E(\ln u_{0,\epsilon}) + k\mathring{h}a_{11}\left\{\gamma_1 - k^{-1}\sum_{s=0}^{k-1} E\,(u_{s,\epsilon})\right\} &(75)\end{aligned}$$

as in (27) and (28). Following the proof of Lemma 2, we can obtain that

$$k^{-1}\sum_{s=0}^{k-1} E\,(u_{s,\epsilon}) \le \gamma_1 + \epsilon' \tag{76}$$

for $\epsilon' > 0$ and all sufficiently large k by replacing (27), (28) and $(z^1_k, r_{1\sigma}, \beta_1)$ with (74), (75) and $(u_{k,\epsilon}, r_{1\sigma} - a_{12}\epsilon, \gamma_1)$, respectively.
Similarly, replacing (28) and (z^1_k, β_1) in (45)–(47) with (75) and $(u_{k,\epsilon}, \gamma_1)$, respectively, we can obtain that for $\epsilon' > 0$ and all sufficiently large k

$$\gamma_1 - \epsilon' \le k^{-1}\textstyle\sum_{s=0}^{k-1} E\,(u_{s,\epsilon}),$$

with which (76) gives the desired result. □

Theorem 3. *Let x^i_k and β_1 be the solutions of (3) and (26), respectively for $i = 1, 2$.*

If $r_{1\sigma} \ge 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$, then $\lim_{k\to\infty} \overline{E}(x^1_k) = \beta_1$ and $\lim_{k\to\infty} x^2_k = 0$ a.s.

Proof. It follows from Lemma 3-(b)-(i), Lemma 4, Theorem 1 and Remark 2 that

$$\lim_{k\to\infty} x_k^2 = 0 \quad a.s.$$

Using Lemma 5-(a) and Lemma 4, we obtain that for $\epsilon > 0$ and all sufficiently large k

$$0 < u_{k,\epsilon} \leq x^1_{N_\epsilon+k} \leq z^1_{N_\epsilon+k}. \tag{77}$$

Lemma 5-(c) and Lemma 3-(b) give

$$\lim_{k\to\infty} \overline{E}\left(u_{k,\epsilon}\right) = a_{11}^{-1}\left(r_{1\sigma} - a_{12}\epsilon\right), \ \lim_{k\to\infty} \overline{E}\left(z_k^1\right) = a_{11}^{-1} r_{1\sigma}, \tag{78}$$

where the first and second equalities are valid under the conditions $r_{1\sigma} - a_{12}\epsilon > 0$ and $r_{1\sigma} \geq 0$, respectively. Therefore using (77), (78) and Remark 8, we obtain the desired result. □

Remark 10. By Theorems 2 and 3, we find that the value $r_{1\sigma}$ is the threshold between the extinction and persistence for the prey population. In addition, although the prey population converges to a non-extinction state in the mean when $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$, the predators dies out when the diffusion coefficient σ_2 is large enough and then

$$-r_{2\sigma} = -r_2 + 0.5\left\{\sigma_2 \cdot (1 - \eta_\varsigma)\right\}^2$$

becomes too large.

Remark 11. We can establish one condition for the extinction of the prey and the persistence of the predator as follows. Lemmas 4 and 3-(a)-(ii) yield

$$\text{if } r_{1\sigma} < 0 \text{ and } r_{2\sigma} \geq 0, \text{ then } \lim_{k\to\infty} k^{-1} \sum_{s=0}^{k-1} E\left(x_s^2\right) \leq a_{22}^{-1} r_{2\sigma}. \tag{79}$$

For finding a lower function of x_k^2, we consider the solution $v_{k,\epsilon}$ of the equation

$$v_{k+1,\epsilon} = v_{k,\epsilon}(1 + \hat{r}_2 - \hat{a}_{21}\epsilon - \hat{a}_{22} v_{k,\epsilon} + \tilde{\sigma}_2 \xi^2_{N_\epsilon+k+1}), \ v_{0,\epsilon} = x^2_{N_\epsilon}, \tag{80}$$

in which ϵ satisfies that for some positive integer N_ϵ and all $k \geq N_\epsilon$

$$0 < x_k^1 \leq \epsilon, \tag{81}$$
$$\hat{r}_2 - \hat{a}_{21}\epsilon + \tilde{\sigma}_2 \varsigma_* < 1, \tag{82}$$
$$\hat{a}_{21}\epsilon + \tilde{\sigma}_2 \varsigma < \tilde{\sigma}_2 \varsigma_*. \tag{83}$$

The inequality (81) is possible under the condition $r_{1\sigma} < 0$ due to Lemma 3-(a). Replacing (64)–(67), $r_{1\sigma} > 0$, $r_{2\sigma} + a_{21}\beta_1 < 0$ and $(u_{k,\epsilon}, r_1, a_{11}, a_{12}, \xi^1)$ in the proof of Lemma 5 with (80)–(83), $r_{1\sigma} < 0$, $r_{2\sigma} > 0$ and $(v_{k,\epsilon}, r_2, a_{22}, a_{21}, \xi^2)$, we can obtain that

$$v_{k,\epsilon} \leq x^2_{N_\epsilon+k}, \ \lim_{k\to\infty} k^{-1} \sum_{s=0}^{k-1} E\left(v_{s,\epsilon}\right) = a_{22}^{-1}\left(r_{2\sigma} - a_{21}\epsilon\right), \tag{84}$$

if $r_{2\sigma} - a_{21}\epsilon > 0$. Therefore (79) and (84) give the desired result:

$$\text{if } r_{1\sigma} < 0 \text{ and } r_{2\sigma} > 0, \text{ then } \lim_{k\to\infty}\left(x_k^1, \overline{E}(x_k^2)\right) = \left(0, a_{22}^{-1} r_{2\sigma}\right) \quad a.s.$$

Now, it remains to establish one condition for persistence of the prey and the predator. Define the matrix A and the constants D_i as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \end{pmatrix} = A \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \tag{85}$$

which give

$$|A| = a_{11}a_{22} + a_{12}a_{21} > 0, \ \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = A^{-1} \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \end{pmatrix} = |A|^{-1} \begin{pmatrix} a_{22}r_{1\sigma} - a_{12}r_{2\sigma} \\ a_{11}\left(r_{2\sigma} + a_{21}\beta_1\right) \end{pmatrix} \geq 0 \tag{86}$$

under the conditions $r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma} + a_{21}\beta_1 \geq 0$.
Using (85), the system (23) can be written as the matrix equation

$$\begin{pmatrix} E\left(\ln x_k^1\right) \\ E\left(\ln x_k^2\right) \end{pmatrix} = \begin{pmatrix} E\left(\ln x_0^1\right) \\ E\left(\ln x_0^2\right) \end{pmatrix} + k\mathring{h}A \begin{pmatrix} D_1 - \overline{E}\left(x_k^1\right) \\ D_2 - \overline{E}\left(x_k^2\right) \end{pmatrix} \tag{87}$$

and multiplying the matrix $|A|A^{-1}$ to (87), we have

$$a_{22}E(\ln x_k^1) - a_{12}E(\ln x_k^2) = C_1 + k\mathring{h}|A|\left\{D_1 - \overline{E}\left(x_k^1\right)\right\}, \tag{88}$$

$$a_{21}E(\ln x_k^1) + a_{11}E(\ln x_k^2) = C_2 + k\mathring{h}|A|\left\{D_2 - \overline{E}\left(x_k^2\right)\right\}, \tag{89}$$

where $C_1 = a_{22}E(\ln x_0^1) - a_{12}E(\ln x_0^2)$ and $C_2 = a_{21}E(\ln x_0^1) + a_{11}E(\ln x_0^2)$.

Lemma 6. *Let x_k^1 and β_1 be the solutions of (3) and (26), respectively. If $r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma} + a_{21}\beta_1 \geq 0$, then for $\epsilon > 0$ and all sufficiently large k*

$$\overline{E}(x_k^1) \leq D_1 + \epsilon, \tag{90}$$

where D_1 is defined in (85).

Proof. Suppose that (90) is false, which means that there exist a constant $\epsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ satisfying both for all k_m

$$k_m^{-1} \sum_{s=0}^{k_m-1} E\left(x_s^1\right) > D_1 + \epsilon_0, \tag{91}$$

and for all k with $k \neq k_m$

$$k^{-1} \sum_{s=0}^{k-1} E\left(x_s^1\right) \leq D_1 + \epsilon_0. \tag{92}$$

Replace (z_k^1, β_1), (31), (32), (28) and (27) in the proof of Lemma 2 with (x_k^1, D_1), (91), (92), (88) and (22), respectively, where we apply (22) with $i = 1$. Then using the boundedness of x_k^1 and following the proof for (37), we can obtain that for all sufficiently large k

$$k^{-1} \sum_{s=0}^{k-1} E\left(x_s^1\right) > D_1 + \epsilon_0. \tag{93}$$

Combining (93) and (88) gives

$$a_{22}E(\ln x_k^1) - a_{12}E(\ln x_k^2) < C_1 + k\mathring{h}|A|(-\epsilon_0). \tag{94}$$

Applying Theorem 1 to (22) with $i=2$, we obtain

$$\sup_{k\geq 0} E(\ln x_k^2) < \infty$$

and then (94) yields

$$\lim_{k\to\infty} E\left(\ln x_k^1\right) = -\infty. \tag{95}$$

Substituting (95) into (22) with $i=1$ and using the boundedness of x_k^1, we obtain

$$\lim_{k\to\infty} \ln x_k^1 = -\infty \quad a.s.,$$

which implies

$$\lim_{k\to\infty} x_k^1 = 0 \quad a.s.$$

Hence the dominated convergence theorem with Theorem 1 leads to

$$\lim_{k\to\infty} E(x_k^1) = 0,$$

which is contradictory to (93) due to $D_1 + \varepsilon_0 > 0$. This completes the proof. □

Remark 12. The equation (90) with (87) gives that for $\epsilon > 0$ and all sufficiently large k

$$E(\ln x_k^2) \leq E(\ln x_0^2) + k\mathring{h}a_{22}\left\{a_{22}^{-1}a_{21}\epsilon + D_2 - \overline{E}(x_k^2)\right\}. \tag{96}$$

Following the proof of Lemma 6 with (96), we can obtain that

$$\text{if } r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma} \text{ and } r_{2\sigma} + a_{21}\beta_1 \geq 0, \text{ then } \overline{E}(x_k^2) \leq a_{22}^{-1}a_{21}\epsilon + D_2 + \epsilon' \tag{97}$$

for $\epsilon' > 0$ and all sufficiently large k by replacing (x_k^1, D_1) and (88) in the proof of Lemma 6 with $(x_k^2, a_{22}^{-1}a_{21}\epsilon + D_2)$ and (96), respectively.

Theorem 4. *Let x_k^i and β_i be the solutions of (3) and (26), respectively for $i=1,2$.*

$$\textit{If } r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma} \textit{ and } r_{2\sigma} + a_{21}\beta_1 \geq 0, \textit{ then } \lim_{k\to\infty} \overline{E}(x_k^i) = D_i,$$

where D_i are defined in (85).

Proof. Substituting (62) into (89) gives that for $\epsilon' > 0$ and all sufficiently large k

$$\epsilon' \geq D_2 - \overline{E}(x_k^2). \tag{98}$$

Combining (98) and (97), we have

$$\lim_{k\to\infty} \overline{E}(x_k^2) = D_2. \tag{99}$$

Applying (99) to (89) with (62) yields

$$\lim_{k\to\infty} k^{-1}E(\ln x_k^1) = \lim_{k\to\infty} k^{-1}E(\ln x_k^2) = 0,$$

with which (88) gives the desired result $\lim\limits_{k\to\infty} \overline{E}(x_k^1) = D_1$. □

Remark 13. Let (x_k, y_k) be the solutions of DDEs (3) with $\sigma_1 = \sigma_2 = 0$ in [35].

(i) If $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 \leq 0$, then $\lim_{k\to\infty}(x_k, y_k) = (a_{11}^{-1}r_1, 0)$.

(ii) If $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 > 0$, then $\lim_{k\to\infty}(x_k, y_k) = (D_x, D_y)$, where (D_x, D_y) is equal to (D_1, D_2) with $\sigma_1 = \sigma_2 = 0$.

Note that the sign of r_2 in the DDE model is fixed to $r_2 < 0$. Adding the noise to the DDEs, we have from Theorems 3 and 4 that

(i)′ If $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} < 0$, then $\lim_{k\to\infty}\left(\overline{E}(x_k^1), x_k^2\right) = \left(a_{11}^{-1}r_{1\sigma}, 0\right)$ *a.s.*

(ii)′ If $r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma}+a_{21}a_{11}^{-1}r_{1\sigma} \geq 0$, then $\lim_{k\to\infty}\left(\overline{E}(x_k^1), \overline{E}(x_k^2)\right) = (D_1, D_2)$.

Hence we demonstrate that the solutions of the DDEs and the DSDEs with small noise have similar asymptotic behavior by comparing (i), (ii) and (i)′, (ii)′, respectively. In addition, when comparing $r_2 + a_{21}a_{11}^{-1}r_1 > 0$ in (ii) and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} < 0$ in (i)′, we understand the effect of strong noise, which changes the behavior of the predator population from non-extinction into extinction. Therefore the main difference between the deterministic and stochastic models is that large stochastic perturbation may result in the extinction of the predator population.

Remark 14. Let (x, y) be the solutions of the SDE model (2), which is a special model in [25] with zero time delays. Note that the sign of r_2 in the SDE model is also negative.

(i) If $r_1 - 0.5\sigma_1^2 < 0$ and $r_2 - 0.5\sigma_2^2 < 0$, then $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$ *a.s.*

(ii) If $r_1 - 0.5\sigma_1^2 > 0$, $r_2 - 0.5\sigma_2^2 < 0$ and $(r_2 - 0.5\sigma_2^2) + a_{21}a_{11}^{-1}(r_1 - 0.5\sigma_1^2) < 0$, then x is stable in the mean and y goes to extinction:

$$\lim_{t\to\infty} t^{-1}\int_0^t x(s)ds = a_{11}^{-1}r_{1\sigma}, \ \ \lim_{t\to\infty} y(t) = 0 \ \ a.s.$$

(iii) If $r_2 - 0.5\sigma_2^2 < 0$ and $(r_2 - 0.5\sigma_2^2) + a_{21}a_{11}^{-1}(r_1 - 0.5\sigma_1^2) > 0$, then both x and y are stable in the mean:

$$\lim_{t\to\infty}\left(t^{-1}\int_0^t x(s)ds, \ t^{-1}\int_0^t y(s)ds\right) = (D_1, D_2) \ \ a.s.$$

Since $r_2 < 0$ in the SDE model (2), the sign of $r_2 - 0.5\sigma_2^2$ in (2) is also negative, which is the reason why the condition $r_2 - 0.5\sigma_2^2 < 0$ is assumed in (i)–(iii). The three results, (i), (ii) and (iii) in this remark, are corresponding to Theorem 2-(b), (i)′ and (ii)′ in Remark 13, respectively. Hence, when replacing the stability of $(x(t), y(t))$ in the mean with the stability of $\left(\overline{E}(x_k^1), \overline{E}(x_k^2)\right)$, we demonstrate that the sufficient conditions for the almost sure global stability of the SDE model (2) also suffice to give the same global stability of the DSDE model (3). In this case, note that there is no constraint on the sign of r_2 in the DSDE model. Therefore we show that the DSDE model (3) is a good discrete model for the corresponding SDE model (2).

6. Numerical examples

In this section, we provide some simulations that illustrate the results in Theorems 1, 2, 3 and 4 with truncation constants $(\varsigma, \varsigma_*) = (19.9, 20)$ in (5) and (12). In this case, we have $0 < \eta_\varsigma < 10^{-85}$, so that we can ignore the effect of the term η_ς when using the values of parameters in the following three examples, where the conditions (12)–(14) are satisfied. In Figures 1, 2 and 3, the DSDE model (3) is simulated 1000 times at each time kh for calculating the expectation values $E\left(x_k\right)$ and $E\left(y_k\right)$, where x_k and y_k denote the

solutions x_k^1 and x_k^2, respectively. We compare our results for the DSDE model (3) with the results for the DDE model in [35], which is the model (3) with $\sigma_1 = \sigma_2 = 0$.

Example 1. Let $h = 0.0001, r_1 = 0.8, r_2 = -0.1, a_{11} = 0.4, a_{12} = 0.001, a_{21} = 0.1, a_{22} = 0.3, \sigma_1^2 = 2.5$ and $\sigma_2^2 = 0.1$. Since $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 > 0$, the solutions x_k and y_k of the DDE model converge to the positive numbers D_x and D_y in Remark 13-(ii), respectively, as displayed in Figure 1-(a). However, since $r_{i\sigma} < 0$ $(i = 1, 2)$, the noises have a large effect on the convergence and, as a result, the solutions of the stochastically perturbed model (3) go to extinction, which are shown in Figures 1-(b) and (c), as in Theorem 2-(a) and (b), respectively. Therefore Figures 1 demonstrates the important role of noise.

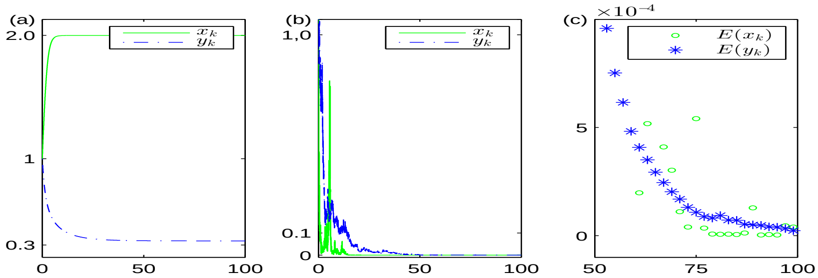

Figure 1: All the x-axes denote time kh. (a) Curves of the solutions of the DDE model. (b) Two realizations of the solutions x_k and y_k of the DSDE model, which converge to zero. (c) Expectation values of the solutions x_k and y_k of the DSDE model, which converge to zero in the mean.

Example 2. Let $h = 0.001, r_1 = 2, r_2 = -2, a_{11} = 1.0, a_{12} = 0.4, a_{21} = a_{22} = 0.3, \sigma_1^2 = 0.2$ and $\sigma_2^2 = 4$. Figure 2-(a) shows that the solutions x_k and y_k of the DDE model converge to $a_{11}^{-1}r_1$ and 0, respectively, as in Remark 13-(i) when $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 \leq 0$. The noises satisfy both $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} < 0$, which are the conditions in Theorem 3. Then Figures 2-(b), (c) and (d) show that the stochastically perturbed model (3) behaves similarly to the DDE model in the sense that $k^{-1}\sum_{i=0}^{k-1} E(x_i)$ and y_k converge to $a_{11}^{-1}r_{1\sigma}$ and 0, respectively, which confirms Theorem 3.

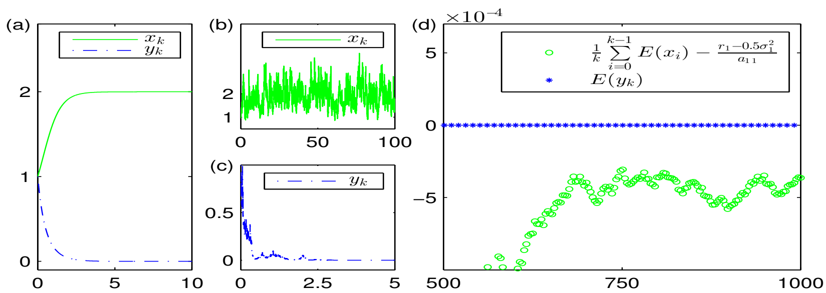

Figure 2: All the x-axes denote time kh. (a) Curves of the solutions of the DDE model. Curves in (b) and (c) are realizations of the solutions x_k and y_k of the DSDE model, respectively. (d) Convergence of average of expectation values of x_k to non-zero and convergence of y_k to zero in the mean.

Example 3. Let $h = 0.001, r_1 = 2.0, r_2 = -0.1, a_{11} = a_{12} = 0.4, a_{21} = 1, a_{22} = 0.3$ and $\sigma_1^2 = \sigma_2^2 = 0.02$, which give that $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 > 0$. Thus Figure 3-(a)

shows that the solutions x_k and y_k of the DDE model converge to D_x and D_y in Remark 13-(ii), respectively, as displayed in Figure 1-(a) in Example 1. However, the condition $r_{1\sigma} > 0$ is different from that in Example 1. Realizations of the solutions of the DSDE model are given in Figures 3-(b) and (c). Since $r_{1\sigma} > a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} > 0$, Figure 3-(d) shows that the DSDE model behaves similarly to the DDE model in the sense that $k^{-1}\sum_{i=0}^{k-1} E(x_i)$ and $k^{-1}\sum_{i=0}^{k-1} E(y_i)$ converge to positive D_1 and D_2, respectively, which demonstrate Theorem 4.

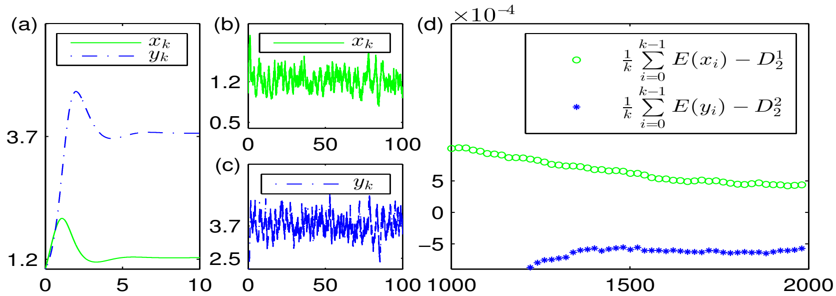

Figure 3: All the x-axes denote time kh. (a) Curves of the solutions of the DDE model. Curves in (b) and (c) are realizations of the solutions x_k and y_k of the DSDE model, respectively. The symbols D_2^1 and D_2^2 in (d) denote D_1 and D_2 defined in (85).

7. Conclusion

In this paper, we have considered a system of discrete-time stochastic difference equations for predator-prey interactions and established sufficient conditions for extinction and non-extinction of the two species. Our results show that if the positive equilibrium point of the deterministic difference system is globally stable, then the stochastic difference model will preserve the nice property in mean provided that the noise is sufficiently small. It is shown, however, that large noise can change the behavior of the predator population from non-extinction into extinction.

Our new discrete Itô formula has played an important role in the two-dimensional DSDE model. In addition we can apply the new formula for the n-dimensional DSDE model

$$x_{k+1}^i = x_k^i \left\{1 + h\left(r_i + \sum_{j=1}^{i-1} a_{ij}x_k^j - \sum_{j=i}^{n} a_{ij}x_k^j\right) + h^{0.5}\sigma_i\xi_{k+1}^i\right\}$$

for $1 \le i \le n$ and $k \ge 0$. Therefore it is a further study to establish sufficient conditions for the extinction and non-extinction of the n species.

Appendix

A.1. The proof of Lemma 1

By Taylor expansion,

$$\varphi(1+x) = \varphi(1) + \varphi'(1)x + 2^{-1}\varphi''(1)x^2 + 6^{-1}\varphi'''(\theta)x^3 \tag{100}$$

with θ lying between 1 and x. Let $x = hf + h^{0.5}g\xi$. Since f, g are $\mathcal{G}$-measurable and ξ is $\mathcal{G}$-independent with $E(\xi) = 0$, we have

$$E\left(x \middle| \mathcal{G}\right) = E\left(hf \middle| \mathcal{G}\right) + E\left(h^{0.5}g\xi \middle| \mathcal{G}\right) = hf + h^{0.5}gE(\xi) = hf \tag{101}$$

and further

$$\begin{aligned} E\left(x^2 \middle| \mathcal{G}\right) &= E\left((hf)^2 \middle| \mathcal{G}\right) + E\left(2hfh^{0.5}g\xi \middle| \mathcal{G}\right) + E\left(hg^2\xi^2 \middle| \mathcal{G}\right) \\ &= (hf)^2 + hg^2 \cdot (1-\mu) \\ &\leq hfM_5h^{\varepsilon} + hg^2 \cdot (1-\mu) \end{aligned} \tag{102}$$

due to $E(\xi^2) = 1 - \mu$ and (18). Using Lemma 1-(ii) gives

$$\left|E\left(6^{-1}\varphi'''(\theta)x^3 \,\middle|\, \mathcal{G}\right)\right| \leq 6^{-1}M_3E\left(\left|x^3\right| \middle| \mathcal{G}\right) \tag{103}$$

and expanding $x^3 = (hf + h^{0.5}g\xi)^3$ yields

$$\begin{aligned} E\left(\left|x^3\right| \middle| \mathcal{G}\right) &\leq hf\left\{(hf)^2 + 3hg^2 \cdot (1-\mu)\right\} + hg^2M_1h^{0.5}g \\ &\leq hf\left(M_5h^{\varepsilon}\right)^2\left\{1 + 3(1-\mu)\right\} + hg^2M_1M_4h^{\varepsilon} \end{aligned} \tag{104}$$

because of (18) and (16). Inserting (101)–(104) into (100), we have

$$E\left(\varphi(1+x) \middle| \mathcal{G}\right) \tag{105}$$
$$= \varphi(1) + \varphi'(1)hf + 2^{-1}\varphi''(1)hg^2 \cdot (1-\mu) + hfO_1\left(h^{\varepsilon}\right) + hg^2O_2\left(h^{\varepsilon}\right), \tag{106}$$

in which the two big O notations denote

$$\begin{aligned} O_1\left(h^{\varepsilon}\right) &= 2^{-1}\varphi''(1)M_5h^{\varepsilon} + 6^{-1}M_3\left(M_5h^{\varepsilon}\right)^2\left\{1 + 3(1-\mu)\right\}, \\ O_2\left(h^{\varepsilon}\right) &= M_1M_5h^{\varepsilon}. \end{aligned}$$

Now it remains to show

$$E\left(\phi\left(1 + hf + h^{0.5}g\xi\right) - \varphi\left(1 + hf + h^{0.5}g\xi\right) \,\middle|\, \mathcal{G}\right) = hg^2O\left(h^{\varepsilon}\right).$$

Let $c_1 = 1 + hf$ and $c_2 = h^{0.5}g$. Then the disintegration formula for conditional expectations with respect to $\mathcal{G}$ gives

$$\begin{aligned} &E\left(\phi\left(1 + hf + \sqrt{h}g\xi\right) - \varphi\left(1 + hf + \sqrt{h}g\xi\right) \middle| \mathcal{G}\right) \\ &\quad = \int_{\mathbb{R}} \left\{\phi\left(c_1 + c_2x\right) - \varphi\left(c_1 + c_2x\right)\right\} p(x)\ dx \end{aligned} \tag{107}$$

due to Lemma 1-(iii) and the fact that f, g are $\mathcal{G}$-measurable, ξ is $\mathcal{G}$ -independent, ϕ is almost everywhere continuous and φ is also continuous (see Theorem 5.4 in [44] for the disintegration formula). Let $U_\delta = [1-\delta, 1+\delta]$ and $s = c_1 + c_2x$. Then (107) becomes

$$\int_{\mathbb{R}-U_\delta} \left\{\phi\left(s\right) - \varphi\left(s\right)\right\} p\left(\frac{s-c_1}{c_2}\right) \frac{ds}{|c_2|} \tag{108}$$

because of Lemma 1-(i). Here p is the probabilty density function of ξ.

Lemma 1-(iii) gives that

$$\begin{aligned}
&\left|\int_{\mathbb{R}-U_\delta} \{\phi(s)-\varphi(s)\}\, p\left(\frac{s-c_1}{c_2}\right)\frac{ds}{|c_2|}\right| \\
&\quad\le \left\{\int_{\mathbb{R}-U_\delta} |\phi(s)-\varphi(s)|\frac{ds}{|c_2|}\right\}\sup_{s\notin U_\delta}\left\{p\left(\frac{s-c_1}{c_2}\right)\frac{1}{|c_2|}\right\} \\
&\quad\le M_4|c_2|^2\sup_{s\notin U_\delta}\left\{p\left(\frac{s-c_1}{c_2}\right)\frac{1}{|c_2|^3}\right\} \\
&\quad= M_4hg^2\sup_{s\notin U_\delta}\left\{p\left(\frac{s-1-hf}{h^{0.5}g}\right)\frac{1}{|h^{0.5}g|^3}\right\}.
\end{aligned}$$

Since there exists some δ_0 such that for $s \notin U_\delta$ and all sufficiently small $h > 0$

$$|s-1-hf| > |s-1| - h|f| > \delta - M_5h^\epsilon > \delta_0 > 0, \tag{109}$$

letting $y = (s-1-hf)/(h^{0.5}g)$ yields

$$|y| = \frac{|s-1-hf|}{h^{0.5}\,|g|} > \frac{\delta_0}{M_5h^\varepsilon} \tag{110}$$

and further

$$\sup_{s\notin U_\delta}\left\{p\left(\frac{s-1-hf}{h^{0.5}g}\right)\frac{1}{|h^{0.5}g|^3}\right\} = \sup_{s\notin U_\delta}\frac{p(y)\,|y|^3}{|s-1-hf|^3}.$$

Hence it follows from (17), (109) and (110) that

$$\sup_{s\notin U_\delta}\frac{p(y)\,|y|^3}{|s-1-hf|^3} < M_2\sup_{s\notin U_\delta}\frac{|y|^{-1}}{|s-1-hf|^3} < M_2\frac{M_5}{\delta_0^2}h^\varepsilon,$$

which gives

$$\left|\int_{\mathbb{R}-U_\delta}\{\phi(s)-\varphi(s)\}\, p\left(\frac{s-c_1}{c_2}\right)\frac{ds}{|c_2|}\right| < hg^2\cdot M_4M_2\frac{M_5}{\delta_0^2}h^\varepsilon. \tag{111}$$

Therefore using (105), (108) and (111), we obtain the desired result.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the 2014 Research Fund of University of Ulsan.

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WEIGHTED SUPERPOSITION OPERATORS FROM ZYGMUND SPACES TO μ-BLOCH SPACES

ZHI JIE JIANG, TING WANG, JUAN LIU, TING LUO, TING SONG

Abstract. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Let φ be an entire function on $\mathbb{C}$ and $u \in H(\mathbb{D})$. The boundedness and compactness of the operators $S_{u,\varphi} : f \mapsto u \cdot \varphi \circ f$ from Zygmund spaces to μ-Bloch spaces are characterized.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$ and $H^\infty(\mathbb{D})$ the space of bounded analytic functions. Let φ be a complex-valued function on $\mathbb{C}$ and $u \in H(\mathbb{D})$. We introduce a class of nonlinear operators by

$$S_{u,\varphi} f = u \cdot \varphi \circ f, \quad f \in H(\mathbb{D}).$$

This operator can be regarded as a generalization of the superposition operator $S_\varphi f = \varphi \circ f$ and the multiplication operator $M_u f = u \cdot f$.

Suppose that X and Y are two metric spaces of analytic functions on $\mathbb{D}$. Note that if X contains the linear functions and S_φ maps X into Y, then φ must be an entire function. In recent years, the following natural questions of the superposition operators are considered.

(a) When does φ induce a superposition operator from X into Y?

(b) When is a superposition operator from X into Y bounded?

(c) When is a superposition operator from X into Y compact?

Although analogous concepts also make sense in the context of real-valued functions and their theory has a long history (see [2]), the study of such natural questions on analytic function spaces has only begun fairly recently. The operators S_φ that map Bergman spaces into area Nevanlinna classes were characterized in [6], which have been extended by other authors to some other analytic function spaces, where it is remarkable the works of Vukotić et. al. in [1], [4] and [5]. It must be mentioned that the authors of [4] gave a very interesting geometric construction of simple connected domain in several analytic function spaces. This technique has been used by many authors; in particular, Xu used it to study the superposition operators from α-Bloch spaces into β-Bloch spaces in [20] and Xiong used it to characterize the superposition operators from Q_p spaces into α-Bloch spaces with $0 < \alpha < 1$ in [18]. It should be noted that quite recently, Castillo et.al. and Ramos Fernández have studied the superposition operators from Bloch-Orlicz spaces into α-Bloch spaces and between weighted Banach spaces of analytic functions in [7] and [14], respectively. In this paper we characterize the boundedness and compactness of the operators $S_{u,\varphi}$ from weighted Zygmund spaces to μ-Bloch spaces. We also consider the superposition operators from weighted Zygmund spaces to weighted Bloch spaces.

Now we present the needed spaces and some facts. The Zygmund space $\mathcal{Z}$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)|f''(z)| < \infty.$$

2000 *Mathematics Subject Classification.* Primary 47H38; Secondary 46E15, 47B38.

Key words and phrases. Weighted Zygmund spaces, μ-Bloch spaces, superposition operators.

With the norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|,$$

it is a Banach space. By Zygmund's theorem (see Theorem 5.3 in [9]), we know that $f \in \mathcal{Z}$ if and only if f is continuous on $\overline{D}$ and

$$\sup_{h>0,\theta\in\mathbb{R}} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$

In closed subspaces of $\mathcal{Z}$, the little Zygmund space $\mathcal{Z}_0$ is usually considered, which is defined by

$$\mathcal{Z}_0 = \big\{f \in \mathcal{Z} : \lim_{|z|\to 1}(1-|z|^2)|f''(z)| = 0\big\}.$$

Let $\alpha \in (0,\infty)$. The weighted Zygmund space $\mathcal{Z}_\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f''(z)| < +\infty.$$

With the norm

$$\|f\|_{\mathcal{Z}_\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f''(z)|,$$

$\mathcal{Z}_\alpha$ is also a Banach space. For the weighted Zygmund spaces and the operators from them into some other spaces, see, e.g., [10], [12] and [15].

Suppose that μ is a positive continuous radial function on $\mathbb{D}$ (that is, $\mu(z) = \mu(|z|)$) and decreasing on $[0,1)$ with $\lim_{r\to 1}\mu(r) = 0$. Let μ be a weight. The μ-Bloch space $\mathcal{B}_\mu$ consists of all $f \in H(\mathbb{D})$ such that $\sup_{z\in\mathbb{D}}\mu(z)|f'(z)| < \infty$. With

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{D}}\mu(z)|f'(z)|,$$

$\mathcal{B}_\mu$ is a Banach space. When $\mu(z) = 1-|z|^2$, the space $\mathcal{B}_\mu$ is just Bloch space and denoted by $\mathcal{B}$; while when $\mu(z) = (1-|z|^2)^\alpha$ with $\alpha > 0$, the space $\mathcal{B}_\mu$ becomes the weighted Bloch space $\mathcal{B}_\alpha$. The μ-Bloch spaces appear in the literature in a natural way when one considers properties of some operators in certain spaces of analytic functions; for example, if $\mu(z) = (1-|z|)\log\frac{2}{1-|z|}$, Attele in [3] proved that the Hankel operator on Bergman spaces induced by a function f is bounded if and only if $f \in \mathcal{B}_\mu$. The logarithmic Bloch type space has been defined and studied in [16]. Recently, the Bloch-Orlicz spaces have been introduced by Ramos-Fernandez in [13].

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \simeq b$ means that there is a positive constant C such that $a/C \le b \le Ca$.

2. The operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$

First we enumerate several useful lemmas. The first one below is well-known.

Lemma 2.1 *There is a positive constant C_α depending only on α such that for any $z \in \mathbb{D}$ and $f \in \mathcal{Z}_\alpha$*

(i)

$$\big|f(z)\big| \le \begin{cases} C_\alpha\|f\|_{\mathcal{Z}_\alpha}, & 0<\alpha<2,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}\log\frac{2}{1-|z|^2}, & \alpha = 2,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}(1-|z|^2)^{2-\alpha}, & \alpha>2.\end{cases}$$

(ii)

$$\big|f'(z)\big| \le \begin{cases} C_\alpha\|f\|_{\mathcal{Z}_\alpha}, & 0<\alpha<1,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}\log\frac{2}{1-|z|^2}, & \alpha = 1,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}(1-|z|^2)^{1-\alpha}, & \alpha>1.\end{cases}$$

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Let $a \in \mathbb{D}$ and $1/\sqrt{2} < |a| < 1$, define

$$f(z) = (z-1)\Big(\Big(1+\log\frac{1}{1-z}\Big)^2+1\Big)$$

and

$$g_a(z) = \frac{f(\bar{a}z)}{\bar{a}}\Big(\log\frac{1}{1-|a|^2}\Big)^{-1}.$$

The function g_a is called the test function with the following property (see [11]).

Lemma 2.2 *The function g_a belongs to $\mathcal{Z}$ and $\|g_a\|_{\mathcal{Z}} \simeq 1$.*

The following result can be found in [17].

Lemma 2.3 *Let $\alpha \in (0,1]$. Then for every bounded sequence $\{f_n\}$ in $\mathcal{Z}_\alpha$ and $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$, we have*

(i) if $\alpha = 1$, then $\lim_{n\to\infty}\sup_{z\in\mathbb{D}}\big|f_n(z)\big| = 0$.

(ii) if $0 < \alpha < 1$, then $\lim_{n\to\infty}\sup_{z\in\mathbb{D}}\big|f_n'(z)\big| = 0$.

The next result is often used in dealing with the compactness of operators on analytic function spaces. Since the proof is standard (see Proposition 3.11 in [8]), it is omitted .

Lemma 2.4 *Let $u \in H(\mathbb{D})$ and φ an entire function. Then the bounded operator $S_{u,\varphi} : \mathcal{Z}_\alpha \to \mathcal{B}_\mu$ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathcal{Z}_\alpha$ such that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$, it follows that $\lim_{n\to\infty}\|S_{u,\varphi}f_n\|_{\mathcal{B}_\mu} = 0$.*

Now we characterize the boundedness of the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$.

Theorem 2.1 *Let $u \in H(\mathbb{D})$ and φ an entire function with $\varphi'(0) \neq 0$. Then the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded if and only if $u \in \mathcal{B}_\mu$ and*

$$L := \sup_{z\in\mathbb{D}}\mu(z)|u(z)|\log\frac{2}{1-|z|^2} < \infty.$$

Proof. Suppose that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded. By taking f_1 the constant function, we obtain $u \in \mathcal{B}_\mu$. Since operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded, for the function $f_2 = g_a$ there exists a positive constant C such that

$$\begin{aligned}\infty > C\|S_{u,\varphi}\| &\geq \|S_{u,\varphi}f_2\|_{\mathcal{B}_\mu} \geq \mu(a)\big|(S_{u,\varphi}f_2)'(a)\big|\\ &= \mu(a)\big|u'(a)\varphi(f_2(a)) + u(a)\varphi'(f_2(a))f_2'(a)\big|\\ &\geq \mu(a)\big(\big|u(a)\big|\big|\varphi'(f_2(a))\big|\big|f_2'(a)\big| - \big|u'(a)\big|\big|\varphi(f_2(a))\big|\big).\end{aligned}$$

From this, we get

$$\mu(a)\big|u'(a)\big|\big|\varphi(f_2(a))\big| + C\|S_{u,\varphi}\| \geq \mu(a)\big|u(a)\big|\big|\varphi'(f_2(a))\big|\big|f_2'(a)\big|.$$

Set $M = C_\alpha\|f_2\|_{\mathcal{Z}}$ and $M_1 = \max_{|z|=M}|\varphi(z)|$. By Lemma 2.1 (i), we have

$$\begin{aligned}M_1\|u\|_{\mathcal{B}_\mu} + C\|S_{u,\varphi}\| &\geq \mu(a)\big|u'(a)\big|\big|\varphi(f_2(a))\big| + C\|S_{u,\varphi}\|\\ &\geq \mu(a)\big|u(a)\big|\big|\varphi'(f_2(a))\big|\big|f_2'(a)\big|\\ &= \mu(a)\big|u(a)\big|\big|\varphi'(g_a(a))\big|\log\frac{1}{1-|a|^2}\\ &\geq \frac{1}{2}\mu(a)\big|u(a)\big|\big|\varphi'(g_a(a))\big|\log\frac{2}{1-|a|^2},\end{aligned}$$

where we have used that when $|a| > 1/\sqrt{2}$,

$$\log\frac{1}{1-|a|^2} \geq \frac{1}{2}\log\frac{2}{1-|a|^2}.$$

It is easy to see that $g_a(a) \to 0$ as $|a| \to 1$. Therefore from this and the fact that

$$\lim_{|a|\to 1} \big|\varphi'(g_a(a))\big| = |\varphi'(0)| \neq 0,$$

we obtain

$$\sup_{1/2<|z|<1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} < \infty.$$

It is clear that

$$\sup_{|z|\le 1/2} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} < \infty.$$

Consequently, we obtain $L < \infty$.

Now let $u \in \mathcal{B}_\mu$ and $L < \infty$. Let $f \in \mathcal{Z}$ and $\|f\|_{\mathcal{Z}} \le M$. Set $M_1 = \max\limits_{|z|=C_\alpha M} |\varphi(z)|$ and $M_2 = \max\limits_{|z|=C_\alpha M} |\varphi'(z)|$. Then by Lemma 2.1, we have

$$\begin{aligned}
\|S_{u,\varphi} f\|_{\mathcal{B}_\mu} &= \big|u(0)\varphi(f(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|(S_{u,\varphi} f)'(z)\big| \\
&= \big|u(0)\varphi(f(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u'(z)\varphi(f(z)) + u(z)\varphi'(f(z))f'(z)\big| \\
&\le C_\alpha M\|u\|_{\mathcal{B}_\mu} + \sup_{z\in\mathbb{D}} \mu(z)\big|u'(z)\big|\big|\varphi(f(z))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u(z)\big|\big|\varphi'(f(z))\big|\big|f'(z)\big| \\
&\le C_\alpha M\|u\|_{\mathcal{B}_\mu} + M_1\|u\|_{\mathcal{B}_\mu} + C_\alpha M M_2 \sup_{z\in\mathbb{D}} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \\
&\le (C_\alpha M + M_1)\|u\|_{\mathcal{B}_\mu} + C_\alpha L M M_2 \\
&< \infty.
\end{aligned}$$

This shows that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded. □

There are a lot of examples satisfying the conditions of Theorem 2.1. Here we take the following two examples. Since the first is clear, its proof is omitted.

Example 2.1 *Let $u(z) = a_0+a_1z+a_2z^2+\cdots+a_nz^n$ and $\varphi(z) = b_0+b_1z+b_2z^2+\cdots+b_mz^m$, where $b_1 \neq 0$. Then the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded.*

Example 2.2 *Let $u(z) = \lambda\frac{a-z}{1-\bar{a}z}$ be the automorphism of $\mathbb{D}$ and $\varphi(z) = e^z$. Then $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded.*

Proof. Since $\|u\|_\infty \le 1$ and it is easy to see that

$$|u'(z)| = \frac{1-|a|^2}{|1-\bar{a}z|^2} \le \frac{2}{1-|a|},$$

we get $u \in \mathcal{B}_\mu$ and $L < \infty$. By Theorem 2.1, the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded. □

From the proof of Theorem 2.1, we can obtain the following sufficient condition of boundedness for the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$.

Theorem 2.2 *Let $u \in H(\mathbb{D})$ and φ an entire function. If $u \in \mathcal{B}_\mu$ and $L < \infty$, then $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded.*

We begin to study when the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is compact.

Theorem 2.3 *Let $u \in H(\mathbb{D})$ and φ an entire function with $\varphi(0) = 0$ and $\varphi'(0) \neq 0$. Then the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is compact if and only if $u \in \mathcal{B}_\mu$ and*

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} = 0.$$

Proof. Suppose that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is compact. Of course, it is bounded, and then $u \in \mathcal{B}_\mu$. Now let us suppose, by the way of contradiction, that

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \neq 0.$$

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Then there exists some $\varepsilon_0 > 0$ and a sequence $\{z_n\} \subseteq \mathbb{D}$ such that $|z_n| \to 1$ as $n \to \infty$ and

$$\mu(z_n)\big|u(z_n)\big| \log \frac{2}{1-|z_n|^2} \geq \varepsilon_0.$$

For each $n \in \mathbb{N}$, take the function $f_n = g_{z_n}$. From Lemma 2.2 it follows that $\|f_n\|_{\mathcal{Z}} \leq C$. One can easily check that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$. Thus it follows from Lemma 2.4 that

$$\begin{aligned}
\|S_{u,\varphi} f_n\|_{\mathcal{B}_\mu} &\geq \mu(z_n)\big|(S_{u,\varphi} f_n)'(z_n)\big| \\
&= \mu(z_n)\big|u'(z_n)\varphi(f_n(z_n)) + u(z_n)\varphi'(f_n(z_n))f_n'(z_n)\big| \\
&\geq \mu(z_n)\big(\big|u(z_n)\varphi'(f_n(z_n))f_n'(z_n)\big| - \big|u'(z_n)\varphi(f_n(z_n))\big|\big) \\
&= \mu(z_n)\big|u(z_n)\big|\big|\varphi'(f_n(z_n))\big|\big|f_n'(z_n)\big| - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f_n(z_n))\big| \\
&\geq \mu(z_n)\big|u(z_n)\big|\big|\varphi'(f_n(z_n))\big| \log \frac{1}{1-|z_n|^2} - \|u\|_{\mathcal{B}_\mu}\big|\varphi(f_n(z_n))\big| \\
&\geq \frac{1}{2}\mu(z_n)\big|u(z_n)\big|\big|\varphi'(f_n(z_n))\big| \log \frac{2}{1-|z_n|^2} - \|u\|_{\mathcal{B}_\mu}\big|\varphi(f_n(z_n))\big| \\
&\geq \frac{1}{2}\big|\varphi'(f_n(z_n))\big|\varepsilon_0 - \|u\|_{\mathcal{B}_\mu}\big|\varphi(f_n(z_n))\big|.
\end{aligned}$$

From this and since Lemma 2.3 (i) implies that $|\varphi(f_n(z_n))| = 0$ as $n \to \infty$, we get

$$0 = \lim_{n\to\infty} \|S_{u,\varphi} f_n\|_{\mathcal{B}_\mu} \geq \frac{1}{2}\big|\varphi'(0)\big|\varepsilon_0,$$

which arrives at a contradiction.

Conversely, by the definition of limit we have that for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} < \varepsilon$$

for all $z \in \{z \in \mathbb{D} : \delta < |z| < 1\}$. Let $M_0 > 0$ and $\|f_n\|_{\mathcal{Z}} \leq M_0$ and $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$. By the Cauchy integral formula and an easy calculation, it is clear that $\{f_n'\}$ also uniformly converges to zero on every compact subset of $\mathbb{D}$ as $n \to \infty$. Let $M = \max_{|z|=C_\alpha M_0} |\varphi'(z)|$. By Lemma 2.1 and Lemma 2.3 (i), we have

$$\begin{aligned}
\|S_{u,\varphi} f_n\|_{\mathcal{B}_\mu} &= \big|u(0)\varphi(f_n(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|(S_{u,\varphi} f_n)'(z)\big| \\
&= \big|u(0)\varphi(f_n(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u'(z)\varphi(f_n(z)) + u(z)\varphi'(f_n(z))f_n'(z)\big| \\
&\leq \big|u(0)\varphi(f_n(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u'(z)\big|\big|\varphi(f_n(z))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u(z)\big|\big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\
&\leq \big|u(0)\varphi(f_n(0))\big| + \|u\|_{\mathcal{B}_\mu} \sup_{z\in\mathbb{D}} \big|\varphi(f_n(z))\big| + \sup_{|z|\leq\delta} \mu(z)\big|u(z)\big|\big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\
&\quad + \sup_{\delta<|z|<1} \mu(z)\big|u(z)\big|\big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\
&\leq \big|u(0)\varphi(f_n(0))\big| + \|u\|_{\mathcal{B}_\mu} \sup_{z\in\mathbb{D}} \big|\varphi(f_n(z))\big| + M \max_{|z|\leq\delta} \mu(z)\big|u(z)\big| \max_{|z|\leq\delta} \big|f_n'(z)\big| \\
&\quad + C_\alpha M_0 M \sup_{\delta<|z|<1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2}.
\end{aligned}$$

Taking the limit as $n \to \infty$ in this inequality, we obtain $\lim_{n\to\infty} \|S_{u,\varphi} f_n\|_{\mathcal{B}_\mu} = 0$. By Lemma 2.4, the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is compact. $\square$

Remark 2.1 Considering Theorem 2.3, we have a reason to regard as the limit

$$\lim_{|z|\to 1^-} \mu(z) \log \frac{2}{1-|z|^2}$$

as an important factor for the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ to be compact.

Theorem 2.4 *Let $u \in H(\mathbb{D})$ and φ an entire function with $\varphi(0) = 0$ and $\varphi'(0) \neq 0$. Then $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is bounded if and only if $u \in \mathcal{B}_{\mu,0}$ and*

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} = 0.$$

Proof. Suppose that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is bounded, then by taking f the constant function we have $u \in \mathcal{B}_{\mu,0}$. Now let us suppose, by the way of contradiction, that

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \neq 0.$$

Then there exist some $\varepsilon_0 > 0$ and a sequence $\{z_n\} \subseteq \mathbb{D}$ with $|z_n| \to 1$ such that

$$\mu(z_n)\big|u(z_n)\big| \log \frac{2}{1-|z_n|^2} \geq \frac{2}{|\varphi'(0)|}\varepsilon_0.$$

Take the function $f = g_{z_n}$. Since $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is bounded, $S_{u,\varphi}f \in \mathcal{B}_{\mu,0}$, that is,

$$\lim_{|z|\to 1} \mu(z)\big|(S_{u,\varphi}f)'(z)\big| = 0;$$

in particular,

$$\lim_{n\to\infty} \mu(z_n)\big|(S_{u,\varphi}f)'(z_n)\big| = 0.$$

Letting $n \to \infty$ in

$$\begin{aligned}
\mu(z_n)\big|(S_{u,\varphi}f)'(z_n)\big| &= \mu(z_n)\big|u'(z_n)\varphi(f(z_n)) + u(z_n)\varphi'(f(z_n))f'(z_n)\big| \\
&\geq \mu(z_n)\big|u(z_n)\big|\big|\varphi'(f(z_n))\big|\big|f'(z_n)\big| - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f(z_n))\big| \\
&\geq \frac{1}{2}\mu(z_n)\big|u(z_n)\big| \log \frac{2}{1-|z_n|^2}\big|\varphi'(f(z_n)\big| - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f(z_n))\big| \\
&\geq \frac{\big|\varphi'(f(z_n))\big|}{|\varphi'(0)|}\varepsilon_0 - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f(z_n))\big|
\end{aligned}$$

arrives at a contradiction.

Conversely, by Theorem 2.1, we know that $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu}$ is bounded. It is enough to prove that for any $f \in \mathcal{Z}$, it holds $S_{u,\varphi}f \in \mathcal{B}_{\mu,0}$. Let $f \in \mathcal{Z}$, $M_1 = \max_{|z|=\|f\|_{\mathcal{Z}}} |\varphi(z)|$ and $M_2 = \max_{|z|=\|f\|_{\mathcal{Z}}} |\varphi'(z)|$. Then for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mu(z)|u'(z)| < \frac{\varepsilon}{2M_1}$$

and

$$\mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} < \frac{\varepsilon}{2M_2\|f\|_{\mathcal{Z}}}$$

for all $z \in \{z \in \mathbb{D} : \delta < |z| < 1\}$. So for $z \in \{z \in \mathbb{D} : \delta < |z| < 1\}$, it follows that

$$\begin{aligned}
\mu(z)\big|(S_{u,\varphi}f)'(z)\big| = \mu(z)\big|(S_{u,\varphi}f)'(z)\big| &= \mu(z)\big|u'(z)\varphi(f(z)) + u(z)\varphi'(f(z))f'(z)\big| \\
&\leq M_1\mu(z)\big|u'(z)\big| + M_2\|f\|_{\mathcal{Z}}\mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \\
&< \varepsilon.
\end{aligned}$$

This shows that $S_{u,\varphi}f \in \mathcal{B}_{\mu,0}$. □

Theorem 2.5 *Let $u \in H(\mathbb{D})$ and φ an entire function with $\varphi(0) = 0$ and $\varphi'(0) \neq 0$. Then the bounded operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is compact if and only if $u \in \mathcal{B}_{\mu,0}$ and*

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} = 0.$$

Proof. Similarly as in the proof of Theorem 2.3, this result is true. □

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3. The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$

Although we can obtain some results of the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ from the preceding discussions, we still will individually consider this operator.

Theorem 3.1 *Let $\alpha \in (0,1)$ and φ an entire function. Then the following assertions hold:*

(i) The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded .

(ii) If $\varphi(0) = 0$, then the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.

Proof. We first prove (i). Let $M > 0$, $f \in \mathcal{Z}_\alpha$ and $\|f\|_{\mathcal{Z}_\alpha} \le M$. Set $M_1 = \max_{|z|=C_\alpha M} |\varphi'(z)|$. Then we have

$$(1-|z|^2)^\beta \big|(S_\varphi f)'(z)\big| = (1-|z|^2)^\beta \big|\varphi'(f(z))\big|\big|f'(z)\big| \le C_\alpha M M_1 (1-|z|^2)^\beta < \infty.$$

This means that the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded.

Now we prove (ii). Suppose that $\|f_n\|_{\mathcal{Z}_\alpha} \le M$ and $\{f_n\}$ uniformly converges to zero on every compact subset of $\mathbb{D}$ as $n \to \infty$, then

$$\begin{aligned}\|S_\varphi f_n\|_{\mathcal{B}_\beta} &= |\varphi(f_n(0))| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \big|(S_\varphi f_n)'(z)\big| \\ &= |\varphi(f_n(0))| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\ &\le |\varphi(f_n(0))| + M_1 \sup_{z\in\mathbb{D}} |f_n'(z)|,\end{aligned}$$

where $M_1 = \max_{|z|=C_\alpha M} |\varphi'(z)|$. By $\varphi(0) = 0$ and Lemma 2.3 (ii), we know that $\lim_{n\to\infty} \|S_\varphi f_n\|_{\mathcal{B}_\beta} = 0$. By Lemma 2.4, the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact. □

When $\alpha = 1$, from Theorem 2.1 and Theorem 2.2 we can obtain characterizations of the boundedness and compactness of the operator $S_\varphi : \mathcal{Z} \to \mathcal{B}_\beta$. It is unnecessary to go into details here.

Theorem 3.2 *Let $\alpha \in (1,2)$ and φ an entire function. We have the following assertions:*

(1) If $\alpha \le 1+\beta$, then (i) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded, and
(ii) when $\varphi(0) = 0$, the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.
(2) If $\alpha > 1+\beta$, then the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if φ is a constant function.

Proof. We first prove the assertion (i) of (1). Let $M > 0$, $f \in \mathcal{Z}_\alpha$ and $\|f\|_{\mathcal{Z}_\alpha} \le M$. Set $M_1 = \max_{|z|=C_\alpha M} |\varphi'(z)|$. Then we have

$$(1-|z|^2)^\beta \big|(S_\varphi f)'(z)\big| = (1-|z|^2)^\beta \big|\varphi'(f(z))\big|\big|f'(z)\big| \le CMM_1 (1-|z|^2)^{1-\alpha+\beta} < \infty.$$

This shows that the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded. As the proof of Theorem 3.1 (ii), the assertion (ii) follows.

Note that we have the relation $\mathcal{Z}_\alpha = \mathcal{B}_{\alpha-1}$. By this and Theorem 4 in [5], the assertion (2) is true. □

Theorem 3.3 *Let $\alpha = 2$ and φ an entire function.*

(1) When $\beta > 1$, (i) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if φ is a polynomial of degree $s \le 1$, and
(ii) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.
(2) When $\beta = 1$, (i) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if φ is a linear function, and
(ii) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.
(3) When $0 < \beta < 1$, the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only φ is a constant function.

Proof. By Theorem 7 of [5], the assertions (i) of (1) and (i) of (2) hold. Also from Theorem 4 of [5], the assertion (3) follows. Now we want to prove the assertion (ii) of (1). Let the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ be compact. From the assertion (i) of (1), we know that, if φ is not a constant function, then $\varphi(z) = az + b$ with $a \neq 0$. Therefore, it is enough to show that $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact when $\varphi(z) = az$. At this time, S_φ is just the multiplication operator M_a defined by $M_a f = a \cdot f$. Thus, by Theorem 3.1 of [19], we know that $M_a : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact. Similar to the proof of the assertion (ii) of (1), the assertion (ii) of (2) is right. $\square$

Theorem 3.4 *Let $\alpha > 2$, $\beta > 1$ and φ an entire function.*

(1) The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if
(i) *when $\alpha > \beta$, φ is a constant.*
(ii) *when $\alpha = \beta$, φ is a linear function.*
(iii) *when $\alpha < \beta$, φ is a polynomial of degree $s \leq \frac{\beta-1}{\alpha-2}$.*

(2) The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact if and only if φ is a polynomial of degree $s < \frac{\beta-1}{\alpha-2}$.

Proof. Note that when $\alpha > 2$, it follows that $\mathcal{Z}_\alpha = \mathcal{B}_{\alpha-1} = H_{\alpha-2}$, where $H_{\alpha-2}$ is called the weighted Banach space of analytic functions defined by

$$H_{\alpha-2} = \{f \in H(\mathbb{D}) : (1 - |z|^2)^{\alpha-2} |f(z)| < \infty\}.$$

Then (1) and (2) follow from Theorem 4.2 of [14] and Proposition 3.1 of [4]. $\square$

Acknowledgments. This work is supported by the Key Fund Project of Sichuan Provincial Department of Education (No.15ZA0221), the Cultivation Project of Sichuan University of Science and Engineering (No.2015PY04) and the innovation foundation for the university students (No.cx20141202).

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Dynamical Analysis Of The Rational Difference Equation $x_{n+1} = \frac{\alpha x_{n-3}}{A+Bx_{n-1}x_{n-3}}$

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ABSTRACT

This article is concerned with the following rational difference equation $x_{n+1} = \frac{\alpha x_{n-3}}{A+Bx_{n-1}x_{n-3}}$ with the initial conditions, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$ are arbitrary real numbers, α, A and B are arbitrary constants. A detailed analytical study of the convergence of the solutions including their dependence on parameters and initial conditions is investigated. The local stability and global attractivity of the difference equation's equilibrium points are discussed. The existence of periodic solutions in the proposed difference equation is also verified analytically. Moreover, numerical simulations are carried out to verify the correctness of the analytical results.

Keywords: Difference equations, Recursive sequences, Analytical study, Infinite products, Convergence, Periodic solution.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Difference equations arise from the study of the evolution of natural phenomena. The applications of difference equations are rapidly increasing to various fields such as economics [1], [12]-[14], mathematical, biology [15]-[16] physics and engineering [7]. Indeed, difference equations represent chief tools of investigating the qualitative behaviors of dynamical systems [33]. Consequently, studying the solutions of difference equations and its qualitative behaviors have become focal topics for research [1]-[36].

In recent years, difference equations have been investigated by many authors. For some results: In [3], Aloqeili found the solution of the difference equation $x_{n+1} = \frac{dx_{n-1}x_{n-k}}{b-cx_{n-s}}$. Cinar [5] obtained the solution of the difference equation $x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}$. In [9], Elabbasy *et al.* discussed the solution and the periodicity character of the difference equations $x_{n+1} = ax_n - \frac{bx_n}{cx_n-dx_{n-1}}$.
In this paper, we study to the following sequence defined recursively by

$$x_{n+1} = \frac{\alpha x_{n-3}}{A+Bx_{n-1}x_{n-3}}, \tag{1}$$

with the initial data: $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$.
Note first that, if $\alpha = 0$, then for all $n \in \mathbb{N}$, $x_n = 0$. Then we will consider that $\alpha \neq 0$. Although we can (by dividing the numerator and denominator by α) obtain a more simply form of such sequences, we will keep them in order to study of the behaviors with respect to α.
Note also that, if one or more of the initial data a, b, c and d is zero, then it will be seen that one or more of the subsequences of $(x_n)_n$ modulo 4 vanish, so that we will suppose that $abcd \neq 0$.
The cases $A = 0$ and $B = 0$ are a trivial, therefore we will assume that $A \neq 0$ and $B \neq 0$. Finally, we will consider the convention: if $(a_p)_p$ is a sequence of complex numbers, and $n > m$, in $\mathbb{Z}$, then $\prod_{p=n}^{m} a_p = 1$.

2. DEFINITIONS AND PRELIMINARIES.

A difference equation of order k is an equation of the form

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-(k-1)}), n = 0, 1, ..., \tag{2}$$

where F is a function that maps on some set I^k into I. A solution of Eq. (2) is a sequence x_n that satisfies Eq. (2) for all $n \geq 0$. With each solution x_n of the Eq. (1), we associate the vector of initial conditions $v_0(x) = (x_0, x_{-1}, ..., x_{-k+1}) \in I^k$.
The norm of the vector $u \in I^k$ will be defined as $\|u\| = \sum_{i=-k+1}^{0} |u_i|$.

Definition 1. (Equilibrium point)
A point $\bar{x} \in \mathbb{R}$ is called an equilibrium point of Eq. (2), if

$$\bar{x} = F(\bar{x}, \bar{x}, ..., \bar{x}).$$

Let $\bar{x} \in \mathbb{R}$ be an equilibrium point of Eq. (2), and denote by $v(\bar{x}) \in I^k$ the vector $v(\bar{x}) = (\bar{x}, \bar{x}, ..., \bar{x})$.
Suppose that the function F is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Consider the linearized equation of Eq. (2) about the equilibrium point $\bar{x}$:

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + ... + q_{k-1} y_{n-(k-1)}, \tag{3}$$

where $q_i = \frac{\partial F}{\partial x_i}(\bar{x}, \bar{x}, ..., \bar{x})$, $i = 0, 1, ..., k-1$, and the characteristic equation of Eq. (3) about $\bar{x}$:

$$\lambda^k - q_0 \lambda^{k-1} - ... - q_{k-2}\lambda - q_{k-1} = 0. \tag{4}$$

Definition 2.

1. When all the roots of Eq. (4) have absolute value less than one, then the equilibrium point of Eq. (2) is locally asymptotically stable.
2. If at least a root of Eq. (4) have absolute value greater than one, then the equilibrium point of Eq. (2) is unstable.

Definition 3.

1. An equilibrium point $\bar{x}$ of Eq. (2) is called hyperbolic if no root of Eq. (4) has absolute value equal one.
2. If there exists a root of Eq. (4) with absolute value equal to one, then the equilibrium point $\bar{x}$ is called nonhyperbolic.
3. An equilibrium point $\bar{x}$ of Eq. (2) is called saddle if there exists a root of Eq. (4) has absolute value less than one. and another root of Eq. (4) greater than one.
4. An equilibrium point $\bar{x}$ of Eq. (2) is called a repeller if all roots of Eq. (4) has absolute value greater than one.
5. A solution x_n of Eq. (2) is called nonoscillatory about $\bar{x}$ or simply nonoscillatory if there exists $N \geq -k$ such that either $x_n \geq \bar{x}$, $\forall n \geq N$ or $x_n \leq \bar{x}$, $\forall n \geq N$. Otherwise, the solution x_n is called oscillatory about $\bar{x}$, or simply oscillatory.
6. A solution x_n of Eq. (2) is called periodic with period p if there exists an integer p, such that

$$x_{n+p} = x_n, \qquad \forall n \geq -k. \tag{5}$$

A solution is called periodic with prime period p if p is the smallest positive integer for which Eq. (5) holds.

3. ANALYTICAL EXPRESSIONS OF $(X_N)_N$

The following Theorem gives an analytical expression of the sequence $(x_n)_n$.

Theorem 1. *Let $(x_n)_n$ be the sequence given by (1) and the initial data that follow, then For all $n \geq 2$*

$$x_{4n-3} = \frac{d\alpha^n \prod_{p=0}^{n-2} \left(A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}, \quad x_{4n-2} = \frac{c\alpha^n \prod_{p=0}^{n-2} \left(A^{2p+2} + Bac \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+1} + Bac \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}. \tag{6}$$

$$x_{4n-1} = \frac{b\alpha^n \prod_{p=0}^{n-1} \left(A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}, \qquad x_{4n} = \frac{a\alpha^n \prod_{p=0}^{n-1} \left(A^{2p+1} + Bac \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+2} + Bac \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}. \tag{7}$$

Proof. By induction, we prove the result for x_{4n-3}. Take $n \geq 2$, and assume that the results hold for the step n, then prove the result for the step $n+1$, we get:

$$\begin{aligned}
x_{4(n+1)-3} &= \frac{\alpha x_{4n-3}}{A + Bx_{4n-1}x_{4n-3}} \\
&= \frac{d\alpha^{n+1} \prod_{p=0}^{n-1} \left(A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right) \left[A \Big(A^{2n} + Bbd \sum_{i=0}^{2n-1} A^i \alpha^{2n-1-i} \Big) + Bbd\alpha^{2n} \right]} \\
&= \frac{d\alpha^{n+1} \prod_{p=0}^{n-1} \left(A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right) \Big(A^{2n+1} + Bbd \Big(\sum_{i=1}^{2n} A^i \alpha^{2n-i} + \alpha^{2n} \Big) \Big)}.
\end{aligned}$$

Hence, we obtain

$$x_{4(n+1)-3} = \frac{d\alpha^{n+1} \prod_{p=0}^{n-1} \left(A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n} \left(A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}.$$

Similarly, the expression for x_{4n-2}, x_{4n-1}, x_{4n} can be easily proved.

Notation. If we denote by $(P_n)_n$ the sequence of two variables polynomials defined for every $n \in \mathbb{N}$, x and y as,

$$P_n(x, y) = (A - \alpha + Bxy)A^n - Bxy\alpha^n.$$

The following Corollary gives a simplified analytic expression when $A \neq \alpha$.

Corollary 1. *Consider the sequence $(x_n)_n$ defined by the Eq. (1) for $A \neq \alpha$, the subsequences can be written as:*

$$x_{4n-3} = \frac{d\alpha^n (A - \alpha) \prod_{p=0}^{n-2} P_{2p+2}(b, d)}{\prod_{p=0}^{n-1} P_{2p+1}(b, d)}, \quad x_{4n-2} = \frac{c\alpha^n (A - \alpha) \prod_{p=0}^{n-2} P_{2p+2}(a, c)}{\prod_{p=0}^{n-1} P_{2p+1}(a, c)},$$

$$x_{4n-1} = \frac{b\alpha^n \prod_{p=0}^{n-1} P_{2p+1}(b,d)}{\prod_{p=0}^{n-1} P_{2p+2}(b,d)}, \qquad and \qquad x_{4n} = \frac{a\alpha^n \prod_{p=0}^{n-1} P_{2p+1}(a,c)}{\prod_{p=0}^{n-1} P_{2p+2}(a,c)}.$$

Proof. It is sufficient to use the binomial identity $x^{p+1}-y^{p+1} = (x-y)\sum_{k=0}^{p} x^k y^{p-k}$ in the analytical expression of the subsequences defined by Eq. (6) and (7).

Corollary 2. *Consider the sequence $(x_n)_n$ defined by the Eq. (1). For $A = \alpha \neq 0$, the sequence can be expressed in Gamma form as*

$$x_{4n-3} = \frac{A2^{2n-2}\Gamma^2(\frac{A}{2Bbd}+n)\Gamma(\frac{A}{Bbd}+1)}{Bb\Gamma^2(\frac{A}{2Bbd}+1)\Gamma(\frac{A}{Bbd}+2n)}, \qquad x_{4n-2} = \frac{A2^{2n-2}\Gamma^2(\frac{A}{2Bac}+n)\Gamma(\frac{A}{Bac})}{Ba\Gamma^2(\frac{A}{2Bac}+1)\Gamma(\frac{A}{Bac}+2n)},$$

$$x_{4n-1} = \frac{b\Gamma(\frac{A}{Bbd}+2n+1)\Gamma^2(\frac{A}{2Bbd}+1)}{2^{2n}\Gamma(\frac{A}{Bbd}+1)\Gamma^2(\frac{A}{2Bbd}+n+1)}, \qquad x_{4n} = \frac{a\Gamma(\frac{A}{Bac}+2n+1)\Gamma^2(\frac{A}{2Bac}+1)}{2^{2n}\Gamma(\frac{A}{Bac}+1)\Gamma^2(\frac{A}{2Bac}+n+1)},$$

where Γ is the Euler's Gamma function.

Proof. Using Eq. (6) we have:

$$\begin{aligned}
x_{4n-3} &= \frac{dA^n \prod_{p=0}^{n-2}\Big(A^{2p+2} + Bbd\sum_{i=0}^{2p+1} A^{2p+1}\Big)}{\prod_{p=0}^{n-1}\Big(A^{2p+1} + Bbd\sum_{i=0}^{2p} A^{2p}\Big)}, \\
&= \frac{dA\prod_{p=0}^{n-2} Bbd\Big(\frac{A}{Bbd}+2p+2\Big)}{\prod_{p=0}^{n-1} Bbd\Big(\frac{A}{Bbd}+2p+1\Big)} = \frac{A\Big[\prod_{p=1}^{n-1} 2\Big(\frac{A}{2Bbd}+p\Big)\Big]^2}{Bb\prod_{p=1}^{2n-1}\Big(\frac{A}{Bbd}+p\Big)} \\
&= \frac{A2^{2n-2}\Gamma^2\Big(\frac{A}{2Bbd}+n\Big)\Gamma(\frac{A}{Bbd}+1)}{Bb\,\Gamma\Big(\frac{A}{Bbd}+2n\Big)\Gamma^2\Big(\frac{A}{2Bbd}+1\Big)}.
\end{aligned}$$

Similarly, one can prove the other relations. This ended the proof.

Remark 1.

1. A common hypothesis in the study of rational difference equations is the choice of positive coefficients and initial data. Therefore, all the solutions will be automatically well defined. It is, in general a problem of great difficulty to determine the good set of initial conditions without finding the analytical expression of the considered sequence.
2. According to the Corollaries 1 and 2, the good set G of the sequence $(x_n)_n$ is given as

   (a) When $A \neq \alpha$,

   $$G = \left\{(a,b,c,d) \in \mathbb{R}^4 \text{ such that } bd,\ ac \in \mathbb{R} - \left\{\frac{-(A-\alpha)A^n}{B(A^n-\alpha^n)},\quad n \in \mathbb{N}\right\}\right\}.$$

(b) When $A = \alpha$, $G = \{(a, b, c, d) \in \mathbb{R}^4$ such that $\frac{A}{Bbd}, \frac{A}{Bac} \notin 2\mathbb{Z}_-\}$.

3. If we choose for example $\alpha = A = B$, we obtain the expression of the general term which can be written and in gamma form as

$$x_{4n-3} = \frac{2^{2n-2}\Gamma^2(\frac{1}{2bd}+n)\Gamma(\frac{1}{bd})}{b\Gamma^2(\frac{1}{2bd}+1)\Gamma(\frac{1}{bd}+2n)}, \qquad x_{4n-2} = \frac{2^{2n-2}\Gamma^2(\frac{1}{2ac}+n)\Gamma(\frac{1}{ac})}{a\Gamma^2(\frac{1}{2ac}+1)\Gamma(\frac{1}{ac}+2n)},$$

$$x_{4n-1} = \frac{b\Gamma(\frac{1}{bd}+2n+1)\Gamma^2(\frac{1}{2bd}+1)}{2^{2n}\Gamma(\frac{1}{bd}+1)\Gamma^2(\frac{1}{2bd}+n+1)}, \qquad x_{4n} = \frac{a\Gamma(\frac{1}{ac}+2n+1)\Gamma^2(\frac{1}{2ac}+1)}{2^{2n}\Gamma(\frac{1}{ac}+1)\Gamma^2(\frac{1}{2ac}+n+1)}.$$

In the following section we will study the convergence of sequence $(x_n)_n$. This will depend evidently on the parameters α, A, B and the initial data.

4. CONVERGENCE OF SOLUTIONS OF *EQ. (1)*

Consider the function F defined on $\mathbb{R}^4$ as: $F(u_0, u_1, u_2, u_3) = \frac{\alpha u_3}{A+Bu_1u_3}$. Using the function F, Eq. (1) can be written as $x_{n+1} = F(x_n, x_{n-1}, x_{n-2}, x_{n-3})$.

Theorem 2. The following statements are true:

(1) For $B(A-\alpha) \geq 0$, Eq.(1) has a unique equilibrium point $\overline{x} = 0$, then

(a) If $A = \alpha$, the equilibrium point is nonhyperbolic.

(b) If $\frac{A}{\alpha} > 1$, the equilibrium point is locally asymptotically stable.

(2) For $B(A-\alpha) < 0$, then

(a) The Eq. (1) has exactly three equilibrium points which are

$$\overline{x}_1 = 0,\ \overline{x}_2 = \sqrt{\tfrac{\alpha-A}{B}},\ \overline{x}_3 = -\sqrt{\tfrac{\alpha-A}{B}}. \tag{8}$$

(b) If $0 < A < \alpha$, then

(i) The equilibrium point $\overline{x}_1 = 0$ is a repeller.

(ii) The equilibrium points $\overline{x}_2$, $\overline{x}_3$ are hyperbolic.

Proof. (1) For $B(A-\alpha) \geq 0$, $\bar{x}$ is an equilibrium point is equivalent to

$$\bar{x} = \frac{\alpha\bar{x}}{A+B\bar{x}^2} \ \Rightarrow\ B\bar{x}^3 + (A-\alpha)\bar{x} = 0 \ \Rightarrow\ \bar{x}(B\bar{x}^2 + A - \alpha) = 0.$$

This shows clearly that if $B(A-\alpha) \geq 0$, $\overline{x} = 0$ is the unique equilibrium point of Eq. (1).
$q_i = \frac{\partial F}{\partial u_i}(0,0,0,0)$, then $q_0 = q_1 = q_2 = 0$ and $q_3 = -\frac{\alpha}{A}$, the characteristic equation of the linearized equation associated with Eq. (1) is then all real roots have absolute value equal to one, so the equilibrium points is nonhyperbolic.$\bar{x}$ is an equilibrium point is equivalent to

$$\lambda^4 - \frac{\alpha}{A} = 0. \tag{9}$$

(a) Suppose that $A = \alpha$, then all real roots have absolute value equal to one, so the equilibrium points is nonhyperbolic.

(b) Suppose that $\frac{A}{\alpha} > 1$, so all the roots of Eq. (9) have absolute value less than one, according the linearized stability Theorem, the equilibrium point $\overline{x} = 0$ is locally asymptotically stable.

(2) For $B(A-\alpha) < 0$, the equation $\bar{x}(B\bar{x}^2 + A - \alpha) = 0$ has exactly three solutions which are the equilibrium points in Eqs. (8).

(a) The characteristic equation about $\overline{x}_1 = 0$ is $\lambda^4 - \frac{\alpha}{A} = 0$, since $0 < A < \alpha$ then all roots has absolute value greater than one and $\overline{x}_1 = 0$ is repeller.

(b) The characteristic equation about $\overline{x}_2$ is $\lambda^4 + \frac{\alpha - A}{A}\lambda^2 - \frac{A}{\alpha} = 0$. The real roots of this equation are $\sqrt{\frac{A}{\alpha}}$ and $-\sqrt{\frac{A}{\alpha}}$, they are less than one, so the equilibrium point $\overline{x}_2$ is hyperbolic. The proof for $\overline{x}_3$ can be similarly obtained.

As it is expected, the convergence of $(x_n)_n$ depends on the parameters α, A, B, and the initial data. We will distinguish the following cases:

(i) Case $|\frac{A}{\alpha}| > 1$.

Theorem 3. *Assume that* $|\frac{A}{\alpha}| > 1$

(1) If $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then every solution of Eq. (1) converges toward zero.

(2) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then the solution of Eq. (1) converges iff $a = b = c = d = \pm\sqrt{\frac{\alpha - A}{B}}$.

(3) If $(A - \alpha + Bbd)(A - \alpha + Bac) = 0$ but not both terms of the product are zero, then every solution of Eq. (1).

Proof. (1) Suppose that $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then Corollary 1 implies that

$$\begin{aligned} x_{4n-3} &= \frac{d\alpha^n (A-\alpha)\prod_{p=0}^{n-2}\Big(A^{2p+2}(A-\alpha+Bbd) - Bbd\alpha^{2p+2}\Big)}{\prod_{p=0}^{n-1}\Big(A^{2p+1}(A-\alpha+Bbd) - Bbd\alpha^{2p+1}\Big)} \\ &= \frac{d\alpha^n (A-\alpha)A^{n-1}\prod_{p=0}^{n-2}\Big(1 - \frac{Bbd}{A-\alpha+Bbd}(\frac{\alpha}{A})^{2p+2}\Big)}{(A-\alpha+Bbd)A^{2n-1}\prod_{p=0}^{n-2}\Big(1 - \frac{Bbd}{A-\alpha+Bbd}(\frac{\alpha}{A})^{2p+1}\Big)}. \end{aligned}$$

Denote by $\beta = \frac{Bbd}{A-\alpha+Bbd}$ and by $(U_p)_p$ the sequence defined as $U_p = \frac{1-\beta(\frac{\alpha}{A})^{2p+2}}{1-\beta(\frac{\alpha}{A})^{2p+1}}$, we get

$$x_{4n-3} = \frac{d(\frac{\alpha}{A})^n (A-\alpha)}{(A-\alpha+Bbd)\Big(1-\beta(\frac{\alpha}{A})^{2n-1}\Big)}\prod_{p=0}^{n-2} U_p.$$

We have either: for $p \in \mathbb{N}$ big enough, $U_p > 1$ or for $p \in \mathbb{N}$ big enough, $0 < U_p < 1$.
Using Taylor expansion of the U_p, we obtain

$$U_p = (1 - \beta(\frac{\alpha}{A})^{2p+2})(1 + \beta(\frac{\alpha}{A})^{2p+1} + o(\frac{\alpha}{A})^{2p+1}) = 1 + \beta(\frac{\alpha}{A})^{2p+1} + o(\frac{\alpha}{A})^{2p+1},$$

then U_p is equivalent to $1 + \beta(\frac{\alpha}{A})^{2p+1}$ which is the general term of a convergent infinite product.
We can easily deduce that $(x_{4n-3})_n$ converges toward zero. same discussion can be obtained for the other subsequences.

(2) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then by the proof of (1), the subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants: $x_{4n-3} = d$ and $x_{4n-1} = b$, also the subsequences $x_{4n-2} = c$ and $x_{4n} = a$. Thus every solution of Eq. (1) converges to a real number l if and only if $a = b = c = d = l$.

(3) Consider for instance the case $A - \alpha + Bbd = 0$ and $A - \alpha + Bac \neq 0$, by (2), the subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants $x_{4n-3} = d$ and $x_{4n-1} = b$, in other hand and also by the proof of case (1), the subsequences $(x_{4n-2})_n$ and $(x_{4n})_n$ converge to zero, then the sequence $(x_n)_n$ diverges. The proof is completed.

(ii) Case $|\frac{A}{\alpha}| = 1$.

Theorem 4. *Assume that* $|\frac{A}{\alpha}| = 1$. We distinguish two subcases, $A = \alpha$ and $A = -\alpha$.

(1) If $A = \alpha$, and let sequence $(x_n)_n$ be the sequence given by the formula (1), then the sequence $(x_n)_n$ converges toward zero.

(2) If $A = -\alpha$, and let sequence $(x_n)_n$ be the sequence given by the formula (1), then we have $x_{4n-1} = \frac{b}{dx_{4n-3}}$, $x_{4n-2} = \frac{c}{ax_{4n}}$ and the sequence $(x_n)_n$ is divergent.

Proof. (1) For $A = \alpha$, let δ the parameter $\delta = \frac{A}{Bbd}$. In the proof of Corollary 2, we find that

$$x_{4n-3} = \frac{A}{Bb(\delta+1)} \prod_{p=1}^{n-1} \Big(\frac{\frac{\delta}{2p}+1}{\frac{\delta+1}{2p}+1} \Big).$$

Denote by $(W_p)_p$ the sequence defined as $W_p = \frac{\frac{\delta}{2p}+1}{\frac{\delta+1}{2p}+1}$, then we get:
For p big enough, we have $0 < W_p < 1$. The Taylor expansion for W_p gives:
$W_p = (1+\frac{\delta}{2p})(1-\frac{\delta+1}{2p}+o(\frac{1}{p})) = 1-\frac{1}{2p}+o(\frac{1}{p})$,
which is a general term of divergent infinite product. Since for p big enough, $0 < W_p < 1$, then $\lim_{n\to\infty} \Pi_{p=1}^{n-1} W_p = 0$. So, we get $\lim_{n\to\infty} x_{4n-3} = 0$. Similarly, one can easily prove that the other subsequences converge to zero, therefore the sequence $(x_n)_n$ converges to zero.

(2) To prove the second part, we replace α by $(-A)$ in the expression of x_{4n-3} of Eq. (6), we obtain

$$x_{4n-3} \quad = \quad \frac{d(-A)^n \prod_{p=0}^{n-2} A^{2p+1}\Big(A+Bbd\sum_{k=0}^{2p+1}(-1)^k\Big)}{\prod_{p=0}^{n-1} A^{2p}\Big(A+Bbd\sum_{k=0}^{2p}(-1)^k\Big)} = \frac{d}{(-1-\delta^{-1})^n}, \ .$$

In other hand, If we replace α by $(-A)$ in the first term of Eq. (7), we obtain

$$x_{4n-1} \quad = \quad b(-A)^n \prod_{p=0}^{n-1} \Big(\frac{A^{2p}(A+Bbd)}{A^{2p+2}}\Big) = b(-1-\delta^{-1})^n.$$

Thus $x_{4n-1} = \frac{b}{dx_{4n-3}}$, hence

(a) If $|1+\delta^{-1}| > 1$, then the subsequence $(x_{4n-3})_n$ converges to zero, so $(|x_{4n-1}|)_n$ goes to infinity.

(b) If $|1+\delta^{-1}| < 1$, then the subsequence $(|x_{4n-3}|)_n$ goes to infinity.

This completed the proof.

(iii) Case $|\frac{A}{\alpha}| < 1$.

Theorem 5. *Let $(x_n)_n$ be the sequence given by the formula (1), then For $|\frac{A}{\alpha}| < 1$, then the subsequences $(x_{4n-3})_n$, $(x_{4n-1})_n$, $(x_{4n-2})_n$ and $(x_{4n})_n$ converge.*

Proof. We need to prove that $(x_{4n-3})_n$ converges. Using Corollary (1), we obtain

$$x_{4n-3} \quad = \quad \frac{d\alpha^n(A-\alpha)\prod_{p=0}^{n-2}\Big(A^{2p+2}(A-\alpha+Bbd)-Bbd\alpha^{2p+2}\Big)}{\prod_{p=0}^{n-1}\Big(A^{2p+1}(A-\alpha+Bbd)-Bbd\alpha^{2p+1}\Big)} = \frac{\alpha-A}{Bb(1-\gamma\lambda^{2n-1})}\prod_{p=0}^{n-2} V_p,$$

where $\gamma = \frac{A-\alpha+Bbd}{Bbd}$, $\lambda = \frac{A}{\alpha}$ and $(V_p)_p$ is the sequence defined by $V_p = \frac{1-\gamma\lambda^{2p+2}}{1-\gamma\lambda^{2p+1}}$. For $p \in \mathbb{N}$ big enough, we have two cases; either $V_p > 1$ or $0 < V_p < 1$. Applying the transformation of infinite product of positive terms to infinite series, and assuming p_0 to be big enough, we get

$$x_{4n-3} = \frac{\alpha - A}{Bb(1-\gamma\lambda^{2n-1})}\Big(\prod_{p=0}^{p_0} V_p\Big)\exp\Big(\sum_{p=p_0+1}^{n-2}\ln(V_p)\Big).$$

It is clear that the sequence $\Big(\frac{\alpha-A}{Bb(1-\gamma\lambda^{2n-1})}\Big)_n$ converges toward $\frac{\alpha-A}{Bb}$. The Taylor expansion of V_p to the first order gives

$$V_p = \frac{1-\gamma\lambda^{2p+2}}{1-\gamma\lambda^{2p+1}} = 1+\gamma(1-\lambda)\lambda^{2p+1}+o(\lambda^{2p+1}).$$

So $\ln(V_p)$ is equivalent to $\gamma(1-\lambda)\lambda^{2p+1}$, which is the general term of a convergent infinite series, then the sequence $(x_{4n-3})_n$ is convergent. Similarly, one can prove that the other subsequences are convergent.

Remark 2. (Commentary on the convergence of $(x_n)_n$ in the case $|\frac{A}{\alpha}| < 1$).
Suppose that $|\frac{A}{\alpha}| < 1$, according to Theorem 5, the subsequences $(x_{4n-3})_n$, $(x_{4n-1})_n$, $(x_{4n-2})_n$ and $(x_{4n})_n$ converge, denote by: l_3, l_2, l_1 and l_0 their limits respectively.
The subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are related by the equations:

$$x_{4(n+1)-3} = \frac{\alpha x_{4n-3}}{A+Bx_{4n-1}x_{4n-3}}, \tag{10}$$

$$x_{4(n+1)-1} = \frac{\alpha x_{4n-1}}{A+Bx_{4(n+1)-3}x_{4n-1}}. \tag{11}$$

Passing to the limit as n goes to infinity in Eq. (10), we obtain $l_3 = \frac{\alpha l_3}{A+Bl_3 l_1}$, then $(S_1): \begin{cases} l_3 = 0, \\ or \\ l_3 \neq 0 \text{ and } l_1 = \frac{\alpha - A}{Bl_3}. \end{cases}$

Passing to the limit as n goes to infinity in Eq. (11), we obtain $l_1 = \frac{\alpha l_1}{A+Bl_3 l_1}$, then $(S_2): \begin{cases} l_1 = 0, \\ or \\ l_1 \neq 0 \text{ and } l_3 = \frac{\alpha - A}{Bl_1}. \end{cases}$

Combining systems (S_1) and (S_2), since $\alpha \neq A$, we obtain

$$\begin{cases} l_3 = l_1 = 0 \\ or \\ l_1 \neq 0, \quad l_3 \neq 0 \text{ and } (S): \begin{cases} l_3 = \frac{\alpha - A}{Bl_1}, \\ and \\ l_1 = \frac{\alpha - A}{Bl_3}. \end{cases} \end{cases}$$

The proposition $l_3 = l_1 = 0$ contradicts the fact that the infinite product $\prod_{p\geq 0} V_p$ converges, in fact if $\lim_{n\to\infty} \prod_{p=0}^{n} V_p = 0$, then $\lim_{n\to\infty} \sum_{p=p_0}^{n} \ln(V_p) = -\infty$, and this is absurd. Hence the only possibility is that

$$l_1 \neq 0, \quad l_3 \neq 0 \text{ and } (S): \begin{cases} l_3 = \frac{\alpha - A}{Bl_1}, \\ and \\ l_1 = \frac{\alpha - A}{Bl_3}. \end{cases}$$

One can easily see that (S) is equivalent to $l_3 = \frac{\alpha - A}{Bl_1}$. Let f be the function defined on $\mathbb{R}^*$ as $f(x) = \frac{\alpha - A}{Bx}$, we have $fof = Id$ and, l_1 and l_3 are related by $f(l_1) = l_3$.

$$f(x) = x \Leftrightarrow \frac{\alpha - A}{Bx} = x \Leftrightarrow x = \mp\sqrt{\frac{\alpha - A}{B}}.$$

Hence: f has fixed points if and only if $\frac{\alpha - A}{B} > 0$.
The numerical example (Figure 4) given in the end of this paper confirm that even we chose $\frac{\alpha - A}{B} > 0$ and $|\frac{A}{\alpha}| < 1$, l_1 and l_3 may be different, which implies the sequence $(x_n)_n$ may converge or diverge.
Finally based on the preview discussion of all preview cases, The following Theorem is now proved.
Theorem 6. (Boundedness of $(x_n)_n$). *The Eq. (1) has an unbounded solutions if and only if $A = -\alpha$.*

5. PERIODICITY CHARACTER OF SOLUTIONS OF *EQ. (1)*

In the sequel, we need the following lemma, which describes sufficient conditions for Eq. (1) to have a periodic solution.
Lemma 1. *Let $(x_n)_{n\geq -3}$ be a solution of Eq. (1) and the initial data that follow. Suppose that there are real numbers l_3, l_2, l_1, l_0 such that $\lim_{n\to\infty} x_{4n-j} = l_j$ for $j = 0, ..., 3$.*

Let $(y_n)_{n\geq -3}$ be the period-4 sequence such that $y_{-j} = l_j$, for all $j = 0, ..., 3$, then the sequence $(y_n)_{n\geq -3}$ is a period-4 solution of Eq. (1). The periodicity results are given by the following Theorem

Theorem 7. *Let $(x_n)_{n\geq -3}$ be a solution of Eq. (1) and the initial data that follow, then*

(1) For $|\frac{A}{\alpha}| > 1$,

(a) If $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then then Eq. (1) has no periodic solutions.

(b) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then the solution of Eq. (1) is a periodic-4 solution.

(c) If either $A - \alpha + Bbd$ or $A - \alpha + Bac$ equals zero but not both of them, then Eq. (1) has a periodic-4 solution.

(2) For $|\frac{A}{\alpha}| = 1$, Eq. (1) has no periodic solutions.

(3) For $|\frac{A}{\alpha}| < 1$, Eq. (1) has periodic-4 solutions.

Proof. (1) Suppose that $|\frac{A}{\alpha}| > 1$,

(a) If $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then by Theorem 3, every solution of Eq. (1) converges to zero, hence, the solutions are not allowed to be periodic (since the solutions are not identically zero).

(b) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then by Theorem 3, the subsequences of $(x_n)_n$ $(x_{4n-j})_n$, $j = 0, .., 3$ are constants: $x_{4n-3} = d$, $x_{4n-2} = c$, $x_{4n-1} = b$ and $x_{4n} = a$, and the sequence $d, c, b, a, d, c, b, a...$ is a periodic-4 solution of Eq. (1).

(c) Consider for instance the case $A - \alpha + Bbd = 0$ and $A - \alpha + Bac \neq 0$, by the proof of Theorem 3, the subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants and equal d and b respectively. Also according to the proof of Theorem 3, the subsequences $(x_{4n-2})_n$ and $(x_{4n})_n$ converge to zero. Applying Lemma 1, the sequence $d, 0, b, 0, d, 0, b, 0, ...$ is a periodic-4 solution of Eq. (1).

(2) The case $A = \alpha$ is similar to (1) (a).
If $A = -\alpha$, then every solution of Eq. (1) is unbounded, so Eq. (1) has no periodic solutions.

(3) If $|\frac{A}{\alpha}| < 1$, then by Theorem 5, there are real numbers l_3, l_2, l_1 and l_0, such that $\lim_{n\to\infty} x_{4n-j} = l_j$ for all $j = 0, .., 3$.
Applying Lemma 1, the sequence $l_3, l_2, l_1, l_0, l_3, l_2, l_1, l_0...$ is a periodic-4 solution of Eq. (1).
This completes the proof.

Remark 3.

(1) Note that if $|\frac{A}{\alpha}| > 1$, $A - \alpha + Bbd = A - \alpha + Bac = 0$, $a = c$, $b = d$, then Eq. (1) has periodic-2 solution $a, b, a, b, ...$

(2) If $|\frac{A}{\alpha}| < 1$, $A - \alpha + Bbd = A - \alpha + Bac = 0$, then, by the proof of Theorem 7, we deduce that the values of the limits of the subsequences are $l_3 = d$, $l_2 = c$, $l_1 = b$ and $l_0 = a$.

6. NUMERICAL SIMULATION

Example 1. Figure (1) illustrates the case $|\frac{A}{\alpha}| > 1$, $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, we choose $a = 2$, $b = -3$, $c = 2$, $d = -2$, $B = 2$, $A = 1.1$ and $\alpha = 1$. We notice that the solution is oscillating about zero with a decreasing amplitude. In fact, according to Theorem 3, the solution has to converge to zero.

Example 2. In order illustrate the case $|\frac{A}{\alpha}| > 1$, $A - \alpha + Bbd = A - \alpha + Bac = 0$, we choose $a = c = 2$, $b = d = -2$, $B = -3$, $A = 13$ and $\alpha = 1$. Figure (2) depicts that the obtained solution is a 2-prime periodic solution. This is coherent with Remark 3.

Example 3. The case $|\frac{A}{\alpha}| > 1$, $A - \alpha + Bbd = 0$ and $A - \alpha + Bac \neq 0$ is illustrated in figure (3), in which we set $a = c = 1$, $b = d = -2$, $B = -2$, $A = 9$ and $\alpha = 1$. The subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants $(x_{4n-3})_n = d$ and $(x_{4n-1})_n = b$, and the subsequences $(x_{4n-2})_n$ and $(x_{4n})_n$ converge to zero. by Lemma 1, the sequence $d, 0, b, 0, d, 0, b, 0, ...$ is a periodic-4 solution of Eq. (1).

Example 4. Figure (4) illustrates the case $|\frac{A}{\alpha}| < 1$, we choose $a = -1$, $b = 0.5$, $c = -0.2$, $d = 0.8$, $B = 1$, $A = 0.5$ and $\alpha = 1$. the subsequences $(x_{4n-3})_n$, $(x_{4n-1})_n$, $(x_{4n-2})_n$ and $(x_{4n})_n$ converge.

Example 5. To illustrate the case $A = \alpha$, we choose $a = 0.1$, $b = 0.2$, $c = 0.3$, $d = -0.4$, $B = 1$, $\alpha = 0.5$ and $A = 0.5$. We notice in the figure (5), that the solution converges to zero (which is coherent to Theorem 4 part (1)), and the Eq. (1) has no periodic solutions (which is coherent to Theorem 7 part (2)).

Example 6. In figure (6) (case $A = -\alpha$), we choose $a = 0.2$, $b = 0.3$, $c = 0.1$, $d = -0.3$, $B = 2$, $\alpha = -0.4$ and $A = 0.4$. We notice that the solution is oscillating about zero with an increasing amplitude and the solution is unbounded, which is coherent to Theorem 4 part (2).

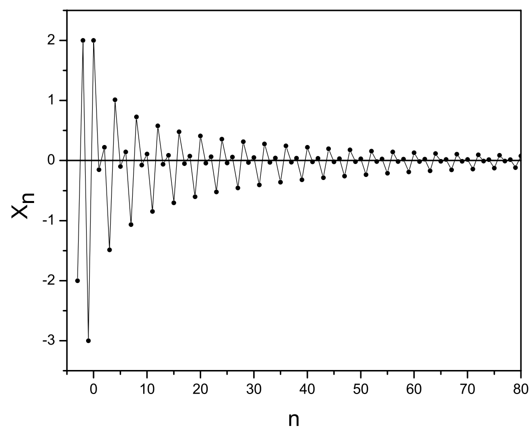

Figure 1.

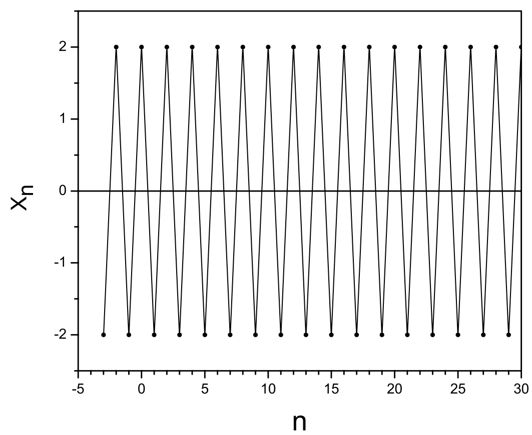

Figure 2.

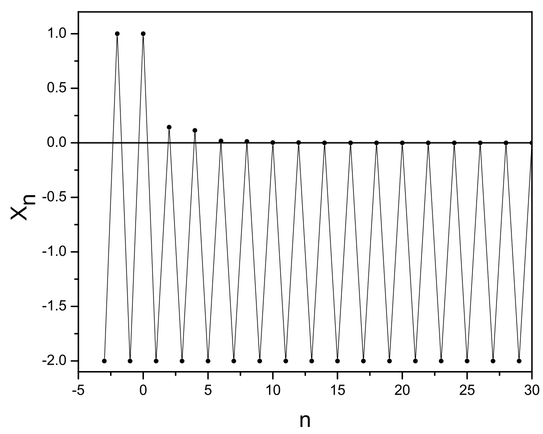

Figure 3.

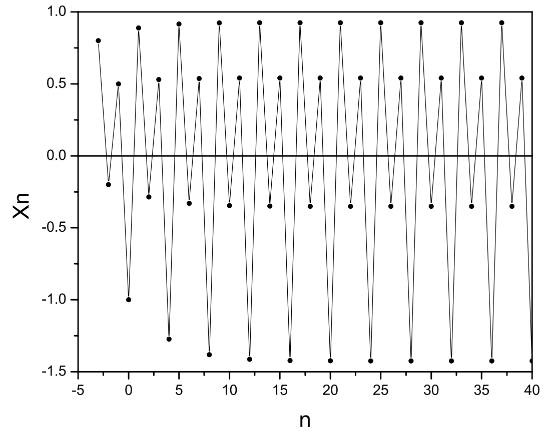

Figure 4.

Conclusion

In this work, some dynamical behaviors of the rational difference equation $x_{n+1} = \frac{\alpha x_{n-3}}{A + B x_{n-1} x_{n-3}}$ with the initial conditions, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$ are arbitrary real numbers, A and B are arbitrary constants, have been investigated. A detailed analytical study of the convergence of the solutions including their dependence on parameters and initial conditions has been illustrated. The local stability and global attractivity of the difference equation's equilibrium points have been demonstrated. The existence of periodic solutions in the proposed difference equation has also been shown analytically. Finally, numerical simulations have been carried out to match the analytical results.

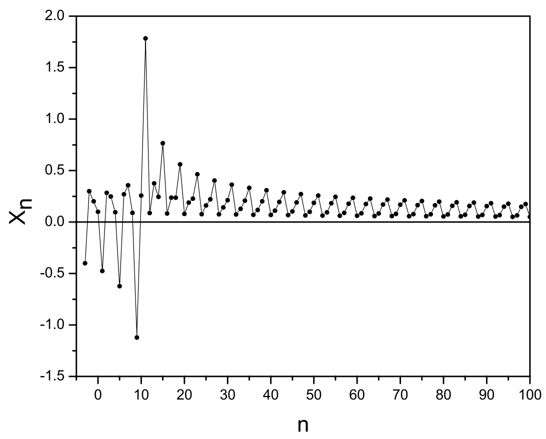

Figure 5.

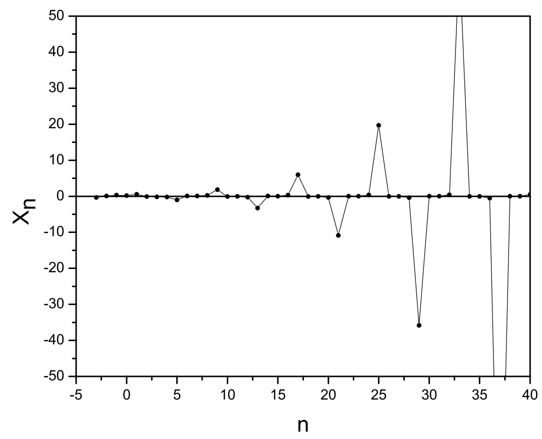

Figure 6.

Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

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QUADRATIC ρ-FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper, we solve the quadratic ρ-functional equations

$$\begin{aligned} &f(x+y)+f(x-y)-2f(x)-2f(y) \\ &\quad = \rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right), \end{aligned} \tag{0.1}$$

where ρ is a fixed non-Archimedean number or a fixed real or complex number with $\rho \neq -1, 2$, and

$$\begin{aligned} &2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y) \\ &\quad = \rho(f(x+y)+f(x-y)-2f(x)-2f(y)), \end{aligned} \tag{0.2}$$

whereρ is a fixed non-Archimedean number or a fixed real or complex number with $\rho \neq -1, \frac{1}{2}$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic ρ-functional equations (0.1) and (0.2) in non-Archimedean Banach spaces and in Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. Gajda [11] following the same approach as in Rassias [22], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [11], as well as by Rassias and Šemrl [21] that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [11], as well as of Rassias and Šemrl [21] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. Găvruta [12], who among others studied the Hyers-Ulam stability of functional equations (cf. the books of Czerwik [8, 9], Hyers, Isac and Th.M. Rassias [14]). The hyperstability of the Cauchy equation was proved by Brzdek [4].

The functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [24] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. See [1, 5, 6, 10, 16, 17, 18, 19, 20, 23] for more

2010 *Mathematics Subject Classification.* Primary 46S10, 39B62, 39B52, 47S10, 12J25.

Key words and phrases. Hyers-Ulam stability; non-Archimedean normed space; quadratic ρ-functional equation.

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functional equations. The survey on the Hyers-Ulam stability of functional equations was given by Brillouet-Bulluot, Brzdek and Cieplinski [3].

The functional equation

$$2f\left(\frac{x+y}{2}\right)+2\left(\frac{x-y}{2}\right)=f(x)+f(y)$$

is called a *Jensen type quadratic equation.*

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s|\le|r|+|s|,\qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s|\le\max\{|r|,|s|\},\qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\le 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0|=0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([15]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|:X\to[0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;
(ii) $\|rx\|=|r|\|x\|\qquad (r\in K, x\in X)$;
(iii) the strong triangle inequality

$$\|x+y\|\le\max\{\|x\|,\|y\|\},\qquad \forall x,y\in X$$

holds. Then $(X,\|\cdot\|)$ is called a *non-Archimedean normed space.*

In Section 2, we solve the quadratic functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the quadratic functional equation (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the quadratic functional equation (0.2) in vector spaces and prove the Hyers-Ulam stability of the quadratic functional equation (0.2) in non-Archimedean Banach spaces.

In Section 4, we prove the Hyers-Ulam stability of the quadratic functional equation (0.1) in Banach spaces.

In Section 5, we prove the Hyers-Ulam stability of the quadratic functional equation (0.2) in Banach spaces.

2. Quadratic ρ-functional equation (0.1) in non-Archimedean Banach spaces

Throughout Sections 2 and 3, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2|\ne 1$ and let ρ be a fixed non-Archimedean number with $\rho\ne -1,2$.

Lemma 2.1. *Let X and Y be vector spaces. A mapping $f:X\to Y$ satisfies*

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0 \tag{2.1}$$

for all $x, y \in X$ if and only if the mapping $f : X \to Y$ satisfies

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) = 0 \tag{2.2}$$

for all $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $f(0) = 0$.

Letting $y = x$ in (2.1), we get $f(2x) - 4f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$. So $f : X \to Y$ satisfies (2.2).

Assume that $f : X \to Y$ satisfies (2.2).

Letting $x = y = 0$ in (2.2), we get $f(0) = 0$.

Letting $y = 0$ in (2.2), we get $4f\left(\frac{x}{2}\right) = f(x)$ for all $x \in X$. and so $f(2x) = 4f(x)$ for all $x \in X$. So $f : X \to Y$ satisfies (2.1). □

We solve the quadratic ρ-functional equation (0.1) in vector spaces.

Lemma 2.2. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$\begin{aligned} f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ = \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \end{aligned} \tag{2.3}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.3).

Letting $x = y = 0$ in (2.3), we get $-2f(0) = 2\rho f(0)$. So $f(0) = 0$.

Letting $y = x$ in (2.3), we get

$$f(2x) - 4f(x) = 0$$

and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.4}$$

for all $x \in X$.

It follows from (2.3) and (2.4) that

$$\begin{aligned} f(x+y) &+ f(x-y) - 2f(x) - 2f(y) \\ &= \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \\ &= \frac{\rho}{2}(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the quadratic ρ-functional equation (2.3) in non-Archimedean Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\lim_{j\to\infty} |4|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) = 0, \tag{2.5}$$

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y) \\ &-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\| \le \varphi(x,y) \end{aligned} \tag{2.6}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x)-h(x)\| \le \sup_{j\in\mathbb{N}}\left\{|4|^{j-1}\varphi\left(\frac{x}{2^j},\frac{x}{2^j}\right)\right\} \tag{2.7}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.6), we get

$$\|f(2x)-4f(x)\| \le \varphi(x,x) \tag{2.8}$$

for all $x \in X$. So

$$\left\|f(x)-4f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2},\frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\|4^l f\left(\frac{x}{2^l}\right)-4^m f\left(\frac{x}{2^m}\right)\right\| \\ &\le \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right)-4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right)-4^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\le \max\left\{|4|^l\left\|f\left(\frac{x}{2^l}\right)-4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right)-4f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\le \sup_{j\in\{l,l+1,\cdots\}}\left\{|4|^j\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}}\right)\right\} \end{aligned} \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.7).

It follows from (2.5) and (2.6) that

$$\begin{aligned} &\|h(x+y)+h(x-y)-2h(x)-2h(y) \\ &-\rho\left(2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-h(x)-h(y)\right)\| \\ &= \lim_{n\to\infty}|4|^n\left\|f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right. \\ &\qquad \left.-\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty}|4|^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)=0 \end{aligned}$$

for all $x, y \in X$. So

$$h(x+y)+h(x-y)-2h(x)-2h(y)=\rho\left(2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-h(x)-h(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.7). Then we have

$$\begin{aligned}\|h(x) - T(x)\| &= \left\|4^q h\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \max\left\{\left\|4^q h\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\|, \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\|\right\} \\ &\le \sup_{j\in\mathbb{N}}\left\{|4|^{q+j-1}\varphi\left(\frac{x}{2^{q+j}}, \frac{x}{2^{q+j}}\right)\right\},\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (2.7). □

Corollary 2.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}&\|f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ &-\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\| \le \theta(\|x\|^r + \|y\|^r)\end{aligned} \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \frac{2\theta}{|2|^r}\|x\|^r$$

for all $x \in X$.

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (2.6) and*

$$\lim_{j\to\infty}\left\{\frac{1}{|4|^j}\varphi(2^{j-1}x, 2^{j-1}y)\right\} = 0$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \sup_{j\in\mathbb{N}}\left\{\frac{1}{|4|^j}\varphi(2^{j-1}x, 2^{j-1}x)\right\} \tag{2.11}$$

for all $x \in X$.

Proof. It follows from (2.8) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{|4|}\varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned}&\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| \\ &\le \max\left\{\left\|\frac{1}{4^l}f\left(2^l x\right) - \frac{1}{4^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{4^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{4^m}f\left(2^m x\right)\right\|\right\} \\ &\le \max\left\{\frac{1}{|4|^l}\left\|f\left(2^l x\right) - \frac{1}{4}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|4|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{4}f\left(2^m x\right)\right\|\right\} \\ &\le \sup_{j\in\{l,l+1,\cdots\}}\left\{\frac{1}{|4|^{j+1}}\varphi(2^j x, 2^j x)\right\}\end{aligned} \tag{2.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.12) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.12), we get (2.11).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.10). *Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2\theta}{|4|}\|x\|^r$$

for all $x \in X$.

3. Quadratic ρ-functional equation (0.2) in non-Archimedean Banach spaces

Let $|2| \ne 1$ and let ρ be a fixed non-Archimedean number with $\rho \ne -1, \frac{1}{2}$.

We solve the quadratic ρ-functional equation (0.2) in vector spaces.

Lemma 3.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$\begin{aligned} 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned} \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $2f(0) = -2\rho f(0)$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$4f\left(\frac{x}{2}\right) - f(x) = 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4} f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} &\frac{1}{2}(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \\ &= 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &= \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the quadratic ρ-functional equation (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\lim_{j\to\infty} \left\{ |4|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)\right\} = 0, \\ &\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &-\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \le \varphi(x, y) \end{aligned} \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \sup_{j\in\mathbb{N}} \left\{ |4|^{j-1} \varphi\left(\frac{x}{2^{j-1}}, 0\right)\right\} \tag{3.4}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi(x, 0) \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned} &\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \\ &\le \max\left\{ \left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \right\} \\ &\le \max\left\{ |4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1} \left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\| \right\} \\ &\le \sup_{j\in\{l,l+1,\cdots\}} \left\{ |4|^j \varphi\left(\frac{x}{2^j}, 0\right)\right\} \end{aligned} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.3. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &-\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \le \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{3.7}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \theta\|x\|^r$$

for all $x \in X$.

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (3.3) and*

$$\lim_{j\to\infty} \left\{ \frac{1}{|4|^j} \varphi(2^j x, 2^j y)\right\} = 0$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \sup_{j\in\mathbb{N}} \left\{ \frac{1}{|4|^j}\varphi(2^j x, 0) \right\} \tag{3.8}$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{|4|}\varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \\ &\leq \max\left\{ \left\| \frac{1}{4^l} f\left(2^l x\right) - \frac{1}{4^{l+1}} f\left(2^{l+1}x\right) \right\|, \cdots, \left\| \frac{1}{4^{m-1}} f\left(2^{m-1}x\right) - \frac{1}{4^m} f\left(2^m x\right) \right\| \right\} \\ &\leq \max\left\{ \frac{1}{|4|^l} \left\| f\left(2^l x\right) - \frac{1}{4} f\left(2^{l+1}x\right) \right\|, \cdots, \frac{1}{|4|^{m-1}} \left\| f\left(2^{m-1}x\right) - \frac{1}{4} f\left(2^m x\right) \right\| \right\} \\ &\leq \sup_{j\in\{l+1,l+2,\cdots\}} \left\{ \frac{1}{|4|^j}\varphi(2^j x, 0) \right\} \end{aligned} \tag{3.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proof of Theorems 2.3. □

Corollary 3.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying (3.7). Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{|2|^r \theta}{|4|} \|x\|^r$$

for all $x \in X$.

4. Quadratic ρ-functional equation (0.1) in Banach spaces

Throughout Sections 4 and 5, assume that X is a normed space and that Y is a Banach space. Let ρ be a fixed real or complex number with $\rho \neq -1, 2$.

We prove the Hyers-Ulam stability of the quadratic ρ-functional equation (2.3) in Banach spaces.

Theorem 4.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{4.1}$$

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ &-\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\| \leq \varphi(x, y) \end{aligned} \tag{4.2}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{4}\Psi(x, x) \tag{4.3}$$

for all $x \in X$.

Proof. Letting $y = x$ in (4.2), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x, x) \tag{4.4}$$

for all $x \in X$. So

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned} \tag{4.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.5) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.5), we get (4.3).

Now, let $T : X \to Y$ be another quadratic mapping satisfying (4.3). Then we have

$$\begin{aligned}\|Q(x) - T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le \frac{4^q}{2}\Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q.

It follows from (4.1) and (4.2) that

$$\begin{aligned}&\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) \\ &-\rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\| \\ &= \lim_{n\to\infty} 4^n \left\|f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right. \\ &\qquad \left. -\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0\end{aligned}$$

for all $x, y \in X$. So

$$Q(x+y)+Q(x-y)-2Q(x)-2Q(y)=\rho\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic. □

Corollary 4.2. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}&\|f(x+y)+f(x-y)-2f(x)-2f(y)\\&-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\|\le\theta(\|x\|^r+\|y\|^r)\end{aligned}\tag{4.6}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\|\le\frac{2\theta}{2^r-4}\|x\|^r$$

for all $x \in X$.

Theorem 4.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (4.2) and*

$$\Psi(x,y):=\sum_{j=0}^{\infty}\frac{1}{4^j}\varphi(2^jx,2^jy)<\infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\|\le\frac{1}{4}\Psi(x,x)\tag{4.7}$$

for all $x \in X$.

Proof. It follows from (4.4) that

$$\left\|f(x)-\frac{1}{4}f(2x)\right\|\le\frac{1}{4}\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^lx)-\frac{1}{4^m}f(2^mx)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^jx\right)-\frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\|\\ &\le \sum_{j=l}^{m-1}\frac{1}{4^{j+1}}\varphi(2^jx,2^jx)\end{aligned}\tag{4.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.8) that the sequence $\{\frac{1}{4^n}f(2^nx)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^nx)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x):=\lim_{n\to\infty}\frac{1}{4^n}f(2^nx)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.7).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (4.6). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{4 - 2^r}\|x\|^r$$

for all $x \in X$.

5. Quadratic ρ-functional equation (0.2) in Banach spaces

Let ρ be a fixed real or complex number with $\rho \neq -1, \frac{1}{2}$.

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional equation (3.1) in Banach spaces.

Theorem 5.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty,$$

$$\begin{aligned} &\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &-\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \leq \varphi(x, y) \end{aligned} \tag{5.1}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \Psi(x, 0) \tag{5.2}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (5.1), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| = \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x, 0) \tag{5.3}$$

for all $x \in X$. So

$$\begin{aligned} \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j}, 0\right) \end{aligned} \tag{5.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.4) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.4), we get (5.2).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 5.2. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &-\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \leq \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{5.5}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2^r\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Theorem 5.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (5.1) and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \Psi(x, 0) \tag{5.6}$$

for all $x \in X$.

Proof. It follows from (5.3) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| &\le \sum_{j=l+1}^{m} \left\|\frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l+1}^{m} \frac{1}{4^j}\varphi(2^j x, 0) \end{aligned} \tag{5.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.7) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.7), we get (5.6).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 5.4. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (5.5). Then there exists a unique quadartic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2^r\theta}{4 - 2^r}\|x\|^r$$

for all $x \in X$.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.3, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

ON MODIFIED DEGENERATE GENOCCHI POLYNOMIALS AND NUMBERS

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Abstract. In this paper, we consider the modified partially degenerate Genocchi polynomials and investigate some properties of these polynomials. From these properties, we give some new and interesting identities of them.

1. Introduction

The Genocchi polynomials are defined by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} \qquad \text{(see [2, 3, 7, 9, 12, 14, 17, 19, 27, 28])}. \tag{1}$$

When $x=0$, $G_n=G_n(0)$ are called the Genocchi numbers. From (1), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} &= \left(\frac{2t}{e^t+1}\right)e^{xt}\\ &= \left(\sum_{l=0}^{\infty}G_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}x^m\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}G_lx^{n-l}\right)\frac{t^n}{n!}. \end{aligned}\tag{2}$$

1991 *Mathematics Subject Classification.* 05A10, 11B68, 11S80, 05A19.
Key words and phrases. Euler polynomials, Genocchi polynomials, degenerate Genocchi polynomials, modified degenerate Genocchi polynomials.

Thus, by comparing the coefficients on both sides of (2), we get

$$G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}. \tag{3}$$

From (1), we can derive the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} &= -\frac{-2t}{e^{-t}+1}e^{-(1-x)t}\\ &= \sum_{n=0}^{\infty}(-1)^{n-1}G_n(1-x)\frac{t^n}{n!}.\end{aligned} \tag{4}$$

By comparing the coefficients on both sides of (4), we get

$$G_n(x) = (-1)^{n-1}G_n(1-x). \tag{5}$$

The gamma and beta function are defined by the following definite integrals: for $(\alpha > 0, \beta > 0)$,

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt \tag{6}$$

and

$$\begin{aligned}B(\alpha,\beta) &= \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt\\ &= \int_0^{\infty}\frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}dt\end{aligned} \tag{7}$$

(see [15,23,24]).Thus by (6) and (7), we get

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{8}$$

The classical Genocchi numbers, a sequence of integers introduced by Angelo Genocchi (1817-1889), have been studied in various context in such diverse areas of mathematics and physics as number theory, combinatorics, complex analysis, topology, and quantum physics. In recent years, Genocchi polynomials and numbers have received considerable attention and many researchers have worked on them, their extensions and their connections with some combinatorial counting.

The degenerate Bernoulli polynomials, the rst degenerate version of well-known families of polynomials, were introduced by Carlitz and rediscovered by Ustinov under the name of Korobov polynomials of the second kind. On the other hand, Korobov polynomials (of the rst kind) are the degenerate version of the Bernoulli polynomials of the second kind. Recently, many researchers began to study various kinds of degenerate versions of the familiar polynomials like Bernoulli, Euler, Genocchi, falling factorial and Bell polynomials by using generating functions, umbral calculus, and p-adic integrals.

The goal of this paper is to introduce the modified degenerate Genocchi polynomials and numbers, a degenerate version of the classical Genocchi polynomials and numbers, in order to study their properties and obtain several new and interesting identities involving them. More precisely, we give some properties, explicit formulas, several identities, a connection with Genocchi polynomials, and some integral formulas. Here they were named as the modified degenerate Genocchi polynomials, since there existed what are called the degenerate Genocchi polynomials whose definition is slightly different from ours (see [1, 4-6, 8, 11-16, 18, 20, 21, 22-26, 28]).

2. Modified degenerate Genocchi polynomials

First, we note that

$$e^t = \lim_{\lambda\to 0}(1+\lambda)^{\frac{t}{\lambda}}, \qquad t = loge^t = \lim_{\lambda\to 0} log(1+\lambda)^{\frac{t}{\lambda}} = \lim_{\lambda\to 0}\frac{t}{\lambda}log(1+\lambda). \tag{9}$$

From (1) and (9), we define the modified degenerate Genocchi polynomials as

$$\frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}(1+\lambda)^{\frac{tx}{\lambda}} = \sum_{n=0}^{\infty} g_{n,\lambda}(x)\frac{t^n}{n!} \tag{10}$$

When $x=0$, $g_{n,\lambda} = g_{n,\lambda}(0)$ are called the modified degenerate Genocchi numbers. From (10), we get

$$\begin{aligned} 2t &= \left((1+\lambda)^{\frac{t}{\lambda}}+1\right)\left(\frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}\right) \\ &= \frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}(1+\lambda)^{\frac{t}{\lambda}} + \frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1} \\ &= \sum_{n=0}^{\infty} g_{n,\lambda}(1)\frac{t^n}{n!} + \sum_{n=0}^{\infty} g_{n,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}(g_{n,\lambda}(1)+g_{n,\lambda})\frac{t^n}{n!}. \end{aligned} \tag{11}$$

By comparing the coefficients on both sides of (11), we get

$$\begin{cases} g_{0,\lambda} = 0 \\ g_{n,\lambda}(1) + g_{n,\lambda} = 2\delta_{1,n}, \end{cases} \tag{12}$$

where $\delta_{1,n}$ is the Kronecker delta. From (10), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} g_{n,\lambda}(x)\frac{t^n}{n!} &= \left(\sum_{m=0}^{\infty} g_{m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}\left(\frac{log(1+\lambda)}{\lambda}\right)^m x^m\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}g_{n-m,\lambda}\left(\frac{log(1+\lambda)}{\lambda}\right)^m x^m\right)\frac{t^n}{n!}. \end{aligned} \tag{13}$$

Thus, by comparing the coefficients on both sides of (13), we obtain the following theorem.

Theorem 2.1. *Let $n \in \mathbb{N}\cup\{0\}$. Then we have*

$$g_{n,\lambda}(x) = \sum_{m=0}^{n}\binom{n}{m}g_{n-m,\lambda}\left(\frac{log(1+\lambda)}{\lambda}\right)^m x^m. \tag{14}$$

From (10), we derive the following equation:

$$\begin{aligned} \sum_{n=0}^{\infty} g_{n,\lambda}(x)\frac{t^n}{n!} &= -\frac{-2t}{(1+\lambda)^{\frac{-t}{\lambda}}+1}(1+\lambda)^{\frac{-(1-x)t}{\lambda}} \\ &= \sum_{n=0}^{\infty}(-1)^{n-1}g_{n,\lambda}(1-x)\frac{t^n}{n!}. \end{aligned} \tag{15}$$

By comparing the coefficients on both sides of (15),

$$g_{n,\lambda}(x) = (-1)^{n-1}g_{n,\lambda}(1-x) \quad (n \geq 0). \tag{16}$$

By (10), we see that

$$\frac{d}{dx}g_{n,\lambda}(x)=g_{n-1,\lambda}(x)\left(\frac{log(1+\lambda)}{\lambda}\right)n \quad (n\geq 1). \tag{17}$$

From (17), we get

$$\begin{aligned}\frac{g_{n+1,\lambda}(1)-g_{n+1,\lambda}}{n+1} &= \int_0^1 \frac{d}{dx}\frac{g_{n+1,\lambda}(x)}{n+1}dx \\ &= \int_0^1 g_{n,\lambda}(x)\left(\frac{log(1+\lambda)}{\lambda}\right)dx. \quad (n\geq 1).\end{aligned} \tag{18}$$

By (18), we obtain the following theorem.

Theorem 2.2. *Let $n\in\mathbb{N}\cup\{0\}$. Then we have*

$$\frac{g_{n+1,\lambda}(1)-g_{n+1,\lambda}}{n+1}=\int_0^1 g_{n,\lambda}(x)\left(\frac{log(1+\lambda)}{\lambda}\right)dx. \tag{19}$$

We note that the Stirling numbers of the first kind are defined as

$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l, \quad (n\geq 0), \quad (\text{see } [1,4-6,8,11-16,18,20,21,22-26,28]). \tag{20}$$

where $(x)_n=x(x-1)\cdots(x-n+1)(n\geq 1)$, and $(x)_0=1$. By (10), we see that

$$\begin{aligned}&\frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}(1+\lambda)^{\frac{tx}{\lambda}} \\ =& \left(\sum_{k=0}^{\infty}g_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{tx}{\lambda}\right)_m\lambda^m\right) \\ =& \left(\sum_{k=0}^{\infty}g_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m}S_1(m,l)\left(\frac{tx}{\lambda}\right)^l\right)\frac{\lambda^m}{m!} \\ =& \left(\sum_{k=0}^{\infty}g_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\left(\sum_{m=l}^{\infty}S_1(m,l)\left(\frac{x}{\lambda}\right)^l\frac{\lambda^m}{m!}l!\right)\frac{t^l}{l!}\right) \\ =& \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=l}^{\infty}\binom{n}{l}g_{n-l,\lambda}S_1(m,l)\left(\frac{x}{\lambda}\right)^l\frac{\lambda^m}{m!}l!\right)\frac{t^n}{n!}\end{aligned} \tag{21}$$

From (21), we obtain the following theorem.

Theorem 2.3. *Let $n\in\mathbb{N}\cup\{0\}$. Then we have*

$$g_{n,\lambda}(x)=\sum_{l=0}^{n}\sum_{m=l}^{\infty}\binom{n}{l}g_{n-l,\lambda}S_1(m,l)\left(\frac{x}{\lambda}\right)^l\frac{\lambda^m}{m!}l!. \tag{22}$$

Let d be an odd integer. Then we see that

$$\begin{aligned}&2t\sum_{l=0}^{d-1}(-1)^l(1+\lambda)^{\frac{lt}{\lambda}} \\ =& \frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}}\left(1-\left(-(1+\lambda)^{\frac{t}{\lambda}}\right)^d\right)\end{aligned}$$

$$\begin{aligned}
&= \frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}}\left(1+(1+\lambda)^{\frac{dt}{\lambda}}\right)\\
&= \frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}}+\frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}}(1+\lambda)^{\frac{dt}{\lambda}}\\
&= \sum_{n=1}^{\infty} g_{n,\lambda}\frac{t^n}{n!}+\sum_{n=1}^{\infty} g_{n,\lambda}(d)\frac{t^n}{n!}\\
&= \sum_{n=1}^{\infty}(g_{n,\lambda}+g_{n,\lambda}(d))\frac{t^n}{n!}\\
&= t\sum_{n=0}^{\infty}\left(\frac{g_{n+1,\lambda}+g_{n+1,\lambda}(d)}{n+1}\right)\frac{t^n}{n!}. \qquad (23)
\end{aligned}$$

Also, we see that

$$\begin{aligned}
&2\sum_{l=0}^{d-1}(-1)^l(1+\lambda)^{\frac{lt}{\lambda}}\\
&= 2\sum_{l=0}^{d-1}\left(\sum_{n=0}^{\infty}(-1)^l\left(\frac{log(1+\lambda)}{\lambda}\right)^n l^n\right)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(2\sum_{l=0}^{d-1}(-1)^l\left(\frac{log(1+\lambda)}{\lambda}\right)^n l^n\right)\frac{t^n}{n!}. \qquad (24)
\end{aligned}$$

From (23) and (24), we obtain the following theorem.

Theorem 2.4. *Let $n \in \mathbb{N}\cup\{0\}$. Then we have*

$$2\sum_{l=0}^{d-1}(-1)^l\left(\frac{log(1+\lambda)}{\lambda}\right)^n l^n = \frac{g_{+1n,\lambda}+g_{n+1,\lambda}(d)}{n+1}. \qquad (25)$$

From (10) and (14), we note that

$$\begin{aligned}
\int_0^1 y^n g_{n,\lambda}(x+y)dy &= \sum_{m=0}^{n}\binom{n}{m}g_{n-m,\lambda}\left(\frac{log(1+\lambda)}{\lambda}\right)^m\int_0^1 y^{n+m}dy\\
&= \sum_{m=0}^{n}\binom{n}{m}\frac{g_{n-m,\lambda}(x)}{n+m+1}\left(\frac{log(1+\lambda)}{\lambda}\right)^m \qquad (26)
\end{aligned}$$

By (16), we get

$$\begin{aligned}
&\int_0^1 y^n g_{n,\lambda}(x+y)dy = (-1)^{n-1}\int_0^1 y^n g_{n,\lambda}(1-(x+y))dy\\
&= (-1)^{n-1}\sum_{m=0}^{n}\binom{n}{m}g_{n-m}(-x)\left(\frac{log(1+\lambda)}{\lambda}\right)^m\int_0^1 y^n(1-y)^m dy\\
&= \sum_{m=0}^{n}\binom{n}{m}(-1)^m g_{n-m,\lambda}(1+x)\left(\frac{log(1+\lambda)}{\lambda}\right)^m B(n+1,m+1)\\
&= \sum_{m=0}^{n}\binom{n}{m}(-1)^m\frac{g_{n-m,\lambda}(1+x)}{n+m+1}\left(\frac{log(1+\lambda)}{\lambda}\right)^m\binom{n+m}{m}^{-1} \qquad (27)
\end{aligned}$$

By (26) and (27), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} & \sum_{m=0}^{n} \binom{n}{m} \frac{g_{n-m,\lambda}(x)}{n+m+1} \left(\frac{log(1+\lambda)}{\lambda} \right)^m \\ = & \sum_{m=0}^{n} \binom{n}{m} (-1)^m \frac{g_{n-m,\lambda}(1+x)}{n+m+1} \left(\frac{log(1+\lambda)}{\lambda} \right)^m \binom{n+m}{m}^{-1} \end{aligned} \tag{28}$$

From (17), we note that

$$\begin{aligned} & \int_0^1 y^n g_{n,\lambda}(x+y)dy \\ = & \frac{g_{n,\lambda}(x+1)}{n+1} - \frac{n}{n+1} \frac{log(1+\lambda)}{\lambda} \int_0^1 y^{n+1} g_{n-1,\lambda}(x+y)dy \\ = & \frac{g_{n,\lambda}(x+1)}{n+1} - \frac{g_{n-1,\lambda}(x+1)}{n+1} \frac{n}{n+2} \frac{log(1+\lambda)}{\lambda} \\ & +(-1)^2 \frac{n(n-1)}{(n+1)(n+2)} \left(\frac{log(1+\lambda)}{\lambda} \right)^2 \int_0^1 y^{n+2} g_{n-2,\lambda}(x+y)dy \\ = & \frac{g_{n,\lambda}(x+1)}{n+1} - \frac{g_{n-1,\lambda}(x+1)}{n+1} \frac{n}{n+2} \frac{log(1+\lambda)}{\lambda} \\ & +(-1)^2 \frac{g_{n-2,\lambda}(x+1)}{n+1} \frac{n(n-1)}{(n+2)(n+3)} \left(\frac{(log(1+\lambda)}{\lambda} \right)^2 \\ & +(-1)^3 \frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)} \left(\frac{(log(1+\lambda)}{\lambda} \right)^3 \int_0^1 y^{n+3} g_{n-3,\lambda}(x+y)dy \end{aligned} \tag{29}$$

By continuing this process, we have

$$\begin{aligned} \int_0^1 y^n g_{n,\lambda}(x+y)dy = & \frac{g_{n,\lambda}(x+1)}{n+1} \\ & + \sum_{m=1}^{n-1} (-1)^m \frac{n(n-1)\cdots(n-m+1)}{(n+1)(n+2)\cdots(n+m+1)} \left(\frac{(log(1+\lambda)}{\lambda} \right)^m g_{n-m,\lambda}(x+1) \end{aligned} \tag{30}$$

Therefore by (26) and (30), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{N}$, we have*

$$\sum_{m=0}^{n} \binom{n}{m} \frac{g_{n-m,\lambda}(x)}{n+m+1} = \sum_{m=0}^{n-1} (-1)^m \frac{\binom{n}{m}}{\binom{n+m}{m}} \frac{g_{n-m,\lambda}(x+1)}{n+m+1} \left(\frac{(log(1+\lambda)}{\lambda} \right)^m \tag{31}$$

Taking $x = 0$, From (16) and (31), we obtain the following corollary.

Corollary 2.7. *For $n \in \mathbb{N}$, we have*

$$\sum_{m=0}^{n} \binom{n}{m} \frac{g_{n-m,\lambda}}{n+m+1} = \sum_{m=0}^{n-1} (-1)^m \frac{\binom{n}{m}}{\binom{n+m}{m}} \frac{g_{n-m,\lambda}(1)}{n+m+1} \left(\frac{(log(1+\lambda)}{\lambda} \right)^m \tag{32}$$

For $n \in \mathbb{N}$, we observe that

$$\int_0^1 y^n g_{n,\lambda}(x+y)dy$$

$$
\begin{aligned}
&= \frac{\lambda}{log(1+\lambda)}\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{\lambda}{log(1+\lambda)}\frac{n}{n+1}\int_0^1 y^{n-1}g_{n+1,\lambda}(x+y)dy \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n-1}(-1)^n g_{n+1,\lambda}(1-(x+y))dy\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}g_{n+1-l,\lambda}(-x)(-1)^n\int_0^1 y^{n-1}(1-y)^l dy\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}g_{n+1-l,\lambda}(-x)(-1)^n B(n,l+1)\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{1}{n+1}\sum_{l=0}^{n+1}\frac{\binom{n+1}{l}}{\binom{n+l}{l}}(-1)^n g_{n+1-l,\lambda}(-x)\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{1}{n+1}\sum_{l=0}^{n+1}\frac{\binom{n+1}{l}}{\binom{n+l}{l}}(-1)^l g_{n+1-l,\lambda}(1+x)\right)
\end{aligned}
$$

(33)

Therefore, by (30) and (33), we obtain the following theorem.

Theorem 2.8. *For $n \in \mathbb{N}$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n-1}(-1)^l\frac{\binom{n}{l}}{\binom{n+l}{l}}\frac{g_{n-l,\lambda}(1+x)}{n+l+1}\left(\frac{log(1+\lambda)}{\lambda}\right)^{l+1} \\
&= \frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{1}{n+1}\sum_{l=0}^{n+1}(-1)^l\frac{\binom{n+1}{l}}{\binom{n+l}{l}}g_{n+1-l,\lambda}(1+x) \qquad (34)
\end{aligned}
$$

Replacing λ by $e-1$ and t by $(e-1)t$ in (10), we get

$$
\begin{aligned}
\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} &= \frac{2t}{e^t+1}e^{xt} \\
&= \sum_{n=0}^{\infty}g_{n,e-1}(x)(e-1)^{n-1}\frac{t^n}{n!}, \qquad (35)
\end{aligned}
$$

where $G_n(x)$ are the Genocchi polynomials. By comparing both sides of (35), we obtain the following theorem.

Theorem 2.9. *For $n \in \mathbb{N}\cup\{0\}$, we have*

$$
G_n(x) = g_{n,e-1}(x)(e-1)^{n-1}. \qquad (36)
$$

By (12) and (18), we get

$$
\begin{aligned}
\int_0^1 g_{n,\lambda}(x)dx &= \frac{\lambda}{log(1+\lambda)}(n+1)^{-1}\int_0^1 \frac{d}{dx}g_{n+1,\lambda}(x)dx \\
&= \frac{\lambda}{log(1+\lambda)}(n+1)^{-1}(g_{n+1,\lambda}(1) - g_{n+1,\lambda}(0)) \\
&= \frac{(-2)\lambda}{log(1+\lambda)}(n+1)^{-1}g_{n+1,\lambda} \qquad (37)
\end{aligned}
$$

where $n \in \mathbb{N}$. Also, we have

$$\begin{aligned}
&\int_0^1 g_{n,\lambda}(x)g_{m,\lambda}(x)dx \\
= \;& \frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}g_{n+1,\lambda}(x)g_{m,\lambda}(x)\mid_0^1 - \frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}\int_0^1 g_{n+1,\lambda}(x)\frac{d}{dx}g_{m,\lambda}(x)dx \\
= \;& \frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}(g_{n+1,\lambda}(1)g_{m,\lambda}(1) - g_{n+1,\lambda}(0)g_{m,\lambda}(0) \\
&- \frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}\frac{log(1+\lambda)}{\lambda}m\int_0^1 g_{n+1,\lambda}(x)g_{m-1,\lambda}(x)dx \\
= \;& -\frac{m}{n+1}\int_0^1 g_{n+1,\lambda}(x)g_{m-1,\lambda}(x)dx \\
= \;& (-1)^2\frac{m(m-1)}{(n+1)(n+2)}\int_0^1 g_{n+2,\lambda}(x)g_{m-2,\lambda}(x)dx \qquad (38)
\end{aligned}$$

By continuing this process, we obtain the following theorem.

Theorem 2.10. *For $m, n \in \mathbb{N}$, we have*

$$\begin{aligned}
&\int_0^1 g_{n,\lambda}(x)g_{m,\lambda}(x)dx \\
= \;& (-1)^{m-2}\frac{m(m-1)\cdots 3}{(n+1)(n+2)\cdots(n+m-2)}\int_0^1 g_{n+m-2,\lambda}(x)g_{2,\lambda}(x)dx. \qquad (39)
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\int_0^1 g_{n+m-2,\lambda}(x)g_{2,\lambda}(x)dx \\
= \;& -\frac{2}{n+m-1}\int_0^1 g_{n+m-1,\lambda}(x)g_{1,\lambda}(x)dx \\
= \;& -\frac{2}{n+m-1}\frac{g_{n+m,\lambda}(x)}{n+m}\frac{\lambda}{log(1+\lambda)}|_0^1 \\
= \;& -\frac{2}{n+m-1}\frac{\lambda}{log(1+\lambda)}\frac{-2g_{n+m,\lambda}}{n+m}. \qquad (40)
\end{aligned}$$

By (41), we obtain the following theorem.

Theorem 2.11. *For $m, n \in \mathbb{N}$, we have*

$$\begin{aligned}
&\int_0^1 g_{n,\lambda}(x)g_{m,\lambda}(x)dx \\
= \;& (-1)^m 2\binom{n+m}{m}^{-1}\frac{\lambda}{log(1+\lambda)}g_{n+m,\lambda}. \qquad (41)
\end{aligned}$$

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Hesitant fuzzy implicative filters in BE-algebras

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Abstract. The notion of hesitant fuzzy implicative filter of a BE-algebra is introduced and related properties are investigated. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy implicative filter. Also, as a generalization of hesitant fuzzy implicative filter, we consider the hesitant fuzzy n-fold implicative filter. Characterizations of hesitant fuzzy n-fold implicative filter are discussed.

1. Introduction

In 2007, Kim and Kim [5] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Song et al. [8] considered the fuzzification of ideals in BE-algebras. They introduced the notion of fuzzy ideals in BE-algebras, and investigated related properties. They gave characterizations of a fuzzy ideal in BE-algebras.

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [9] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [11, 12, 13, 14, 15]), and is applied to residuated lattices and MTL-algebras (see [4, 6]).

In this paper, we introduce the notion of hesitant fuzzy implicative filter of a BE-algebra, and investigate some properties of it. We consider characterizations of hesitant fuzzy implicative filter of a BE-algebra. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy implicative filter. Also, as a generalization of hesitant fuzzy implicative filter, we consider the hesitant fuzzy n-fold implicative filter. We discuss characterizations of hesitant fuzzy n-fold implicative filter.

2. Preliminaries

[0]**2010 Mathematics Subject Classification**: 06F35; 03G25; 06D72.

[0]**Keywords**: filter; implicative filter; hesitant fuzzy (implicative) filter.

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By a *BE-algebra* ([5]) we mean a system $(X;*,1)$ of type $(2,0)$ which the following axioms hold:

(2.1) $(\forall x \in X)\,(x * x = 1)$,
(2.2) $(\forall x \in X)\,(x * 1 = 1)$,
(2.3) $(\forall x \in X)\,(1 * x = x)$,
(2.4) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)$ (exchange).

We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$.

A BE-algebra $(X;*,1)$ is said to be *transitive* ([5]) if it satisfies: for any $x, y, z \in X$, $y * z \leq (x * y) * (x * z)$. A BE-algebra $(X;*,1)$ is said to be *self distributive* ([5]) if it satisfies: for any $x, yz \in X$, $x * (y * z) = (x * y) * (x * z)$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general ([5]).

Every self distributive BE-algebra $(X;*,1)$ satisfies the following properties:

(2.5) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z)$,
(2.6) $(\forall x, y \in X)\,(x * (x * y) = x * y)$,
(2.7) $(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))$,

Definition 2.1.([5]) Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* of X if

(F1) $1 \in F$;
(F2) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

F is an *implicative filter* of X if it satisfies (F1) and

(F3) $(\forall x, y, z \in X)(x * (y * z), x * y \in F \Rightarrow x * z \in F)$.

Definition 2.2.([9]) Let E be a reference set. A *hesitant fuzzy set* on E is defined in terms of a function that when applied to E returns a subset of $[0,1]$, which can be viewed as the following mathematical representation:

$$H_E := \{(e, h_E(e)) | e \in E\}$$

where $h_E : E \to \mathscr{P}([0,1])$.

Definition 2.3. Given a non-empty subset A of X, a hesitant fuzzy set

$$H_X := \{(x, h_X(x)) | x \in X\}$$

on satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A \tag{2.8}$$

is called a *hesitant fuzzy set related to* A (briefly, A-*hesitant fuzzy set*) on X, and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a mapping from X to $\mathscr{P}([0,1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

For a hesitant set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of a BE-algebra X and a subset γ of $[0,1]$, the hesitant fuzzy γ-inclusive set of H_X, denoted by $H_X(\gamma)$, is defined to be the set

$$H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}.$$

For any hesitant fuzzy set $H_X = \{(x, h_X(x)|x \in X\}$ and $G_X = \{(x, g_X(x))|x \in X\}$, we call H_X a *hesitant fuzzy subset* of G_X, denoted by $H_X \widetilde{\subseteq} G_X$, if $h_X(x) \subseteq g_X(x)$ for all $x \in X$. The *hesitant fuzzy union* of H_X and G_X, denoted by $H_X \widetilde{\cup} G_X$, is defined to be the hesitant fuzzy set $(h_X \widetilde{\cup} g_X)(x) = h_X(x) \cup g_X(x)$ for all $x \in X$. The *hesitant fuzzy intersection* of H_X and G_X, denoted by $H_X \widetilde{\cap} G_X$, is defined to be the hesitant fuzzy set $(h_X \widetilde{\cap} g_X)(x) = h_X(x) \cap g_X(x)$ for all $x \in X$.

3. **Hesitant fuzzy implicative filters**

Definition 3.1.([3]) Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy filter of X related to A* (briefly, *A-hesitant fuzzy filter* of X) if it satisfies the following condition:

$$(\forall x \in A)\,(h_A(x) \subseteq h_A(1))\,, \tag{3.1}$$

$$(\forall x, y \in A)\,(h_A(x * y) \cap h_A(x) \subseteq h_A(y))\,. \tag{3.2}$$

An A-hesitant fuzzy filter of X with $A = X$ is called a *hesitant fuzzy filter* of X.

Proposition 3.2.([3]) *Let $H_A := \{(x, h_A(x))|x \in X\}$ be an A-hesitant fuzzy filter of X where A is a subalgebra of X. Then the following assertions are valid.*

(i) $(\forall x, y \in A)(x \leq y \Rightarrow h_A(x) \subseteq h_A(y))$,
(ii) $(\forall x, y, z \in A)(h_A(x * (y * z)) \cap h_A(y) \subseteq h_A(x * z))$,
(iii) $(\forall a, x \in A)(h_A(a) \subseteq h_A((a * x) * x)$.

Definition 3.3. Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy implicative filter of X related to A* (briefly, *A-hesitant fuzzy implicative filter* of X) if it satisfies (3.1) and

$$(\forall x, y, z \in A)\,(h_A(x * (y * z)) \cap h_A(x * y) \subseteq h_A(x * z))\,. \tag{3.3}$$

An A-hesitant fuzzy implicative filter of X with $A = X$ is called a *hesitant fuzzy implicative filter* of X.

Example 3.4. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	1	a
d	1	1	a	1	1	a
0	1	1	1	1	1	1

For a subalgebra $A = \{1, a, b\}$ of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X defined by

$$H_A = \left\{(1, [0, 1]), (a, (0, \tfrac{1}{2}]), (b, (0, \tfrac{1}{2})), (c, (0, \tfrac{1}{4})), (d, \emptyset), (0, \emptyset)\right\}$$

It is easy to check that H_A is an A-hesitant fuzzy implicative filter of X.

Proposition 3.5. *Every A-hesitant fuzzy implicative filter of a BE-algebra X is an A-hesitant fuzzy filter of X.*

Proof. Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy implicative filter of X. It follows from (2.4) and (3.3) that

$$\begin{aligned} h_A(y * (x * z)) \cap h_A(x * y) &= h_A(x * (y * z)) \cap h_A(x * y) \\ &\subseteq h_A(x * z) \end{aligned} \tag{3.4}$$

for any $x, y, z \in X$. Setting $x := 1$ in (3.4), we have $h_A(y * z) \cap h_A(y) \subseteq h_A(z)$. Therefore $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy filter of X. □

The converse of Proposition may not be true in general (see Example 3.6).

Example 3.6. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra as in Example 3.4. Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined as follows:

$$h_X : X \to \mathscr{P}([0, 1]),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of $[0, 1]$ with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that H_X is a hesitant fuzzy filter of X. But it is not a hesitant fuzzy implicative filter of X, since $h_X(d * (a * 0)) \cap h_X(d * a) = \gamma_2 \nsubseteq \gamma_1 = h_X(d * 0)$.

We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy implicative filter.

Proposition 3.7. *Let X be a self distributive BE-algebra. Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy filter of X satisfying*

$$(\forall x, y, z \in X)(h_X(x * (y * (y * z))) \cap h_X(y * x)) \subseteq h_X(y * z). \tag{3.5}$$

Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X.

Proof. Since $x*(y*z) = y*(x*z) \leq (x*y)*(x*(x*z)) = x*(y*(x*z)) = y*(x*(x*z))$ for all $x, y \in X$, we have $h_X(x*(y*z)) \subseteq h_X(y*(x*(x*z)))$ by Proposition 3.2(i). It follows from (3.5) that $h_X(x*z) \supseteq h_X(y*(x*(x*z)) \cap h_X(x*y) \supseteq h_X(x*(y*z)) \cap h_X(x*y)$. Thus $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

Theorem 3.8. *Let X be a transitive BE-algebra. For any hesitant fuzzy filter $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X, the following are equivalent:*

(i) $H_X := \{(x, h_X(x)) \mid x \in X\}$ *is a hesitant fuzzy implicative filter,*
(ii) $(\forall x, y \in X)\,(h_X(x*(x*y)) \subseteq h_X(x*y))$,
(iii) $(\forall x, y, z \in X)\,(h_X(x*(y*z)) \subseteq h_X((x*y)*(x*z)))$.

Proof. (i)⇒(ii) Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. Setting $z := y, y := x$ in (3.3), we get

$$\begin{aligned} h_X(x*y) \supseteq & h_X(x*(x*y)) \cap h_X(x*x) \\ = & h_X(x*(x*y)) \cap h_X(1) \\ = & h_X(x*(x*y)). \end{aligned}$$

Hence (ii) holds.
(ii)⇒(iii) Suppose that (ii) holds. Since $x*(y*z) \leq x*((x*y)*(x*z)) = x*(x*((x*y)*z))$, by Proposition 3.2(i) we have $h_X(x*((x*y)*(x*z))) = h_X(x*(x*((x*y)*z))) \supseteq h_X(x*(y*z))$. It follows from (ii) that

$$\begin{aligned} h_X((x*y)*(x*z)) = & h_X(x*((x*y)*z)) \\ \supseteq & h_X(x*(x*((x*y)*z))) \\ \supseteq & h_X(x*(y*z)). \end{aligned}$$

Thus (iii) holds.
(iii)⇒(ii) Assume that (iii) holds. By (3.2) and (iii), we have

$$\begin{aligned} h_X(x*z) \supseteq & h_X((x*y)*(x*z)) \cap h_X(x*y) \\ \supseteq & h_X(x*(y*z)) \cap h_X(x*y). \end{aligned}$$

Therefore $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

Theorem 3.9. *Let X be a self distributive BE-algebra. Then the hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X is a hesitant fuzzy implicative filter of X if and only if it is a hesitant fuzzy filter of X.*

Proof. By Proposition 3.5, every hesitant fuzzy implicative filter of X is a hesitant fuzzy filter of X.

Conversely, assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy filter of X. For any $x, y, z \in X$, by (3.2) we have

$$\begin{aligned} h_X(x * z) \supseteq & h_X((x * y) * (x * z)) \cap h_X(x * y) \\ = & h_X(x * (y * z)) \cap h_X(x * y). \end{aligned}$$

Hence $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

For any element x and y of a BE-algebra X and positive integer n, let $x^n * y$ denote $x * (\cdots * (x * (x * y)) \cdots)$ in which x occurs n times, and $x^0 * y = 1$.

Definition 3.10. Let X be a BE-algebra and let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X. Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is called a *hesitant fuzzy n-fold implicative filter* of X if it satisfies (3.1) and

(3.6) $(\forall x, y, z \in X) \left(h_X(x^n * (y * z)) \cap h_X(x^n * y)\right) \subseteq h_X(x^n * z))$.

Note that a hesitant fuzzy 1-fold implicative filter of X is a hesitant fuzzy implicative filter of X.

Example 3.11. Let $X := \{1, a, b, c, d, 0\}$ is a transitive BE-algebra ([11]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined as follows:

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x \in \{1, b, c\} \\ \gamma_1 & \text{if } x \in \{a, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy n-fold implicative filter of X.

Theorem 3.12. *Every hesitant n-fold fuzzy implicative filter of X is a hesitant fuzzy filter of X.*

Proof. Taking $x := 1$ in (3.6) and (2.3), we have $h_X(z) \supseteq h_X(y * z) \cap h_X(y)$. Hence $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy filter of X. □

The converse of Theorem 3.12 may not be not true in general (see Example 3.13).

Example 3.13. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra as in Example 3.11. Let H_X be a hesitant fuzzy set on X defined as follows:

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that H_X is a hesitant fuzzy filter of X. But it is not a hesitant fuzzy 1-fold implicative filter of X, since $h_X(d * c) = h_X(b) = \gamma_1 \not\supseteq \gamma_2 = h_X(1) = h_X(d * (b * c)) \cap h_X(d * b)$.

Theorem 3.14. *Let X be a transitive BE-algebra. For any hesitant fuzzy filter $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X, the following are equivalent:*

(i) $H_X := \{(x, h_X(x)) \mid x \in X\}$ *is a hesitant fuzzy n-fold implicative filter,*
(ii) $(\forall x, y \in X)\,(h_X(x^{n+1} * y) \subseteq h_X(x^n * y))$,
(iii) $(\forall x, y, z \in X)\,(h_X(x^n * (y * z)) \subseteq h_X((x^n * y) * (x^n * z)))$.

Proof. (i)⇒(ii) Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy n-fold implicative filter of X. Setting $z := y, y := x$ in (3.6), we have

$$\begin{aligned} h_X(x^n * y) \supseteq& h_X(x^n * (x * y)) \cap h_X(x^n * x) \\ =& h_X(x^{n+1} * y) \cap h_X(1) \\ =& h_X(x^{n+1} * y). \end{aligned}$$

Hence (ii) holds.

(ii)⇒(iii) Suppose that (ii) holds. Since $x^n * (y * z) \leq x^n * ((x^n * y) * (x^n * z))$, we have $h_X(x^n * ((x^n * y) * (x^n * z))) \supseteq h_X(x^n * (y * z))$. Since $x^{n+1} * (x^{n-1} * ((x^n * y) * z)) = x^n * (x^n * ((x^n * y) * z))) = x^n * ((x^n * y)) * (x^n * z))$ and using (ii), we have

$$\begin{aligned} h_X(x^{n+1} * (x^{n-2} * ((x^n * y) * z)) =& h_X(x^n * (x^{n-1} * ((x^n * y) * z)) \\ \supseteq& h_X(x^{n+1} * (x^{n-1} * ((x^n * y) * z))) \\ =& h_X(x^n * ((x^n * y) * (x^n * z))) \\ \supseteq& h_X(x^n * (y * z)). \end{aligned} \tag{3.7}$$

By (ii) and (3.7), we have

$$\begin{aligned} h_X(x^{n+1} * (x^{n-3} * ((x^n * y) * z))) =& h_X(x^n * (x^{n-2} * ((x^n * y) * z))) \\ \supseteq& h_X(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) \\ \supseteq& h_X(x^n * (y * z)). \end{aligned}$$

Continuing this process, we conclude that

$$\begin{aligned} h_X((x^n * y) * (x^n * z)) =& h_X(x^n * ((x^n * y) * z)) \\ \supseteq& h_X(x^n * (y * z)). \end{aligned}$$

(iii)⇒(i) Let $x, y, z \in X$. It follows from (iii) that

$$\begin{aligned} h_X(x^n * z) \supseteq & h_X((x^n * y) * (x^n * z)) \cap h_X(x^n * y) \\ \supseteq & h_X((x^n * (y * z)) \cap h_X(x^n * y). \end{aligned}$$

Hence $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant n-fold fuzzy implicative filter □

Definition 3.15. Let n be a positive integer. A BE-algebra X is said to be *n-fold implicative* if it satisfies the equality $x^{n+1} * y = x^n * y$ for all $x, y \in X$.

Corollary 3.16. *In an n-fold implicative BE-algebra, the notion of hesitant fuzzy filters and hesitant fuzzy n-fold implicative filters coincide.*

Proof. Straightforward. □

Theorem 3.17. *A hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of a BE-algebra X is a hesitant fuzzy implicative filter of X if and only if the hesitant fuzzy γ-inclusive set $H_X(\gamma)$ is an implicative filter of X for all $\gamma \in \mathscr{P}([0, 1])$ with $H_X(\gamma) \neq \emptyset$.*

The filter $H_X(\gamma)$ in Theorem 3.17 is called the *γ-inclusive filter* of X.

Proof. Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. Let $x, y, z \in X$ and $\gamma \in \mathscr{P}([0, 1])$ be such that $x * (y * z) \in H_X(\gamma)$ and $x * y \in H_X(\gamma)$. Then $\gamma \subseteq h_X(x * (y * z))$ and $\gamma \subseteq h_X(x * y)$. Using (3.1) and (3.3), we have $\gamma \subseteq h_X(1)$ and $\gamma \subseteq h_X(x * (y * z) \cap h_X(x * y) \subseteq h_X(x * z)$ for $x, y, z \in X$. Hence $1 \in H_X(\gamma)$ and $x * z \in H_X(\gamma)$. Thus $H_X(\gamma)$ is an implicative filter of X.

Conversely, suppose that $H_X(\gamma)$ is an implicative filter of X for all $\gamma \in \mathscr{P}([0, 1])$ with $H_X(\gamma) \neq \emptyset$. For any $x \in X$, let $h_X(x) = \gamma$. Since $H_X(\gamma)$ is an implicative filter of X, we have $1 \in H_X(\gamma)$ and so $h_X(x) = \gamma \subseteq h_X(1)$. For any $x, y, z \in X$, let $h_X(x*(y*z)) = \gamma_{x*(y*z)}$ and $h_X(x*y) = \gamma_{x*y}$. Take $\gamma = \gamma_{x*(y*z)} \cap \gamma_{x*y}$. Then $x * (y * z) \in H_X(\gamma)$ and $x * y \in H_X(\gamma)$ which imply that $x * z \in H_X(\gamma)$. Hence

$$h_X(x * z) \supseteq \gamma = \gamma_{x*(y*z)} \cap \gamma_{x*y} = h_X(x * (y * z)) \cap h_X(x * y).$$

Thus $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

Theorem 3.18. *Every hesitant fuzzy implicative filter of a BE-algebra can be represented as a hesitant fuzzy γ-inclusive set of a hesitant fuzzy implicative filter.*

Proof. Let F be an implicative filter of a BE-algebra X. For a subset γ of $[0, 1]$, define a hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X by

$$h_X : X \to \mathscr{P}([0, 1]),\ x \mapsto \begin{cases} \gamma & \text{if } x \in F, \\ \emptyset & \text{if } x \notin F. \end{cases}$$

Obviously, $F = H_X(\gamma)$. We now prove that H_X is a hesitant fuzzy implicative filter of X. Since $1 \in F = H_X(\gamma)$, we have $h_X(1) = \gamma \supseteq h_X(x)$ for all $x \in X$. Let $x, y, z \in X$. If $x*(y*z), x*y \in F$, then

$x*z \in F$ because F is an implicative filter of X. Hence $h_X(x*(y*z)) = h_X(x*y) = h_X(x*z) = \gamma$, and so $h_X(x*(y*z)) \cap h_X(x*y) \subseteq h_X(x*z)$. If $x*(y*z) \in F$ and $x*y \notin F$, then $h_X(x*(y*z)) = \gamma$ and $h_X(x*y) = \emptyset$ which imply that

$$h_X(x*(y*z)) \cap h_X(x*y) = \gamma \cap \emptyset = \emptyset \subseteq h_X(x*z).$$

Similarly, if $x*(y*z) \notin F$ and $x*y \in F$, then $h_X(x*(y*z)) \cap h_X(x*y) \subseteq h_X(x*z)$. Obviously, if $x*(y*z) \notin F$ and $x*y \notin F$, then $h_X(x*(y*z)) \cap h_X(x*y) \subseteq h_X(x*z)$. Therefore $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

For two elements a and b of X, consider a hesitant fuzzy set $H_X^{a,b} := \{(x, h_X^{a,b} x)) \mid x \in X\}$ where

$$h_X^{a,b} : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_1 & \text{if } a*(b*x) = 1, \\ \gamma_2 & \text{otherwise,} \end{cases}$$

where γ_1 and γ_2 are subsets of X with $\gamma_2 \subsetneq \gamma_1$.

There exist $a, b \in X$ such that $H_X^{a,b}$ is not a hesitant fuzzy implicative filter of X (see Example 3.19).

Example 3.19. Consider the BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given in Example 3.4. Then $H_X^{1,a}$ is not a hesitant fuzzy implicative filter of X since

$$h_X^{1,a}(1*(a*b)) \cap h_X^{1,a}(1*a) = \gamma_1 \nsubseteq h_X^{1,a}(1*b) = \gamma_2.$$

Now we provide a condition for the hesitant fuzzy set $H_X^{a,b}$ to be a hesitant fuzzy implicative filter of X for all $a, b \in X$.

Theorem 3.20. *If X is a self distributive BE-algebra, then the hesitant fuzzy set $H_X^{a,b}$ is a hesitant fuzzy implicative filter of X for all $a, b \in X$.*

Proof. Let $a, b \in X$. Obviously, $h_X^{a,b}(1) \supseteq h_X^{a,b}(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $a*(b*(x*(y*z))) \neq 1$ or $a*(b*(x*y)) \neq 1$. Then $h_X^{a,b}(x*(y*z)) = \gamma_2$ or $h_X^{a,b}(x*y) = \gamma_2$. Hence

$$h_X^{a,b}(x*(y*z)) \cap h_X^{a,b}(x*y) = \gamma_2 \subseteq h_X^{a,b}(x*z).$$

Assume that $a*(b*(x*(y*z)))=1$ and $a*(b*(x*y))=1$. Then

$$\begin{aligned}1&=a*(b*(x*(y*z)))\\&=a*(b*((x*y)*(x*z)))\\&=a*((b*(x*y))*(b*(x*z)))\\&=(a*(b*(x*y)))*(a*(b*(x*z)))\\&=1*(a*(b*(x*z)))\\&=a*(b*(x*z)),\end{aligned}$$

and so $h_X^{a,b}(x*(y*z))\cap h_X^{a,b}(x*y)=\gamma_1=h_X^{a,b}(x*z)$. Therefore $H_X^{a,b}$ is a hesitant fuzzy implicative filter of X for all $a,b\in X$. □

Theorem 3.21. *If H_X and G_X are hesitant fuzzy implicative filters of a BE-algebra X, then the hesitant fuzzy intersection $H_X\tilde{\cap}G_X$ of H_X and G_X is a hesitant fuzzy implicative filter of X.*

Proof. For any $x\in X$, we have

$$(h_X\tilde{\cap}g_X)(1)=h_X(1)\cap g_X(1)\supseteq h_X(x)\cap g_X(x)=(h_X\tilde{\cap}g_X)(x).$$

Let $x,y,z\in X$. Then

$$\begin{aligned}(h_X\tilde{\cap}g_X)(x*z)&=h_X(x*z)\cap g_X(x*z)\\&\supseteq(h_X(x*(y*z))\cap h_X(x*y))\cap(g_X(x*(y*z))\cap g_X(x*y))\\&=(h_X(x*(y*z))\cap g_X(x*(y*z)))\cap(h_X(x*y)\cap g_X(x*y))\\&=(h_X\tilde{\cap}g_X)(x*(y*z))\cap(h_X\tilde{\cap}g_X)(x*y).\end{aligned}$$

Hence $H_X\tilde{\cap}G_X$ is a hesitant fuzzy implicative filter of X. □

The hesitant fuzzy union of hesitant fuzzy implicative filters of a BE-algebra X may not be a hesitant fuzzy implicative filter of X as the following example.

Example 3.22. Let $X=\{1,a,b,c,d\}$ is a BE-algebra with the following Cayley table ([5]):

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let H_X and G_X be hesitant fuzzy sets of X defined, respectively, as follows:

$$h_X:X\to\mathscr{P}([0,1]),\ x\mapsto\begin{cases}\gamma_3 & \text{if } x\in\{1,b\}\\ \gamma_1 & \text{if } x\in\{a,c,d\}\end{cases}$$

and

$$g_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_4 & \text{if } x \in \{1, a, c\} \\ \gamma_2 & \text{if } x \in \{b, d\} \end{cases}$$

where γ_1, γ_2, γ_3, and γ_4 are subsets of $[0, 1]$ with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3 \subsetneq \gamma_4$. It is easy to check that H_X and G_X are hesitant fuzzy implicative filters of X. But $H_X \tilde{\cup} G_X$ is not a hesitant fuzzy implicative filter of X, since

$$\begin{aligned}(h_X \tilde{\cup} g_X)(1 * (c * d)) \cap (h_X \tilde{\cup} g_X)(1 * c) &= (h_X \tilde{\cup} g_X)(b) \cap (h_X \tilde{\cup} g_X)(c) \\ &= (h_X(b) \cup g_X(b)) \cap (h_X(c) \cup g_X(c)) \\ &= \gamma_3 \cap \gamma_4 = \gamma_3 \nsubseteq \gamma_2 = \gamma_1 \cup \gamma_2 \\ &= h_X(1 * d) \cup g_X(1 * d) = (h_X \tilde{\cup} g_X)(1 * d).\end{aligned}$$

Let H_X be a hesitant fuzzy set set of a BE-algebra X. For any $a, b \in X$ and $k \in \mathbb{N}$, consider the set

$$h_X[a^k; b] := \left\{x \in X \mid h_X\left(a^k * (b * x)\right) = h_X(1)\right\}$$

where $h_X(a^k * x) = h_X(a * (a * (\cdots * (a * (a * x)) \cdots)))$ in which a appears k-times. Note that $a, b, 1 \in h_X[a^k; b]$ for all $a, b \in X$ and $k \in \mathbb{N}$.

Proposition 3.23. *Let H_X be a hesitant fuzzy set of a BE-algebra X such that the condition (3.1) and $h_X(x*y) = h_X(x) \cup h_X(y)$ for all $x, y \in X$. For any $a, b \in X$ and $k \in \mathbb{N}$, if $x \in h_X[a^k; b]$, then $y * x \in h_X[a^k; b]$ for all $y \in X$.*

Proof. Assume that $x \in h_X[a^k; b]$. Then $h_X(a^k * (b * x)) = h_X(1)$, and so

$$\begin{aligned}h_X(a^k * (b * (y * x))) &= h_X(a^k * (y * (b * x))) \\ &= h_X(y * (a^k * (b * x))) \\ &= h_X(y) \cup h_X(a^k * (b * x)) \\ &= h_X(y) \cup h_X(1) = h_X(1)\end{aligned}$$

for all $y \in X$ by the exchange property of the operation $*$. Hence $y*x \in h_X[a^k; b]$ for all $y \in X$. □

Proposition 3.24. *For any hesitant fuzzy set H_X of a BE-algebra X, let $a \in X$ satisfy the following condition $a * x = 1$ for all $x \in X$. Then $h_X[a^k; b] = X = h_X[b^k; a]$ for all $b \in X$ and $k \in \mathbb{N}$.*

Proof. For any $x \in X$, we have

$$\begin{aligned} h_X(a^k * (b * x)) &= h_X(a^{k-1} * (a * (b * x))) \\ &= h_X(a^{k-1} * (b * (a * x))) \\ &= h_X(a^{k-1} * (b * 1)) \\ &= h_X(1), \end{aligned}$$

and so $x \in h_X[a^k; b]$. Similarly, $x \in h_X[b^k; a]$. □

Proposition 3.25. *Let X be a self distributive BE-algebra and let H_X be an order-preserving soft set of X with the property (3.1). If $b \leq c$ in X, then $h_X[a^k; c] \subseteq h_X[a^k; b]$ for all $a \in X$ and $k \in \mathbb{N}$.*

Proof. Let $a, b, c, \in X$ be such that $b \leq c$. For any $k \in \mathbb{N}$, if $x \in h_X[a^k; c]$, then

$$\begin{aligned} h_X(1) = h_X(a^k * (c * x)) &= h_X(c * (a^k * x)) \\ &\subseteq h_X(b * (a^k * x)) = h_X(a^k * (b * x)) \end{aligned}$$

by (2.5), Proposition 3.2(i) and (2.4), and so $h_X(a^k * (b * x)) = h_X(1)$. Thus $x \in h_X[a^k; b]$, which completes the proof. □

The following example shows that there exists a hesitant fuzzy set H_X of X, $a, b \in X$ and $k \in \mathbb{N}$ such that $h_X[a^k; b]$ is not a filter of X.

Example 3.26. Let $X = \{1, a, b, c\}$ is a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let H_X be a hesitant fuzzy set of X U defined as follows:

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. Then it is a hesitant fuzzy set of X. But $h_X[c; b] = \{x \in X | h_X(c*(b*x)) = h_X(1)\} = \{1, a, b\}$ is not an implicative filter, since $1*(a*c) = a \in h_X[c; b]$, $1 * a = a \in h_X[c; b]$ and $1 * c = c \notin h_X[c; b]$.

We provide conditions for a set $h_X[a^k; b]$ to be an implicative filter.

Theorem 3.27. *Let H_X be a hesitant fuzzy set of a self distributive BE-algebra X. If h_X is injective, then $h_X[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.*

Proof. Assume that X is a self distributive BE-algebra and h_X is injective. Obviously, $1 \in h_X[a^k; b]$. Let $a, b, x, y, z \in X$ and $k \in \mathbb{N}$ be such that $x * (y * z) \in h_X[a^k; b]$ and $x * y \in h_X[a^k; b]$. Then $h_X(a^k * (b * (x * (y * z)))) = h_X(1)$ which implies that $a^k * (b * (x * (y * z))) = 1$ since h_X is injective. Since X is a self distributive BE-algebra, we have

$$\begin{aligned}
h_X(1) &= h_X(a^k * (b * (x * (y * z)))) \\
&= h_X(a^{k-1} * (a * (b * (x * (y * z))))) \\
&= h_X(a^{k-1} * (a * ((b * (x * y)) * (b * (x * z))))) \\
&= \cdots \\
&= h_X((a^k * (b * (x * y))) * (a^k * (b * (x * z)))) \\
&= h_X(1 * (a^k * (b * (x * z)))) \\
&= h_X(a^k * (b * (x * z))),
\end{aligned}$$

which implies that $x * z \in h_X[a^k; b]$. Therefore $h_X[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

Theorem 3.28. *Let H_X be a hesitant fuzzy set of a self distributive B-algebra X satisfying the condition (3.1) and*

$$(\forall x, y \in X)\,(h_X(x * y) = h_X(x) \cap h_X(y))\,. \tag{3.8}$$

Then $h_X[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.

Proof. Let $a, b \in X$ and $k \in \mathbb{N}$. Obviously, $1 \in h_X[a^k; b]$. Let $x, y, z \in X$ be such that $x * (y * z) \in h_X[a^k; b]$ and $x * y \in h_X[a^k; b]$. Then $h_X\left(a^k * (b * (x * (y * z)))\right) = h_X(1)$, which implies from (3.8) and (3.1) that

$$\begin{aligned}
h_X(1) &= h_X(a^k * (b * (x * (y * z)))) \\
&= h_X(a^{k-1} * (a * (b * (x * (y * z))))) \\
&= h_X(a^{k-1} * (a * ((b * (x * y)) * (b * (x * z))))) \\
&= \cdots \\
&= h_X((a^k * (b * (x * y))) * (a^k * (b * (x * z)))) \\
&= h_X(a^k * (b * (x * y))) \cap h_X(a^k * (b * (x * z))) \\
&= h_X(1) \cap h_X(a^k * (b * (x * z))) \\
&= h_X(a^k * (b * (x * z))).
\end{aligned}$$

Hence $x * z \in h_X[a^k; b]$ and therefore $h_X[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.3, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

A new quadratic functional equation version and its stability and superstability

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Abstract. Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces. It is shown that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation

$$\begin{aligned} f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \\ = f(x) + f(y) + f(z) \end{aligned} \tag{0.1}$$

if and only if $f : \mathcal{X} \to \mathcal{Y}$ is a quadratic mapping.

Furthermore, we prove the superstability and the Hyers-Ulam stability for the quadratic functional equation (0.1) by using a direct method.

Keywords: Hyers-Ulam stability; quadratic functional equation; fixed point method; quadratic functional inequality; orthogonality space.

1. Introduction and preliminaries

Studying functional equations focusing on their approximate and exact solutions, conduces to one of the most substantial significant study brunches in functional equations, what we would call *"the theory of stability of functional equations"*. This theory specifically analyzes relationships between approximate and exact solutions of functional equations. Actually a functional equation is considered to be *stable*, if one can find an exact solution for any approximate solution of that certain functional equation. Another related and close term in this area is *superstability*, which has a similar nature and concept to the stability problem. As a matter of fact, superstability for a given functional equation occurs when any approximate solution is an exact solution too. In such this situation the functional equation is called *superstable*.

In 1940, the most preliminary form of stability problems was proposed by Ulam [40]. He gave a talk and asked the following: "when and under what conditions does an exact solution of a functional equation near an approximately solution of that exist?"

In 1941, this question that today is considered as the source of the stability theory, was formulated and solved by Hyers [14] for the Cauchy's functional equation in Banach spaces. Then the result of Hyers was generalized by Aoki [1] for additive mappings and by Rassias [32] for linear mappings by considering an unbounded Cauchy difference. In 1994, Găvruţa [13] provided a further generalization of Rassias' theorem in which he replaced the unbounded Cauchy difference by a general control function for the existence of a unique linear mapping. For more epochal information and various aspects about the stability of functional equations theory, we refer the

[0]2010 Mathematics Subject Classification: 39B52.
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reader to the monographs [15, 28, 33, 35], which also include many interesting results concerning the stability of different functional equations in many various spaces.

Consider the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

The function $f(x)=cx^2$ is a solution for the quadratic functional equation and obviously every satisfied function in this equation is said to be a quadratic function. A stability problem for this equation was first proved by Skof [39] and then was generalized by Cholewa [9], Czerwik [7, 8] and others [2, 4, 30, 31, 33, 34]. Moreover, there are some other different types of quadratic functional equations that their stability problems have been investigated by many authors. We refer the readers to the papers [3, 5, 6, 10, 11, 12, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 36, 37, 38, 41].

This paper is organized as follows: In Section 2, we consider the superstability of the quadratic functional equation (0.1) and in Sections 3 and 4, we prove two types of stability problems for the quadratic functional equation (0.1).

2. Superstability of the functional equation (0.1)

To commence proving the superstability of the quadratic functional equation (0.1), we first solve it and then will give a superstability theorem.

Proposition 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces. A mapping $f:\mathcal{X}\to\mathcal{Y}$ satisfies* (0.1) *if and only if the mapping $f:\mathcal{X}\to\mathcal{Y}$ is a quadratic mapping.*

Proof. Sufficiency. Assume that $f:\mathcal{X}\to\mathcal{Y}$ satisfies (0.1).

Letting $x=y=z=0$ in (0.1), we have $4f(0)=3f(0)$. So $f(0)=0$.

Letting $y=z=0$ in (0.1), we get

$$\begin{aligned} 2f\left(\frac{x}{2}\right)+2f\left(-\frac{x}{2}\right)&=f(x), \\ 2f\left(-\frac{x}{2}\right)+2f\left(\frac{x}{2}\right)&=f(-x) \end{aligned} \tag{2.1}$$

for all $x\in\mathcal{X}$, which imply that $f(x)=f(-x)$ for all $x\in\mathcal{X}$.

It follows from (2.1) that $4f\left(\frac{x}{2}\right)=f(x)$ and so $f(2x)=4f(x)$ for all $x\in\mathcal{X}$.

Putting $z=0$ in (0.1), we see that

$$\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)=f(x)+f(y)$$

for all $x,y\in\mathcal{X}$, which means that $f:\mathcal{X}\to\mathcal{Y}$ is a quadratic mapping.

Necessity. Assume that $f:\mathcal{X}\to\mathcal{Y}$ is quadratic.

By (1.1), one can easily get $f(0)=0$, $f(x)=f(-x)$ and $f(2x)=4f(x)$ for all $x\in\mathcal{X}$. So by applying (1.1), we obtain

$$\begin{aligned} &f\Big(\frac{x+y+z}{2}\Big)+f\Big(\frac{x-y-z}{2}\Big)+f\Big(\frac{y-x-z}{2}\Big)+f\Big(\frac{z-x-y}{2}\Big) \\ &=\Big[2f\Big(\frac{x}{2}\Big)+2f\Big(\frac{y+z}{2}\Big)\Big]+\Big[2f\Big(-\frac{x}{2}\Big)+2f\Big(\frac{y-z}{2}\Big)\Big] \\ &=4f\left(\frac{x}{2}\right)+f\Big(\frac{y+z+y-z}{2}\Big)+f\Big(\frac{y+z-y+z}{2}\Big) \\ &=f(x)+f(y)+f(z) \end{aligned}$$

for all $x,y,z\in\mathcal{X}$, which is the functional equation (0.1) and the proof is complete. □

Theorem 2.2. *Let $\mathcal{X},\mathcal{Y}$ be normed spaces with norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$, respectively. Let δ be a nonnegative real number and $\varphi:\mathcal{X}^3\to[0,\infty)$ be a function with*

$$\varphi(0,0,0)=0, \qquad \varphi(x,y,3x+y)=0$$

for all $x,y\in\mathcal{X}$. Suppose that $f:\mathcal{X}\to\mathcal{Y}$ is a mapping such that

$$\begin{aligned}\Big\|f\Big(\frac{x+y+z}{2}\Big)+f\Big(\frac{x-y-z}{2}\Big)+f\Big(\frac{y-x-z}{2}\Big)-f(y)-f(z)\Big\|_{\mathcal{Y}}\\ \le \Big\|f(x)-f\Big(\frac{z-x-y}{2}\Big)\Big\|_{\mathcal{Y}}+\delta\cdot\varphi(x,y,z)\end{aligned} \tag{2.2}$$

for all $x,y,z\in\mathcal{X}$. Then f is a quadratic mapping.

Proof. Putting $x=y=z=0$ in (2.2), we get

$$\big\|f(0)\big\|_{\mathcal{Y}}\le\big\|0\big\|_{\mathcal{Y}}+\delta\cdot\varphi(0,0,0)=0.$$

So $f(0)=0$.

Replacing x,y,z by $0,x,x$ in (2.2), respectively, we obtain

$$\big\|f(-x)-f(x)\big\|_{\mathcal{Y}}\le\big\|0\big\|_{\mathcal{Y}}+\delta\cdot\varphi(0,x,x)=0.$$

So $f(x)=f(-x)$ for all $x\in\mathcal{X}$.

Replacing x,y and z by $x,-3x$ and 0, and then by $2x,-3x$ and $3x$ in (2.2), respectively, we have

$$[f(x)-f(3x)]+2f(2x)=0,$$
$$2[f(x)-f(3x)]+f(4x)=0,$$

which result that $f(2x)=4f(x)$ and $f(3x)=9f(x)$ for all $x\in\mathcal{X}$.

Letting $x=v-u, y=2u-v$ and $z=2v-u$ and then $x=u+v, y=-3v$ and $z=3u$ in (2.2), respectively, we get the equalities

$$\begin{aligned}f(2u-v)+f(2v-u)&=f(u)+f(v)+f(2u-2v),\\ f(2u-v)+f(2v-u)&=f(3u)+f(3v)-f(2u+2v).\end{aligned}$$

Thus

$$f(u)+f(v)+4f(u-v)=9f(u)+9f(v)-4f(u+v),$$

which is simplified to

$$f(u+v)+f(u-v)=2f(u)+2f(v)$$

for all $u,v\in\mathcal{X}$. So f is quadratic. □

Theorem 2.2 covers several other cases for $\varphi:\mathcal{X}^3\to[0,\infty)$. For example, we can define φ satisfying the mentioned conditions with $\varphi(x,y,z):=\|y\|_{\mathcal{X}}-\|3x-z\|_{\mathcal{X}}$ or $\varphi(x,y,z):=\|3x+y-z\|_{\mathcal{X}}$. In addition, to make a simpler result, one can put $\delta=0$.

3. Hyers-Ulam stability of the functional equation (0.1): Type A

In this section, we prove the Hyers-Ulam stability of the quadratic functional equation (0.1). We will suppose that $\mathcal{X}$ is a normed space and $\mathcal{Y}$ is a complete normed space with norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$, respectively.

Theorem 3.1. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function with $\varphi(0,0,0) = 0$ and the following condition holds:*

$$\text{if} \quad \begin{cases} \|x\|_{\mathcal{X}} \le \|x'\|_{\mathcal{X}}, & \text{or} \\ \|y\|_{\mathcal{X}} \le \|y'\|_{\mathcal{X}}, & \text{or} \\ \|z\|_{\mathcal{X}} \le \|z'\|_{\mathcal{X}}, \end{cases} \qquad \Longrightarrow \qquad \varphi(x,y,z) \le \varphi(x',y',z') \tag{3.1}$$

for all $x, y, z, x', y', z' \in \mathcal{X}$. Denote by ϕ a function such that

$$\phi(x,y,z) := \sum_{n=0}^{\infty} 2^{2n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) < \infty \tag{3.2}$$

for all $x, y, z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying

$$\begin{aligned} \left\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) - f(y) - f(z) \right\|_{\mathcal{Y}} &\le \left\| f(x) - f\Big(\frac{z-x-y}{2}\Big) \right\|_{\mathcal{Y}} \\ &+ \varphi(x,y,z) \end{aligned} \tag{3.3}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\| f(x) - \mathcal{Q}(x) \right\|_{\mathcal{Y}} \le 2\phi(x,x,x) \tag{3.4}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x = y = z = 0$ in (3.3), we get

$$\left\| f(0) \right\|_{\mathcal{Y}} \le \left\| 0 \right\|_{\mathcal{Y}} + \varphi(0,0,0) = 0.$$

So $f(0) = 0$.

Replacing x, y, z by $x, x, 4x$ and $x, 0, 3x$ in (3.3), respectively, and then using (3.1), we obtain

$$\begin{aligned} \left\| f(3x) + 2f(2x) - f(x) - f(4x) \right\|_{\mathcal{Y}} &\le \varphi(x,x,4x) \le \varphi(4x,4x,4x), \\ \left\| 2f(2x) + f(x) - f(3x) \right\|_{\mathcal{Y}} &\le \varphi(x,0,3x) \le \varphi(4x,4x,4x) \end{aligned}$$

for all $x \in \mathcal{X}$. These inequalities give

$$\left\| 4f(2x) - f(4x) \right\|_{\mathcal{Y}} \le \left\| f(3x) + 2f(2x) - f(x) - f(4x) \right\|_{\mathcal{Y}} + \left\| 2f(2x) + f(x) - f(3x) \right\|_{\mathcal{Y}} \le 2\varphi(4x,4x,4x).$$

Thus

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\|_{\mathcal{Y}} \le 2\varphi(x,x,x) \tag{3.5}$$

for all $x \in \mathcal{X}$. Using the induction method, we show that

$$\left\| 4^n f\left(\frac{x}{2^n}\right) - f(x) \right\|_{\mathcal{Y}} \le \sum_{s=0}^{n-1} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right) \tag{3.6}$$

for all $n \ge 1$ and all $x \in \mathcal{X}$. The case $n = 1$ is the inequality (3.5). For the case $n+1$, by (3.5) and (3.6), we have

$$\begin{aligned} \left\| 4^{n+1} f\left(\frac{x}{2^{n+1}}\right) - f(x) \right\|_{\mathcal{Y}} &\le 4^n \left\| 4f\left(\frac{1}{2}\left(\frac{x}{2^n}\right)\right) - f\left(\frac{x}{2^n}\right) \right\|_{\mathcal{Y}} + \left\| 4^n f\left(\frac{x}{2^n}\right) - f(x) \right\|_{\mathcal{Y}} \\ &\le 4^n \cdot 2\varphi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right) + \sum_{s=0}^{n-1} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right) = \sum_{s=0}^{n} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right) \end{aligned}$$

for all $x \in \mathcal{X}$, which ends the induction method.

Assume that m, l are positive integers with $m > l$. From (3.6), it follows that

$$\left\| 4^m f\left(\frac{x}{2^m}\right) - 4^l f\left(\frac{x}{2^l}\right) \right\|_{\mathcal{Y}} = 4^l \left\| 4^{m-l} f\left(\frac{1}{2^{m-l}}\left(\frac{x}{2^l}\right)\right) - f\left(\frac{x}{2^l}\right) \right\|_{\mathcal{Y}} \le \sum_{s=l}^{m-1} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right)$$

for all $x \in \mathcal{X}$, in which by (3.2) the right-hand side tends to zero as $m, l \to \infty$. This clarifies that the sequence $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is Cauchy in the complete space $\mathcal{Y}$ and therefore convergent in it. So we can define for all $x \in \mathcal{X}$, the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{Q}(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right).$$

Now passing the limit $n \to \infty$ in (3.6) and then using (3.2), we obtain (3.4).

To end the proof, we show that $\mathcal{Q}$ is a unique quadratic mapping. It follows from (3.3) that

$$\begin{aligned}
&\left\|\mathcal{Q}\Big(\frac{x+y+z}{2}\Big) + \mathcal{Q}\Big(\frac{x-y-z}{2}\Big) + \mathcal{Q}\Big(\frac{y-x-z}{2}\Big) - \mathcal{Q}(y) - \mathcal{Q}(z)\right\|_{\mathcal{Y}} \\
&\quad = \lim_{n\to\infty} 4^n \left\| f\Big(\frac{x+y+z}{2^{n+1}}\Big) + f\Big(\frac{x-y-z}{2^{n+1}}\Big) + f\Big(\frac{y-x-z}{2^{n+1}}\Big) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right) \right\|_{\mathcal{Y}} \\
&\quad \le \lim_{n\to\infty} \left\| 4^n f\left(\frac{x}{2^n}\right) - 4^n f\Big(\frac{z-x-y}{2^{n+1}}\Big)\right\|_{\mathcal{Y}} + \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$, in which by (3.2), the second term of the right-hand side tends to zero as $n \to \infty$, and therefore we obtain

$$\left\|\mathcal{Q}\Big(\frac{x+y+z}{2}\Big) + \mathcal{Q}\Big(\frac{x-y-z}{2}\Big) + \mathcal{Q}\Big(\frac{y-x-z}{2}\Big) - \mathcal{Q}(y) - \mathcal{Q}(z)\right\|_{\mathcal{Y}} \le \left\|\mathcal{Q}(x) - \mathcal{Q}\Big(\frac{z-x-y}{2}\Big)\right\|_{\mathcal{Y}}$$

for all $x, y, z \in \mathcal{X}$. Now by applying Theorem 2.2 (with $\delta = 0$), we conclude that $\mathcal{Q}$ is a quadratic mapping.

Let $\mathcal{Q}' : \mathcal{X} \to \mathcal{Y}$ be another quadratic mapping satisfying (3.4). Then we have

$$\begin{aligned}
\left\|\mathcal{Q}(x) - \mathcal{Q}'(x)\right\|_{\mathcal{Y}} &\le 4^n \left\|\mathcal{Q}\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_{\mathcal{Y}} + 4^n \left\|\mathcal{Q}'\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_{\mathcal{Y}} \\
&\le 2 \cdot 4^n \cdot 2\phi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right) = 4\sum_{s=n}^{\infty} 2^{2s+1} \varphi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right)
\end{aligned}$$

for all $x \in \mathcal{X}$. By (3.2), the right-hand side tends to zero as $n \to \infty$, and thus $\mathcal{Q}(x) = \mathcal{Q}'(x)$ for all $x \in \mathcal{X}$. This means the uniqueness of $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ and so the proof is complete. □

Theorem 3.2. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function satisfying $\varphi(0, 0, 0) = 0$ and (3.1). Denote by ϕ a function such that*

$$\phi(x, y, z) := \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \varphi\Big(2^n x, 2^n y, 2^n z\Big) < \infty \tag{3.7}$$

for all $x, y, z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying (3.3). Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ satisfying (3.4).

Proof. As in the proof of Theorem 3.1, we can first get the inequality (3.5), and then by replacing x by $2x$ in (3.5), we obtain

$$\left\|\frac{1}{4} f(2x) - f(x)\right\|_{\mathcal{Y}} \le \frac{1}{2}\varphi(2x, 2x, 2x)$$

for all $x \in \mathcal{X}$.

Using the induction method, we get

$$\left\|\frac{1}{4^n} f(2^n x) - f(x)\right\|_{\mathcal{Y}} \le \sum_{s=1}^{n} \frac{1}{2^{2s-1}} \varphi(2^s x, 2^s x, 2^s x) \tag{3.8}$$

for all $n \ge 1$ and all $x \in \mathcal{X}$.

Now by the same method which was done in the proof of Theorem 3.1, we have the Cauchy sequence $\left\{\frac{1}{4^n} f(2^n x)\right\}$, and then the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ defined by

$$\mathcal{Q}(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in \mathcal{X}$.

And finally we can conclude the inequality (3.4) by (3.7) and (3.8).

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.3. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1, p_2, p_3 > 2$ or $p_1, p_2, p_3 < 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\begin{aligned}\Big\|f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) - f(y) - f(z)\Big\|_{\mathcal{Y}} \\ \le \Big\|f(x) - f\Big(\frac{z-x-y}{2}\Big)\Big\|_{\mathcal{Y}} + \delta\big(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3}\big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le \sum_{i=1}^{3} \frac{2^{p_i+1}}{\big|2^{p_i} - 4\big|}\delta\|x\|_{\mathcal{X}}^{p_i}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x, y, z) = \delta\big(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3}\big)$ and applying Theorem 3.1 for the case $p_1, p_2, p_3 > 2$, and Theorem 3.2 for the case $p_1, p_2, p_3 < 2$, we get the desired results. □

Corollary 3.4. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1+p_2+p_3 \neq 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\begin{aligned}\Big\|f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) - f(y) - f(z)\Big\|_{\mathcal{Y}} \\ \le \Big\|f(x) - f\Big(\frac{z-x-y}{2}\Big)\Big\|_{\mathcal{Y}} + \delta\big(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3}\big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le \frac{2^{p_1+p_2+p_3+1}}{\big|2^{p_1+p_2+p_3} - 4\big|}\delta\|x\|_{\mathcal{X}}^{p_1+p_2+p_3}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x, y, z) = \delta\big(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3}\big)$ and applying Theorem 3.1 for the case $p_1 + p_2 + p_3 > 2$, and Theorem 3.2 for the case $p_1 + p_2 + p_3 < 2$, we get the desired results. □

4. Hyers-Ulam stability of the functional equation (0.1): Type B

In this section, we bring another type of stability theorems for the quadratic functional equation (0.1) which is more prevalent in considering stability problems rather than the given type in the previous section.

First of all, for convenience, we define for a given mapping $f : \mathcal{X} \to \mathcal{Y}$, the difference operator:

$$\begin{aligned}\mathcal{D}f(x, y, z) =: f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \\ - f(x) - f(y) - f(z)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$.

Theorem 4.1. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function satisfying $\varphi(0, 0, 0) = 0$ and (3.1). Denote by ϕ a function such that*

$$\phi(x, y, z) := \sum_{n=0}^{\infty} \frac{9^n}{4^n}\varphi\left(\frac{2^n}{3^n}x, \frac{2^n}{3^n}y, \frac{2^n}{3^n}z\right) < \infty \tag{4.1}$$

for all $x, y, z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying

$$\big\|\mathcal{D}f(x, y, z)\big\|_{\mathcal{Y}} \le \varphi(x, y, z) \tag{4.2}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq \phi(x, x, x) \tag{4.3}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x = y = z = 0$ in (4.2), we get $f(0) = 0$.

Replacing x, y, z by $0, x, 3x$ and then by $2x, 2x, 2x$ in (4.2), respectively, we obtain

$$\begin{aligned} \big\|2f(2x) + f(x) - f(3x)\big\|_{\mathcal{Y}} &\leq \varphi(0, x, 3x) \leq \varphi(3x, 3x, 3x), \\ \Big\|f(2x) - f(x) - \frac{1}{3}f(3x)\Big\|_{\mathcal{Y}} &\leq \frac{1}{3}\varphi(2x, 2x, 2x) \leq \frac{1}{3}\varphi(3x, 3x, 3x). \end{aligned}$$

Adding the above inequalities, we conclude that $\big\|3f(2x) - \frac{4}{3}f(3x)\big\|_{\mathcal{Y}} \leq \frac{4}{3}\varphi(3x, 3x, 3x)$ and therefore

$$\Big\|\frac{9}{4}f\left(\frac{2}{3}x\right) - f(x)\Big\|_{\mathcal{Y}} \leq \varphi(x, x, x)$$

for all $x \in \mathcal{X}$.

By the induction method, we can show that

$$\Big\|\frac{9^n}{4^n}f\left(\frac{2^n}{3^n}x\right) - f(x)\Big\|_{\mathcal{Y}} \leq \sum_{s=0}^{n-1}\frac{9^s}{4^s}\varphi\left(\frac{2^s}{3^s}x, \frac{2^s}{3^s}x, \frac{2^s}{3^s}x\right) \tag{4.4}$$

for all $x \in \mathcal{X}$.

Now similar to the method in the proof of Theorem 3.1, we have the Cauchy sequence $\left\{\frac{9^n}{4^n}f\left(\frac{2^n}{3^n}x\right)\right\}$, and then the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ defined by

$$\mathcal{Q}(x) := \lim_{n\to\infty}\frac{9^n}{4^n}f\left(\frac{2^n}{3^n}x\right)$$

for all $x \in \mathcal{X}$. This definition and the inequality (4.4) lead us to the inequality (4.3).

It follows from (4.1) and (4.2) that

$$\big\|\mathcal{D}\mathcal{Q}(x, y, z)\big\|_{\mathcal{Y}} \leq \lim_{n\to\infty}\frac{9^n}{4^n}\Big\|\mathcal{D}f\left(\frac{2^n}{3^n}x, \frac{2^n}{3^n}y, \frac{2^n}{3^n}z\right)\Big\|_{\mathcal{Y}} \leq \lim_{n\to\infty}\frac{9^n}{4^n}\varphi\left(\frac{2^n}{3^n}x, \frac{2^n}{3^n}y, \frac{2^n}{3^n}z\right) = 0.$$

Hence $\mathcal{D}\mathcal{Q}(x, y, z) = 0$ for all $x, y, z \in \mathcal{X}$. Now Proposition 2.1 signifies that $\mathcal{Q}$ is a quadratic mapping.

The proof of the uniqueness of $\mathcal{Q}$ is similar to the proof of Theorem 3.1. □

Theorem 4.2. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function satisfying $\varphi(0, 0, 0) = 0$ and (3.1). Denote by ϕ a function such that*

$$\phi(x, y, z) := \sum_{n=0}^{\infty}\frac{4^n}{9^n}\varphi\left(\frac{3^n}{2^n}x, \frac{3^n}{2^n}y, \frac{3^n}{2^n}z\right) < \infty$$

for all $x, y, z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying (4.2). Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ satisfying (4.3).

Proof. The proof is similar to the proof of the previous theorem and thus we omit it. □

Corollary 4.3. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1, p_2, p_3 > 2$ or $p_1, p_2, p_3 < 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\big\|\mathcal{D}f(x, y, z)\big\|_{\mathcal{Y}} \leq \delta(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3})$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq \sum_{i=1}^{3}\frac{2^{p_i-2}}{\left|\frac{2^{p_i}}{9} - \frac{3^{p_i}}{4}\right|}\delta\|x\|_{\mathcal{X}}^{p_i}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x,y,z) = \delta\left(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3}\right)$ and applying Theorem 4.1 for the case $p_1, p_2, p_3 > 2$, and Theorem 4.2 for the case $p_1, p_2, p_3 < 2$, we get the desired results. □

Corollary 4.4. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1+p_2+p_3 \neq 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\left\|\mathcal{D}f(x,y,z)\right\|_{\mathcal{Y}} \leq \delta(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3})$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\|f(x) - \mathcal{Q}(x)\right\|_{\mathcal{Y}} \leq \frac{2^{p_1+p_2+p_3-2}}{\left|\frac{2^{p_1+p_2+p_3}}{9} - \frac{3^{p_1+p_2+p_3}}{4}\right|}\theta\|x\|_{\mathcal{X}}^{p_1+p_2+p_3}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x,y,z) = \delta\left(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3}\right)$ and applying Theorem 4.1 for the case $p_1 + p_2 + p_3 > 2$, and Theorem 4.2 for the case $p_1 + p_2 + p_3 < 2$, we get the desired results. □

This paper is just a start for the quadratic functional equation (0.1). Actually this functional equation and its stability problems can be studied more in various mathematical structures and spaces. Such this studied approach can cause to have a deeper description of this equation's unknown properties which will probably be more interesting and remarkable.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.3, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Some New Results on Preconditioned Generalized Mixed-Type Splitting Iterative Methods

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Abstract

In this paper, we present three preconditioned generalized mixed-type splitting (GMTS) methods for solving the weighted linear least square problem. We compare the spectral radii of the iteration matrices of the preconditioned and the original methods. The comparison results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. Finally, we give two numerical examples to confirm our theoretical results.

Keywords: Preconditioning, GMTS method, linear system, convergence, comparison.

2000 AMS Classification: 65F10.

1. Introduction

We consider the following weighted least squares problem

$$(1.1)\qquad \min_{x\in R^n} (Ax-b)^T W^{-1} (Ax-b)\,,$$

where $A \in R^{n\times n}$ is nonsingular, $b \in R^n$, $W \in R^{n\times n}$ is a symmetric positive definite matrix, see [1,4,9].

In order to solve it, one has to solve a nonsingular linear system as

$$(1.2)\qquad Hy = f,$$

where

$$(1.3)\qquad H = A^T W^{-1} A = \begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix} \in R^{n\times n}$$

is an invertible matrix with

$$B = (b_{ij})_{p\times p}\,,\ C = (c_{ij})_{q\times q}\,,\ L = (l_{ij})_{q\times p}\,,\ U = (u_{ij})_{p\times q}\,,$$

$p+q=n$ and $f = A^T W^{-1} b \in R^n$, see [1,4].

Throughout the paper, we consider the following decomposition for the matrix H,

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$H = \hat{D} - \hat{L} - \hat{U}$, in which

$$\hat{D} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} 0 & 0 \\ -L & 0 \end{pmatrix}, \quad \hat{U} = \begin{pmatrix} B & -U \\ 0 & C \end{pmatrix}. \tag{1.4}$$

In [1], authors established a generalized AOR(GAOR) method to solve systems of linear equations (1.2). In paper [2, 3] , authors studied the preconditioned GAOR methods. In [4], authors presented a generalized mixed-type splitting (GMTS) iterative method which is generalized GAOR method. And they studied the preconditioned generalized mixed-type splitting iterative methods to solve (1.2). They showed that the preconditioned GMTS methods converge faster than the GMTS method, whenever the GMTS method is convergent.

In this paper, we propose three new preconditioners and give the comparison theorems between the preconditioned and original methods. These results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. And we prove that in the case that the GMTS method is convergent, using the third preconditioned GMTS method leads to the better convergence rate than the first and the second preconditioned GMTS methods. In Section 4, we give two examples to confirm our theoretical results. And we know that the preconditioned GMTS methods with preconditioners in this paper have the better converge rate than the preconditioned GMTS method with preconditioner P^*.

2. Preliminaries

2.1 Definition [5] $A \in R^{n\times n}$ is called a Z-matrix if $a_{ij} \leq 0$ for $i, j = 1, 2, ..., n$ $(i \neq j)$.

2.2 Definition [5] Let A be a Z-matrix with positive diagonal elements. Then the matrix A is called an M-matrix if A is nonsingular and $A^{-1} \geq 0$.

2.3 Definition [6] The splitting $A = M - N$ is called

(1) a regular splitting of A if $M^{-1} \geq 0$ and $N \geq 0$;
(2) a nonnegative splitting of A if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$;
(3) a weak nonnegative splitting of A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ (the first type) or $NM^{-1} \geq 0$ (the second type);
(4) a convergent splitting of A if $\rho(M^{-1}N) < 1$.

2.1. Lemma. [4] *Let A be a Z-matrix. Moreover, suppose that $A = M - N$ is a weak nonnegative splitting of the first type. Then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.*

2.2. Lemma. [7] *Let $A = M - N$ be a regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and A^{-1} is nonnegative.*

2.3. Lemma. [8] *Let matrix $A = (a_{ij})_{n\times n}$ be given such that*

(1) *$a_{ij} \leq 0$ for $i, j = 1, 2, ..., n$ $(i \neq j)$,*
(2) *A is nonsingular,*
(3) *$A^{-1} \geq 0$.*

Then,

(1) *$a_{ii} > 0$ for $i = 1, 2, ..., n$, i.e., A is an M-matrix,*
(2) *$\rho(B) < 1$ where $B = I - D^{-1}A$, where $D = diag\{a_{11}, ..., a_{nn}\}$.*

2.4. Lemma. [6] *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak nonnegative splittings of A, where $A^{-1} \geq 0$, if $M_1^{-1} \geq M_2^{-1}$ then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$.*

3. Comparison results

Consider the linear system (1.2), the generalized mixed-type splitting (GMTS) iterative method is given as follows:

$$(\hat{D} + D_1 + L_1 - \hat{L})y^{(k+1)} = (D_1 + L_1 + \hat{U})y^{(k)} + f \tag{3.1}$$

where $\hat{D}$, $\hat{L}$ and $\hat{U}$ are defined by (1.4), and D_1 is an auxiliary nonnegative block diagonal matrix, L_1 is an auxiliary strictly nonnegative block lower triangular matrix such that $0 \leq D_1 \leq \hat{D}$ and $0 \leq L_1 \leq \hat{L}$. Evidently, the iteration matrix of the GMTS iterative method is given as follow:

$$T = (\hat{D} + D_1 + L_1 - \hat{L})^{-1}(D_1 + L_1 + \hat{U}).$$

In this paper, we propose the new preconditioners as follows,

$$P_i^* = \begin{pmatrix} I + S_i & 0 \\ 0 & I + V_i \end{pmatrix}, \quad i = 1, 2, 3 \tag{3.2}$$

where

$$S_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{p-1,1} & 0 & \cdots & 0 & 0 \\ b_{p1} & 0 & \cdots & 0 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & b_{12} & \cdots & b_{1,p-1} & b_{1p} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & b_{12} & \cdots & b_{1,p-1} & b_{1p} \\ b_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{p-1,1} & 0 & \cdots & 0 & 0 \\ b_{p1} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ c_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{q-1,1} & 0 & \cdots & 0 & 0 \\ c_{q1} & 0 & \cdots & 0 & 0 \end{pmatrix}, \; V_2 = \begin{pmatrix} 0 & c_{12} & \cdots & c_{1,q-1} & c_{1q} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0 & c_{12} & \cdots & c_{1,q-1} & c_{1q} \\ c_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{q-1,1} & 0 & \cdots & 0 & 0 \\ c_{q1} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then $P_i^* H$ can be expressed by

$$P_i^* H = \begin{pmatrix} I - B_i^* & U_i^* \\ L_i^* & I - C_i^* \end{pmatrix},$$

where $B_i^* = B - S_i(I - B)$, $C_i^* = C - V_i(I - C)$, $L_i^* = (I + V_i)L$, $U_i^* = (I + S_i)U$.

Let us consider the corresponding splitting for the preconditioned GMTS method, that is the generalized mixed-type splitting for the $\bar{H}_i = P_i^* H = \bar{M}_i - \bar{N}_i$, where

$$\bar{M}_i = \hat{D}_i^* + \bar{D}_1 + \bar{L}_1 - \hat{L}_i^*, \quad \bar{N}_i = \bar{D}_1 + \bar{L}_1 + \hat{U}_i^*$$

and

$$\hat{D}_i^* = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},\ \hat{L}_i^* = \begin{pmatrix} 0 & 0 \\ -L_i^* & 0 \end{pmatrix},\ \hat{U}_i^* = \begin{pmatrix} B_i^* & -U_i^* \\ 0 & C_i^* \end{pmatrix},\ i = 1, 2, 3,$$

$\bar{D}_1$ is an auxiliary nonnegative block diagonal matrix with $0 \le \bar{D}_1 \le \hat{D}_i^*$, $\bar{L}_1$ is an auxiliary strictly nonnegative block lower triangular matrix with $0 \le \bar{L}_1 \le \hat{L}_i^*$.

The iteration matrix of the preconditioned GMTS method is

$$T_i^* = (\hat{D}_i^* + \bar{D}_1 + \bar{L}_1 - \hat{L}_i^*)^{-1}(\bar{D}_1 + \bar{L}_1 + \hat{U}_i^*).$$

3.1. Lemma. [4] *Assume that $L \le 0, U \le 0, B \ge 0, C \ge 0$ and H in (1.2) is irreducible. If D_1 is nonsingular, then the iteration matrix of the GMTS method is irreducible.*

3.2. Lemma. [4] *Assume that $L \le 0, U \le 0, B \ge 0, C \ge 0$, then the corresponding splitting of GMTS method is a regular splitting for the matrix H.*

Similar to the proof of Lemma 3.2, we can prove the following lemma.

3.3. Lemma. *Assume that $L \le 0, U \le 0, B \ge 0, C \ge 0$, then the corresponding splitting of PGMTS method is a regular splitting for the matrix $P_i^* H$ $(i = 1, 2, 3)$.*

3.4. Theorem. *Let H be an M-matrix, then $P_i^* H$ $(i = 1, 2, 3)$ is an M-matrix.*

Proof. Consider the following splitting for H, $H = M_1 - N_1$,
where $M_1 = (P_1^*)^{-1}$, $N_1 = (P_1^*)^{-1}(\hat{L}^* + \hat{U}^*)$,
and $\hat{L}^* = \begin{pmatrix} 0 & 0 \\ -L_1^* & 0 \end{pmatrix}, \hat{U}^* = \begin{pmatrix} B_1^* & -U_1^* \\ 0 & C_1^* \end{pmatrix}$.

We can see that $M_1^{-1}N_1 = \hat{L}^* + \hat{U}^*$ and $M_1^{-1} \ge 0$. Then$H = M_1 - N_1$ is a weak nonnegative splitting of the first type. By the assumption H is an M-matrix, hence Lemma 2.1 implies that $\rho(M_1^{-1}N_1) < 1$. Let us assume that $P_1^* H = I - \hat{L}^* - \hat{U}^*$, using the fact that $\rho(\hat{L}^* + \hat{U}^*) = \rho(M_1^{-1}N_1) < 1$, by Lemma 2.2 and Lemma 2.3, it is easy to know that $P_1^* H$ is an M-matrix. The similar results can be gotten when $i = 2, 3$. $\square$

Now, we will show that in the case that the GMTS method converges, the preconditioned GMTS methods converge faster.

3.5. Theorem. *Let T and T_1^* be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively, assume that the matrix H is irreducible, $L \le 0, U \le 0, B \ge 0, C \ge 0, 0 \le D_1 \le \hat{D}$, $\quad 0 \le \bar{D}_1 \le \hat{D}_1^*, 0 \le L_1 \le \hat{L}$, $0 \le \bar{L}_1 \le \hat{L}_1^*, b_{i,1} > 0, c_{j,1} > 0$, for some $i \in \{2, 3, ..., p\}, j \in \{2, 3, ..., q\}$. If $\rho(T) < 1, \bar{D}_1 \le D_1$ and $\bar{L}_1 \le L_1$, then $\rho(T_1^*) \le \rho(T)$.*

Proof. As the matrix H is irreducible, so the $P_1^* H$ is irreducible. And by Lemma 3.1, we know that T and T_1^* are irreducible. Consider the GMTS splitting for the matrix $H = M - N$, where $M = \hat{D} + D_1 + L_1 - \hat{L}$, $N = D_1 + L_1 + \hat{U}$.

Obviously, $H = M - N$ is a regular splitting, and by the assumption $\rho(M^{-1}N) < 1$, we can get that H is an M-matrix. From Theorem 3.4, we know that P_1^*H is also an M-matrix. Thus, from Lemma 3.3, we know that $\bar{H}_1 = \bar{M}_1 - \bar{N}_1$ is a regular splitting. Therefore, as H is an M-matrix, we can get $\rho(T_1^*) = \rho(\bar{M}_1^{-1}\bar{N}_1) < 1$.

Now, we define the following splitting for the matrix H, $H = M_1^* - N_1^*$, in which $M_1^* = (I + \bar{S}_1)^{-1}\bar{M}_1$, $N_1^* = (I + \bar{S}_1)^{-1}\bar{N}_1$ and

$$\bar{S}_1 = \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}.$$

Consider the iteration matrix of the GMTS method $T = M^{-1}N$, it is easy to see that

$$M - \bar{M}_1 = \begin{pmatrix} D_{11} - D_{11}^* & 0 \\ L_{21} + L - L_{21}^* - L_1^* & D_{22} - D_{22}^* \end{pmatrix},$$

where

$$D_1 = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix} \leq \hat{D},\ \bar{D}_1 = \begin{pmatrix} D_{11}^* & 0 \\ 0 & D_{22}^* \end{pmatrix} \leq \hat{D}_1^*,$$

$L_1 = \begin{pmatrix} 0 & 0 \\ L_{21} & 0 \end{pmatrix} \leq \hat{L}$ and $\bar{L}_1 = \begin{pmatrix} 0 & 0 \\ L_{21}^* & 0 \end{pmatrix} \leq \hat{L}_1$.

It is known that $L_1^* = (I + V_1)L$, hence $L_1^* - L = V_1 L \leq 0$.

By computations, we know that $\bar{M}_1 \leq M$, so $\bar{M}_1^{-1} \geq M^{-1}$. Consequently,

$$M^{-1} \leq \bar{M}_1^{-1} \leq \bar{M}_1^{-1}(I + \bar{S}_1) = (M_1^*)^{-1}.$$

From Lemma 2.4, we deduce that

$$\rho(\bar{M}_1^{-1}\bar{N}_1) = \rho((M_1^*)^{-1}N_1^*) \leq \rho(M^{-1}N),$$

so $\rho(T_1^*) \leq \rho(T)$. □

Similar to the proof of Theorem 3.5, we can get the following two theorems.

3.6. Theorem. *Let T and T_2^* be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively. Assume that the matrix H is irreducible, $L \leq 0, U \leq 0, B \geq 0, C \geq 0, 0 \leq D_2 \leq \hat{D}$, $\quad 0 \leq \bar{D}_2 \leq \hat{D}_2^*, 0 \leq L_2 \leq \hat{L}$, $0 \leq \bar{L}_2 \leq \hat{L}_2^*, b_{1,i} > 0, c_{1,j} > 0$, for some $i \in \{2, 3, ..., p\}, j \in \{2, 3, ..., q\}$. If $\rho(T) < 1, \bar{D}_2 \leq D_2$ and $\bar{L}_2 \leq L_2$, then $\rho(T_2^*) \leq \rho(T)$.*

3.7. Theorem. *Let T and T_3^* be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively. Assume that the matrix H is irreducible, $L \leq 0, U \leq 0, B \geq 0, C \geq 0$, $0 \leq D_3 \leq \hat{D}$, $0 \leq \bar{D}_3 \leq \hat{D}_3^*$, $0 \leq L_3 \leq \hat{L}$, $0 \leq \bar{L}_3 \leq \hat{L}_3^*$, $b_{i,1} > 0, c_{j,1} > 0, b_{1,i} > 0, c_{1,j} > 0$, for some $i \in \{2, 3, ..., p\}, j \in \{2, 3, ..., q\}$. If $\rho(T) < 1, \bar{D}_3 \leq D_3$ and $\bar{L}_3 \leq L_3$, then $\rho(T_3^*) \leq \rho(T)$.*

Now, we prove that in the case that the GMTS method is convergent, using the third preconditioned GMTS method leads to the better convergence rate than the first and the second preconditioned GMTS methods.

3.8. Theorem. *Suppose that the matrix H is irreducible, $L \leq 0, U \leq 0, B \geq 0, C \geq 0$, $b_{i,1} > 0, c_{j,1} > 0, b_{1,i} > 0, c_{1,j} > 0$, for some $i \in \{2, 3, ..., p\}, j \in$*

$\{2,3,...,q\}$, *the auxiliary block diagonal matrices are chosen as* $\alpha_i I$ *and the auxiliary block lower triangular matrices as* $\beta_i L_i^*$ *for* $i = 1, 3,$ $0 \le \alpha_3 \le \alpha_1 \le 1, 0 \le \beta_1 \le \beta_3 \le 1$. *Then* $\rho(T_3^*) \le \rho(T_1^*)$ *if* $\rho(T) < 1$.

Proof. By the assumption $\rho(T) < 1$, and according the Lemma 2.1, H is an M-matrix. Assume that $P_i^* H = \widetilde{M}_i - \widetilde{N}_i, i = 1, 3$ where

$$\widetilde{M}_i = \begin{pmatrix} I + D_{11}^i & 0 \\ L_{21}^i + L_i^* & I + D_{22}^i \end{pmatrix}, \qquad \widetilde{N}_i = \begin{pmatrix} B_i^* + D_{11}^i & -U_i^* \\ L_{21}^i & C_i^* + D_{22}^i \end{pmatrix},$$

and $L_{21}^i = -\beta_i L_i^*, D_{11}^i = \alpha_i I_p, D_{22}^i = \alpha_i I_q$ for $i = 1, 3$.

Now, we define the following splitting for the matrix H, i.e. $H = M_i - N_i (i = 1, 3)$ such that $M_i = (I + \widetilde{S}_i)^{-1} \widetilde{M}_i$ and $N_i = (I + \widetilde{S}_i)^{-1} \widetilde{N}_i$,

where $\widetilde{S}_i = \begin{pmatrix} S_i & 0 \\ 0 & V_i \end{pmatrix}$.

Since

$$L_{21}^1 - L_{21}^3 = -\beta_1 L_1^* + \beta_3 L_3^* \ge \beta_1 L_3^* - \beta_1 L_1^* = -\beta_1 (L_1^* - L_3^*),$$

so

$$L_{21}^1 - L_{21}^3 + L_1^* - L_3^* \ge (1 - \beta_1)(L_1^* - L_3^*),$$

then

$$\begin{aligned} \widetilde{M}_1 - \widetilde{M}_3 &= \begin{pmatrix} D_{11}^1 - D_{11}^3 & 0 \\ L_{21}^1 - L_{21}^3 + L_1^* - L_3^* & D_{22}^1 - D_{22}^3 \end{pmatrix} \\ &\ge \begin{pmatrix} (\alpha_1 - \alpha_3) I_p & 0 \\ (1 - \beta_1)(L_1^* - L_3^*) & (\alpha_1 - \alpha_3) I_q \end{pmatrix}, \end{aligned}$$

as $L_1^* - L_3^* = (V_1 - V_3) L \ge 0$, then $\widetilde{M}_1 \ge \widetilde{M}_3$.

Notice that $\widetilde{M}_1^{-1} \ge 0, \widetilde{M}_3^{-1} \ge 0$, hence $\widetilde{M}_1^{-1} \le \widetilde{M}_3^{-1}$ and

$$\begin{aligned} M_1^{-1} &= \widetilde{M}_1^{-1}(I + \widetilde{S}_1) \\ &= \widetilde{M}_1^{-1} + \widetilde{M}_1^{-1} \widetilde{S}_1 \\ &\le \widetilde{M}_3^{-1} + \widetilde{M}_1^{-1}(\widetilde{S}_1 - \widetilde{S}_3) + \widetilde{M}_1^{-1} \widetilde{S}_3 \\ &\le \widetilde{M}_3^{-1} + \widetilde{M}_3^{-1} \widetilde{S}_3 \\ &= \widetilde{M}_3^{-1}(I + \widetilde{S}_3) = M_3^{-1}. \end{aligned}$$

Since H is an M-matrix, Lemma 2.4 implies that

$$\rho(M_3^{-1} N_3) \le \rho(M_1^{-1} N_1).$$

According $M_i^{-1} N_i = \widetilde{M}_i^{-1} \widetilde{N}_i$ for $i = 1, 3$, we can conclude that

$$\rho(T_3^*) \le \rho(T_1^*).$$

□

Similar to the proof of Theorem 3.8, we can get the following theorem.

3.9. Theorem. *Suppose that the matrix* H *is irreducible,* $L \le 0, U \le 0, B \ge 0, C \ge 0,$ $b_{i,1} > 0, c_{j,1} > 0, b_{1,i} > 0, c_{1,j} > 0,$ *for some* $i \in \{2,3,...,p\}, j \in \{2,3,...,q\}$, *the auxiliary block diagonal matrices are chosen as* $\alpha_i I$ *and the auxiliary block lower triangular matrices as* $\beta_i L_i^*$ *for* $i = 2, 3,$ $0 \le \alpha_3 \le \alpha_2 \le 1, 0 \le \beta_2 \le \beta_3 \le 1$. *Then* $\rho(T_3^*) \le \rho(T_2^*)$ *if* $\rho(T) < 1$.

4. Examples

4.1 Example Consider

$$H = \begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix},$$

where $B = (b_{ij})_{p\times p}, C = (c_{ij})_{(n-p)\times(n-p)}, L = (l_{ij})_{(n-p)\times p}$ and $U = (u_{ij})_{p\times(n-p)}$ with

$$b_{ii} = \frac{1}{10\times(i+1)}, \quad i = 1,2,\cdots,p,$$

$$b_{ij} = \frac{1}{30} - \frac{1}{30\times j+i}, \quad i<j, \quad i = 1,2,\cdots,p-1, \quad j = 2,\cdots,p,$$

$$b_{ij} = \frac{1}{30} - \frac{1}{30\times(i-j+1)+i}, \quad i>j, \quad i = 2,\cdots,p,\ j = 1,2,\cdots,p-1,$$

$$c_{ii} = \frac{1}{10\times(p+i+1)}, \quad i = 1,2,\cdots,n-p,$$

$$c_{ij} = \frac{1}{30} - \frac{1}{30\times(p+j)+p+i}, \ i<j,\ i = 1,2,\cdots,n-p-1,\ j = 2,\cdots,n-p,$$

$$c_{ij} = \frac{1}{30} - \frac{1}{30\times(i-j+1)+p+i}, \ i>j,\ i = 2,\cdots,n-p,\ j = 1,2,\cdots,n-p-1,$$

$$l_{ij} = \frac{1}{30\times(p+i-j+1)+p+i} - \frac{1}{30}, \ i = 1,2,\cdots,n-p,\ j = 1,2,\cdots,p,$$

$$u_{ij} = \frac{1}{30\times(p+j)+i} - \frac{1}{30}, \ i = 1,2,\cdots,p,\ j = 1,2,\cdots,n-p.$$

In the experiments, the auxiliary matrices are chosen such that

$$D_1 = 0.5(\frac{1}{\omega}-1)I,\ \overline{D}_1 = 0.5(\frac{1}{\omega}-1)I,\ L_1 = 0.5(1-\frac{\gamma}{\omega})\widehat{L_i},\ \overline{L}_1 = 0.5(1-\frac{\gamma}{\omega})\widehat{L}_i^*.$$

From Table 1, we see that these results accord with Theorems 3.5 - 3.9.

Table 1. The spectral radii of the GMTS and preconditioned GMTS iteration matrices

n	ω	r	p	$\rho(T)$	$\rho(T_1^*)$	$\rho(T_2^*)$	$\rho(T_3^*)$
10	0.9	0.8	5	0.2352	0.2156	0.2140	0.2048
20	0.8	0.6	5	0.5736	0.5609	0.5605	0.5568
20	0.8	0.6	10	0.5551	0.5413	0.5404	0.5334
25	0.8	0.6	8	0.7164	0.7074	0.7070	0.7033
30	0.9	0.7	10	0.8680	0.8635	0.8633	0.8613
30	0.9	0.7	20	0.8676	0.8630	0.8627	0.8605

In [4], the authors considered the following preconditioner

$$P^* = \begin{pmatrix} I+S & 0 \\ 0 & I+V \end{pmatrix}, \tag{4.1}$$

where

$$S = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \frac{b_{p1}}{\alpha} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \frac{c_{q1}}{\beta} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Table 2. The spectral radii of the preconditioned GMTS iteration matrices

n	ω	r	p	$\alpha=\beta$	$\rho(T^*)$	$\rho(T_1^*)$	$\rho(T_2^*)$	$\rho(T_3^*)$
10	0.9	0.8	5	3	0.2335	0.2156	0.2140	0.2048
20	0.8	0.6	5	2	0.5729	0.5609	0.5605	0.5568
20	0.8	0.6	10	2	0.5542	0.5413	0.5404	0.5334
25	0.8	0.6	8	3	0.7161	0.7074	0.7070	0.7033
30	0.9	0.7	10	2	0.8678	0.8635	0.8633	0.8613
30	0.9	0.7	20	2	0.8673	0.8630	0.8627	0.8605

Here, T^* is the GMTS iteration matrix for solving $P^*Hy = P^*f$.

From Table 2, we see that the preconditioned GMTS methods with preconditioners in this paper have better converge rates than the preconditioned GMTS method with preconditioner P^*.

4.2 Example The coefficient matrix H in Equation (1.2) is given by

$$H = \begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix},$$

where

$$B = \begin{pmatrix} b_{11} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} -\frac{1}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & 0 \end{pmatrix}.$$

Table 3 displays the spectral radii of the corresponding iteration matrices with $\omega = 0.9, \gamma = 0.8$ and different values of b_{11} and c_{11}.

From Table 3, we can see that $\rho(T_i^*) \leq \rho(T)$ for $i = 1,2,3$ and $\rho(T_3^*) \leq \rho(T_i^*)$ for $i = 1,2$ when $\rho(T) < 1$. These numerical results are in accordance with the theoretical results given in Theorems 3.5- 3.9.

Table 3. The spectral radii of the GMTS and preconditioned GMTS iteration matrices

b_{11}	c_{11}	$\rho(T)$	$\rho(T_1^*)$	$\rho(T_2^*)$	$\rho(T_3^*)$
0	0	0.6804	0.6303	0.6381	0.6140
0	0.3	0.7657	0.7253	0.7323	0.7071
0.2	0.2	0.7614	0.7186	0.7265	0.6987
0.2	0.5	0.8860	0.8677	0.8713	0.8596
0.5	0.5	0.9553	0.9483	0.9499	0.9453

5. Conclusion

In this paper, we propose three new preconditioners and give comparison theorems between the preconditioned and original methods. These results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. Finally, we give two examples to confirm our theoretical results.

6. Acknowledgments

The authors would like to thank the referees for their valuable comments and suggestions, which greatly improved the original version of this paper.

This work was supported by the National Natural Science Foundation of China (Grant No. 11401333) and the Natural Science Foundation of Shandong Province (No. ZR2012AL09).

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A Linear Adaptive time-stepping Method for Solving Vibration Problems with Damping Terms

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Abstract

A linear adaptive time-stepping method is devised for linear or nonlinear damping vibration analysis, which has wide applications in civil engineering. In the time direction, the underlying problem is discretized by a linear C^0-continuous discontinuous Galerkin method combined with the technique of linearization. By means of the energy method, some optimal a posteriori error estimates are established for linear vibration problems. Motivated by these estimates, we design an adaptive time-stepping strategy for actual computation. Numerical results are performed to illustrate the efficiency of the adaptive method.

Keywords. Time-stepping method, Vibration, Damping, A posteriori error analysis, Adaptive algorithm

1 Introduction

This paper aims to design and analyze an adaptive time-stepping method for solving the following problem:

For any real number $T > 0$, find $\mathbf{u} : [0, T] \to \mathbb{R}^d$ (with d the spatial dimension) such that

$$\begin{cases} \mathbf{M}\mathbf{u}''(t) + \mathbf{F}\big(t, \mathbf{u}(t), \mathbf{u}'(t)\big) = 0, & 0 < t < T, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{v}_0, \end{cases} \tag{1.1}$$

where $(\cdot)'$ and $(\cdot)''$ denote respectively the first and second order derivatives in time; $\mathbf{M}$ is a given $(d \times d)$ matrix and $\mathbf{F}$ is a given vector-valued function from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{R}^d$; $\mathbf{u}_0$ and $\mathbf{v}_0$ are two given vectors in $\mathbb{R}^d$.

The above problem is frequently encountered in structure analysis of dynamical transient response (cf. [5]). Concretely speaking, the mathematical models for structure analysis are described by a system of second-order linear/nonlinear evolution equations, which give rise to the problem (1.1), after spatial discretization by finite element methods, finite difference methods or spectral methods (cf. [2, 9, 11, 16, 17, 21, 22]).

When the vector-valued function $\mathbf{F}$ is linear with respect to $\mathbf{u}$ and $\mathbf{u}'$, there are various numerical methods for solving the problem (1.1). The most widely used may be classified as modal superposition (cf. [6, 14]) and direct-time integration methods including the Runge-Kutta, central difference, Houbolt, Newmark-β and Wilson-θ methods (see [11] and the references therein for details). The space-time finite element method (cf. [7, 12, 13]) is another widely developed approach for solving second order time evolution equations. One typical way is using the time-discontinuous Galerkin (TDG) method (cf. [7,15]) in the time direction for the displacement and velocity fields together, but it has the disadvantage that an ill-conditioned (4×4) block system must be solved at each time step, which is time consuming. To overcome this difficulty, some linear C^0-continuous time-stepping methods were used in [18], where only the primal variables are involved and only a (1×1) block system

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should be solved at each time step. Moreover, an adaptive method was proposed in [18] for solving second order abstract evolution equations, where the optimal a posteriori error estimates are established, which, in conjunction with the error equidistribution strategy and some ideas implied in the Runge-Kutta-Felberg method, leads to an adaptive time-stepping method.

In this paper, we intend to use some ideas in [18] to develop an adaptive time-stepping method for solving the problem (1.1). In the time direction, the problem (1.1) is discretized by a linear C^0-continuous discontinuous Galerkin method combined with the technique of linearization (including three linearization methods). Then, by means of the energy method, some optimal a posteriori error estimates are established for linear vibration problems via some ideas in [18]. It deserves to emphasize that the mathematical argument developed here is greatly simplified by using the Lagrange basis functions instead of the Legendre polynomials. Motivated by these estimates, we construct a posteriori error estimates for nonlinear problems, based on which we design an adaptive time-stepping strategy for actual computation. Numerical results are performed to illustrate the efficiency of the adaptive method.

The rest of this paper is organized as follows. In Section 2, we present a time-stepping finite element method for the problem (1.1), and the detailed implementation of the previous method is also developed for actual computation. In Section 3, a posteriori error analysis is established in detail for linear vibration problems. In Section 4, we propose an adaptive algorithm based on some a posteriori error estimates. A series of numerical results are performed in the final section.

2 A linear time-stepping finite element method

2.1 The formulation of a linear time-stepping finite element method

Throughout this paper, we assume that Problem (1.1) has a unique solution and the matrix $\mathbf{M}$ is symmetric positive definite. We use a standard time-stepping method to discretize Problem (1.1) (cf. [10, 18, 19]). To this end, we first partition the time interval $I := (0, T)$ with the nodes

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

to get the following subintervals:

$$J_n = (t_{n-1}, t_n], \quad k_n = t_n - t_{n-1}, \quad 1 \le n \le N.$$

Define

$$\mathcal{V}_1 = \left\{ \mathbf{v} : \bar{I} \to \mathbb{R}^d;\ \mathbf{v} \in C(\bar{I}),\ \mathbf{v}|_{J_n}(t) = \sum_{j=0}^{1} t^j \mathbf{w}_j,\ \mathbf{w}_j \in \mathbb{R}^d,\ 1 \le n \le N \right\},$$

$$\mathcal{W}_2 = \left\{ \mathbf{v} : \bar{I} \to \mathbb{R}^d;\ \mathbf{v} \in C^1(\bar{I}),\ \mathbf{v}|_{J_n}(t) = \sum_{j=0}^{2} t^j \mathbf{w}_j,\ \mathbf{w}_j \in \mathbb{R}^d,\ 1 \le n \le N \right\},$$

$$\mathcal{H}_q = \left\{ \mathbf{v} : \bar{I} \to L^2(I);\ \mathbf{v}|_{J_n}(t) = \sum_{j=0}^{q} t^j \mathbf{w}_j,\ \mathbf{w}_j \in \mathbb{R}^d,\ 1 \le n \le N \right\}, \quad q = 0, 1.$$

Let $\mathcal{V}_1(J_n)$ and $\mathcal{W}_2(J_n)$ be the restrictions of $\mathcal{V}_1$ and $\mathcal{W}_2$ to J_n, respectively. Similarly, denote by $\mathcal{H}_q(J_n)$ the restriction of $\mathcal{H}_q$ to J_n. Thus, our time-stepping method for (1.1) is

to find $\mathbf{U} \in \mathcal{V}_1$ such that

$$\begin{cases} \int_{J_n} \left(\langle \mathbf{U}'', \ \mathbf{w}' \rangle_{\mathbf{M}} + \langle \mathbf{F}(t, \mathbf{U}, \mathbf{U}'), \ \mathbf{w}' \rangle \right) \mathrm{d}t + \left\langle \dot{\mathbf{U}}_+^{n-1} - \dot{\mathbf{U}}_-^{n-1}, \ \dot{\mathbf{w}}_+^{n-1} \right\rangle_{\mathbf{M}} = 0, \\ \mathbf{U}^0 = \mathbf{u}_0, \quad \dot{\mathbf{U}}_-^0 = \mathbf{v}_0, \qquad \mathbf{w} \in \mathcal{V}_1(J_n), \quad 1 \le n \le N, \end{cases} \tag{2.1}$$

where

$$\langle \mathbf{a}, \mathbf{b} \rangle := \mathbf{b}^{\mathrm{T}} \mathbf{a}, \quad \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{A}} := \mathbf{b}^{\mathrm{T}} \mathbf{A} \mathbf{a}, \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^d, \ \mathbf{A} \in \mathbb{R}^{d \times d}, \tag{2.2}$$
$$\dot{\mathbf{w}}_\pm^{n-1} := \lim_{s \to 0^+} \mathbf{w}'(t_{n-1} \pm s), \quad \mathbf{w}^{n-1} := \mathbf{w}(t_{n-1}).$$

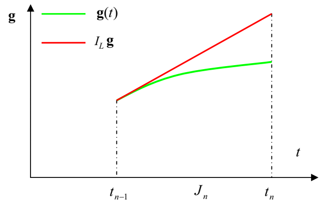
2.2 Implementation of the time-stepping method

Since $\mathbf{U} \in \mathcal{V}_1$, we have by a direct manipulation that, for any $t \in J_n$,

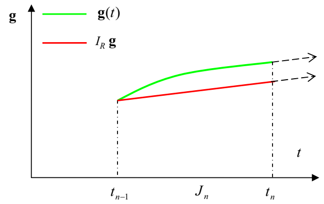
$$\mathbf{U}(t) = \mathbf{U}^{n-1} + (t - t_{n-1}) \dot{\mathbf{U}}_-^n, \qquad \mathbf{U}'(t) = \dot{\mathbf{U}}_-^n, \qquad \mathbf{U}''(t) = \mathbf{0}. \tag{2.3}$$

To implement the method (2.1) in actual computation, we require to linearize the nonlinear function $\mathbf{F}(\mathbf{t}, \mathbf{U}, \mathbf{U}')$ with respect to $\mathbf{U}$. As shown in Figure 1, for a given function $g(t)$, its linearization over J_n are usually the interpolants given by

$$\mathcal{I}_L \mathbf{g}(t) = \mathbf{g}(t_{n-1}) + (t - t_{n-1}) \mathbf{g}'(t_{n-1}) \quad \text{or} \quad \mathcal{I}_R \mathbf{g}(t) = \mathbf{g}(t_{n-1}) + (t - t_{n-1}) \mathbf{g}'(t_n), \qquad t \in J_n.$$



(a) $\mathcal{I}_L$



(b) $\mathcal{I}_R$

Figure 1: Diagrams of the (local) interpolate operators $\mathcal{I}_L$ and $\mathcal{I}_R$.

Note that the function $\mathbf{F} = \mathbf{F}(\mathbf{t}, \mathbf{U}, \mathbf{U}')$ is discontinuous at the interior node t_n. Recalling the expression (2.3), we have by the direct computation that the right limit of $\mathbf{F}$ at $t = t_{n-1}$ can be expressed as

$$\mathbf{F}_+^{n-1} = \mathbf{F}(t_{n-1}, \mathbf{U}^{n-1}, \dot{\mathbf{U}}_+^{n-1}) = \mathbf{F}(t_{n-1}, \mathbf{U}^{n-1}, \dot{\mathbf{U}}_-^n). \tag{2.4}$$

Using the chain rule for differentiation and (2.3), we find that, at $t = t_n$, the left limit of the full derivative of $\mathbf{F}(t, \mathbf{U}, \mathbf{U}')$ with respect to t is given as follows:

$$\begin{aligned} \dot{\mathbf{F}}_-^n &= \frac{\partial \mathbf{F}}{\partial t}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) + \frac{\partial \mathbf{F}}{\partial \mathbf{U}}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) \dot{\mathbf{U}}_-^n + \frac{\partial \mathbf{F}}{\partial \mathbf{U}'}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) \mathbf{0} \\ &= \frac{\partial \mathbf{F}}{\partial t}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) + \frac{\partial \mathbf{F}}{\partial \mathbf{U}}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) \dot{\mathbf{U}}_-^n \\ &=: \left. \frac{\partial \mathbf{F}}{\partial t} \right|_{t_-^n} + \left. \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \right|_{t_-^n} \dot{\mathbf{U}}_-^n. \end{aligned}$$

Similarly, we have

$$\dot{\mathbf{F}}_+^{n-1} := \left. \frac{\partial \mathbf{F}}{\partial t} \right|_{t_+^{n-1}} + \left. \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \right|_{t_+^{n-1}} \dot{\mathbf{U}}_-^n.$$

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With these results in mind, we have by the definitions of the interpolation operators $\mathcal{I}_L$ and $\mathcal{I}_R$ that

$$\text{Left side Scheme}: \ \mathbf{F}\big(t, \mathbf{U}(t), \mathbf{U}'(t)\big) \ \approx \mathcal{I}_L\mathbf{F} = \mathbf{F}_+^{n-1} + (t - t_{n-1})\dot{\mathbf{F}}_+^{n-1}, \tag{2.5}$$

$$\text{Right side Scheme}: \ \mathbf{F}\big(t, \mathbf{U}(t), \mathbf{U}'(t)\big) \ \approx \mathcal{I}_R\mathbf{F} = \mathbf{F}_+^{n-1} + (t - t_{n-1})\dot{\mathbf{F}}_-^{n}. \tag{2.6}$$

Now, inserting (2.3) and (2.6) into the first equation of (2.1) and taking $\dot{\mathbf{w}}$ to be $\mathbf{w}^*$ or $(t - t_{n-1})\mathbf{w}^*$, where $\mathbf{w}^*$ is any constant vector in $\mathbb{R}^d$, we find that the method (2.1) is equivalent to finding $\{\dot{\mathbf{U}}_-^n\}_{n=0}^N$ such that

$$\left(\mathbf{M} + \frac{k_n^2}{2}\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\bigg|_{t_-^n}\right)\dot{\mathbf{U}}_-^n + \frac{1}{2}k_n^2\frac{\partial \mathbf{F}}{\partial t}\bigg|_{t_-^n} + k_n\mathbf{F}_+^{n-1} = \mathbf{M}\dot{\mathbf{U}}_-^{n-1}, \quad 1 \le n \le N. \tag{2.7}$$

Note that the quantities $\frac{\partial \mathbf{F}}{\partial t}\big|_{t_-^n}$, $\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\big|_{t_-^n}$ and $\mathbf{F}_+^{n-1}$ are all the functions of the unknown vector $\dot{\mathbf{U}}_-^n$, so the above scheme is implicit. However, if we use the linearization formulation (2.5) instead of (2.6), then the system (2.1) reduces to

$$\left(\mathbf{M} + \frac{k_n^2}{2}\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\bigg|_{t_+^{n-1}}\right)\dot{\mathbf{U}}_-^n + \frac{1}{2}k_n^2\frac{\partial \mathbf{F}}{\partial t}\bigg|_{t_+^{n-1}} + k_n\mathbf{F}_+^{n-1} = \mathbf{M}\dot{\mathbf{U}}_-^{n-1}, \quad 1 \le n \le N. \tag{2.8}$$

It is noted that in most vibration problems, it suffices for us to deal with the linear damping case, indicating that the function $\mathbf{F}$ is linear with respect to the independent variable $\mathbf{u}'$. In this case, since the quantities $\frac{\partial \mathbf{F}}{\partial t}\big|_{t_+^{n-1}}$ and $\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\big|_{t_+^{n-1}}$ in (2.8) do not depend on $\dot{\mathbf{U}}_-^n$, the system (2.8) is essentially a linear system of the unknown vector $\dot{\mathbf{U}}_-^n$. Hence, we can work out $\dot{\mathbf{U}}_-^n$ with much less computational cost, compared to the method (2.7).

In order to balance the efficiency and stability of the time-stepping method, it is very natural to split the nonlinear term $\mathbf{F}$ into two parts $\mathbf{F}_L$ and $\mathbf{F}_R$, which correspond to the non-stiff and the stiff terms of the original system (1.1), respectively. Then, it is better for us to use $\mathcal{I}_L\mathbf{F}_L + \mathcal{I}_R\mathbf{F}_R$ to approximate $\mathbf{F}$ in (2.1). In other words, we have

$$\text{Semi} - \text{side Scheme}: \ \mathbf{F} \ \approx \mathcal{I}_L\mathbf{F}_L + \mathcal{I}_R\mathbf{F}_R = \mathbf{F}_+^{n-1} + (t - t_{n-1})\big(\dot{\mathbf{F}}_{L+}^{n-1} + \dot{\mathbf{F}}_{R-}^{n}\big). \tag{2.9}$$

It is noted that for the linear damping system, the semi-side scheme also yields a linear system for getting the unknown vector $\dot{\mathbf{U}}_-^n$.

Now, let us present the solution process of the method (1.1) in detail. Once we obtain $\mathbf{U}$ in J_{n-1}, we can get $\dot{\mathbf{U}}_-^n$ by solving the system (2.7) or (2.8). Then the function $\mathbf{U}$ over J_n is completely determined using the formulation $\mathbf{U}(t) = \mathbf{U}^{n-1} + (t - t_{n-1})\dot{\mathbf{U}}_-^n$ for all $t \in J_n$. On implementing this computation recursively, we can thereby determine the function $\mathbf{U}$ completely.

In the last part of this subsection, we give the solution process explicitly for the vibration analysis related to linear transient dynamic response. At this moment, we can reformulate the problem (1.1) as follows.

For any real number $T > 0$, find $\mathbf{u} : [0, T] \to \mathbb{R}^d$ such that

$$\begin{cases} \mathbf{M}\mathbf{u}'' + \mathbf{C}\mathbf{u}' + \mathbf{K}\mathbf{u} = \mathbf{f}, & 0 < t < T, \\ \mathbf{u}(0) = \mathbf{u}_0, \\ \mathbf{u}'(0) = \mathbf{v}_0, \end{cases} \tag{2.10}$$

where $\mathbf{C}$ and $\mathbf{K}$ are the $(d \times d)$ damping and stiffness matrices of the dynamic system, respectively. We assume that $\mathbf{C}$ and $\mathbf{K}$ are symmetric and semi-definite. Observing that

$$\mathbf{F}\big(t, \mathbf{u}(t), \mathbf{u}'(t)\big) = \mathbf{C}\mathbf{u}' + \mathbf{K}\mathbf{u} - \mathbf{f},$$

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we have from the variational formulation (2.1) that

$$\left(\frac{k_n^2}{2}\mathbf{K} + k_n\mathbf{C} + \mathbf{M}\right)\dot{\mathbf{U}}_-^n = \mathbf{M}\dot{\mathbf{U}}_-^{n-1} - k_n\mathbf{K}\mathbf{U}^{n-1} + \mathbf{f}^n, \quad 1 \le n \le N, \tag{2.11}$$

where $\mathbf{f}^n := \int_{J_n} \mathbf{f} \mathrm{d}t$.

3 A posteriori error analysis for linear problems

For the numerical method (2.1) for the linear vibration problem (2.10), following the similar arguments leading to Theorem 2.5 in [18], we can derive some stability estimates to the numerical solution $\mathbf{U}$ and then establish the required a priori error estimates. Another way to derive such estimates is to use the mathematical argument due to [24]. Since the objective of this article is to develop efficient adaptive time stepping method for the linear vibration problem (2.10) and the generalized problem (1.1), we will focus on in this section a posteriori error analysis for the problem (2.10) discretized by the method (2.1). Motivated by such an analysis, we will heuristically mention in the next section some error estimators for the nonlinear problem (1.1) and then devise the corresponding adaptive time stepping method.

3.1 Reconstruction

As shown in [18], in order to get efficient a posteriori error estimates for the method (2.1), we require to construct a higher order reconstruction $\widetilde{\mathbf{U}}$ from the approximate solution $\mathbf{U}$. So let us first recall such a reconstruction given in [18]. Introduce an invertible linear operator $\widetilde{I}_2 : \mathcal{V}_1 \to \mathcal{W}_2$ as follows. With any $\mathbf{w} \in \mathcal{V}_1$ we associate an element $\widetilde{\mathbf{w}} := \widetilde{I}_2\mathbf{w} \in \mathcal{W}_2$ defined by locally interpolating $\mathbf{w}$ in each subinterval $J_n (1 \le n \le N)$, i.e., $\widetilde{\mathbf{w}}|_{J_n} \in \mathcal{W}_2(J_n)$ is uniquely determined by

$$\widetilde{\mathbf{w}}(t) = \widetilde{\mathbf{w}}(t_{n-1}) + k_n\dot{\mathbf{w}}_-^{n-1}\Phi_0\left(\frac{t-t_{n-1}}{k_n}\right) + k_n\dot{\mathbf{w}}_-^n\Phi_1\left(\frac{t-t_{n-1}}{k_n}\right), \qquad 1 \le n \le N, \tag{3.1}$$

and the initial values $\widetilde{\mathbf{w}}(0) = \mathbf{w}(0)$, $\widetilde{\mathbf{w}}'(0) = \mathbf{w}'(0)$. In (3.1), the definition of Φ_0, Φ_1 are given as

$$\Phi_0(\xi) = -\frac{1}{2}\xi^2 + \xi, \qquad \Phi_1(\xi) = \frac{1}{2}\xi^2. \tag{3.2}$$

We call $\widetilde{\mathbf{w}}$ a time reconstruction of $\mathbf{w}$, as shown in Figure 2. It is easy to check by the

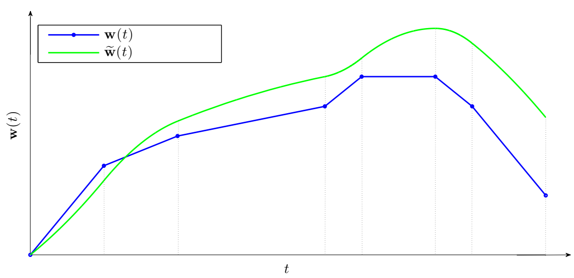

Figure 2: Diagram of $\widetilde{I}_2 w$.

above construction that

$$\widetilde{\mathbf{w}}'(t_n) = \dot{\mathbf{w}}_-^n, \quad 1 \le n \le N. \tag{3.3}$$

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Thus, for an approximate solution U, the reconstructed function we hope to find is $\widetilde{\mathbf{U}} \in \mathcal{W}_2$, defined by

$$\widetilde{\mathbf{U}}(t) = \widetilde{\mathbf{U}}(t_{n-1}) + k_n \dot{\mathbf{U}}_-^{n-1} \Phi_0(\frac{t-t_{n-1}}{k_n}) + k_n \dot{\mathbf{U}}_-^n \Phi_1(\frac{t-t_{n-1}}{k_n}), \qquad 1 \le n \le N. \tag{3.4}$$

By a direct computation we have

$$\widetilde{\mathbf{U}}''(t) = \frac{1}{k_n}(\dot{\mathbf{U}}_-^n - \dot{\mathbf{U}}_-^{n-1}), \qquad 1 \le n \le N. \tag{3.5}$$

Observing that the function $\mathbf{U}(t)$ can be rewritten as

$$\mathbf{U}(t) = \mathbf{U}(t_{n-1}) + k_n \dot{\mathbf{U}}_+^{n-1} \Phi_0(\frac{t-t_{n-1}}{k_n}) + k_n \dot{\mathbf{U}}_-^n \Phi_1(\frac{t-t_{n-1}}{k_n}), \quad t \in J_n,$$

subtracting which from (3.4) we know

$$\mathbf{U}(t) - \widetilde{\mathbf{U}}(t) = \mathbf{U}^{n-1} - \widetilde{\mathbf{U}}^{n-1} + k_n(\dot{\mathbf{U}}_+^{n-1} - \dot{\mathbf{U}}_-^{n-1})\Phi_0(\frac{t-t_{n-1}}{k_n}), \quad t \in J_n. \tag{3.6}$$

Hence,

$$\mathbf{U}^n - \widetilde{\mathbf{U}}^n = \mathbf{U}^{n-1} - \widetilde{\mathbf{U}}^{n-1} + \frac{1}{2}k_n^2 \widetilde{\mathbf{U}}'', \quad t \in J_n,$$

i.e.,

$$\mathbf{U}^n - \widetilde{\mathbf{U}}^n = \frac{1}{2}\sum_{m=1}^{n} k_m^2 \widetilde{\mathbf{U}}''|_{J_m}, \quad t \in J_n. \tag{3.7}$$

Moreover, by integration by parts and (3.3), it follows that

$$\int_{J_n} \langle \widetilde{\mathbf{U}}'',\ \mathbf{w}' \rangle_{\mathbf{M}} \mathrm{d}t = \int_{J_n} \langle \mathbf{U}'',\ \mathbf{w}' \rangle_{\mathbf{M}} \mathrm{d}t + \langle \dot{\mathbf{U}}_+^{n-1} - \dot{\mathbf{U}}_-^{n-1}, \dot{\mathbf{w}}_+^{n-1} \rangle_{\mathbf{M}}, \quad \mathbf{w} \in \mathcal{V}_1(J_n),$$

and use the variational equation in (2.1) we further have

$$\int_{J_n} (\langle \widetilde{\mathbf{U}}'',\ \mathbf{w}' \rangle_{\mathbf{M}} + \langle \mathbf{C}\mathbf{U}' + \mathbf{K}\mathbf{U} - \mathbf{f},\ \mathbf{w}' \rangle)\ \mathrm{d}t = 0, \quad \mathbf{w} \in \mathcal{V}_1 \quad 1 \le n \le N,$$

i.e.,

$$\mathbf{M}\widetilde{\mathbf{U}}'' + P_0(\mathbf{C}\mathbf{U}' + \mathbf{K}\mathbf{U} - \mathbf{f}) = 0, \qquad t \in J_n, \tag{3.8}$$

where P_q $(q = 0, 1)$ stands for the (local) L^2 orthogonal projection operator on to $\mathcal{H}_q(J_n)$ (cf. [1]), defined by

$$\int_{J_n} \langle P_q\mathbf{v} - \mathbf{v},\ \mathbf{w} \rangle \mathrm{d}t = 0, \qquad \mathbf{w} \in \mathcal{H}_q(J_n). \tag{3.9}$$

3.2 Error estimates

Let $\|\cdot\|$, $\|\cdot\|_{\mathbf{M}}$, $\|\cdot\|_{\mathbf{C}}$ and $\|\cdot\|_{\mathbf{K}}$ be the norms (or seminorms) over $\mathbb{R}^d$, defined by the inner products (2.2), respectively. We further define

$$\|\mathbf{v}\|_{L^\infty_{\mathbf{M}}(G)} = \operatorname*{ess\,sup}_{t\in G} \|\mathbf{v}(t)\|_{\mathbf{M}}, \qquad \|\mathbf{v}\|_{L^\infty_{\mathbf{M}^{-1}}(G)} = \operatorname*{ess\,sup}_{t\in G} \|\mathbf{v}(t)\|_{\mathbf{M}^{-1}}, \tag{3.10}$$

where $\mathbf{M}^{-1}$ is the inverse of the matrix $\mathbf{M}$. We assume that for the given function f, the linear problem (2.10) has a unique solution satisfying that

$$\mathbf{u} \in C([0,T];\ \mathbb{R}^d) \cap C^1([0,T];\ \mathbb{R}^d).$$

Let $\widetilde{\mathbf{e}} := \mathbf{u} - \widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{R}}$ be the residual of $\widetilde{\mathbf{U}}$ given by

$$\widetilde{\mathbf{R}}(t) := \mathbf{M}^{-1}(\mathbf{M}\widetilde{\mathbf{U}}''(t) + \mathbf{C}\widetilde{\mathbf{U}}'(t) + \mathbf{K}\widetilde{\mathbf{U}}(t) - \mathbf{f}(t)), \qquad t \in J_n,\ 1 \le n \le N. \tag{3.11}$$

Theorem 3.1 *Let* $\mathbf{u}$ *and* $\mathbf{U}$ *be the solution of* (2.10) *and* (2.1), *respectively. Let* $\widetilde{\mathbf{U}}$ *be the reconstruction of* $\mathbf{U}$ *by* (3.1). *Then for any* $t \in [0, T]$, *there holds*

$$\max_{0\le\tau\le t} \|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} \le 2\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s, \tag{3.12}$$

where $\widetilde{\mathbf{R}}$ *is given by* (3.11).

Proof. Subtracting (3.11) from (2.10) gives

$$\mathbf{M}\widetilde{\mathbf{e}}''(t) + \mathbf{C}\widetilde{\mathbf{e}}'(t) + \mathbf{K}\widetilde{\mathbf{e}}(t) = -\mathbf{M}\widetilde{\mathbf{R}}(t). \tag{3.13}$$

Then, we test (3.13) by $\widetilde{\mathbf{e}}'$ and integrate over $t \in [0,\ \tau]$ to get

$$\begin{aligned}&\int_0^\tau \big(\langle\widetilde{\mathbf{e}}''(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{M}} + \langle\widetilde{\mathbf{e}}'(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{C}} + \langle\widetilde{\mathbf{e}}(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{K}}\big)\,\mathrm{d}s\\ &\quad= \int_0^\tau \langle -\widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{M}}\,\mathrm{d}s.\end{aligned} \tag{3.14}$$

Moreover, using integration by parts and noting that $\widetilde{\mathbf{e}}(0) = \widetilde{\mathbf{e}}'(0) = 0$, we arrive at

$$\frac{1}{2}\|\widetilde{\mathbf{e}}'(\tau)\|^2_{\mathbf{M}} + \int_0^\tau \|\widetilde{\mathbf{e}}'(s)\|^2_{\mathbf{C}}\,\mathrm{d}s + \frac{1}{2}\|\widetilde{\mathbf{e}}(\tau)\|^2_{\mathbf{K}} = \int_0^\tau \langle -\widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{M}}\,\mathrm{d}s,\ \tau\in[0,t]. \tag{3.15}$$

Hence, it follows from (3.15) and the Cauchy-Schwarz inequality that

$$\begin{aligned}\frac{1}{2}(\max_{0\le\tau\le t}\|\widetilde{\mathbf{e}}'(\tau)\|_{\mathbf{M}})^2 &\le \max_{0\le\tau\le t}\int_0^\tau |\langle\widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{M}}|\,\mathrm{d}s\\ &\le \int_0^t |\langle\widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{M}}|\,\mathrm{d}s \ \le\ \max_{0\le\tau\le t}\|\widetilde{\mathbf{e}}'(\tau)\|_{\mathbf{M}}\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s,\end{aligned}$$

which readily yields

$$\max_{0\le\tau\le t}\|\widetilde{\mathbf{e}}'(\tau)\|_{\mathbf{M}} \le 2\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s, \tag{3.16}$$

as required. ■

Now, we proceed with the efficiency of the above a posteriori error estimates.

Lemma 3.1 *For* $t \in J_n$, $1 \le n \le N$,

$$\mathbf{U}(t) - P_0\mathbf{U}(t) = (t - t_{n-1} - \frac{1}{2}k_n)\dot{\mathbf{U}}^{\mathbf{n}}_{-}. \tag{3.17}$$

Moreover, for $1 \le n \le N$,

$$\|(\mathbf{U}-\widetilde{\mathbf{U}})'\|_{L^\infty_{\mathbf{M}}(J_n)} = k_n\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)}. \tag{3.18}$$

Furthermore, there holds

$$\begin{aligned}2\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s \le \sum_{m-1}^{n}\bigg(&\frac{2}{3}k_m^3\|\mathbf{K}\widetilde{\mathbf{U}}^{(3)}\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)} + tk_m^2\|\mathbf{K}\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)}\\ &+\frac{1}{2}k_m^2\|\mathbf{K}\mathbf{U}'\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)} + k_m^2\|\mathbf{C}\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)}\\ &+2\int_{J_m}\|\mathbf{f}(s) - P_0\mathbf{f}(s)\|_{\mathbf{M}^{-1}}\,\mathrm{d}s\bigg).\end{aligned} \tag{3.19}$$

Proof. First of all, recalling the definition of (Local) L^2 projection (3.9), we can deduce that

$$P_0\mathbf{U}(t) = \frac{1}{k_n}\int_{J_n}\mathbf{U}(s)\mathrm{d}s = \frac{1}{k_n}\int_{J_n}\big(\mathbf{U}^{n-1} + (s-t_{n-1})\dot{\mathbf{U}}_-^n\big)\mathrm{d}s$$
$$= \mathbf{U}^{n-1} + \frac{1}{2}k_n\dot{\mathbf{U}}_-^n, \qquad t \in J_n,$$

so

$$\mathbf{U}(t) - P_0\mathbf{U}(t) = (t - t_{n-1} - \frac{1}{2}k_n)\dot{\mathbf{U}}_-^n, \qquad t \in J_n. \tag{3.20}$$

On the other hand, differentiating (3.6) with respect to the variable t directly yields

$$(\mathbf{U} - \widetilde{\mathbf{U}})'(t) = -(t - t_{n-1})\widetilde{\mathbf{U}}'', \qquad t \in J_n, \tag{3.21}$$

which implies (3.18).

Moreover, we have by (3.8) and (3.11) that

$$\mathbf{M}\widetilde{\mathbf{R}} = \mathbf{K}(\widetilde{\mathbf{U}} - P_0\mathbf{U}) + \mathbf{C}\big(\widetilde{\mathbf{U}}' - P_0(\mathbf{U}')\big) - (\mathbf{f} - P_0\mathbf{f}). \tag{3.22}$$

Write

$$\mathbf{K}(\widetilde{\mathbf{U}} - P_0\mathbf{U}) = \mathbf{K}(\widetilde{\mathbf{U}} - \mathbf{U}) + \mathbf{K}(\mathbf{U} - P_0\mathbf{U}),$$

and owing to the fact that$P_0(\mathbf{U}') = \mathbf{U}'$ we know

$$\mathbf{C}\big(\widetilde{\mathbf{U}}' - P_0(\mathbf{U}')\big) = \mathbf{C}(\widetilde{\mathbf{U}} - \mathbf{U})'.$$

Hence, the equation (3.22) can be reformulated as

$$\mathbf{M}\widetilde{\mathbf{R}}(s) = \mathbf{K}(\widetilde{\mathbf{U}} - \mathbf{U})(s) + \mathbf{K}(\mathbf{U} - P_0\mathbf{U})(s) + \mathbf{C}(\widetilde{\mathbf{U}} - \mathbf{U})'(s) - (\mathbf{f} - P_0\mathbf{f})(s),$$

which, in conjunction with (3.6), (3.20) and (3.21), yields the estimate (3.19). ■

Now, let us continue to discuss the lower and upper a posteriori error bound for the method (2.1).

Theorem 3.2 (lower and upper bounds) *Let* $\mathbf{u}$ *and* $\mathbf{U}$ *be the solution of* (2.10) *and* (2.1)*, respectively. Let* $\widetilde{\mathbf{U}}$ *be the reconstruction of* $\mathbf{U}$ *by* (3.1)*. Then for* $t \in [0,T]$, $1 \le n \le N$,

$$\begin{aligned}\max_{1\le m\le n} k_m^2\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} &\le \|(\mathbf{u}-\mathbf{U})'\|_{L^\infty_{\mathbf{M}}(0,t)} + \max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} \\ &\le \max_{1\le m\le n} k_m\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} + 4\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s, \end{aligned} \tag{3.23}$$

where the a posteriori term $\widetilde{\mathbf{R}}$ *is given by* (3.11)*.*

Proof. Using the triangle inequality and (3.18), we obtain

$$\begin{aligned}\max_{1\le m\le n} k_m^2\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} &= \|(\mathbf{U}-\widetilde{\mathbf{U}})'\|_{L^\infty_{\mathbf{M}}(0,t)} \\ &\le \|(\mathbf{u}-\mathbf{U})'\|_{L^\infty_{\mathbf{M}}(0,t)} + \max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}}, \end{aligned} \tag{3.24}$$

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which implies the left side estimate of (3.23). Again, by the triangle inequality, (3.18) and (3.16), we have

$$\begin{aligned}\|(\mathbf{u}-\mathbf{U})'\|_{L^\infty_{\mathbf{M}}(0,t)} &\le \max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} + \|(\mathbf{U}-\widetilde{\mathbf{U}})'(\tau)\|_{L^\infty_{\mathbf{M}}(0,t)} \\ &\le \max_{1\le m\le n} k_m\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} + 2\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s. \end{aligned}\tag{3.25}$$

This together with (3.16) and (3.24) yields

$$\begin{aligned}&\max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} + \|(\mathbf{u}-\mathbf{U})'(\tau)\|_{L^\infty_{\mathbf{M}}(0,t)} \\ &\le \max_{1\le m\le n} k_m\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} + 4\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s,\end{aligned}$$

which leads to the right side estimate of (3.23). ■

4 An adaptive algorithm

Motivated by Theorem 3.2 (cf. the estimate (3.25)), we are tempted to introduce a posteriori error estimator of the time-stepping method (2.1) for solving even a nonlinear problem (1.1) heuristically. That means, let

$$\eta := \max_{1\le n\le N} k_n\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)} + 2\int_0^T \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\mathrm{d}s, \tag{4.1}$$

where $\widetilde{\mathbf{R}}$ is the residual of a nonlinear problem, defined by

$$\widetilde{\mathbf{R}}(t) = \mathbf{M}^{-1}\Big(\mathbf{M}\widetilde{\mathbf{U}}''(t) + \mathbf{F}\big(t, \widetilde{\mathbf{U}}(t), \widetilde{\mathbf{U}}'(t)\big)\Big), \qquad t\in J_n,\ 1\le n\le N.$$

Then the quantity η may be viewed as a posteriori error estimator for the method (2.1). Until now, it is beyond our power to develop reliability and efficiency estimates for such an estimator.

Based on the above error estimator, using the error equidistribution strategy as used in [4, 20], we can construct the error indicator corresponding to the subinterval J_n as

$$\Theta := 2\ \max\{\ \Theta_1,\ \ \Theta_2\ \}, \tag{4.2}$$

where

$$\Theta_1 := k_n\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)}, \qquad \Theta_2 := 2\frac{T}{k_n}\int_{J_n}\|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\mathrm{d}s.$$

The magnitude of Θ affects the choice of k_n, the length of the subinterval J_n.

Next, let us study how to compute the quantities Θ_1 and Θ_2 after we get $\dot{\mathbf{U}}^n_-$ at each time step by (2.1). First of all, from (3.5) and the definition of Θ_1, we have

$$\Theta_1 = \|\dot{\mathbf{U}}^n_- - \dot{\mathbf{U}}^{n-1}_-\|_{\mathbf{M}}\ . \tag{4.3}$$

For deriving Θ_2, we should obtain $\widetilde{\mathbf{R}}(t)$ in advance. It follows from (3.4) that

$$\begin{aligned}&\widetilde{\mathbf{U}}(t) = \widetilde{\mathbf{U}}(t_{n-1}) + k_n\dot{\mathbf{U}}^{n-1}_-\Phi_0(\xi) + k_n\dot{\mathbf{U}}^n_-\Phi_1(\xi),\\ &\widetilde{\mathbf{U}}'(t) = \dot{\mathbf{U}}^{n-1}_-(1-\xi) + \dot{\mathbf{U}}^n_-\xi, \qquad \widetilde{\mathbf{U}}''(t) = \frac{1}{k_n}(-\dot{\mathbf{U}}^{n-1}_- + \dot{\mathbf{U}}^n_-),\end{aligned}\tag{4.4}$$

where $\xi = (t - t_{n-1})/k_n$ and $\Phi_0,\ \Phi_1$ are defined as in (3.2).

Furthermore, in actual computation, we will use the Gaussian quadrature formula (cf. [23]) to evaluate Θ_2 numerically. In other words, for $t \in J_n$, $1 \le n \le N$,

$$\int_{J_n} \|\widetilde{\mathbf{R}}(t)\|_{\mathbf{M}} \mathrm{d}t \ \approx \ \sum_{j=1}^{N_g} k_n \omega_j \|\widetilde{\mathbf{R}}(t_{n-1} + k_n \zeta_j)\|_{\mathbf{M}}, \tag{4.5}$$

where ζ_j and ω_j $(1 \le j \le N_g)$ are the Gaussian quadrature points and weights on reference interval $[0,1]$, respectively.

Remark 4.1 *Let us discuss the cost of computing Θ_2 briefly. It is evident that the cost is taken in numerical integration by Gaussian quadrature formula* (4.5)*. Since the quadrature method is highly accurate, very few nodes are enough for actual computation (with the number ≤ 10). Next, we have to evaluate $\|\widetilde{\mathbf{R}}(\cdot)\|_{\mathbf{M}}$ at the quadrature nodes, the main cost of which corresponds to numerical solution of a linear system with $\mathbf{M}$ as a coefficient matrix. Generally speaking, the mass matrix $\mathbf{M}$ is a well-conditioned symmetric positive definite matrix, so the linear system can be solved by the conjugate gradient method very efficiently. According to the above analysis, we find that the cost for computing Θ_2 is inexpensive.*

With the help of the previous preparations and using some ideas implied in the Runge-Kutta-Felberg method (cf. [23]), we are ready to present the following Algorithm 1 to compute the numerical solution of the problem (1.1) by using the adaptive time-stepping strategy.

Algorithm 1 Adaptive Time Stepping Method

Given a tolerance ϵ, a parameter $\delta \in (0,1)$, and the max (min) time step size $k_{\max}$ ($k_{\min}$) by user

· **Step 0:** Initialize $n = 1$, $t_0 = 0$, $k_1 = k_{\max}$, $\mathbf{U}^0 = \mathbf{u}_0$, $\dot{\mathbf{U}}^0_- = \mathbf{v}_0$
· **WHILE** $t_{n-1} < T$
· **Step 1:** Given t_{n-1}, k_n, $\mathbf{U}^{n-1}$, $\dot{\mathbf{U}}^{n-1}_-$
· **1(a):** Get the numerical solution $\mathbf{U}^n$, $\dot{\mathbf{U}}^n_-$ by (2.7)
· **1(b):** Get the approximation $\widetilde{\mathbf{U}}^n$ by (3.4)
· **1(c):** Evaluate Θ_1 by (4.3)
· **1(d):** Get $\widetilde{\mathbf{R}}(t)$ at Gaussian quadrature points by (4.4) and (3.11)
· **1(e):** Summation to get the value of Θ_2 by (4.5)
· **1(f):** Get Θ by (4.2)
· **Step 2:** If $\delta\epsilon \le \Theta \le \epsilon$, $k_{n+1} = k_n$, go to **Step 5**
· **Step 3:** If $\Theta < \delta\epsilon$, $k_{n+1} = \min\{2k_n,\ k_{\max}\}$, go to **Step 5**
· **Step 4:** If $\Theta > \epsilon$, $k_n = \max\{k_n/2,\ k_{\min}\}$, go to **Step 1**
· **Step 5:** Let $t_n = t_{n-1} + k_n$, $n = n + 1$, go to loop condition judgment
· **END WHILE**

Remark 4.2 *Similar to the Runge-Kutta-Felberg method (cf. [23]), the parameter $\delta \in (0,1)$ in Algorithm 1 is used to determine how to enlarge the step size during the computation process (see Step 3 in Algorithm 1). The choice of δ is very technical. If δ is chosen too small, the over-refined meshes would be used in time, deteriorating the efficiency of Algorithm 1. If it is chosen too large, the algorithm would enlarge the step size more frequently, increasing the extra computational cost remarkably. From our numerical experience, it's better to choose δ such that $1/32 \le \delta \le 1/2$.*

5 Numerical experiments

5.1 Efficiency of the estimators

Example 5.1 (Nonlinear lumped mass system) *For illustrating the effectiveness of the a posteriori error estimates developed in the previous sections, we first study the vibration of a multi-structure model, a similar one as given in [3]. As shown in Figure 3, the structure consists of two rigid elements (vehicles) with lumped masses equal to m_1 and m_2, respectively; these elements are connected with each other by soften, classical and harden springs with linear damping. And the restoring force of these springs are given as follows:*

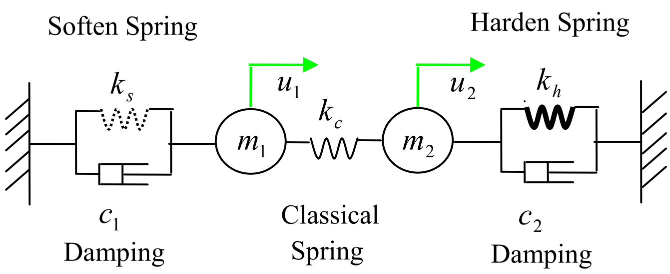


Figure 3: Example: 5.1: The nonlinear dynamic system.

$$\begin{aligned}
&\text{Classical Spring } (k_c): && f_c = -\kappa_1 u, && (5.1)\\
&\text{Softening Spring } (k_s): && f_s = -\kappa_2 \tanh(u), && (5.2)\\
&\text{Hardening Spring } (k_h): && f_h = -\kappa_3 u(1+\kappa_4 u^2). && (5.3)
\end{aligned}$$

In our actual computation, we choose $m_1 = m_2 = 1$, and choose the spring stiffness as $\kappa_1 = \kappa_2 = \kappa_3 = 1$. The damping coefficients are taken as $c_1 = c_2 = 1$. Hence by d Alembert's principle, we can get the following system of nonlinear dynamic equations,

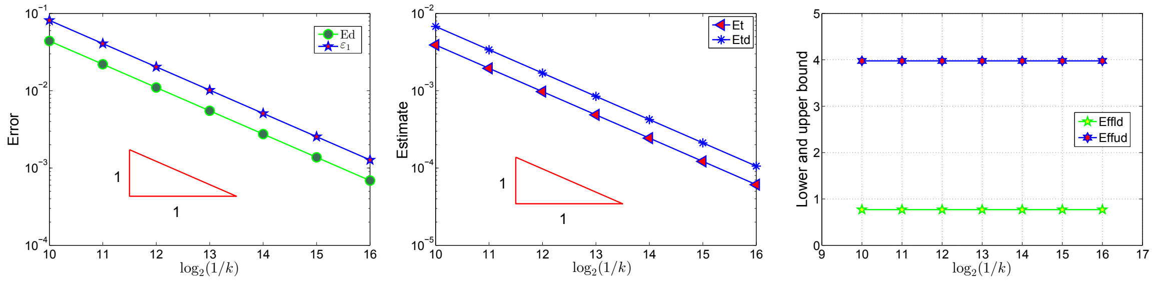
$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}'' = \begin{pmatrix} c_1 u_1'(t) + f_s\big(u_1(t)\big) + f_c\big(u_1(t)\big) - f_c\big(u_2(t)\big) - f_1(t) \\ c_2 u_2'(t) + f_h\big(u_2(t)\big) + f_c\big(u_2(t)\big) - f_c\big(u_1(t)\big) - f_2(t) \end{pmatrix}, \tag{5.4}$$

where f_1 and f_2 are the external forces. We choose $T = 1$ and the exact solution to be $\mathbf{u}(t) = (u_1, u_2)^{\mathrm{T}} = (\sin(\pi t), \sin(2\pi t))^{\mathrm{T}}$, so the force term $\mathbf{f}$ can be computed by the equations (5.4). We solve the solution of the dynamical system by the method (2.1) combined with the right side scheme (2.7).

In our numerical computation, for a given natural number N, we adopt the uniform partition in time with the mesh size $k = T/N$, $1 \le n \le N$. To show the computational performance of our method, define

$$\begin{aligned}
&\mathrm{Ed} = \max_{0\le\tau\le T} \|(\mathbf{u}-\mathbf{U})'(\tau)\|_{\mathbf{M}} , && \mathrm{Et} = \max_{0\le\tau\le T} \|(\mathbf{u}-\widetilde{\mathbf{U}})(\tau)\|_{\mathbf{M}} ,\\
&\mathrm{Etd} = \max_{0\le\tau\le T} \|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} , && \varepsilon_1 = 2\int_0^T \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s ,\\
&\varepsilon_2 = \max_{0\le n\le N} k_n \|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)} , && \varepsilon_3 = \eta = 2\varepsilon_1 + \varepsilon_2 .\\
&\mathrm{Effld} = \frac{\varepsilon_2}{\mathrm{Ed}+\mathrm{Etd}}, && \mathrm{Effud} = \frac{\varepsilon_3}{\mathrm{Ed}+\mathrm{Etd}}.
\end{aligned}$$

In Figure 4(a) we present the values of Et *and* ε_1 *as well as their orders (which are* 1*). In Figure 4(b) we give the estimates of the reconstruction solution Et and Etd as well as*



(a) Order of Ed and ε_1 (b) Order of Et and Etd (c) Lower and Upper bound

Figure 4: Example 5.1. Numerical results corresponding to estimators in Theorem 3.12 and Theorem 3.2.

their orders. Moreover, we present the values of these effectivity indices in Figure 4(c), from which we can observe that $0.77 \approx \text{Effld} < 1 < \text{Effud} \approx 3.98$*. Therefore, our a posteriori error estimator* (4.1) *is rather efficient.*

5.2 Efficiency of the adaptive algorithm

Example 5.2 (Nonlinear Klein-Gordon equation) *In order to test the effectiveness of our adaptive Algorithm 1, we consider the nonlinear Klein-Gordon equations (cf. [8]),*

$$u_{tt}(\mathbf{x},t) - \Delta u(\mathbf{x},t) + \beta u_t(\mathbf{x},t) + u^2(\mathbf{x},t) = f(\mathbf{x},t),$$

equipped with the homogeneous Dirichlet boundary condition and the initial conditions. After the discretization by P_1 *conforming element in the space direction, we obtain the following system of nonlinear ODEs,*

$$\begin{cases} \mathbf{M}\mathbf{u}''(t) + \mathbf{C}\mathbf{u}'(t) + \mathbf{K}\mathbf{u}(t) + \mathbf{M}\mathbf{u}^2(t) = \mathbf{f}(t), & 0 < t < T, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{v}_0, \end{cases} \tag{5.5}$$

where $\mathbf{u}$ *is the vector representation of the finite element solution* u_h *in terms of the shape basis functions* $\{\varphi_i\}$*, i.e.,* $u_h(\mathbf{x},t) = \sum_{i=1}^{M}\{\mathbf{u}(t)\}_i\varphi_i(\mathbf{x})$*. The mass matrix* $\mathbf{M}$*, the stiff matrix* $\mathbf{K}$*, the damping matrix* $\mathbf{C}$ *and the force* $\mathbf{F}$ *are defined respectively by* $[\mathbf{M}]_{ij} = \int_\Omega \varphi_j\varphi_i \mathrm{d}\Omega$, $[\mathbf{K}]_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \mathrm{d}\Omega$, $[\mathbf{C}]_{ij} = \beta\int_\Omega \varphi_j\varphi_i \mathrm{d}\Omega$ *and* $\{\mathbf{f}\}_i = \int_\Omega f(t)\varphi_i \mathrm{d}\Omega$*. In the numerical computation, we choose the damping coefficient* $\beta = 0.05$ *and the terminal time* $T = 1.0$*. Consider the* 1*-dim case of the above problem with the force* f *given such that the exact solution is*

$$u(x,t) = \mathrm{e}^{-t/2}x(1-x)\sin((1.5\pi + \mathrm{actan}(500(2t-1))x), \qquad 0 < x < 1,$$

which varies rapidly around $t = 0.5$*. After the discretization in space direction with a fine uniform mesh* $h = 1/5000$*, we solve the semi-discrete problem by using Algorithm 1 combined with the semi-side scheme* (2.9) *with* $\mathbf{F}$ *split into* $\mathbf{F}^R := \mathbf{C}\mathbf{u}' + \mathbf{K}\mathbf{u} - \mathbf{f}$ *and* $\mathbf{F}^L := \mathbf{M}\mathbf{u}^2$*, so that we only require to solve a linear system at each time subinterval. When implementing Algorithm 1 in this example, we set the related parameters by* $\epsilon = 2.5\mathrm{e}-1$, $\delta = 1/2$, $k_{\max} = 1\mathrm{e}-1$ *and* $k_{\min} = 2\mathrm{e}-4$*.*

To show the efficiency of Algorithm 1, we also carry out the numerical simulation using the uniform time stepping method with the same number of subintervals as for the adaptive method. The numerical solution obtained by the uniform time stepping method with $k = k_{\min}/100$ *is used as a reference solution.*

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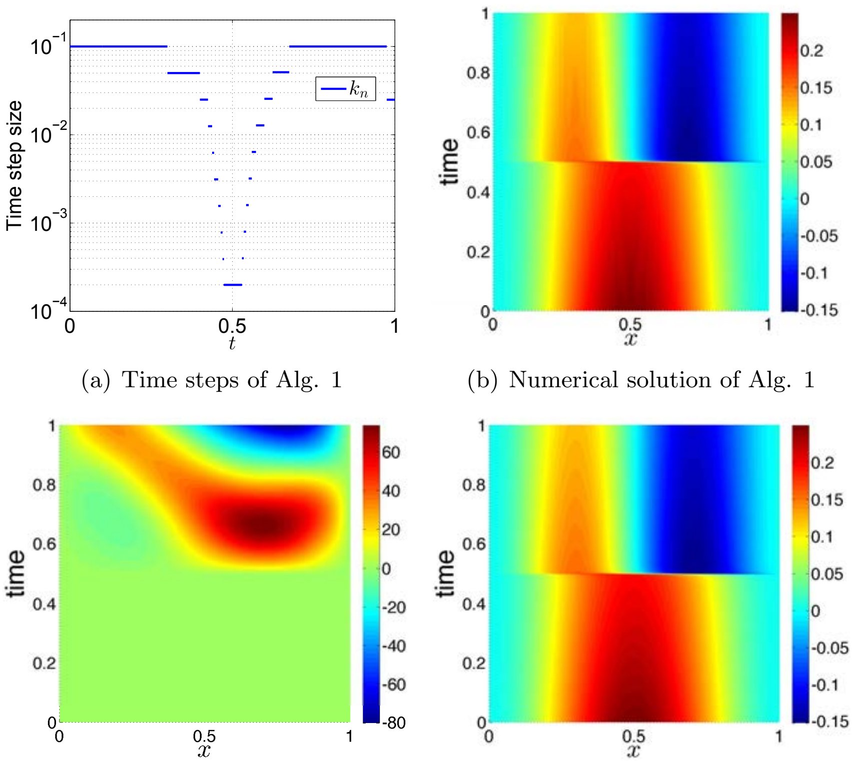

(a) Time steps of Alg. 1

(b) Numerical solution of Alg. 1

(c) Numerical solution with uniform stepsize

(d) Reference solution

Figure 5: Example 5.2. Comparison of numerical results.

From Figure 5(a) we can see the time step size becomes extremely small around $t = 0.5$ *in order to capture the rapid change of the solution, and the step size will become large automatically when the solution varies slowly, which illustrates the efficiency of Algorithm 1. The numerical results with Algorithm 1 and the uniform time stepping method, and the reference solution are shown in Figures 5(b), 5(c) and 5(d), respectively, from which we may find that the adaptive method can approximate the exact solution very well even if it varies rapidly, but the uniform time stepping method fails. We mention further that for the adaptive method in this example, the total CPU time used is approximately* 147.1 s, *while the one for computing* Θ *is only* 7.4 s, *only covers a very small amount of the total time.*

Acknowledgments

The authors would like to thank an anonymous referee for valuable suggestions and comments leading to great improvement of the early version of the paper.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.3, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

A fractional Means inequality

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Abstract

Here we produce an interesting fractional means scalar inequality.

2010 AMS Subject Classification: 26A33, 26D10, 26D15, 26D20.
Key Words and Phrases: Means inequality, fractional derivative.

We make

Remark 1 *Let $\nu > 0$, $n := \lceil \nu \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), $f(\cdot, y) \in AC^n([a,b])$, $\forall\, y \in [c,d]$ (it means $\frac{\partial^{n-1} f(\cdot,y)}{\partial x^{n-1}} \in AC([a,b])$, $\forall\, y \in [c,d]$). Then the left Caputo partial fractional derivative with respect to x, is given by (see [1], p. 270)*

$$\frac{\partial_{*a}^{\nu} f(x,y)}{\partial x^{\nu}} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \frac{\partial^n f(t,y)}{\partial x^n} dt, \tag{1}$$

$\forall\, y \in [c,d]$, and it exists almost everywhere for x in $[a,b]$, Γ denotes the gamma function.

Then, we get the left Caputo fractional Taylor formula ([2], p. 54)

$$f(x,y) = \sum_{k=0}^{n-1} \frac{\partial^k f(a,y)}{\partial x^k} (x-a)^k + \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} \frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}} dt, \tag{2}$$

$\forall\, x \in [a,b]$, for each $y \in [c,d]$.

Above $\left(\int_a^x (x-t)^{\nu-1} \frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}} dt\right) \in AC^n([a,b])$, $\forall\, y \in [c,d]$.

Let now $f(x,\cdot) \in AC^n([c,d])$, $\forall\, x \in [a,b]$ (it means $\frac{\partial^{n-1} f(x,\cdot)}{\partial y^{n-1}} \in AC([c,d])$, $\forall\, x \in [a,b]$). Then the left Caputo partial fractional derivative with respect to y, is given by

$$\frac{\partial_{*c}^{\nu} f(x,y)}{\partial y^{\nu}} = \frac{1}{\Gamma(n-\nu)} \int_c^y (y-s)^{n-\nu-1} \frac{\partial^n f(x,s)}{\partial y^n} ds, \tag{3}$$

$\forall\, x \in [a,b]$, and it exists almost everywhere for y in $[c,d]$.

1

Then, we get the left Caputo fractional Taylor formula

$$f(x,y)=\sum_{k=0}^{n-1}\frac{\partial^k f(x,c)}{\partial y^k}(y-c)^k+\frac{1}{\Gamma(\nu)}\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds, \tag{4}$$

$\forall\ y\in[c,d]$, *for each* $x\in[a,b]$.

Above $\left(\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds\right)\in AC^n([c,d])$, $\forall\ x\in[a,b]$.

Assume

$$\frac{\partial^k f(a,y)}{\partial x^k}=0,\quad \text{for } k=1,...,n-1,\ \forall\ y\in[c,d], \tag{5}$$

we get

$$f(x,y)-f(a,y)=\frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}dt. \tag{6}$$

Additionally assume $f(a,y)=0$, $\forall\ y\in[c,d]$, *then*

$$f(x,y)=\frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}dt, \tag{7}$$

$\forall\ y\in[c,d]$, $\forall\ x\in[a,b]$.

Assume

$$\frac{\partial^k f(x,c)}{\partial y^k}=0,\quad \text{for } k=1,...,n-1,\ \forall\ x\in[a,b], \tag{8}$$

we get

$$f(x,y)-f(x,c)=\frac{1}{\Gamma(\nu)}\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds, \tag{9}$$

$\forall\ y\in[c,d]$, $\forall\ x\in[a,b]$.

Additionally assume that $f(x,c)=0$, $\forall\ x\in[a,b]$, *then*

$$f(x,y)=\frac{1}{\Gamma(\nu)}\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds, \tag{10}$$

$\forall\ y\in[c,d]$, $\forall\ x\in[a,b]$.

Assuming (5) and (8), we get

$$2f(x,y)-f(a,y)-f(x,c)=$$

$$\frac{1}{\Gamma(\nu)}\left\{\int_a^x (x-t)^{\nu-1}\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}dt+\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds\right\}, \tag{11}$$

$\forall\ x\in[a,b]$, $\forall\ y\in[c,d]$.

Additionally assume that $f(a,y)=0$, $\forall\ y\in[c,d]$, *and* $f(x,c)=0$, $\forall\ x\in[a,b]$, *we obtain*

$$f(x,y)=\frac{1}{2\Gamma(\nu)}\left\{\int_a^x (x-t)^{\nu-1}\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}dt+\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds\right\}, \tag{12}$$

2

$\forall\ x \in [a,b]\,,\ \forall\ y \in [c,d]\,.$

We can rewrite (11) as follows:

$$f(x,y) - \left(\frac{f(a,y)+f(x,c)}{2}\right) =$$

$$\frac{1}{2\Gamma(\nu)}\left\{\int_a^x (x-t)^{\nu-1}\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}dt + \int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds\right\}, \quad (13)$$

$\forall\ x \in [a,b]\,,\ \forall\ y \in [c,d]\,.$

If $0<\nu<1$, then $n=1$, and (13) is valid without (5) and (8), which in this case are void conditions.

Call

$$\Delta f(x,y) := f(x,y) - \left(\frac{f(a,y)+f(x,c)}{2}\right). \quad (14)$$

Assume $f \in C([a,b]\times[c,d])$, then

$$\int_a^b\int_c^d \Delta f(x,y)\,dxdy = \int_a^b\int_c^d f(x,y)\,dxdy -$$

$$\left(\frac{(b-a)\int_c^d f(a,y)\,dy + (d-c)\int_a^b f(x,c)\,dx}{2}\right). \quad (15)$$

Hence it holds

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \Delta f(x,y)\,dxdy = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy -$$

$$\left(\frac{\frac{1}{(d-c)}\int_c^d f(a,y)\,dy + \frac{1}{(b-a)}\int_a^b f(x,c)\,dx}{2}\right). \quad (16)$$

Assume now that

$$\frac{\partial_{*a}^{\nu} f(x,y)}{\partial x^{\nu}}, \frac{\partial_{*c}^{\nu} f(x,y)}{\partial y^{\nu}} \in C([a,b]\times[c,d]) \quad (17)$$

Clearly, it holds

$$|\Delta f(x,y)| \le$$

$$\frac{1}{2\Gamma(\nu)}\left\{\int_a^x (x-t)^{\nu-1}\left|\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}\right|dt + \int_c^y (y-s)^{\nu-1}\left|\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}\right|ds\right\} \le$$

$$\frac{1}{2\Gamma(\nu)}\left\{\frac{(x-a)^{\nu}}{\nu}\left\|\frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}}\right\|_{\infty} + \frac{(y-c)^{\nu}}{\nu}\left\|\frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}}\right\|_{\infty}\right\} \le \quad (18)$$

$$\frac{1}{2\Gamma(\nu+1)}\left\{(b-a)^{\nu}\left\|\frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}}\right\|_{\infty} + (d-c)^{\nu}\left\|\frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}}\right\|_{\infty}\right\}.$$

3

That is

$$|\Delta f(x,y)| \leq \frac{1}{2\Gamma(\nu+1)} \left\{ (b-a)^{\nu} \left\| \frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}} \right\|_{\infty} + (d-c)^{\nu} \left\| \frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}} \right\|_{\infty} \right\} =: \lambda. \quad (19)$$

Hence

$$\frac{1}{(b-a)(d-c)} \left| \int_a^b \int_c^d \Delta f(x,y)\, dxdy \right| \leq$$

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d |\Delta f(x,y)|\, dxdy \leq \lambda.$$

We have derived:

Theorem 2 *Let* $\nu > 0$*,* $n := \lceil \nu \rceil$*,* $f(\cdot, y) \in AC^n([a,b])$*,* $\forall\ y \in [c,d]$*; and* $f(x,\cdot) \in AC^n([c,d])$*,* $\forall\ x \in [a,b]$*. Assume* $\frac{\partial^k f(a,y)}{\partial x^k} = 0$*, for* $k = 1,...,n-1$*,* $\forall\ y \in [c,d]$*; and* $\frac{\partial^k f(x,c)}{\partial y^k} = 0$*, for* $k = 1,...,n-1$*,* $\forall\ x \in [a,b]$*. Furthermore, assume* $f \in C([a,b] \times [c,d])$ *and* $\frac{\partial_{*a}^{\nu} f(x,y)}{\partial x^{\nu}}, \frac{\partial_{*c}^{\nu} f(x,y)}{\partial y^{\nu}} \in C([a,b] \times [c,d])$*. Then*

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dxdy - \left(\frac{\frac{1}{(b-a)} \int_a^b f(x,c)\, dx + \frac{1}{(d-c)} \int_c^d f(a,y)\, dy}{2} \right) \right|$$

$$\leq \frac{1}{2\Gamma(\nu+1)} \left\{ (b-a)^{\nu} \left\| \frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}} \right\|_{\infty} + (d-c)^{\nu} \left\| \frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}} \right\|_{\infty} \right\}. \quad (20)$$

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Volume 23, Number 4 October 15, 2017
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE
SCOPE OF THE JOURNAL
An international publication of Eudoxus Press, LLC
(fifteen times annually)
Editor in Chief: George Anastassiou
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An international publication of Eudoxus Press, LLC, of TN.

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3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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On A System of Rational Difference Equations

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In this paper, we investigate behaviors of well-defined solutions of the following system

$$x_{n+1} = \frac{A_1 y_{n-(3k-1)}}{B_1 + C_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)}},$$

$$y_{n+1} = \frac{A_2 x_{n-(3k-1)}}{B_2 + C_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)}},$$

where $n \in \mathbb{N}_0$, $k \in \mathbb{Z}^+$ the coefficients $A_1, A_2, B_1, B_2, C_1, C_2$ and the initial conditions are arbitrary real numbers.
Keywords: System of difference equations, Asymptotic behavior, Periodicity, Closed form solution.
AMS Classification: 39A10

1 Introduction

There has been a great effort in studying periodic and asymptotic behaviors of solutions of difference equations (see e.g. [3,6,12,15,18,20-23,27,35,45,46]). Also, studying in system of difference equations has increased considerably (see, e.g. [5,7,8,16,17,19,28-30,32-34,37,38,40,43,47]).

Ozkan et al. [31] gave the solutions of the systems of the difference equations

$$\begin{aligned} x_{n+1} &= \frac{y_{n-2}}{-1 \mp y_{n-2} x_{n-1} y_n}, \\ y_{n+1} &= \frac{x_{n-2}}{-1 \mp x_{n-2} y_{n-1} x_n}, \\ z_{n+1} &= \frac{x_{n-2} + y_{n-2}}{-1 \mp x_{n-2} y_{n-1} x_n}, n \in \mathbb{N}_0. \end{aligned} \tag{1}$$

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In [39] it was showed that the system of difference equations, which is an extension of first and second equations of system (1) with respect to coefficients,

$$
\begin{aligned}
x_n &= \frac{c_n y_{n-3}}{a_n + b_n y_{n-1} x_{n-2} y_{n-3}}, \\
y_n &= \frac{\gamma_n x_{n-3}}{\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3}}, n \in \mathbb{N}_0,
\end{aligned}
\tag{2}
$$

where the sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$, and the initial values $x_i, y_i, i \in \{1,2,3\}$ are real numbers, such that $c_n \neq 0, \gamma_n \neq 0, n \in \mathbb{N}_0$, can be solved in closed form, and for the case when all sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$ are constant it was described the asymptotic behavior of well-defined solutions of the system.

In [41] it was showed that an extension of system (2) with respect to indices

$$
\begin{aligned}
x_n &= \frac{c_n y_{n-(2k-1)}}{a_n + b_n y_{n-(2k-1)} \prod_{i=1}^{k-1} y_{n-(2i-1)} x_{n-2i}}, \\
y_n &= \frac{\gamma_n x_{n-(2k-1)}}{\alpha_n + \beta_n x_{n-(2k-1)} \prod_{i=1}^{k-1} x_{n-(2i-1)} y_{n-2i}},
\end{aligned}
\tag{3}
$$

where $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$, and the initial conditions $x_i, y_i, i \in \{1, 2, ...2k-1\}$ are real numbers, is solved in closed form, and the behavior of its well-defined solutions when all the sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n$ are constant was described. Related rational difference equations are studied, e.g. in [1,2,4,9-11,13,14,24-26,31,36,42,44,48].

In this paper we consider an other extension of system (2)

$$
\begin{aligned}
x_{n+1} &= \frac{A_1 y_{n-(3k-1)}}{B_1 + C_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)}}, \\
y_{n+1} &= \frac{A_2 x_{n-(3k-1)}}{B_2 + C_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)}},
\end{aligned}
\tag{4}
$$

where $n \in \mathbb{N}_0$, k is a positive integer, the initial conditions and the coefficients A_1, A_2, B_1 , B_2, C_1, C_2 are arbitrary real numbers. We will consider only well-defined solutions, that is, $B_1 + C_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)} \neq 0$ and $B_2 + C_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)} \neq 0, n = 0, 1, 2, ...$

2 Special Cases

2.1 The case $A_1 = 0$ or $A_2 = 0$

If $A_1 = 0$, we obtain directly $x_n = 0$ for $n > 0$. By using this, we get $y_n = 0$ for $n > 3k$. If $A_2 = 0$, we obtain directly $y_n = 0$ for $n > 0$. By using this, we get $x_n = 0$ for $n > 3k$. From now on both of A_1 and A_2 will be considered a non-zero real numbers.

System (4) is equivalent to the following system

$$x_{n+1} = \frac{y_{n-(3k-1)}}{b_1 + c_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)}},$$
$$y_{n+1} = \frac{x_{n-(3k-1)}}{b_2 + c_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)}}, \tag{5}$$

where $n \in \mathbb{N}_0$, $b_i = \frac{B_i}{A_i}$ and $c_i = \frac{C_i}{A_i}, i = 1, 2$. So, we will consider system (5) instead of system (4).

2.2 The case $b_1 = 0$ or $b_2 = 0$

If $b_1 = 0$, from the first equation of system (5), we have $x_{n-2k} y_{n-k} x_n = \frac{1}{c_1}$, $n > 0$. Using this, we obtain directly $y_n = \alpha x_{n-3k}$, $n \geq k$, where $\alpha = \frac{c_1}{b_2 c_1 + c_2}$. From this and by the change of variables

$$z_n = \frac{y_n}{x_{n-3k}}, w_n = \frac{y_{n-3k}}{x_n}, n \geq k, \tag{6}$$

system (5) can be transformed into the system

$$w_{n+1} = c - c b_2 z_{n-(k-1)}, z_{n+1} = \alpha, n \geq k - 1, \tag{7}$$

where $c = \frac{c_1}{c_2}$. The solutions are obtained easily as $z_n = w_n = \alpha$, $n \geq k$. This means every solution of system (5) is periodic with $6k$ periods, not necessarily prime period, such that $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n \geq 4k$.

If $b_2 = 0$, we get immediately $y_{n-2k} x_{n-k} y_n = \frac{1}{c_2}$, $n > 0$. From the first equation in system (5) and using this, we obtain $x_n = \beta y_{n-3k}$, $n \geq k$, where $\beta = \frac{c_2}{b_1 c_2 + c_1}$. The change of variables

$$u_n = \frac{x_n}{y_{n-3k}}, t_n = \frac{x_{n-3k}}{y_n}, n \geq k, \tag{8}$$

reduces system (5) to the system

$$t_{n+1} = \bar{c} - \bar{c} b_1 u_{n-(k-1)}, u_{n+1} = \beta, n \geq 2k - 1, \tag{9}$$

where $\bar{c} = \frac{c_2}{c_1}$. The solutions of this system $t_n = u_n = \beta$, $n \geq 2k - 1$, are obtained easily. So, every solution of system (5) is periodic with $6k$ periods, not necessarily prime period, such that $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n \geq 4k$.

Assume that $b_1 = 0$ and $b_2 = 0$. We have $x_{n-2k} y_{n-k} x_n = \frac{1}{c_1}$, $y_{n-2k} x_{n-k} y_n = \frac{1}{c_2}$, $n > 0$. Then, we get immediately $x_n = \frac{c_2}{c_1} y_{n-3k}$, $y_n = \frac{c_1}{c_2} y_{n-3k}$, $n > k$. Thus, we can write $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n > 4k$.

2.3 The case $c_1 = 0$ or $c_2 = 0$

If $c_1 = 0$, we have $x_n = \frac{1}{b_1} y_{n-3k}$, $n > 0$. From this and using the change of variables

$$v_n = \frac{1}{x_{n+3k} y_{n-k} x_n}, n > 0, \tag{10}$$

the second equation of system (5) implies the linear equation

$$v_{n+1} = b_1 b_2 v_{n-(2k-1)} + {b_1}^2 c_2, n = 0, 1, 2, ... \tag{11}$$

We can rewrite the equation (11) in the form of

$$v_{2kn+m} = b_1 b_2 v_{2k(n-1)+m} + b_1^2 c_2, \tag{12}$$

where $n \in \mathbb{N}_0$, $m = 1, 2, ..., k$. Considering the solution of a nonhomogeneous first order difference equation, we can give the solution of the equation (12) such that

$$v_{2kn+m} = (b_1 b_2)^n v_{m-2k} + b_1^2 c_2 \frac{1 - (b_1 b_2)^{n+1}}{1 - b_1 b_2}, n \geq 0. \tag{13}$$

when $b_1 b_2 \neq 1$. If $b_1 b_2 = 1$, the solution of the equation (12) can be written as

$$v_{2kn+m} = v_{m-2k} + (n+1) b_1^2 c_2, n \geq 0. \tag{14}$$

From (10), we have

$$x_{2kn+3k+m} = \frac{v_{2k(n-1)+m}}{v_{2kn+m}} x_{2kn-3k+m}.$$

Considering $x_n = \frac{1}{b_1} y_{n-3k}$, we obtain the solutions of system (5) as

$$x_{6kn+3k+m} = x_{-3k+m} \prod_{r=0}^{n} \frac{v_{6kr-2k+m}}{v_{6kr+m}}, y_{6kn+m} = b_1 x_{-3k+m} \prod_{r=0}^{n} \frac{v_{6kr-2k+m}}{v_{6kr+m}}, \tag{15}$$

$$x_{6kn+5k+m} = x_{-k+m} \prod_{r=0}^{n} \frac{v_{6kr+m}}{v_{6kr+2k+m}}, y_{6kn++2k+m} = b_1 x_{-k+m} \prod_{r=0}^{n} \frac{v_{6kr+m}}{v_{6kr+2k+m}}, \tag{16}$$

$$x_{6kn+7k+m} = x_{k+m} \prod_{r=0}^{n} \frac{v_{6kr+2k+m}}{v_{6kr+4k+m}}, y_{6kn+4k+m} = b_1 x_{k+m} \prod_{r=0}^{n} \frac{v_{6kr+2k+m}}{v_{6kr+4k+m}}, \tag{17}$$

$n \geq 0$ and $m = 1, 2, ..., 2k$.

Suppose that $c_2 = 0$. Then, we have $y_n = \frac{1}{b_2} x_{n-3k}$, $n > 0$. From this and using the change of variable

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$$u_n = \frac{1}{y_{n+3k} y_{n+k} y_{n-k}}, n > 0, \tag{18}$$

the first equation of system (5) implies the linear equation

$$u_{n+1} = b_1 b_2 u_{n-(2k-1)} + b_2^2 c_1, n \geq 0. \tag{19}$$

By similar processes just as we did, we can rewrite the equation (19) as

$$u_{2kn+m} = b_1 b_2 u_{2k(n-1)+m} + b_2^2 c_1, n \geq 0, \tag{20}$$

where $m = 1, 2, ..., k$. We obtain the solution of the equation (20)

$$u_{2kn+m} = (b_1 b_2)^n u_{m-2k} + b_2^2 c_1 \frac{1 - (b_1 b_2)^{n+1}}{1 - b_1 b_2}, n \geq 0, \tag{21}$$

when $b_1 b_2 \neq 1$. When $b_1 b_2 = 1$, the solution of the equation (20)

$$u_{2kn+m} = u_{m-2k} + (n+1) b_2^2 c_1, n \geq 0. \tag{22}$$

From (18), we have

$$y_{n+3k} = \frac{1}{u_n y_{n+k} y_{n-k}}, n > 0,$$

and

$$y_{2kn+3k+m} = \frac{u_{2k(n-1)+m}}{u_{2kn+m}} y_{2kn-3k+m}.$$

Considering $y_n = \frac{1}{b_2} x_{n-3k}$, $n > 0$, we obtain the solutions of system (5) as

$$x_{6kn+m} = b_2 y_{-3k+m} \prod_{r=0}^{n} \frac{u_{6kr-2k+m}}{u_{6kr+m}}, y_{6kn+3k+m} = y_{-3k+m} \prod_{r=0}^{n} \frac{u_{6kr-2k+m}}{u_{6kr+m}}, \tag{23}$$

$$x_{6kn++2k+m} = b_2 y_{-k+m} \prod_{r=0}^{n} \frac{u_{6kr+m}}{u_{6kr+2k+m}}, y_{6kn+5k+m} = y_{-k+m} \prod_{r=0}^{n} \frac{u_{6kr+m}}{u_{6kr+2k+m}}, \tag{24}$$

$$x_{6kn+4k+m} = b_2 y_{k+m} \prod_{r=0}^{n} \frac{u_{6kr+2k+m}}{u_{6kr+4k+m}}, y_{6kn+7k+m} = y_{k+m} \prod_{r=0}^{n} \frac{u_{6kr+2k+m}}{u_{6kr+4k+m}}, \tag{25}$$

$n \geq 0$ and $m = 1, 2, ..., 2k$.

Suppose that both c_1 and c_2 are equal to zero. We get immediately $x_{n+1} = \frac{1}{b_1} y_{n-(3k-1)}$, $y_{n+1} = \frac{1}{b_2} x_{n-(3k-1)}$, $n \geq 0$. From this result, we obtain $x_{n+1} = \frac{1}{b_1 b_2} x_{n-(6k-1)}, y_{n+1} = \frac{1}{b_1 b_2} y_{n-(6k-1)}$, $n \geq 3k$. So, we have $x_{6kn+3k+m} = \frac{1}{b_1 b_2} x_{6kn-3k+m}$, $y_{6kn+3k+m} = \frac{1}{b_1 b_2} y_{6kn-3k+m}$ and from this $x_{6kn+3k+m} = \left(\frac{1}{b_1 b_2}\right)^{n+1} x_{-3k+m}$, $y_{6kn+3k+m} = \left(\frac{1}{b_1 b_2}\right)^{n+1} y_{-3k+m}$, $n \geq 0$, $m = 1, 2, ..., 6k$.

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3 Main Case

In this section, we will need the following results, given in the reference [16], in the proofs of our results.

Consider the first order Riccati difference equation

$$x_{n+1} = \frac{a + bx_n}{c + dx_n}, n = 0, 1, ..., \tag{26}$$

where the parameters and the initial condition x_0 are arbitrary real numbers.

Theorem 1 *The followings are true:*

1) *Eq.*(26) *has a prime period-2 solution if and only if* $b + c = 0$.

2) *Suppose* $b + c = 0$. *Then every solution* $\{x_n\}$ *of Eq.* (26) *with* $x_0 \neq 0$ *is periodic with period 2.*

Theorem 2 *Assume that* $d \neq 0, bc - ad \neq 0,\ b + c \neq 0$ *and* $R = \frac{bc-ad}{(b+c)^2} < \frac{1}{4}$. *Then the forbidden set* F *of Eq.*(26) *is given as follows:*

$$F = \left\{ \frac{b+c}{d} \left(\frac{\lambda_1 \lambda_2^n - \lambda_2 \lambda_1^n}{\lambda_2^n - \lambda_1^n} \right) - \frac{c}{d} : n \geq 1 \right\} .$$

For any well-defined solution $\{x_n\}$ *of Eq.* (26), *we have*

$$x_n = \frac{b+c}{d} \left(\frac{c_1 \lambda_1^{n+1} - c_2 \lambda_2^{n+1}}{c_1 \lambda_1^n - c_2 \lambda_2^n} \right) - \frac{c}{d},$$

for $n = 0, 1, ...,$ *where* $\lambda_1 = \frac{1-\sqrt{1-4R}}{2}$, $\lambda_2 = \frac{1+\sqrt{1-4R}}{2}$, $c_1 = \frac{\lambda_2(b+c)-(dx_0+c)}{(\lambda_2-\lambda_1)(b+c)}$ *and* $c_2 = \frac{(dx_0+c)-\lambda_1(b+c)}{(\lambda_2-\lambda_1)(b+c)}$.

Corollary 1 *Assume that the conditions in Theorem2 hold. Let* $\{x_n\}$ *be a well-defined solution of Eq.* (26). *Then*

$$lim_{n\to\infty} x_n = \frac{\lambda_2(b+c)-c}{d}.$$

Theorem 3 *Assume that* $d \neq 0, bc - ad \neq 0,\ b + c \neq 0$ *and* $R = \frac{bc-ad}{(b+c)^2} = \frac{1}{4}$. *Then the forbidden set* F *of Eq.*(26) *is given as follows:*

$$F = \left\{ \frac{n(b-c)-(b+c)}{2dn} : n \geq 1 \right\}.$$

For any well-defined solution $\{x_n\}$ *of Eq.* (26), *we have*

$$x_n = \frac{b+c}{d} \left(\frac{(b+c)+(n+1)(2dx_0+(c-b))}{2(b+c)+2n(2dx_0+(c-b))} \right) - \frac{c}{d},$$

for $n = 0, 1, ...$

Corollary 2 *Assume that the conditions in Theorem3 hold. Let* $\{x_n\}$ *be a well-defined solution of Eq.*(26). *Then*

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$$lim_{n\to\infty} x_n = \frac{b-c}{2d}.$$

Now we consider the system (5) with b_1, b_2, c_1, c_2 parameters and the initial conditions are non-zero real numbers. By the change of variables (6), the system (5) reduces to

$$z_{n+1} = \frac{w_{n-(k-1)}}{\frac{1}{\alpha} w_{n-(k-1)} - \gamma_2}, w_{n+1} = \frac{1}{\beta} - \gamma_1 z_{n-(k-1)}, n \geq k, \tag{27}$$

where $\gamma_1 = \frac{c_1 b_2}{c_2}, \gamma_2 = \frac{c_2 b_1}{c_1}, \alpha = \frac{c_1}{b_2 c_1 + c_2}, \beta = \frac{c_2}{b_1 c_2 + c_1}$. We can rewrite the system (27) such that

$$z_{n+1} = \frac{\frac{1}{\beta} - \gamma_1 z_{n-(2k-1)}}{\left(\frac{1}{\alpha\beta} - \gamma_2\right) - \frac{\gamma_1}{\alpha} z_{n-(2k-1)}}, w_{n+1} = \frac{\left(\frac{1}{\alpha\beta} - \gamma_1\right) w_{n-(2k-1)} - \frac{\gamma_2}{\beta}}{\frac{1}{\alpha} w_{n-(2k-1)} - \gamma_2}, \tag{28}$$

$n \geq 2k$. Each of the equation in (28)is a $2k$th order Riccati difference equation. Furthermore, the equations in (28) can be rewritten such that

$$\begin{aligned} z_{2kn+1+i} &= \frac{\frac{1}{\beta} - \gamma_1 z_{2k(n-1)+1+i}}{\left(\frac{1}{\alpha\beta} - \gamma_2\right) - \frac{\gamma_1}{\alpha} z_{2k(n-1)+1+i}}, \\ w_{2kn+1+i} &= \frac{\left(\frac{1}{\alpha\beta} - \gamma_1\right) w_{2k(n-1)+1+i} - \frac{\gamma_2}{\beta}}{\frac{1}{\alpha} w_{2k(n-1)+1+i} - \gamma_2}, \end{aligned} \tag{29}$$

$n > 0$, $i = 0, 1, ..., (2k-1)$. Note that the equations in (29) are first order Riccati difference equation in variables z_{2kn+i}, w_{2kn+i}, for $i = 1, 2, ..., 2k$.

Theorem 4 *Assume that $b_1 b_2 = -1$ and $\{x_n, y_n\}$ is a well-defined solution of system (5). Then,*

$$\begin{aligned} x_{2kn+1+i} &= \frac{x_{2k(n-2)+1+i} x_{2k(n-3)+1+i}}{x_{2k(n-5)+1+i}}, \\ y_{2kn+1+i} &= \frac{y_{2k(n-2)+1+i} y_{2k(n-3)+1+i}}{y_{2k(n-5)+1+i}}, \end{aligned}$$

for $n \geq 4$, $i = 0, 1, ..., (2k-1)$.

***Proof* 1** *Consider system (29) and suppose that $b_1 b_2 = -1$. Then, we have*

$$\begin{aligned} \frac{1}{\alpha\beta} - \gamma_1 - \gamma_2 &= \frac{1}{\frac{c_1}{b_2 c_1 + c_2} \frac{c_2}{b_1 c_2 + c_1}} - \frac{c_1 b_2}{c_2} - \frac{c_2 b_1}{c_1} \\ &= \frac{b_1 b_2 c_1 c_2 + c_1^2 b_2 + c_2^2 b_1 + c_1 c_2 - c_1^2 b_2 - c_2^2 b_1}{c_1 c_2} \\ &= \frac{c_1 c_2 (b_1 b_2 + 1)}{c_1 c_2} \\ &= 0. \end{aligned}$$

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So, from Theorem1(2) we conclude that every solution of each equation in system (29) *is periodic with period* $4k$, *that is,*

$$z_{2kn+1+i} = z_{2k(n-2)+1+i}, w_{2kn+1+i} = w_{2k(n-2)+1+i}, \tag{30}$$

for $n \geq 2$, $i = 0, 1, ..., (2k-1)$.

From (6), we have

$$x_n = \frac{z_{n-3k}}{w_n} x_{n-6k}, y_n = \frac{z_n}{w_{n-3k}} y_{n-6k}, \tag{31}$$

for $n \geq 4k$. *System* (31) *can be written such that*

$$x_{2kn+1+i} = \frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}} x_{2kn+1+i-6k}, y_{2kn+1+i} = \frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}} y_{2kn+1+i-6k} \tag{32}$$

for $n \geq 2$, $i = 0, 1, ..., (2k-1)$. *From* (6), (30) *and* (32), *we get*

$$\begin{aligned} x_{2kn+1+i} &= \frac{z_{2k(n-2)+1+i-3k}}{w_{2k(n-2)+1+i}} x_{2kn+1+i-6k} \\ &= \frac{\frac{y_{2k(n-2)+1+i-3k}}{x_{2k(n-2)+1+i-6k}}}{\frac{y_{2k(n-2)+1+i-3k}}{x_{2k(n-2)+1+i}}} x_{2kn+1+i-6k} \\ x_{2kn+1+i} &= \frac{x_{2k(n-2)+1+i} x_{2k(n-3)+1+i}}{x_{2k(n-5)+1+i}} \end{aligned} \tag{33}$$

and similarly

$$y_{2kn+1+i} = \frac{y_{2k(n-2)+1+i} y_{2k(n-3)+1+i}}{y_{2k(n-5)+1+i}} \tag{34}$$

for $n \geq 4$, $i = 0, 1, ..., (2k-1)$.

Theorem 5 *Assume that* $\{x_n, y_n\}$ *is a well-defined solution of system* (5). *Then the followings are true:*

i) *Assume that* $b_1 b_2 = 1$. *Then every solution converges to a periodic solution with period* $6k$.

ii) *Assume that* $b_1 b_2 \neq 1$. *Then,*

a) *If* $b_1 b_2 < -1$ *or* $b_1 b_2 > 1$, *then*

$$lim_{n\to\infty} \frac{x_n}{x_{n-6k}} = lim_{n\to\infty} \frac{y_n}{y_{n-6k}} = \frac{b_2 c_1 + c_2}{b_2 c_2 (b_1 b_2 c_1 + c_2 b_1 + c_1)}.$$

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b) *If* $-1 < b_1b_2 < 1$, *then every solution converges to a periodic solution with period* $6k$.

Proof 2

i) Consider system (29) with $\gamma_1 = \frac{c_1b_2}{c_2}, \gamma_2 = \frac{c_2b_1}{c_1}, \alpha = \frac{c_1}{b_2c_1+c_2}, \beta = \frac{c_2}{b_1c_2+c_1}$. *Suppose that* $b_1b_2 = 1$. *Then, we have*

$$\begin{aligned}\frac{-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}(-\frac{\gamma_1}{\alpha})}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2} &= \frac{\gamma_1\gamma_2}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2} \\ &= \frac{\frac{c_1b_2}{c_2}\frac{c_2b_1}{c_1}}{\left(-\frac{c_1b_2}{c_2}+\frac{1}{\frac{c_1}{b_2c_1+c_2}\frac{c_2}{b_1c_2+c_1}}-\frac{c_2b_1}{c_1}\right)^2} \\ &= \frac{b_1b_2}{(b_1b_2+1)^2} \\ &= \frac{1}{4}.\end{aligned} \tag{35}$$

Similarly, it can be seen that

$$\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2} = \frac{\gamma_1\gamma_2}{(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2)^2} = \frac{1}{4}. \tag{36}$$

So, from (31), (32) and Theorem3, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}} &= \lim_{n\to\infty}\frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}} \\ &= \frac{\frac{-\gamma_1-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{2\left(-\frac{\gamma_1}{\alpha}\right)}}{\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)-(-\gamma_2)}{2\left(\frac{1}{\alpha}\right)}} = 1\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}} &= \lim_{n\to\infty}\frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}} \\ &= \frac{\frac{-\gamma_1-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{2\left(-\frac{\gamma_1}{\alpha}\right)}}{\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)-(-\gamma_2)}{2\left(\frac{1}{\alpha}\right)}} = 1,\end{aligned}$$

$i = 0, 1, ..., (2k-1)$. *Thus, we have* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-6k}$ *and* $\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n-6k}$. *So, theproof of* (i) *is finished.*

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ii)a) Assume that $b_1b_2 < -1$ *or* $b_1b_2 > 1$. *From* (35) *and* (36), *we get that*

$-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}\left(-\frac{\gamma_1}{\alpha}\right))\frac{1}{\left(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2\right)^2} < \frac{1}{4}$

(36)

and

$\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha})\frac{1}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2} < \frac{1}{4}.$

So, from (31), (32) and Theorem2, we obtain that

$$\lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}} = \lim_{n\to\infty}\frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}}$$

$$= \frac{\dfrac{\frac{1+\sqrt{1-4\frac{-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}\left(-\frac{\gamma_1}{\alpha}\right)}{\left(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2\right)^2}}}{2}\left(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2\right)-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{\left(-\frac{\gamma_1}{\alpha}\right)}}{\dfrac{\frac{1+\sqrt{1-4\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2}}}{2}\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)-(-\gamma_2)}{\frac{1}{\alpha}}}$$

$$= \frac{\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)-\left(b_1b_2+1+\frac{c_1b_2}{c_2}\right)}{-\frac{c_1b_2}{c_2}\left(\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)+\frac{c_2b_1}{c_1}\right)} \tag{37}$$

$$= \frac{b_2c_1+c_2}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)}$$

and

$\lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}} = \lim_{n\to\infty}\frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}}$

$=(b_2c_1+c_2)\frac{1}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)},$

$i = 0,1,...,(2k-1)$. *Thus, we have* $\lim_{n\to\infty}\frac{x_n}{x_{n-6k}} = \lim_{n\to\infty}\frac{y_n}{y_{n-6k}} = \frac{b_2c_1+c_2}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)}$.

b) Assume that $-1 < b_1b_2 < 1$. *From (35) and (36), we get that*

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-$\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}(-\frac{\gamma_1}{\alpha}))\frac{1}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2} < \frac{1}{4}$

and

(

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$\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}\right)\frac{1}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2}<\frac{1}{4}$. *So, from (31), (32), (37) and Theorem2, we obtain*

$$\lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}}=\lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}}$$
$$=\frac{\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)-\left(b_1b_2+1+\frac{c_1b_2}{c_2}\right)}{-\frac{c_1b_2}{c_2}\left(\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)+\frac{c_2b_1}{c_1}\right)}$$
$$=\frac{b_1b_2+\frac{c_1b_2}{c_2}}{\frac{c_1b_2}{c_2}\left(1+\frac{c_2b_1}{c_1}\right)}$$
$$=\frac{b_1b_2c_2+c_1b_2}{b_1b_2c_2+c_1b_2}=1,$$

$i=0,1,...,(2k-1)$. *So, we get immediately* $\lim_{n\to\infty}x_n=\lim_{n\to\infty}x_{n-6k}$ *and* $\lim_{n\to\infty}y_n=\lim_{n\to\infty}y_{n-6k}$ *and then the proof of is finished.*

Acknowledgement 1 *The author is grateful to Karamanoglu Mehmetbey University Scientific Research Council (BAP No: BAP-01-M-14).*

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A Numerical Approach Based on Subdivision Schemes for Solving Non-Linear Fourth Order Boundary Value Problems

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Abstract

This paper presents an iterative collocation numerical approach based on interpolating subdivision schemes for the solution of non-linear fourth order boundary value problems involving ordinary differential equations. Numerical evidence suggests that the scheme converges to a smooth approximate solution of non-linear fourth order boundary value problem. The convergence of the approach is also discussed. Main purpose of this article is to explore and seek the applications of subdivision schemes in the field of physics and engineering.

Keywords: Boundary value problem, Subdivision schemes, Collocation algorithm, Approximation, interpolation

AMS Classification: 30E25; 65D07; 97N50

1 Introduction

Boundary value problems arise in several branches of physics and engineering. In recent years, there has been significant progress in solving problems associated with system of linear and nonlinear partial and ordinary differential equations involving boundary conditions. Two point nonlinear boundary value problems often cannot be solved by analytical methods. With increasing interest in finding solutions to nonlinear boundary value problems has come an increasing need for solution techniques.

In this paper, we consider the following type of nonlinear boundary value problem

$$y^{(iv)} = f(x, y, y^{'}) \tag{1.1}$$

with the boundary conditions

$$\begin{cases} y(a) = \alpha_1, & y'(a) = \alpha_2, \\ y(b) = \alpha_3, & y'(b) = \alpha_4, \end{cases} \tag{1.2}$$

where $\alpha_i, i = 1, 2, 3, 4$ are constants. We assume that the problem is well-posed.

A variety of methods have been introduced to solve these problem e.g., shooting methods, splines methods [2, 3, 4, 5, 6], finite difference methods, finite element methods, the collocation methods and other approximation methods. For discrete methods, like shooting and finite differences methods, only discrete approximate values of the unknown $y(x)$ can be obtained. For fitting curve to data we need further data processing techniques. For the case of spline interpolation or approximation methods the unknown function $y(x)$ is assumed to be piecewise polynomial which requires at least piecewise higher order differentiability of the function $f(x, y, y^{'})$. To overcome these disadvantages, Qu and Agarwal

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[8, 9, 10] introduced the subdivision based algorithm for the solution of two point second order boundary value problems. Mustafa and Ejaz [14] solved third order boundary value problem by using subdivision technique. Higher order nonlinear problems have not been solved by subdivision techniques. This motivates us to solve fourth order boundary value problems by subdivision schemes based collocation iterative method. This paper introduces a numerical method based on subdivision technique for the solution of fourth order nonlinear boundary value problem.

In Section 2, some results about subdivision algorithms and basis function are given. In Section 3, a numerical method to solve (1.1) using refinable basis functions is formulated and its convergence properties studied. Error properties are given in Section 4. Numerical examples illustrating the feasibility of our proposed algorithm are given in Section 5.

2 Basis Functions and their Derivatives

Some useful results for the solution of non-linear boundary value problem are discussed in this section. Introduction to the basis functions of subdivision schemes that are used to construct the approximate solutions of proposed problem (1.1) is also part of this section.

2.1 Interpolating subdivision scheme

A mathematical formulation of binary subdivision scheme is defined as:

$$\begin{cases} P_{2i}^{k+1} = \sum\limits_{j\in\mathbb{Z}} a_{-2j} P_{i+j}^{k} \\ \\ P_{2i+1}^{k+1} = \sum\limits_{j\in\mathbb{Z}} a_{i-2j} P_{i+j}^{k} \end{cases} \tag{2.1}$$

The scheme is a stepwise interpolatory scheme if and only if the coefficient a_i satisfy $a_{2i} = \delta_i \quad \forall\ i \in \mathbb{Z}$. We consider the following binary interpolating subdivision scheme [7, 12, 13].

$$\begin{cases} p_{2i}^{k+1} = p_i^k, \\ \\ p_{2i+1}^{k+1} = \frac{35}{65536}\left(p_{i-4}^k + p_{i+5}^k\right) - \frac{405}{65536}\left(p_{i-3}^k + p_{i+4}^k\right) + \frac{567}{16384}\left(p_{i-2}^k + p_{i+3}^k\right) \\ \\ \qquad\qquad - \frac{2205}{16384}\left(p_{i-3}^k + p_{i+4}^k\right) + \frac{19845}{32768}\left(p_{i}^k + p_{i+1}^k\right). \end{cases} \tag{2.2}$$

The scheme (2.2) is C^4-continuous, having support length $(-9, 9)$ and approximation order is ten.

2.2 Basis functions

The basis functions is the limit function resulting from cardinal data, where all vertices of the polygon have value zero except for one. Let $\phi(x)$, $x \in \mathbb{R}$ be the fundamental solution of (2.2) and satisfies the two scale equations

$$\begin{aligned} \phi(x) = \phi(2x) + \frac{1}{65536}&[39690\{\phi(2x-1) + \phi(2x+1)\} - 8820\{\phi(2x-3) \\ &+\phi(2x+3)\} + 2268\{\phi(2x-5) + \phi(2x+5)\} - 405\{\phi(2x-7) \\ &+\phi(2x+7)\} + 35\{\phi(2x-9) + \phi(2x+9)\}], \quad x \in \mathbb{R} \end{aligned} \tag{2.3}$$

and

$$\phi(x) \in C^4, \quad \phi(x) = 0, \quad x \in]-8, 8[, \quad \phi(i) = \delta_0, \quad i \in \mathbb{Z}. \tag{2.4}$$

Furthermore, first derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(i)}(0) \quad = 0, & \phi^{(i)}(\pm 1) = \mp\frac{1914621952}{1159104017}, & \phi^{(i)}(\pm 2) = \pm\frac{530452796}{1159104017}, \\ \\ \phi^{(i)}(\pm 3) = \mp\frac{1470464}{13780629}, & \phi^{(i)}(\pm 4) = \pm\frac{17297069}{1159104017}, & \phi^{(i)}(\pm 5) = \mp\frac{2772992}{5795520085}, \\ \\ \phi^{(i)}(\pm 6) = \mp\frac{1127636}{10431936153}, & \phi^{(i)}(\pm 7) = \mp\frac{4096}{8113728119}, & \phi^{(i)}(\pm 8) = \mp\frac{5}{9272832136}. \end{cases} \tag{2.5}$$

Second derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(ii)}(0) = -\frac{2370618501415}{309077185968}, & \phi^{(ii)}(\pm 1) = \frac{3265310153216}{676106344305}, & \phi^{(ii)}(\pm 2) = -\frac{878265102572}{676106344305}, \\ \phi^{(ii)}(\pm 3) = \frac{734063059456}{2028319032915}, & \phi^{(ii)}(\pm 4) = -\frac{80883901277}{1352212688610}, & \phi^{(ii)}(\pm 5) = \frac{214899200}{135221268861}, \\ \phi^{(ii)}(\pm 6) = \frac{297875188}{405663806583}, & \phi^{(ii)}(\pm 7) = \frac{64000}{19317324123}, & \phi^{(ii)}(\pm 8) = \frac{4375}{618154371936}. \end{cases} \tag{2.6}$$

Third derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(iii)}(0) = 0, & \phi^{(iii)}(\pm 1) = \pm\frac{43317515008}{15295995855}, & \phi^{(iii)}(\pm 2) = \mp\frac{121530512357}{61183983420}, \\ \phi^{(iii)}(\pm 3) = \pm\frac{240606976}{566518365}, & \phi^{(iii)}(\pm 4) = \mp\frac{5285889107}{244735933680}, & \phi^{(iii)}(\pm 5) = \mp\frac{37414144}{3059199171}, \\ \phi^{(iii)}(\pm 6) = \pm\frac{1090169}{453214692}, & \phi^{(iii)}(\pm 7) = \mp\frac{21760}{437028453}, & \phi^{(iii)}(\pm 8) = \mp\frac{2975}{13984910496}. \end{cases} \tag{2.7}$$

Fourth derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(iv)}(0) = \frac{33869667}{457408}, & \phi^{(iv)}(\pm 1) = -\frac{5295054752}{89730585}, & \phi^{(iv)}(\pm 2) = \frac{10404741119}{358922340}, \\ \phi^{(iv)}(\pm 3) = -\frac{74879584}{9970065}, & \phi^{(iv)}(\pm 4) = \frac{295020869}{2871378720}, & \phi^{(iv)}(\pm 5) = \frac{9238624}{17946117}, \\ \phi^{(iv)}(\pm 6) = -\frac{900187}{7976052}, & \phi^{(iv)}(\pm 7) = \frac{71840}{17946117}, & \phi^{(iv)}(\pm 8) = \frac{11225}{328157568}. \end{cases} \tag{2.8}$$

The above derivative values are found by using the left eigenvectors of the subdivision process (2.2). The detailed description about these left eigenvectors and derivatives can be found in [8, 11, 14]. The graphical representations of above mentioned derivatives are shown in Figure 1.

3 Description of Iterative Numerical Method

This section describes the method for the numerical solution of nonlinear boundary value problem (1.1). The detail of the method is given below:

3.1 The collocation method

In this subsection, the collocation method is constructed based on the interpolating subdivision scheme (2.2). Our numerical approach for nonlinear fourth order boundary value problem using collocation method based on subdivision scheme is to seek an approximate solution as

$$Z(x) = \sum_{i=-8}^{N+8} z_i \phi\left(\frac{x - x_i}{h}\right), \quad 0 \leqslant x \leqslant 1 \tag{3.1}$$

where N is the positive integer $N \geqslant 8$, $h = 1/N$ and $x_i = i/N = ih$ and $\{z_i\}$ are the unknowns to be determined for the solution of (1.1). In order to solve the problem, a collocation method $Z(x)$ is considered to be the solution of the above differential equation at $x = x_j$ and we substitute equation (3.1) into equation (1.1). This leads to

$$Z^{(iv)}(x_j) = f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N \tag{3.2}$$

and boundary conditions

$$Z(0) = \alpha_1,\ Z^{'}(0) = \alpha_2,\ Z(N) = \alpha_3,\ Z^{'}(N) = \alpha_4 \tag{3.3}$$

From (3.1), we get

$$Z^{(iv)}(x) = \frac{1}{h^4} \sum_{i=-8}^{N+8} z_i \phi^{(iv)}\left(\frac{x - x_i}{h}\right), \quad 0 \leqslant x \leqslant 1 \tag{3.4}$$

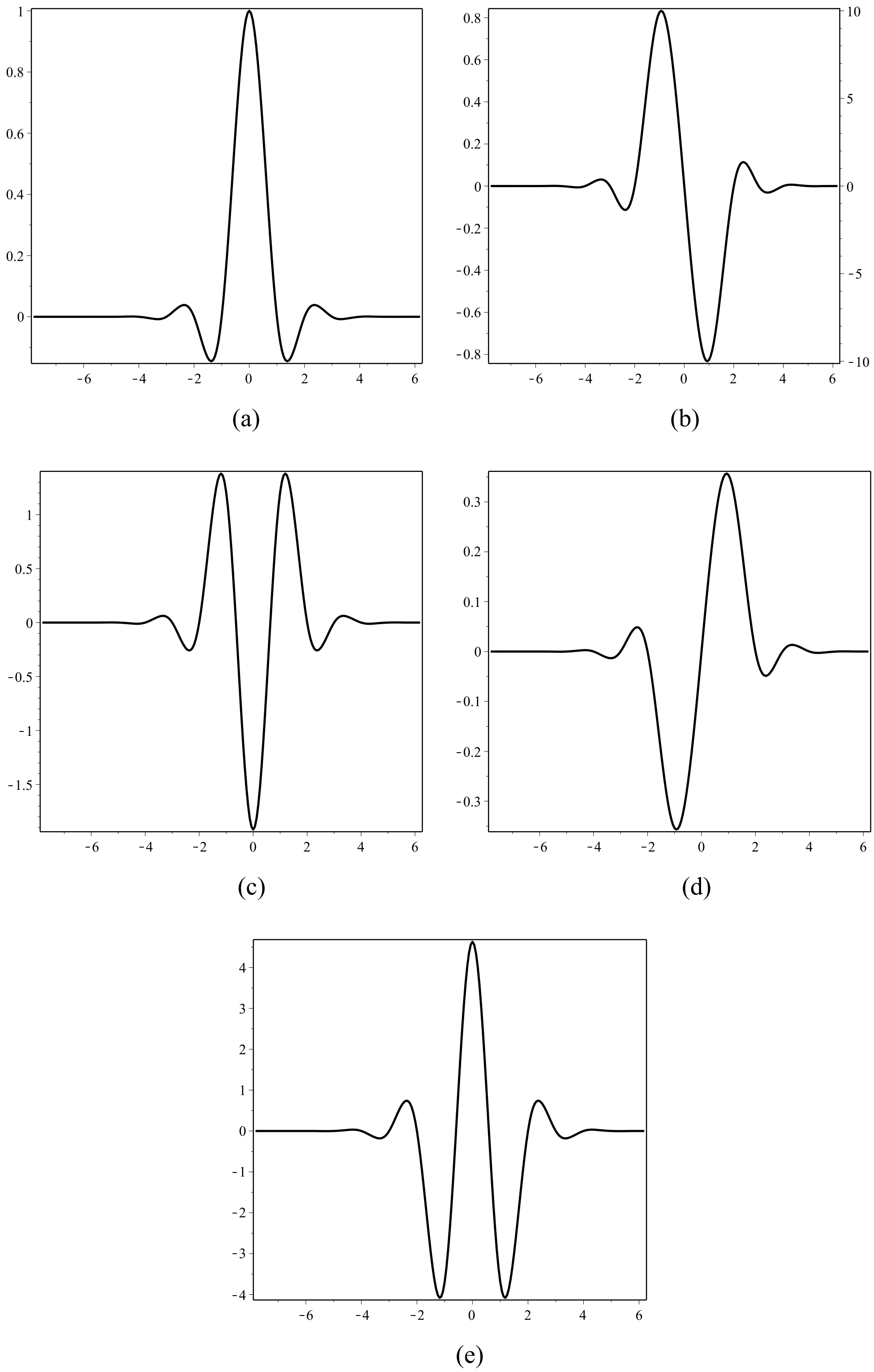

Figure 1: Graphical representation of: (a) basis functions, (b) first, (c) second, (d) third, (e) fourth derivatives of basis function

Substituting (3.4) into (3.2), we obtain

$$\sum_{i=-8}^{N+8} z_i \phi^{(iv)} \left(\frac{x_j - x_i}{h} \right) = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N.$$

This can be written as:

$$\sum_{i=-8}^{N+8} z_i \phi_{j-i}^{(iv)} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N$$

Since $\phi_i^{(iv)} = \phi_{-i}^{(iv)}$, the above system of equations becomes

$$\sum_{i=-8}^{N+8} z_i \phi_{i-j}^{(iv)} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N. \tag{3.5}$$

The nonlinear system of equations (3.5) can be simply in following Theorems 1 and 2.

Theorem 1. *The nonlinear system of equations (3.5) for $j = 0$ becomes*

$$\sum_{i=-8}^{8} z_i \phi_i^{(iv)} = h^4 f(x_0, Z(x_0), Z^{'}(x_0)). \tag{3.6}$$

Proof. By expanding (3.5) for $j = 0$, we obtain

$$\sum_{i=-8}^{N+8} z_i \phi_i^{(iv)} = h^4 f(x_0, Z(x_0), Z^{'}(x_0))$$

$$\begin{aligned} & z_{-8}\phi_{-8}^{iv} + z_{-7}\phi_{-7}^{iv} + z_{-6}\phi_{-6}^{iv} + \cdots + z_7\phi_7^{iv} + z_8\phi_8^{iv} + z_9\phi_9^{iv} + \cdots + z_{N+7}\phi_{N+7}^{iv} \\ & + z_{N+8}\phi_{N+8}^{iv} = h^4 f(x_0, Z(x_0), Z^{'}(x_0)). \end{aligned}$$

Since $\phi^{iv}(i)$ exists only for the interval for $i \in [-8, 8]$ and outside the interval it will be zero. Then above equation can be written as

$$z_{-8}\phi_{-8}^{iv} + z_{-7}\phi_{-7}^{iv} + z_{-6}\phi_{-6}^{iv} + \cdots + z_7\phi_7^{iv} + z_8\phi_8^{iv} = h^4 f(x_0, Z(x_0), Z^{'}(x_0)).$$

□

Theorem 2. *For $j = 1, 2, \cdots, N$, the nonlinear system of equations (3.5) becomes*

$$\sum_{i=-8+j}^{8+j} z_i \phi_{i-j}^{(iv)} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)). \tag{3.7}$$

Proof. By expanding (3.5), for $j = 1, 2, 3, \cdots, N$, we get

$$\begin{aligned} & z_{-8}\phi_{-8-j}^{iv} + z_{-7}\phi_{-7-j}^{iv} + z_{-6}\phi_{-6-j}^{iv} + \cdots + z_7\phi_{7-j}^{iv} + z_8\phi_{8-j}^{iv} + z_9\phi_{9-j}^{iv} + z_{10}\phi_{10-j}^{iv} \\ & + \cdots + z_{N+6}\phi_{N+6-j}^{iv} + z_{N+7}\phi_{N+7-j}^{iv} + z_{N+8}\phi_{N+8-j}^{iv} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)). \end{aligned} \tag{3.8}$$

Substituting $j = 1$ in (3.8), it becomes

$$\begin{aligned} & z_{-8}\phi_{-8-1}^{iv} + z_{-7}\phi_{-7-1}^{iv} + z_{-6}\phi_{-6-1}^{iv} + \cdots + z_7\phi_{7-1}^{iv} + z_8\phi_{8-1}^{iv} + z_9\phi_{9-1}^{iv} + z_{10}\phi_{10-1}^{iv} \\ & + \cdots + z_{N+6}\phi_{N+6-1}^{iv} + z_{N+7}\phi_{N+7-1}^{iv} + z_{N+8}\phi_{N+8-1}^{iv} = h^4 f(x_1, Z(x_1), Z^{'}(x_1)). \end{aligned}$$

This implies

$$\begin{aligned} & z_{-8}\phi_{-9}^{iv} + z_{-7}\phi_{-8}^{iv} + z_{-6}\phi_{-7}^{iv} + \cdots + z_7\phi_6^{iv} + z_8\phi_7^{iv} + z_9\phi_8^{iv} + z_{10}\phi_9^{iv} + \cdots \\ & + z_{N+6}\phi_{N+5}^{iv} + z_{N+7}\phi_{N+6}^{iv} + z_{N+8}\phi_{N+7}^{iv} = h^4 f(x_1, Z(x_1), Z^{'}(x_1)). \end{aligned}$$

Since $\phi^{iv}(i)$ is non-zero only for the interval for $i \in [-8, 8]$ and outside the interval it will be zero. Then above equation becomes

$$z_{-7}\phi^{iv}_{-8} + z_{-6}\phi^{iv}_{-7} + \cdots + z_7\phi^{iv}_6 + z_8\phi^{iv}_7 + z_9\phi^{iv}_8 = h^4 f(x_1, Z(x_1), Z^{'}(x_1)). \tag{3.9}$$

For j=2, (3.8) becomes

$$\begin{aligned} &z_{-8}\phi^{iv}_{-8-2} + z_{-7}\phi^{iv}_{-7-2} + z_{-6}\phi^{iv}_{-6-2} + \cdots + z_7\phi^{iv}_{7-2} + z_8\phi^{iv}_{8-2} + z_9\phi^{iv}_{9-2} + z_{10}\phi^{iv}_{10-2} \\ &+ \cdots + z_{N+6}\phi^{iv}_{N+6-2} + z_{N+7}\phi^{iv}_{N+7-2} + z_{N+8}\phi^{iv}_{N+8-2} = h^4 f(x_2, Z(x_2), Z^{'}(x_2)). \end{aligned}$$

This implies

$$\begin{aligned} &z_{-8}\phi^{iv}_{-10} + z_{-7}\phi^{iv}_{-9} + z_{-6}\phi^{iv}_{-8} + \cdots + z_7\phi^{iv}_5 + z_8\phi^{iv}_6 + z_9\phi^{iv}_7 + z_{10}\phi^{iv}_8 \\ &+ \cdots + z_{N+6}\phi^{iv}_{N+4} + z_{N+7}\phi^{iv}_{N+5} + z_{N+8}\phi^{iv}_{N+6} = h^4 f(x_2, Z(x_2), Z^{'}(x_2)). \end{aligned}$$

By using the definition of ϕ^{iv}_i given in (2.8), above equation yields

$$z_{-6}\phi^{iv}_{-8} + z_{-5}\phi^{iv}_{-7} + \cdots + z_7\phi^{iv}_5 + z_8\phi^{iv}_6 + z_9\phi^{iv}_7 + z_{10}\phi^{iv}_8 = h^4 f(x_2, Z(x_2), Z^{'}(x_2)). \tag{3.10}$$

By using the similar pattern for $j = 1, 2$, we can find the expression for $j = 3, 4, \cdots N$

$$\begin{aligned} &z_{-8+j}\phi^{iv}_{-8-j} + z_{-7+j}\phi^{iv}_{-7-j} + z_{-6+j}\phi^{iv}_{-6-j} + \cdots + z_{7+j}\phi^{iv}_{7-j} + z_{8+j}\phi^{iv}_{8-j} + z_{9+j}\phi^{iv}_{9-j} \\ &+ \cdots + z_{N+6+j}\phi^{iv}_{N+6-j} + z_{N+7+j}\phi^{iv}_{N+7-j} + z_{N+8+j}\phi^{iv}_{N+8-j} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)). \end{aligned} \tag{3.11}$$

□

The nonlinear system of equations (3.5) is equivalent to the following non-linear system of $N + 1$ equations with (N+17) unknowns $\{z_i\}$.

$$AZ = F(z) \tag{3.12}$$

where A is banded matrix of order $(N+1)\times(N+17)$, Z is the unknown vector of order $N+17$ and $F(z)$ is the vector of order $N + 1$ depends on z. The matrix A, vectors Z and $F(z)$ are given explicitly by

$$A = [\phi^{iv}_{pq}(q - p - 8)]_{(N+1)\times(N+17)} \tag{3.13}$$

where $p = 1, 2, 3 \cdots, N + 1$ and $q = 1, 2, 3, \cdots, N + 17$ represent the row and column respectively.

$$F(z) = \Big(h^4 f(x_0, Z(x_0), Z^{'}(x_0)), \cdots, h^4 f(x_N, Z(x_N), Z^{'}(x_N))\Big)^T \tag{3.14}$$

$$Z = (z_{-8}, z_{-7}, z_{-6}, \cdots, z_{N+6}, z_{N+7}, z_{N+8})^T \tag{3.15}$$

$$Z^{'}(x_j) = \sum_{i=-8}^{N+8} z_j \phi^{'}\left(\frac{x_j - x_i}{h}\right)$$

where $\phi'(i)$ is already defined in (2.5) with $\phi(i) = \phi_i$.

3.2 Boundary conditions at end points

For unique solution of the nonlinear system (3.5), we need sixteen more conditions. Four conditions can be attained from given boundary conditions for the nonlinear system of equations and remaining conditions are attained by using some extrapolation method. The details of the given boundary conditions and extrapolation method are given below:

3.2.1 Boundary Conditions

The given boundary conditions are

$$Z(0) = \alpha_1,\ Z^{'}(0) = \alpha_2,\ Z(N) = \alpha_3,\ Z^{'}(N) = \alpha_4$$

The approximation of derivative conditions at ends point is defined as:

$$\begin{aligned} Z'(0) = \left(\frac{N}{2520}\right)&\{-7381z_0 + 25200z_1 - 56700z_2 + 100800z_3 - 132300z_4 + 127008z_5 \\ &-88200z_6 + 43200z_7 - 14175z_8 + 28800z_9 - 252z_{10}\} + O(h^{10}) \end{aligned} \tag{3.16}$$

$$\begin{aligned} Z'(N) = \left(\frac{N}{2520}\right)&\{7381z_N - 25200z_{N-1} + 56700z_{N-2} - 100800z_{N-3} + 132300z_{N-4} \\ &-127008z_{N-5} + 88200z_{N-6} - 43200z_{N-7} + 14175z_{N-8} - 28800z_{N-9} \\ &+252z_{N-10}\} + O(h^{10}). \end{aligned} \tag{3.17}$$

3.2.2 Extrapolation Method

The remaining twelve conditions for the nonlinear systems (3.5) to obtain stable systems for the solution of (1.1) are obtained by using the following extrapolation method.
We define six conditions at left end points and six conditions at the right end points. Since subdivision scheme (2.2) reproduces nine degree (i.e. tenth order) polynomials, so we define boundary conditions of order ten for the solution of (3.5). For simplicity only left end points $z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}$ are discussed and the values of right end points $z_{N+2}, z_{N+3}, z_{N+4}, z_{N+5}, z_{N+6}, z_{N+7}$ can be treated similarly.

The values $z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}$ can be determined by the polynomial $q(x)$ interpolating $(x_i, z_i),\ 2 \leq i \leq 7$. Precisely, we have

$$z_{-i} = q(-x_i),\ i = 2, 3, 4, 5, 6, 7$$

where

$$q(x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} Z(x_{i-j}).$$

From (3.1), $Z_1(x_i) = z_i$, $i = 2, 3, 4, 5, 6, 7$ and replacing x_i by $-x_i$, we have

$$q(-x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} z_{-i+j}.$$

Hence the following boundary conditions can be employed at the left end

$$\sum_{j=0}^{10} \binom{10}{j} (-1)^j z_{-i+j} = 0,\ i = 7, 6, 5, 4, 3, 2. \tag{3.18}$$

Similarly for the right end, we can define $z_i = q(-x_i)$, $i = N+2,\ N+3,\ N+4,\ N+5,\ N+6,\ N+7$ and

$$q(x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} z_{i-j}.$$

So we have the following boundary conditions at the right end

$$\sum_{j=0}^{10} \binom{10}{j} (-1)^j z_{i-j} = 0,\ i = N+2, N+3, N+4, N+5, N+6, N+7. \tag{3.19}$$

Finally, we obtain a new system of $(N + 17)$ linear equations with $(N + 17)$ unknowns $\{z_i\}$. The $N+1$ equations are obtained from (3.5), four equations from boundary conditions (3.3) and twelve from boundary conditions (3.18) and (3.19) for the numerical solution of proposed problem.
Hence the stable nonlinear system of equations is defined as:

$$BZ = R(z) \tag{3.20}$$

where the matrix B is given by

$$B = (C_0^T, A^T, C_1^T)^T \tag{3.21}$$

A is defined in (3.13), C_0, C_1 and the vector R(z) is defined as

$$C_0 = \left(\begin{array}{cccccccccccc}
0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 & -10 & 1 \\
0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 & -10 \\
0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 \\
0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 \\
0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7381N}{2520} & \frac{25200N}{2520} & -\frac{56700N}{2520} & \frac{100800N}{2520} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array} \right.$$

$$\left. \begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-10 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
45 & -10 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-120 & 45 & -10 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
210 & -120 & 45 & -10 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-\frac{132300N}{2520} & \frac{127008N}{2520} & -\frac{88200N}{2520} & \frac{43200N}{2520} & -\frac{14175N}{2520} & \frac{2800N}{2520} & -\frac{252N}{2520} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{array} \right) \tag{3.22}$$

The first six rows of C_0 are obtained from (3.18), second last row is obtained from (3.16) and last row is taken from given boundary conditions $Z_1(0)$ which is defined in (3.3) and

$$C_1 = \left(\begin{array}{cccccccccccccc}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & \frac{N}{10} & -\frac{10N}{9} & \frac{45N}{8} & -\frac{120N}{7} & 35N & -\frac{252N}{5} & \frac{105N}{2} & -40N & \frac{45N}{2} & -10N \\
0 & 0 & \cdots & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45
\end{array} \right.$$

$$\left. \begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{7381N}{2520} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
45 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-120 & 45 & -10 & 1 & 0 & 0 & 0 & 0 & 0 \\
210 & -120 & 45 & -10 & 1 & 0 & 0 & 0 & 0 \\
-252 & 210 & -120 & 45 & -10 & 1 & 0 & 0 & 0 \\
-120 & 210 & -252 & 210 & -120 & 45 & -10 & 1 & 0
\end{array} \right), \tag{3.23}$$

First row of C_1 is obtained from $Z_1(N)$ which is defined in (3.3), second row is obtained from (3.17) and the last six rows are obtained from (3.19), Z which is defined in (3.15) and R_1 is defined as

$$R(z) = (0, 0, 0, 0, 0, 0, Z'(0), Z(1), F^T(z), Z(1), Z'(1), 0, 0, 0, 0, 0, 0)^T, \tag{3.24}$$

where $F(z)$ is defined by (3.14).

3.2.3 Non-singularity of a matrix

We can check the non-singularity of coefficient matrix B defined in (3.21) by different methods. We observe that the determinant of matrix B is non-zero for $N \leq 500$. Hence the non linear system of equations have a solution for $N \leq 500$. We also check the non singularity of matrix by finding eigenvalues up to $N \leq 500$ and we observe that all the eigenvalues are non-zero. Hence by [15] we conclude that the B is non-singular. For large $N > 500$ the matrix may or may not be singular.

3.3 Iterative algorithm and its convergence

An iterative algorithm and its convergence are described in this section.

3.3.1 Iterative algorithm based on basis function

The iterative algorithm based on basis function of the subdivision scheme (2.2) are as defined in the following three steps.

First step: Initial approximation

The initial approximation is important because the numerical solution depends on the initial approximation. We define the process for finding the initial approximation as follows:
Let initial approximate solution Z^0 be the solution of the following linear system

$$BZ^0 = F^0 \tag{3.25}$$

where

$$\begin{cases} F^0 = (0,0,0,0,0,0,y'(a),y(a),f_0,f_1,f_2,\cdots,f_N,y(b),y'(b),0,0,0,0,0,0)^T, \\ f_i = h^4 f(x_i, L_i, D), \quad i = 0,1,2,\cdots N \\ L_i = y(0) + ih\left(\frac{y(b)-y(a)}{b-a}\right) \\ D = y(b) - y(a). \end{cases} \tag{3.26}$$

F^0 is the initial linear approximation of the non-linear vector $R(z)$.

Second step: Numerical solution

The numerical solutions Z^* of the nonlinear system are obtained by using the simple iterative scheme

$$BZ^{(m+1)} = R(Z^m), \quad m = 0,1,2,3,\cdots \tag{3.27}$$

Third step: Stopping condition

The above iterative processes will terminate when the following condition is satisfied

$$||z^{(m)} - z^{(m-1)}|| \leq tol \tag{3.28}$$

where tolerance is supposed value i.e. $tol = 10^{-6}$. The convergence of the above iterative algorithm is guaranteed by the following proposition.

Theorem 3. *The successive solutions $\{Z^{(m)}\}$ generated by the iterative algorithm (3.27) linearly converges to the solution Z^* of the non-linear solution of the system (3.20) provided that the M_0 and M_1 are Lipschitz constants and step size h is small.*
i.e.

$$\|B^{-1}\| \leq \left(M_0 h^4 + \frac{4994220330463}{1460471061420} M_1 h^3\right). \tag{3.29}$$

Proof. Let Z^* and $Z^{(m)}$ be the solutions of the nonlinear system (3.20). Then by definition, for small h we have

$$BZ^* = R(Z^*), \tag{3.30}$$

$$BZ^{m+1} = R(Z^m). \tag{3.31}$$

Let the error vector be defined as $e^{(k)} = Z^k - Z^*$ at kth iteration which satisfies

$$\begin{aligned} BZ^{(m+1)} - BZ^* &= R(Z^k) - R(Z^*), \\ B(Z^{(m+1)} - Z^*) &= R(Z^k) - R(Z^*), \\ Be^{(k+1)} &= R(Z^k) - R(Z^*). \end{aligned} \tag{3.32}$$

For $i = 0, 1, 2, \cdots, N$

$$D_4 e_i^{(k+1)} = (F(Z^k) - F(Z^*))_i.$$

By mean value theorem, which is stated as "If a function $f(x, y, z)$ is continuously differentiable in an open set of $\mathbb{R}^3$ containing points (x_1, y_1, z_1) and (x_2, y_2, z_2) and the line segment connecting them, then an equation

$$f(x_2, y_2, z_2) - f(x_1, y_1, z_1) = f_x^{'}(r, s, t)(x_2 - x_1) + f_y^{'}(r, s, t)(y_2 - y_1) + f_z^{'}(r, s, t)(z_2 - z_1)$$

is valid for the interior point (a, b, c) of the segment.", we have

$$D_4 e_i^{(k+1)} = f(x_i, Z_i^{(k)}, Z'^{(k)}) - f(x_i, Z_i^{(*)}, Z'^{(*)}).$$

The above equation can be written as (by using mean value theorem)

$$D_4 e_i^{(k+1)} = f_x^*(x_i - x_i) + f_y^*(Z_i^{(k)} - Z_i^{(*)}) + f_{y'}^*(Z'^{(k)} - Z'^{(*)})$$

by using the definition of error vector, we have

$$\begin{aligned} D_4 e_i^{(k+1)} &= f_y^* e^{(k)} + f_{y'}^* e'^{(k)}, \\ D_4 e_i^{(k+1)} &= f_y^* e^{(k)} + f_{y'}^* D_1 e^{(k)} \end{aligned}$$

where D_4 and D_1 are the derivative difference operators defined as

$$\begin{aligned} D_1 f_i = \frac{1}{2920942122840h} &[1575(f_{i-8} - f_{i+8}) + 1474560(f_{i-7} - f_{i+7}) \\ &+315738080(f_{i-6} - f_{i+6}) + 1397587968(f_{i-5} - f_{i+5}) \\ &-43588613880(f_{i-4} - f_{i+4}) + 311679549440(f_{i-3} - f_{i+3}) \\ &-1336741045920(f_{i-2} - f_{i+2}) + 4824847319040(f_{i-1} - f_{i+1})] \end{aligned}$$

$$\begin{aligned} D_4 f_i = \frac{1}{183768238080h^4} &[392875(f_{i+8} - f_{i-8}) + 45977600(f_{i+7} - f_{i-7}) \\ &-1296269280(f_{i+6} - f_{i-6}) + 5912719360(f_{i+5} - f_{i-5}) \\ &+1180083476(f_{i+4} - f_{i-4}) - 86261280768(f_{i+3} - f_{i-3}) \\ &+332951715808(f_{i+2} - f_{i-2}) - 677767008256(f_{i+1} - f_{i-1}) \\ &+850467338370 f_i]. \end{aligned}$$

This implies

$$D_4 e_i^{(k+1)} = h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}.$$

Since $e_i = e_{N-i} = 0, \quad i = 0, -1, -2, \cdots, -8$, we have

$$Be_i^{(k+1)} = h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}$$

This can be written as

$$e_i^{(k+1)} = B^{-1}(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}).$$

By taking norm on both sides, we get

$$\|e_i^{(k+1)}\| = \|B^{-1}(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)})\|.$$

This implies

$$\|e_i^{(k+1)}\| = \|B^{-1}\| \|(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)})\|.$$

By using the definition of Lipschitz condition, we get

$$\|e^{(k+1)}\| \leq h^4 M_0 (b-a) \|B^{-1}\| \|e^k\| + h^3 M_1 \|D_1\| \|e^{(k)}\|.$$

This implies

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \leq \|B^{-1}\| \left(h^4 M_0 (b-a) + h^3 M_1 \|D_1\|\right),$$

which is equivalent to

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \approx h^3 M_1 \|B^{-1}\| \|D_1\| \leq h M_1 \|B^{-1}\| \|D_1\|,$$

i-e

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \approx h M_1 \|B^{-1}\| \|D_1\|.$$

The results follows immediately from this inequality and the following fact

$$\|D_1\| = \frac{4994220330463}{1460471061420}. \tag{3.33}$$

A simple approximation of condition by omitting the quatric term is

$$h \leq \frac{1460471061420}{4994220330463} M_1^{-1} \left\|B^{-1}\right\|^{-1}. \tag{3.34}$$

This complete the proof. □

4 Error Estimation

From the approximation properties of the basis function $\phi(x)$, it is shown that the collocation method (3.1) with nonic precision treatments at the end points has at least power of approximation $O(h^3)$. Here we present our main results for error estimation. Proof of these results are similar to the proof of Proposition [14, 8].

Theorem 4. *Suppose the exact solution $y(x) \in C^4[0,1]$ and $\{z_i\}$ are obtained by (3.20) then absolute error by interpolating collocation algorithm is*

$$||err(x)||_\infty = ||Z^{(l)}(x) - y^{(l)}(x)||_\infty = O(h^{3-l}), \; l = 0, 1, 2, 3.$$

where l denotes the order of derivative.

Proof. Since the order of approximation of subdivision scheme (2.2) is ten so by direct calculation (fourth left eigenvector), we can find derivative of smooth function $y(x)$ as

$$y^{iv}(x_j) = \frac{2^4}{183768238080h^4}\{392875y(x_j-8h) + 45977600y(x_j-7h)$$

$$-1296269280y(x_j-6h) + 5912719360y(x_j-5h) + 1180083476y(x_j-4h)$$

$$-86261280786y(x_j-3h) + 332951715808y(x_j-2h) - 677767008256y(x_j-h)$$

$$+850467338370y(x_j) - 677767008256y(x_j+h) + 332951715808y(x_j+2h)$$

$$-86261280786y(x_j+3h) + 1180083476y(x_j+4h) + 5912719360y(x_j+5h)$$

$$-1296269280y(x_j+6h) + 45977600y(x_j+7h) + 392875y(x_j+8h)\} + O(h^{10}).$$

This can be written as

$$y_j^{iv} = \frac{2^4}{183768238080h^4}\{392875y_{j-8} + 45977600y_{j-7} - 1296269280y_{j-6}$$

$$+5912719360y_{j-5} + 1180083476y_{j-4} - 86261280786y_{j-3} + 332951715808y_{j-2}$$

$$-677767008256y_{j-1} + 850467338370y_j - 677767008256y_{j+1} + 332951715808y_{j+2}$$

$$-86261280786y_{j+3} + 1180083476y_{j+4} + 5912719360y_{j+5} - 1296269280y_{j+6}$$

$$+45977600y_{j+7} + 392875y_{j+8}\} + O(h^{10}). \tag{4.1}$$

Similarly, we have

$$Z_j^{iv} = \frac{2^4}{183768238080h^4}\{392875z_{j-8} + 45977600z_{j-7} - 1296269280z_{j-6}$$

$$+5912719360z_{j-5} + 1180083476z_{j-4} - 86261280786z_{j-3} + 332951715808z_{j-2}$$

$$-677767008256z_{j-1} + 850467338370z_j - 677767008256z_{j+1} + 332951715808z_{j+2}$$

$$-86261280786z_{j+3} + 1180083476z_{j+4} + 5912719360z_{j+5} - 1296269280z_{j+6}$$

$$+45977600z_{j+7} + 392875z_{j+8}\} + O(h^{10}). \tag{4.2}$$

If we define error function $e(x) = Z(x) - y(x)$ and error vectors at the nodes by

$$e(x_j) = Z(x_j) - y(x_j + jh),\ -8 \le j \le N+8,$$

or equivalently $e_j = Z_j - y_j,\ -8 \le j \le N+8$, This implies

$$\begin{cases} e'_j = Z'_j - y'_j, \\ e''_j = Z''_j - y''_j, \\ e'''_j = Z'''_j - y'''_j \\ e^{iv}_j = Z^{iv}_j - y^{iv}_j. \end{cases} \tag{4.3}$$

By subtracting (4.2) from (4.1), we get

$$y_j^{iv} - Z_j^{iv} = \frac{2^4}{183768238080h^4}\{392875(y_{j-8} - z_{j-8}) + 45977600(y_{j-7} - z_{j-7})$$

$$-1296269280(y_{j-6} - z_{j-6}) + 5912719360(y_{j-5} - z_{j-5}) + 1180083476(y_{j-4} - z_{j-4})$$

$$-86261280786(y_{j-3} - z_{j-3}) + 332951715808(y_{j-2} - z_{j-2}) - 677767008256(y_{j-1} - z_{j-1})$$

$$+850467338370(y_j - z_j) - 677767008256(y_{j+1} - z_{j+1}) + 332951715808(y_{j+2} - z_{j+2})$$

$$-86261280786(y_{j+3} - z_{j+3}) + 1180083476(y_{j+4} - z_{j+4}) + 5912719360(y_{j+5} - z_{j+5})$$

$$-1296269280(y_{j+6} - z_{j+6}) + 45977600(y_{j+7} - z_{j+7}) + 392875(y_{j+8} - z_{j+8})\} + O(h^{10}).$$

This implies

$$e_j^{iv} = \frac{2^4}{183768238080h^4}\{392875e_{j-8} + 45977600e_{j-7} - 1296269280e_{j-6}$$

$$+5912719360e_{j-5} + 1180083476e_{j-4} - 86261280786e_{j-3} + 332951715808e_{j-2}$$

$$-677767008256e_{j-1} + 850467338370e_j - 677767008256e_{j+1} + 332951715808e_{j+2}$$

$$-86261280786e_{j+3} + 1180083476e_{j+4} + 5912719360e_{j+5} - 1296269280e_{j+6}$$

$$+45977600e_{j+7} + 392875e_{j+8}\} + O(h^{10}). \tag{4.4}$$

From (1.1), (3.1), (4.3) and by assuming the tenth order boundary treatments at the end points, we have

$$e_j^{iv} = a_j e_j + b_j e_j^{'}, \quad 0 \le i \le N \tag{4.5}$$

and

$$e_j = \begin{cases} \max\limits_{0\le k\le 7}\{|e_k|\}O(h^{10}), & -8 \le i \le 0 \\ \max\limits_{N-3\le k\le N}\{|e_k|\}O(h^{10}), & N \le i \le N+8 \end{cases} \tag{4.6}$$

where $j = 0, 1, \cdots N$

$$a_j = f_y(t_j, y_j^*, y_j^{'*}), \quad b_j = f_{y'}(t_j, y_j^*, y_j^{'*}),$$

and

$$y_j^* = y_j + \theta_j e_j, \quad y_j^{'*} = y_j' + \theta_j e_j^{'}, \quad 0 \le \theta_j \le 1.$$

Using the results (4.4) and

$$[1575(z_{i-8} - z_{i+8}) + 1474560(z_{i-7} - z_{i+7}) + 315738080(z_{i-6} - z_{i+6}) + 1397587968$$
$$(z_{i-5} - z_{i+5}) - 43588613880(z_{i-4} - z_{i+4}) + 311679549440(z_{i-3} - z_{i+3}) - 1336741045920$$
$$(z_{i-2} - z_{i+2}) + 4824847319040(z_{i-1} - z_{i+1})] = 2920942122840hZ^{'} + O(h^{10}), \tag{4.7}$$

It can be conclude that relation (4.5)and (4.6) is equivalent to

$$(B + O(h^8) - O(h^4) - D_1 O(h^3))E = O(h^{10})\|E\|,$$

where $E = (e_{-8}, e_{-7}, \cdots, e_7, e_8)$.

Hence for small h, the coefficient matrix $B + O(h)$, will be invertible, thus using the standard result from algebra and effect of $\|B^{-1}\|$, we have the following estimate

$$\|E\| \le \frac{\|B^{-1}\|}{1 - O(h)} O(h^{10}) = O(h^3). \tag{4.8}$$

□

5 Results and Discussions

In this section, we test the proposed method on some nonlinear problems. Numerical results for each of the problems are presented in the tables. These values are very close to the true solutions and the values of the errors are also given in the table.

Example 1. *Consider the following non-linear boundary value problem [1]*

$$y^{iv} - 6\exp(-4y) = -12(1+x)^{-4}, \tag{5.1}$$

with boundary conditions

$$y(0) = 0, y'(0) = 1, y(1) = ln(2) = y'(1) = 0.5.$$

The exact solution of the problem (5.1) is $y = \ln(1+x)$. Using the collocation method described in Section 3 for $N = 10$, $h = 10^{-1}$ and $tol = 10^{-6}$ with tenth order boundary treatment at end points. The numerical results are obtained after third iteration with the condition (3.28). The obtained numerical results for this problem are presented in Table 1. The maximum absolute error obtained by the proposed method is 1.78×10^{-3}. The graphical comparison between exact and approximate solutions is shown in Figure 2.

Table 1: Numerical results of Example 1

x_i	Analytic solution Y_i	Approximate solution Z_i	Error $= \|\|Y_i - Z_i\|\|_\infty$
0.0	0	0	0
0.1	0.0953101798	0.0950147533	0.0002954265
0.2	0.1823215568	0.1814496227	0.0008719341
0.3	0.2623642645	0.2609546573	0.0014096072
0.4	0.3364722366	0.3347370220	0.0017352146
0.5	0.4054651081	0.4036840381	0.0017810699
0.6	0.4700036292	0.4684459279	0.0015577013
0.7	0.5306282511	0.5294932609	0.0011349902
0.8	0.5877866649	0.5871580370	0.0006286279
0.9	0.6418538862	0.6416636708	0.0001902154
1.0	0.6931471806	0.6931471806	0

Example 2. *Consider the non-linear boundary value problem [1]*

$$y^{(iv)} = y^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48 \tag{5.2}$$

subject to the boundary conditions

$$y(0) = y'(0) = 0, y(1) = y'(1) = 1.$$

Using the collocation method described in Section 3 for $N = 10$, $h = 10^{-1}$ and $tol = 10^{-6}$ with tenth order boundary treatment at end points. The numerical results are obtained after third iteration with the condition (3.28). The obtained numerical results for this problem are presented in Table 2. The maximum absolute error obtained by the proposed method is 1.73×10^{-2}. The graphical comparison between exact and approximate solutions is shown in Figure 3.

6 Conclusion

This study has presented a numerical approach based on subdivision collocation algorithm for solving the numerical solution of nonlinear fourth order boundary value problems. The proposed iterative method

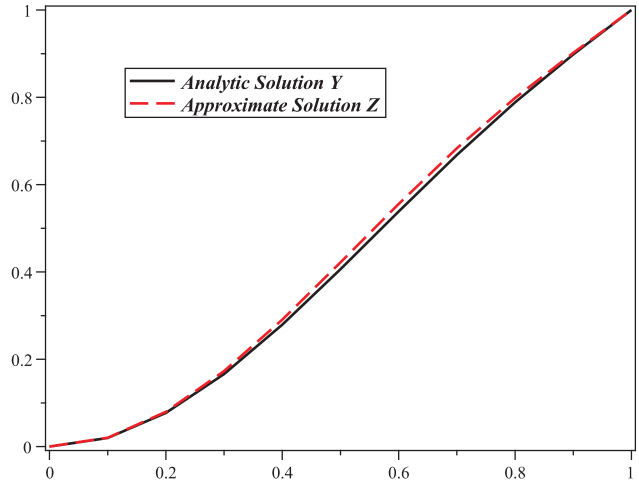

Figure 2: *Comparison of the analytic and approximate solution of Example 1.*

Table 2: Numerical results of Example 2

x_i	Analytic solution Y_i	Approximate solution Z_i	Error $= \|\|Y_i - Z_i\|\|_\infty$
0.0	0	0	0
0.1	0.01981	0.0202195	0.0004095
0.2	0.07712	0.0796952	0.0025752
0.3	0.16623	0.1728732	0.0066432
0.4	0.27904	0.2905995	0.0115595
0.5	0.40625	0.4219208	0.0156708
0.6	0.53856	0.5558846	0.0173246
0.7	0.66787	0.6833406	0.0154706
0.8	0.78848	0.7987412	0.0102612
0.9	0.89829	0.9019417	0.0036517
1.0	1.00000	1.0000000	0

has been applied on different nonlinear fourth order boundary value problems. Numerical results show that the accuracy of the approximate solution is $O(h^3)$. We have also observed that the accuracy of the solution can be improved by choosing different subdivision schemes with the proper adjustment of boundary conditions.

Acknowledgement

This work is supported by NRPU (P. No. 3183) and Indigenous Ph.D. Scholarship Scheme of HEC Pakistan.

Competing interests

The authors declare that they have no competing interests.

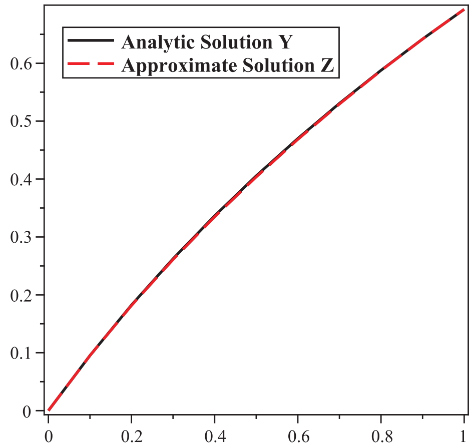

Figure 3: *Comparison of the analytic and approximate solution of Example 2.*

Authorś contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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On Stability of Quintic Functional Equations in Random Normed Spaces

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Abstract. In this paper, using the direct and fixed point methods, we investigate the generalized Hyers-Ulam stability of the quintic functional equation:

$$2f(2x+y)+2f(2x-y)+f(x+2y)+f(x-2y)=20[f(x+y)+f(x-y)]+90f(x)$$

in random normed spaces under the minimum t-norm.

1. Introduction

A classical question in stability of functional equations is as follows:

Under what conditions, is it true that a mapping which approximately satisfies a functional equation (ξ) *must be somehow close to an exact solution of* (ξ)?

We say the functional equation (ξ) is *stable* if any approximate solution of (ξ) is near to a true solution of (ξ).

The study of stability problem for functional equations is related to a question of Ulam [15] concerning the stability of group homomorphisms. The famous Ulam stability problem was partially solved by Hyers [9] for linear functional equation of Banach spaces. Subsequently, the result of Hyers theorem was generalized by Aoki [2] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference. Cădariu and Radu [3] applied the *fixed point method* to investigation of the Jensen functional equation. They could present a short and a simple proof (different from the *direct method* initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of Jensen functional equation and for quadratic functional equation. Their methods are a powerful tool for studying the stability of several functional equations.

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[0]**2000 Mathematics Subject Classification**: 39B52, 39B72, 47H09, 47H47.

[0]**Keywords**: Generalized Hyers-Ulam stability, quintic functional equation, random normed spaces, fixed point theorem.

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On the other hand, the theory of *random normed spaces* (briefly, RN-spaces) is important as a generalization of deterministic result of normed spaces and also in the study of random operator equations. The notion of an RN-space corresponds to the situations when we do not know exactly the norm of the point and we know only probabilities of passible values of this norm. The RN-spaces may provide us the appropriate tools to study the geometry of nuclear physics and have usefully application in quantum particle physics. A number of papers and research monographs have been published on generalizations of the stability of different functional equations in RN-spaces [5, 6, 10, 11, 16].

In the sequel, we use the definitions and notations of a random normed space as in [1, 13, 14].

A function $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ is called a *distribution function* if it is nondecreasing and left-continuous, with $F(0) = 0$ and $F(+\infty) = 1$. The class of all probability distribution functions F with $F(0) = 0$ is denoted by Λ. D^+ is a subset of Λ consisting of all functions $F \in \Lambda$ for which $F(+\infty) = 1$, where $l^-F(x) = \lim_{t\to x^-} F(t)$. For any $a \geq 0$, ϵ_a is the element of D^+, which is defined by

$$\epsilon_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases}$$

Definition 1.1. ([13]) A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a t-norm) if T satisfies the following conditions:
(1) T is commutative and associative;
(2) T is continuous;
(3) $T(a, 1) = a$ for all $a \in [0, 1]$;
(4) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Three typical examples of continuous t-norms are as follows:

$$T_M(a, b) = \min\{a, b\}, \quad T_P(a, b) = ab, \quad T_L(a, b) = \max\{a + b - 1, 0\}.$$

Recall that, if T is a t-norm and $\{x_n\}$ is a sequence of numbers in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n) = T(x_1, \cdots, x_n)$ for each $n \geq 2$ and $T_{i=n}^\infty x_n$ is defined as $T_{i=1}^\infty x_{n+i}$ ([8]).

Definition 1.2. ([14]) Let X be a real linear space, μ be a mapping from X into D^+ (for any $x \in X$, $\mu(x)$ is denoted by μ_x) and T be a continuous t-norm. The triple (X, μ, T) is called a random normed space (briefly RN-space) if μ satisfies the following conditions:
(RN1) $\mu_x(t) = \epsilon_o(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X, \alpha \neq 0$ and all $t \geq 0$;
(RN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

Example 1.1. Every normed space $(X, \|\cdot\|)$ defines a RN-space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ and T_M is the minimum t-norm. This space is called the *induced random normed space.*

Definition 1.3. Let (X, μ, T) be a RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\mu_{x_n - x}(t) > 1 - \lambda$$

whenever $n \geq N$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} \mu_{x_n - x} = 1$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\mu_{x_n - x_m}(t) > 1 - \lambda$$

whenever $n \geq m \geq N$.

(3) The RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. ([13]) *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence of X such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Recently, Cho et. al. [4] was introduced and proved the Hyers-Ulam-Rassias stability of the following quintic functional equations

$$2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) = 20[f(x+y) + f(x-y)] + 90f(x) \tag{1.1}$$

for fixed $k \in \mathbb{Z}^+$ with $k \geq 3$ in quasi-β-normed spaces.

Remark 1.1. (1) If we put $x = y = 0$ in the equation (1.1), then $f(0) = 0$.

(2) $f(2^n x) = 2^{5n} f(x)$ for all $x \in X$ and $n \in \mathbb{Z}^+$.

(3) f is an odd mapping.

Throughout this paper, let X be a real linear space, (Z, μ', T_M) be an RN-space and (Y, μ, T_M) be a complete RN-space. For any mapping $f : X \to Y$, we define

$$\begin{aligned} &Df(x, y) \\ &= 2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) - 20[f(x+y) + f(x-y)] - 90f(x) \end{aligned}$$

for all $x, y \in X$. In this paper, using the direct and fixed point methods, we investigate the generalized Hyers-Ulam stability of the quintic functional equation:

$$2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) = 20[f(x+y) + f(x-y)] + 90f(x)$$

in random normed spaces under the minimum t-norm.

2. Random stability of the functional equation (1.1)

In this section, we investigate the generalized Hyers-Ulam stability problem of the quintic functional equation (1.1) in RN-spaces in the sense of Scherstnev under the minimum t-norm T_M.

Theorem 2.1. *Let $\phi : X^2 \to Z$ be a function such that, for some $0 < \alpha < 2^5$,*

$$\mu'_{\phi(2x,2y)}(t) \geq \mu'_{\alpha\phi(x,y)}(t) \tag{2.1}$$

and $\lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t) = 1$ *for all* $x, y \in X$ *and* $t > 0$. *If* $f : X \to Y$ *is a mapping with* $f(0) = 0$ *such that*

$$\mu_{Df(x,y)}(t) \geq \mu'_{\phi(x,y)}(t) \tag{2.2}$$

for all $x, y \in X$ *and* $t > 0$, *then there exists a unique quintic mapping* $Q : X \to Y$ *such that*

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\Big(2^2(2^5 - \alpha)t\Big) \tag{2.3}$$

for all $x \in X$ *and* $t > 0$.

Proof. Letting $y = 0$ in (2.2), we get

$$\mu_{\frac{f(2x)}{2^5} - f(x)}(t) \geq \mu'_{\phi(x,0)}(128t) \tag{2.4}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^n x$ in (2.4), we get

$$\mu_{\frac{f(2^{n+1}x)}{2^{5(n+1)}} - \frac{f(2^n x)}{2^{5n}}}(t) \geq \mu'_{\phi(x,0)}\Big(\Big(\frac{2^5}{\alpha}\Big)^n 128t\Big)$$

for all $x \in X$ and $t > 0$. Since $\frac{f(2^n x)}{2^{5n}} - f(x) = \sum_{j=0}^{n-1}\Big(\frac{f(2^{j+1}x)}{2^{5(j+1)}} - \frac{f(2^j x)}{2^{5j}}\Big)$,

$$\mu_{\frac{f(2^n x)}{2^{5n}} - f(x)}\Big(\sum_{j=0}^{n-1} \frac{1}{128}\Big(\frac{\alpha}{2^5}\Big)^j t\Big) \geq T_{M\,j=0}^{\,n-1}(\mu'_{\phi(x,0)}(t)) = \mu'_{\phi(x,0)}(t) \tag{2.5}$$

for all $x \in X$ and $t > 0$. Substituting x by $2^m x$ in (2.5), we get

$$\mu_{\frac{f(2^{n+m}x)}{2^{5(n+m)}} - \frac{f(2^m x)}{2^{5m}}}(t) \geq \mu'_{\phi(x,0)}\Big(\frac{t}{\sum_{j=m}^{n+m-1}(\frac{\alpha}{2^5})^j}\Big) \tag{2.6}$$

for all $x \in X$ and $m, n \in \mathbb{Z}$ with $n > m \geq 0$. Since $\alpha < k^3$, the sequence $\{\frac{f(2^n x)}{2^{5n}}\}$ is a Cauchy sequence in the complete RN-space (Y, μ, T_M) and so it converges to some point $Q(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (2.6). Then we get

$$\mu_{\frac{f(2^n x)}{2^{5n}} - f(x)}(t) \geq \mu'_{\phi(x,0)}\left(\frac{128t}{\sum_{j=0}^{n-1}(\frac{\alpha}{2^5})^j}\right),$$

and so, for any $\delta > 0$,

$$\begin{aligned}
&\mu_{Q(x)-f(x)}(\delta + t)\\
&\geq T_M\Big(\mu_{Q(x)-\frac{f(2^n x)}{2^{5n}}}(\delta), \mu_{\frac{f(2^n x)}{2^{5n}} - f(x)}(t)\Big)\\
&\geq T_M\left(\mu_{Q(x)-\frac{f(2^n x)}{2^{5n}}}(\delta), \mu'_{\phi(x,0)}\left(\frac{128t}{\sum_{j=0}^{n-1}(\frac{\alpha}{2^5})^j}\right)\right)
\end{aligned} \tag{2.7}$$

for all $x \in X$ and $t > 0$. Taking the limit as $n \to \infty$ in (2.7), we get

$$\mu_{Q(x)-f(x)}(\delta + t) \geq \mu'_{\phi(x,0)}\left(2^2(2^5 - \alpha)t\right) \tag{2.8}$$

Since δ is arbitrary, by taking $\delta \to 0$ in (2.8), we have

$$\mu_{Q(x)-f(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(2^5 - \alpha)t\right) \tag{2.9}$$

for all $x \in X$ and $t > 0$. Therefore, we conclude that the condition (2.3) holds.

Also, replacing x and y by $2^n x$ and $2^n y$ in (2.2), respectively, we have

$$\mu_{\frac{Df(2^n x, 2^n y)}{2^{5n}}}(t) \geq \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t)$$

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for all $x, y \in X$ and $t > 0$. It follows from $\lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t) = 1$ that Q satisfies the equation (1.1), which implies that Q is a quintic mapping.

To prove the uniqueness of the quintic mapping Q, let us assume that there exists another mapping $\widetilde{Q} : X \to Y$ which satisfies (2.3). Fix $x \in X$. Then $Q(2^n x) = 2^{5n}Q(x)$ and $\widetilde{Q}(2^n x) = 2^{5n}\widetilde{Q}(x)$ for all $n \in \mathbb{Z}^+$. Thus it follows from (2.3) that

$$\begin{aligned} &\mu_{Q(x)-\widetilde{Q}(x)}(t) \\ &= \mu_{\frac{Q(2^n x)}{2^{5n}} - \frac{\widetilde{Q}(2^n x)}{2^{5n}}}(t) \\ &\geq T_M\Big(\mu_{\frac{Q(2^n x)}{2^{5n}} - \frac{f(2^n x)}{2^{5n}}}\Big(\frac{t}{2}\Big), \mu_{\frac{f(2^n x)}{2^{5n}} - \frac{\widetilde{Q}(2^n x)}{2^{5n}}}\Big(\frac{t}{2}\Big)\Big) \\ &\geq \mu'_{\phi(x,0)}\Big(2^2(2^5-\alpha)\Big(\frac{2^5}{\alpha}\Big)^n t\Big). \end{aligned} \tag{2.10}$$

Since $\lim_{n\to\infty} \Big(2^2(2^5-\alpha)\Big(\frac{2^5}{\alpha}\Big)^n t\Big) = \infty$, we have $\mu_{Q(x)-\widetilde{Q}(x)}(t) = 1$ for all $t > 0$. Thus the quintic mapping Q is unique. This completes the proof. □

Theorem 2.2. *Let $\phi : X^2 \to Z$ be a function such that, for some $2^5 < \alpha$,*

$$\mu'_{\phi(\frac{x}{2},\frac{y}{2})}(t) \geq \mu'_{\phi(x,y)}(\alpha t) \tag{2.11}$$

and $\lim_{n\to\infty} \mu'_{2^{5n}\phi(\frac{x}{2^n},\frac{y}{2^n})}(t) = 1$ for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (2.2), then there exists a unique cubic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(\alpha - 2^5)t\right) \tag{2.12}$$

for all $x \in X$ and $t > 0$.

Proof. It follows from (2.2) that

$$\mu_{f(x)-2^5 f(\frac{x}{2})}(t) \geq \mu'_{\phi(x,0)}\left(2^2\alpha t\right) \tag{2.13}$$

for all $x \in X$. Applying the triangle inequality and (2.13), we have

$$\mu_{f(x)-2^{5n}f(\frac{x}{2^n})}(t) \geq \mu'_{\phi(x,0)}\left(\frac{2^2\alpha t}{\sum_{j=m}^{n+m-1}\left(\frac{2^5}{\alpha}\right)^j}\right) \tag{2.14}$$

for all $x \in X$ and $m, n \in \mathbb{Z}$ with $n > m \geq 0$. Then the sequence $\{2^{5n}f(\frac{x}{2^n})\}$ is a Cauchy sequence in the complete RN-space (Y, μ, T_M) and so it converges to some point $Q(x) \in Y$. We can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} 2^{5n} f\Big(\frac{x}{2^n}\Big)$$

for all $x \in X$. Then the mapping Q satisfies (1.1) and (2.12). The remaining assertion follows the similar proof method in Theorem 2.1. This complete the proof. □

Corollary 2.3. *Let θ be a nonnegative real number and z_0 be a fixed unit point of Z. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies*

$$\mu_{Df(x,y)}(t) \geq \mu'_{\theta z_0}(t) \tag{2.15}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $C : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\theta z_0}(124t) \tag{2.16}$$

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for all $x \in X$ and $t > 0$.

Proof. Let $\phi : X^2 \to Z$ be defined by $\phi(x,y) = \theta z_0$. Then, the proof follows from Theorem 2.1 by $\alpha = 1$. This completes the proof. □

Corollary 2.4. *Let $p, q \in \mathbb{R}$ be positive real numbers with $p, q < 5$ and z_0 be a fixed unit point of Z. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies*

$$\mu_{Df(x,y)}(t) \geq \mu'_{(\|x\|^p+\|y\|^q)z_0}(t) \tag{2.17}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\|x\|^p z_0}\Big(2^2(2^5 - 2^p)t\Big) \tag{2.18}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\phi : X^2 \to Z$ be defined by $\phi(x,y) = (\|x\|^p + \|y\|^q)z_0$. Then the proof follows from Theorem 2.1 by $\alpha = 2^p$. This completes the proof. □

Now, we give an example to illustrate that the quintic functional equation (1.1) is not stable for $r = 5$ in Corollary 2.4

Example 2.1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) = \begin{cases} x^5, & \text{for} \quad |x| < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^{5n}}$$

for all $x \in \mathbb{R}$. Then f satisfies the functional inequality

$$\begin{aligned} &|2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) - 20[f(x+y) + f(x-y)] - 90f(x)| \\ &\leq \frac{136 \cdot 32^2}{31}\Big(|x|^5 + |y|^5\Big) \end{aligned} \tag{2.19}$$

for all $x, y \in X$, but there do not exist a quintic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $d > 0$ such that

$$|f(x) - Q(x)| \leq d|x|^5$$

for all $x \in \mathbb{R}$. In fact, it is clear that f is bounded by $\frac{32}{31}$ on $\mathbb{R}$. If $|x|^5 + |y|^5 = 0$, then (2.19) is trivial. If $|x|^5 + |y|^5 \geq \frac{1}{32}$, then

$$|Df(x,y)| \leq \frac{136 \cdot 32}{31} \leq \frac{136 \cdot 32^2}{31}\Big(|x|^5 + |y|^5\Big).$$

Now, suppose that $0 < |x|^5 + |y|^5 < \frac{1}{32}$. Then there exists a positive integer $k \in Z^+$ such that

$$\frac{1}{32^{k+2}} \leq |x|^5 + |y|^5 < \frac{1}{32^{k+1}}$$

and so

$$32^k|x|^5 < \frac{1}{32}, \quad 32^k|y|^5 < \frac{1}{32},$$

$$2^n(2x+y),\ 2^n(2x-y),\ 2^n(x+2y),\ 2^n(x-2y),\ 2^n(x-y),\ 2^n x \in (-1,1)$$

and

$$\begin{aligned}&\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\quad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)\\&=0\end{aligned}$$

for all $n=0,1,\cdots,k-1$. Thus we obtain

$$\begin{aligned}&|Df(x,y)|\\&\le\sum_{n=0}^{\infty}\frac{1}{2^{5n}}|\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\qquad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)|\\&\le\sum_{n=k}^{\infty}\frac{1}{2^{5n}}|\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\qquad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)|\\&\le\frac{136\cdot 32^2}{31}\Big(|x|^5+|y|^5\Big).\end{aligned}$$

Therefore, f satisfies (2.19).

Now, we claim that the quintic functional equation (1.1) is not stable for $r=5$ in Corollary 2.4. Suppose on the contrary that there exists a quintic mapping $Q:\mathbb{R}\to\mathbb{R}$ and constant $d>0$ such that

$$|f(x)-Q(x)|\le d|x|^5$$

for all $x\in\mathbb{R}$. Since f is bounded and continuous for all $x\in\mathbb{R}$, Q is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2.1, Q must have $Q(x)=cx^5$ for all $x\in\mathbb{R}$. So, we obtain

$$|f(x)|\le(d+|c|)|x|^5 \tag{2.20}$$

for all $x\in\mathbb{R}$. Let $m\in\mathbb{Z}^+$ such that $m+1>d+|c|$.

If x is in $(0,2^{-m})$, then $2^nx\in(0,1)$ for $n=0,1,\cdots,m$. For this x, we have

$$f(x)=\sum_{n=0}^{\infty}\frac{\phi(2^n)}{2^{5n}}\ge\sum_{n=0}^{m}\frac{(2^nx)^5}{2^{5n}}=(m+1)x^5>(d+|c|)|x|^5,$$

which contradiction (2.20).

Remark 2.1. In Corollary 2.4, if we assume that

$$\phi(x,y)=\|x\|^r\|y\|^rz_0$$

or

$$\phi(x,y)=(\|x\|^r\|y\|^s+\|x\|^{r+s}+\|y\|^{r+s})z_0,$$

then we have Ulam-Gavuta-Rassias product stability and JMRassias mixed product-sum stability, respectively.

Next, we apply a fixed point method for the generalized Hyer-Ulam stability of the functional equation (1.1) in RN-spaces. The following Theorem will be used in the proof of Theorem 2.6.

Theorem 2.5. ([7]) *Suppose that (Ω, d) is a complete generalized metric space and $J : \Omega \to \Omega$ is a strictly contractive mapping with Lipshitz constant $L < 1$. Then, for each $x \in \Omega$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers $n \geq 0$ or there exists a natural number n_0 such that*

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ is convergent to a fixed point $y*$ of J;*
(3) *$y*$ is the unique fixed point of J in the set $\Lambda = \{y \in \Omega : d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Lambda$.*

Theorem 2.6. *Let $\phi : X^2 \to D^+$ be a function such that, for some $0 < \alpha < 2^5$,*

$$\mu'_{\phi(x,y)}(t) \leq \mu'_{\phi(2x,2y)}(\alpha t) \tag{2.21}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ such that

$$\mu_{D(x,y)}(t) \geq \mu'_{\phi(x,y)}(t) \tag{2.22}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,y)}\big(2^2(2^5 - \alpha)t\big) \tag{2.23}$$

for all $x \in X$ and $t > 0$.

Proof. It follows from (2.22) that

$$\mu_{f(x)-\frac{f(2x)}{2^5}}(t) \geq \mu'_{\phi(x,0)}(128t) \tag{2.24}$$

for all $x \in X$ and $t > 0$. Let $\Omega = \{g : X \to Y, g(x) = 0\}$ and the mapping d defined on Ω by

$$d(g, h) = \inf\{c \in [0, \infty) : \mu_{g(x)-h(x)}(ct) \geq \mu'_{\phi(x,0)}(t), \forall x \in X\}$$

where, as usual, $\inf \emptyset = -\infty$. Then (Ω, d) is a generalized complete metric space (see [10]). Now, let us consider the mapping $J : \Omega \to \Omega$ defined by

$$Jg(x) = \frac{1}{2^5} g(2x)$$

for all $g \in \Omega$ and $x \in X$. Let g, h in Ω and $c \in [0, \infty)$ be an arbitrary constant with $d(g, h) < c$. Then $\mu_{g(x)-h(x)}(ct) \geq \mu'_{\phi(x,0)}(t)$ for all $x \in X$ and $t > 0$ and so

$$\mu_{Jg(x)-Jh(x)}\Big(\frac{\alpha ct}{2^5}\Big) = \mu_{g(2x)-h(2x)}(\alpha ct) \geq \mu'_{\phi(x,0)}(t) \tag{2.25}$$

for all $x \in X$ and $t > 0$. Hence we have

$$d(Jg, Jh) \leq \frac{\alpha c}{2^5} \leq \frac{\alpha}{2^5} d(g, h)$$

for all $g, h \in \Omega$. Then J is a contractive mapping on Ω with the Lipschitz constant $L = \frac{\alpha}{2^5} < 1$. Thus it follows from Theorem 2.5 that there exists a mapping $Q : X \to Y$, which is a unique fixed point of J in the set $\Omega_1 = \{g \in \Omega : d(f, g) < \infty\}$, such that

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{5n}}$$

for all $x \in X$ since $\lim_{n\to\infty} d(J^n f, Q) = 0$. Also, from $\mu_{f(x)-\frac{f(2x)}{2^5}}(t) \geq \mu'_{\phi(x,0)}(128t)$, it follows that $d(f, Jf) \leq \frac{1}{128}$. Therefore, using Theorem 2.5 again, we get

$$d(f, Q) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{2^2(2^5 - \alpha)}.$$

This means that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(2^5-\alpha)t\right)$$

for all $x \in X$ and $t > 0$.

Also, replacing x and y by $2^n x$ and $2^n y$ in (2.22), respectively, we have

$$\mu_{DQ(x,y)}(t) \geq \lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t) = \lim_{n\to\infty} \mu'_{\phi(x,y)}\left(\left(\frac{2^5}{\alpha}\right)^n t\right) = 1$$

for all $x, y \in X$ and $t > 0$. By (RN1), the mapping Q is quintic.

To prove the uniqueness, let us assume that there exists a quintic mapping $Q' : X \to Y$ which satisfies (2.23). Then Q' is a fixed point of J in Ω_1. However, it follows from Theorem 2.5 that J has only one fixed point in Ω_1. Hence $Q = Q'$. This completes the proof. □

Theorem 2.7. *Let $\phi : X^2 \to D^+$ be a function such that, for some $0 < 2^5 < \alpha$,*

$$\mu'_{\phi(x,y)}(t) \leq \mu'_{\phi(\frac{x}{2},\frac{y}{2})}(\alpha t) \tag{2.26}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (2.22), then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(\alpha-2^5)t\right) \tag{2.27}$$

for all $x \in X$ and $t > 0$.

Proof. By a modification in the proofs of Theorem 2.2 and 2.6, we can easily obtain the desired results. This completes the proof. □

Now, we present a corollary that is an application of Theorem 2.6 and 2.7 in the classical case.

Corollary 2.8. *Let X be a Banach space, ϵ and p be positive real numbers with $p \neq 5$. Assume that $f : X \to X$ is a mapping with $f(0) = 0$ which satisfies*

$$\|Df(x,y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\|Q(x) - f(x)\| \leq \frac{\epsilon\|x\|^p}{2^2|2^5 - 2^p|}$$

for all $x \in X$ and $t > 0$.

Proof. Define $\mu : X \times \mathbb{R} \to \mathbb{R}$ by

$$\mu_x(t) = \begin{cases} \frac{t}{t+\|x\|}, & \text{if } t > 0, \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in X$ and $t \in \mathbb{R}$. Then (X, μ, T_M) is a complete RN-space. Denote $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ and $t > 0$. It follows from $\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p)$ that

$$\mu_{Df(x,y)}(t) \geq \mu'_{\phi(x,y)}(t)$$

for all $x, y \in X$ and $t > 0$, where $\mu' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$\mu'_x(t) = \begin{cases} \frac{t}{t+|x|}, & \text{if } t > 0, \\ 0, & \text{otherwise}, \end{cases}$$

is a random norm on $\mathbb{R}$. Then all the conditions of Theorems 2.6 and 2.7 hold and so there exists a unique quintic mapping $Q : X \to X$ such that

$$\begin{aligned} \frac{t}{t + \|Q(x) - f(x)\|} &= \mu_{Q(x)-f(x)}(t) \\ &\geq \mu'_{\phi(x,0)}\Big(2^2|2^5 - \alpha|t\Big) = \frac{2^2|2^5 - \alpha|t}{2^2|2^5 - \alpha|t + \epsilon\|x\|^p}. \end{aligned}$$

Therefore, we obtain the desired result, where $\alpha = 2^p$. This completes the proof. □

Acknowledgments

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under grant no. (18-130-36-HiCi). The authors, therefore, acknowledge with thanks DSR technical and financial support. Also, Yeol Je Cho was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and future Planning (2014R1A2A2A01002100).

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Generalized composition operators on Zygmund type spaces and Bloch type spaces

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Abstract. In this paper, we investigate the boundedness and compactness of generalized composition operators on Zygmund type spaces and Bloch type spaces with normal weight.

MSC 2000: 47B33, 30H10.

Keywords: Generalized composition operator, Bloch type space, Zygmund type space.

1 Introduction

Let k be a positive continuous function on $[0, 1)$. k is called normal, if there exist positive numbers a and b, $0 < a < b$, and $\delta \in [0, 1)$ such that(see [12]),

$$\frac{k(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{k(r)}{(1-r)^a} = 0; \tag{1}$$

$$\frac{k(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{k(r)}{(1-r)^b} = \infty. \tag{2}$$

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Let ω be normal on $[0, 1)$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch type space, denoted by $\mathcal{B}_\omega$, if

$$\|f\|_{\mathcal{B}_\omega} = |f(0)| + \sup_{z\in\mathbb{D}} \omega(|z|)|f'(z)| < \infty.$$

It is easy to see that $\mathcal{B}_\omega$ is a Banach space with the norm $\|\cdot\|_{\mathcal{B}_\omega}$. When $\omega(t) = 1 - t^2$, we get the Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{D})$. See [19] for more information of the Bloch space.

Suppose μ is normal on $[0, 1)$. The Zygmund type space, denoted by $\mathcal{Z}_\mu$, is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)| < \infty.$$

It is also easy to see that $\mathcal{Z}_\mu$ is a Banach space with the norm $\|\cdot\|_{\mathcal{Z}_\mu}$. When $\mu(t) = 1-t^2$, we get the Zygmund space (see [2, 8]).

1

Throughout the paper, $S(\mathbb{D})$ denotes the set of analytic self-map of $\mathbb{D}$. Associated with $\varphi \in S(\mathbb{D})$ is the composition operator C_φ, which is defined by $(C_\varphi f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D})$. We refer the books [1, 19] for the theory of composition operators. Composition operators mapping into the Bloch space on $\mathbb{D}$ were studied in, for example, [1, 4, 11, 14, 15, 18]. See [5, 6, 9, 10] for some results of the composition operator mapping into the Zygmund space.

Motivated by the fact that weighted composition operators naturally come from isometries of some function spaces, for $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, Li and Stević [9] defined the generalized composition operator, denoted by C_φ^g, as follows.

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

They characterized the boundedness and compactness of C_φ^g on the Zygmund space and the Bloch space in [9]. See, for example, [7, 13, 16] for the study of the operator C_φ^g.

In this paper, motivated by [9], we investigate the boundedness and compactness of the generalized composition operator C_φ^g on Zygmund type spaces and Bloch type spaces with normal weight.

In this paper, constants are denoted by C, they are positive and may differ from one occurrence to the next. We say that $A \lesssim B$ if there exists a constant C such that $A \leq CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2 Proof of main results

In this section, we give some auxiliary results which will be used in proving the main results of this paper. They are incorporated in the lemmas which follow.

Lemma 1. [3] *Suppose μ is normal on $[0, 1)$. Then there exists $\mu_* \in H(\mathbb{D})$, such that*

(i) For any $t \in [0, 1)$, $\mu_*(t) \in R^+$, $\mu_*(t)$ is increasing on $[0, 1)$;

(ii) $\inf_{t\in[0,1)} \mu(t)\mu_*(t) > 0$; $\quad \sup_{z\in\mathbb{D}} \mu(|z|)|\mu_*(z)| < \infty$.

In the rest of the paper, we will always use μ_* to denote the analytic function related to μ in Lemma 1. By a calculation, we get the following lemma.

Lemma 2. *Suppose μ is normal on $[0, 1)$. Then the following statements hold.*

(i) There exists a $\delta \in (0, 1)$, such that μ is decreasing on $[\delta, 1)$, $\lim_{t\to 1} \mu(t) = 0$.

(ii) For all $\alpha > 1, \beta \in (0, 1)$, when $t \in (0, 1)$, $s \in (\beta, 1)$,

$$\mu(t) \approx \mu(t^\alpha) \approx \frac{1}{\mu_*(t)}, \quad \int_0^{s^\alpha} \frac{1}{\mu(t)}dt \approx \int_0^s \frac{1}{\mu_(t)}dt.$$

(iii) For any $z \in \mathbb{D}$, $\left|\int_0^z \mu_*(\eta)d\eta\right| \lesssim \int_0^{|z|} \mu_*(t)dt$. If $|\eta| \leq |z|$, $\mu(|z|)|\mu_*(\eta)| < C$.

2

Proof. (i). By the definition of normal function, there exist positive numbers a and b, $0 < a < b$, and $\delta \in [0, 1)$ such that (1) and (2) hold. Since $\mu(t) = \frac{\mu(t)}{(1-t)^a}(1-t)^a$, we see that μ is decreasing on $[\delta, 1)$ and $\lim_{t\to 1}\mu(t) = 0$.

(ii). From $\lim_{t\to 1}\frac{1-t}{1-t^\alpha} = \frac{1}{\alpha} > 0$, for any $t \in [\delta^{\frac{1}{\alpha}}, 1)$,

$$1 > \frac{\mu(t)}{\mu(t^\alpha)} = \frac{\frac{\mu(t)}{(1-t)^b}}{\frac{\mu(t^\alpha)}{(1-t^\alpha)^b}}\frac{(1-t)^b}{(1-t^\alpha)^b} > \frac{(1-t)^b}{(1-t^\alpha)^b} > C.$$

So when $t \in (0, 1)$, $\mu(t) \approx \mu(t^\alpha)$. By Lemma 1, when $t \in (0, 1)$, $\mu(t) \approx \frac{1}{\mu_*(t)}$ is obvious.

When $s \in (\beta, 1)$,

$$\begin{aligned}\int_0^{s^\alpha}\frac{1}{\mu(t)}dt &= \int_0^{\beta^\alpha}\frac{1}{\mu(t)}dt + \int_{\beta^\alpha}^{s^a}\frac{1}{\mu(t)}dt = C + \int_\beta^s\frac{\alpha t^{\alpha-1}}{\mu(t^\alpha)}dt \\ &\approx \int_0^\beta\frac{1}{\mu(t)}dt + \int_\beta^s\frac{1}{\mu(t)}dt = \int_0^s\frac{1}{\mu(t)}dt.\end{aligned}$$

(iii). Since μ_* is analytic, we see that (iii) holds. The proof is completed. □

Lemma 3.[17] *Suppose μ is normal on $[0, 1)$. Then for all $z \in \mathbb{D}$ and $f \in \mathcal{B}_\mu$,*

$$|f(z)| < G_\mu(z)\|f\|_{\mathcal{B}_\mu},\ \textit{where}\ G_\mu(z) = 1 + \int_0^{|z|}\frac{1}{\mu(t)}dt.$$

Remark 1. From the definitions of $\mathcal{Z}_\mu$ and $\mathcal{B}_\mu$, for all $z \in \mathbb{D}$ and $f \in \mathcal{Z}_\mu$,

$$|f'(z)| \le G_\mu(z)\|f'\|_{\mathcal{B}_\mu} \le G_\mu(z)\|f\|_{\mathcal{Z}_\mu}.$$

Lemma 4. [17] *Suppose that μ is normal on $[0, 1)$ such that $\int_0^1\frac{1}{\mu(t)}dt < \infty$. If $\{f_n\}$ is bounded in $\mathcal{B}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$, then*

$$\lim_{n\to\infty}\sup_{z\in\mathbb{D}}|f_n(z)| = 0.$$

The relationship between Zygmund type spaces and Bloch type spaces was established as follows.

Lemma 5. *Suppose that μ is normal on $[0, 1)$. Let $\mu_+(t) = (1-t)\mu(t)$. Then*

(i) *μ_+ is normal on $[0, 1)$, $\lim_{|z|\to 1} G_{\mu_+}(z) = \infty$.*

(ii) *$\mathcal{B}_\mu = \mathcal{Z}_{\mu_+}$ and $\|\cdot\|_{\mathcal{B}_\mu} \approx \|\cdot\|_{\mathcal{Z}_{\mu_+}}$.*

Proof. (i) Obviously, μ_+ is normal on $[0, 1)$. Since μ is normal,there exist positive numbers a and b, $0 < a < b$, and $\delta \in [0, 1)$ such that (1) and (2) holds. Then

$$\int_0^1\frac{1}{\mu_+(t)}dt > \int_\delta^1\frac{1}{(1-t)^{a+1}}\frac{(1-t)^a}{\mu(t)}dt > \frac{(1-\delta)^a}{\mu(\delta)}\int_\delta^1\frac{1}{(1-t)^{1+a}}dt = +\infty,$$

as desired.

(ii) First we prove that $\mathscr{Z}_{\mu_+} \subseteq \mathcal{B}_\mu$. For all $f \in \mathscr{Z}_{\mu_+}$, we have

$$\mu(|z|)|f'(z) - f'(0)| = \mu(|z|)\left|\int_0^z f''(\eta)d\eta\right| \leq \|f\|_{\mathscr{Z}_{\mu+}} \int_0^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt. \tag{3}$$

If $|z| \leq \delta$, $\int_0^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt < C$. If $|z| > \delta$,

$$\begin{aligned}
\int_0^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt &= \left(\int_0^{\delta} \frac{\mu(|z|)}{\mu(t)(1-t)}dt + \int_\delta^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt\right) \\
&\leq \left(C + \int_\delta^{|z|} \frac{\frac{\mu(|z|)}{(1-|z|)^a}}{\frac{\mu(t)}{(1-t)^a}} \frac{(1-|z|)^a}{(1-t)^{a+1}}dt\right) \\
&\leq \left(C + \int_\delta^{|z|} \frac{(1-|z|)^a}{(1-t)^{a+1}}dt\right) \leq C.
\end{aligned}$$

From Lemma 2, $\mu(t)$ is bounded on $[0, 1)$. By (3),

$$\mu(|z|)|f'(z)| \leq C\|f\|_{\mathscr{Z}_{\mu_+}} + \mu(|z|)|f'(0)| \lesssim \|f\|_{\mathscr{Z}_{\mu_+}} + |f'(0)| \leq 2\|f\|_{\mathscr{Z}_{\mu_+}}.$$

Therefore $\|f\|_{\mathcal{B}_\mu} \lesssim \|f\|_{\mathscr{Z}_{\mu_+}}$ and $\mathscr{Z}_{\mu_+} \subseteq \mathcal{B}_\mu$.

Next we prove that $\mathcal{B}_\mu \subseteq \mathscr{Z}_{\mu_+}$. For any $f \in \mathcal{B}_\mu$, by Cauchy's formula,

$$|f''(z)| \leq \frac{2}{1-|z|} \max_{|\eta - z| = \frac{1-|z|}{2}} |f'(\eta)| \leq \frac{2}{1-|z|} \max_{|\eta| = \frac{1+|z|}{2}} |f'(\eta)| \leq \frac{2\|f\|_{\mathcal{B}_\mu}}{\mu(\frac{1+|z|}{2})(1-|z|)}.$$

If $|z| \leq \delta$, $\frac{\mu(|z|)}{\mu(\frac{1+|z|}{2})} < C$ is obvious. When $\delta < |z| < 1$,

$$\frac{\mu(|z|)}{\mu(\frac{1+|z|}{2})} = 2^b \frac{\frac{\mu(|z|)}{(1-|z|)^b}}{\frac{\mu(\frac{1+|z|}{2})}{(1-\frac{1+|z|}{2})^b}} < 2^b.$$

So $\|f\|_{\mathscr{Z}_{\mu_+}} \lesssim \|f\|_{\mathcal{B}_\mu}$ and hence $\mathcal{B}_\mu \subseteq \mathscr{Z}_{\mu_+}$. The proof is completed. □

To study the compactness, we need the following lemma, which can be proved in a standard way (see, for example, Proposition 3.11 in [1]).

Lemma 6. *Suppose that $g \in H(\mathbb{D}), \varphi \in S(\mathbb{D})$, X, Y are Bloch type spaces or Zygmund type spaces. If $C_\varphi^g : X \to Y$ is bounded, then $C_\varphi^g : X \to Y$ is a compact operator if and only if whenever $\{f_n\}$ is bounded in X and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, $\lim_{n\to\infty} \|C_\varphi^g f_n\|_Y = 0$.*

3 The boundness and compactness of $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega(\mathcal{B}_\omega)$

Theorem 1. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0, 1)$. Then $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)|G_\mu(\varphi(z)) < \infty \qquad \text{and} \qquad \sup_{z\in\mathbb{D}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \infty. \tag{4}$$

Proof. Suppose that (4) holds. For any $f \in \mathscr{Z}_\mu$, by Lemma 3 and Remark 1, we have

$$\begin{aligned}\sup_{z\in\mathbb{D}} \omega(|z|)|(C_\varphi^g f)''(z)| &\le \sup_{z\in\mathbb{D}} \omega(|z|)\left|f''(\varphi(z))\varphi'(z)g(z)\right| + \sup_{z\in\mathbb{D}} \omega(|z|)\left|f'(\varphi(z))g'(z)\right| \\ &\le \sup_{z\in\mathbb{D}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}\|f\|_{\mathscr{Z}_\mu} + \sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)|G_\mu(\varphi(z))\|f\|_{\mathscr{Z}_\mu} \\ &< \infty,\end{aligned}$$

and $|(c_\varphi^g f)(0)| + |(C_\varphi^g f)'(0)| = |f'(\varphi(0))g(0)| \le |g(0)|G_\mu(\varphi(0))\|f\|_{\mathscr{Z}_\mu} < \infty$. Hence $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded.

Conversely, suppose $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded. From $z, z^2 \in \mathscr{Z}_\mu$, we see that

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)| < \infty \qquad \text{and} \qquad \sup_{z\in\mathbb{D}} \omega(|z|)|\varphi'(z)g(z)| < \infty. \tag{5}$$

Therefore

$$\sup_{|\varphi(z)|\le\frac{1}{2}} \omega(|z|)|g'(z)|G_\mu(\varphi(z)) < \infty \qquad \text{and} \qquad \sup_{|\varphi(z)|\le\frac{1}{2}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \infty. \tag{6}$$

For any $\xi \in \mathbb{D}$, if $|\varphi(\xi)| > \frac{1}{2}$, let $a = \varphi(\xi)$ and

$$p_a(z) = \int_0^{\overline{a}z} \left(\int_0^{t^2} \mu_*(\eta)d\eta\right)^2 dt - \int_0^{\overline{a}z} \left(\int_0^{\frac{t^3}{|a|^2}} \mu_*(\eta)d\eta\right)^2 dt,$$

$$q_a(z) = \frac{p_a(z)}{\int_0^{|a|} \mu_*(\eta)d\eta}. \tag{7}$$

Then

$$p_a'(z) = \overline{a}\left(\int_0^{(\overline{a}z)^2} \mu_*(\eta)d\eta\right)^2 - \overline{a}\left(\int_0^{\frac{(\overline{a}z)^3}{|a|^2}} \mu_*(\eta)d\eta\right)^2,$$

$$p_a''(z) = 4\overline{a}^3 z\mu_*(\overline{a}^2 z^2)\int_0^{(\overline{a}z)^2} \mu_*(\eta)d\eta - \frac{6\overline{a}^4 z^2}{|a|^2}\mu_*\left(\frac{(\overline{a}z)^3}{|a|^2}\right)\int_0^{\frac{(\overline{a}z)^3}{|a|^2}} \mu_*(\eta)d\eta.$$

By Lemmas 1 and 2,

$$\mu(|z|)|p_a''(z)| \lesssim \left|\int_0^{(\overline{a}z)^2} \mu_*(\eta)d\eta\right| + \left|\int_0^{\frac{(\overline{a}z)^3}{|a|^2}} \mu_*(\eta)d\eta\right| \lesssim \int_0^{|a|} \mu_*(\eta)d\eta$$

So

$$\|q_a\|_{\mathscr{Z}_\mu} = q_a(0) + q_a'(0) + \sup_{z\in\mathbb{D}} \mu(|z|)|q_a''(z)| < C. \tag{8}$$

Hence, when $|\varphi(\xi)| > \frac{1}{2}$,

$$\frac{\omega(|\xi|)|\varphi'(\xi)g(\xi)|}{\mu(|\varphi(\xi)|)} \approx \omega(|\xi|)|(C_\varphi^g q_a)''(\xi)| \le \|C_\varphi^g q_a\|_{\mathscr{Z}_\omega} < \|q_a\|_{\mathscr{Z}_\mu}\|C_\varphi^g\| < \infty. \tag{9}$$

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From (6) and (9), we see that the second inequality in (4) holds.

Let $f_a(z) = \int_0^{\overline{a}z} \int_0^{\eta} \mu_*(w)dwd\eta$. Then

$$f_a'(z) = \overline{a}\int_0^{\overline{a}z} \mu_*(w)dw, \ \ f_a''(z) = \overline{a}^2\mu_*(\overline{a}z), \ \|f_a\|_{\mathscr{Z}_\mu} \le C.$$

By Lemma 2, when $|\varphi(\xi)| > \frac{1}{2}$,

$$\begin{aligned}\omega(|\xi|)|g'(\xi)|\int_0^{|a|}\frac{1}{\mu(t)}dt \approx&\, \omega(|\xi|)\left|(C_\varphi^g f_a)''(\xi) - f_a''(\varphi(\xi))\varphi'(\xi)g(\xi)\right| \\ \le&\, \|C_\varphi^g f_a\|_{\mathscr{Z}_\omega} + \sup_{\xi\in\mathbb{D}}\omega(|\xi|)\mu_*(|\varphi(\xi)|^2)|\varphi'(\xi)g(\xi)| \\ \lesssim&\, \|f_a\|_{\mathscr{Z}_\mu}\|C_\varphi^g\| + \sup_{\xi\in\mathbb{D}}\frac{\omega(|\xi|)|\varphi'(\xi)g(\xi)|}{\mu(|\varphi(\xi)|)}.\end{aligned} \tag{10}$$

From (6) and (10), we see that the first inequality in (4) holds. The proof is completed. □

Theorem 2. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0, 1)$ such that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded. Then the following statements hold:*

(i) When $\lim_{|z|\to 1} G_\mu(z) < \infty$, $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1}\frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0. \tag{11}$$

(ii) When $\lim_{|z|\to 1} G_\mu(z) = \infty$, $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1}\omega(|z|)|g'(z)|G_\mu(\varphi(z)) = 0 \quad \text{and} \quad \lim_{|\varphi(z)|\to 1}\frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0. \tag{12}$$

Proof. Because $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded, (5) holds.

(i). Suppose (11) holds. For any $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$\frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \varepsilon, \text{ when } |\varphi(z)| > \delta. \tag{13}$$

Let $\{f_n\} \subset \mathscr{Z}_\mu$ be bounded and converge to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 4 and Cauchy estimate,

$$\lim_{n\to\infty}\sup_{z\in\mathbb{D}}|f_n'(z)| = 0, \quad \text{and} \quad \lim_{n\to\infty}\sup_{|z|\le\delta}|f_n''(z)| = 0. \tag{14}$$

From Remark 1, (5) and $\sup_{n\in\mathbb{N}}\|f_n\|_{\mathscr{Z}_\mu} < \infty$,

$$\begin{aligned}\|C_\varphi^g f_n\|_{\mathscr{Z}_\omega} =&\, |(C_\varphi^g f_n)'(0)| + \sup_{z\in\mathbb{D}}\omega(|z|)\left|f_n''(\varphi(z))\varphi'(z)g(z) + f_n'(\varphi(z))g'(z)\right| \\ \lesssim&\, |f_n'(\varphi(0))| + \sup_{|\varphi(z)|\le\delta}|f_n''(\varphi(z))| + \sup_{|\varphi(z)|>\delta}\frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} + \sup_{z\in\mathbb{D}}|f_n'(z)|.\end{aligned}$$

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By (13) and (14), $\lim_{n\to\infty} \|C_\varphi^g f_n\|_{\mathscr{Z}_\mu} = 0$. Using Lemma 6, we see that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact.

Conversely, assume that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact. Suppose $\{z_n\} \subset \mathbb{D}$ is a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. Let $a_n = \varphi(z_n)$ and

$$r_n(z) = \mu(|a_n|) \int_0^{\overline{a_n}z} \int_0^t \mu_*^2(\eta) d\eta dt.$$

From Lemma 2, $\{r_n\}$ is bounded in $\mathscr{Z}_\mu$ and $r_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$ when $n \to \infty$. By Lemmas 4 and 6, we have

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |r_n'(z)| = 0 \text{ and } \lim_{n\to\infty} \|C_\varphi^g r_n\|_{\mathscr{Z}_\omega} = 0. \tag{15}$$

Using Lemma 2, (5) and (15),

$$\begin{aligned} \lim_{n\to\infty} \frac{\omega(|z_n|)|\varphi'(z_n)g(z_n)|}{\mu(|\varphi(z_n)|)} &\approx \lim_{n\to\infty} \omega(|z_n|) \left|(C_\varphi^g r_n)''(z_n) - r_n'(a_n)g'(z_n)\right| \\ &\leq \lim_{n\to\infty} \|C_\varphi^g r_n\|_{\mathscr{Z}_\omega} + \lim_{n\to\infty} \omega(|z_n|) \left|r_n'(a_n)g'(z_n)\right| = 0, \end{aligned}$$

which implies that $\lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0$.

(ii). Suppose (12) holds. For any $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$\omega(|z|)|g'(z)|G_\mu(\varphi(z)) < \varepsilon \quad \text{and} \quad \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \varepsilon, \tag{16}$$

when $|\varphi(z)| > \delta$. Let $\{f_n\}$ be a bounded sequence in $\mathscr{Z}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Cauchy estimate,

$$\lim_{n\to\infty} \sup_{|\varphi(w)|\leq\delta} |f_n'(\varphi(w))| = 0, \ \lim_{n\to\infty} \sup_{|\varphi(w)|\leq\delta} |f_n''(\varphi(w))| = 0. \tag{17}$$

From Lemma 3, Remark 1 and (5),

$$\begin{aligned} \|C_\varphi^g f_n\|_{\mathscr{Z}_\omega} &= |(C_\varphi^g f_n)'(0)| + \sup_{z\in\mathbb{D}} \omega(|z|) \left|f_n''(\varphi(z))\varphi'(z)g(z) + f_n'(\varphi(z))g'(z)\right| \\ &\lesssim |f_n'(\varphi(0))| + \sup_{|\varphi(z)|\leq\delta} |f_n''(\varphi(z))| + \sup_{|\varphi(z)|\leq\delta} |f_n'(\varphi(z))| + \\ &\quad \sup_{|\varphi(z)|>\delta} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} \|f_n\|_{\mathscr{Z}_\mu} + \sup_{|\varphi(z)|>\delta} \omega(|z|)|g'(z)|G_\mu(\varphi(z))\|f_n\|_{\mathscr{Z}_\mu} \end{aligned}$$

By (16) and (17), we see that $\lim_{n\to\infty} \|C_\varphi^g f_n\|_{\mathscr{Z}_\mu} = 0$. From Lemma 6, $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact.

Conversely, suppose that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact. Let $\{z_n\} \subset \mathbb{D}$ be a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. Let $a_n = \varphi(z_n)$ and $q_n = q_{a_n}$, where q_a is defined in (7). By (8), $\{q_n\}$ is bounded in $\mathscr{Z}_\mu$. Obviously, $q_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 6, $\lim_{n\to\infty} \|C_\varphi^g q_n\|_{\mathscr{Z}_\omega} = 0$. By (9),

$$\lim_{n\to\infty} \frac{\omega(|z_n|)|\varphi'(z_n)g(z_n)|}{\mu(|\varphi(z_n)|)} \approx \lim_{n\to\infty} \omega(|z_n|)|(C_\varphi^g q_n)''(z_n)| \leq \lim_{n\to\infty} \|C_\varphi^g q_n\|_{\mathscr{Z}_\omega} = 0,$$

which implies that $\lim\limits_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0$.

Let

$$k_n(z) = \frac{\int_0^{\overline{a_n}z} \left(\int_0^t \mu_*(s)ds\right)^2 dt}{\int_0^{|a_n|} \mu_*(s)ds}. \tag{18}$$

Then

$$k_n'(z) = \frac{\overline{a_n}\left(\int_0^{\overline{a_n}z} \mu_*(s)ds\right)^2}{\int_0^{|a_n|} \mu_*(s)ds}, k_n''(z) = \frac{2(\overline{a_n})^2\mu_*(\overline{a_n}z)\int_0^{\overline{a_n}z} \mu_*(s)ds}{\int_0^{|a_n|} \mu_*(s)ds}.$$

By Lemma 2, $\{k_n\}$ is bounded in $\mathscr{Z}_\mu$ and $k_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. From Lemma 6, $\lim\limits_{n\to\infty} \|C_\varphi^g k_n\|_{\mathscr{Z}_\omega} = 0$. By Lemma 2,

$$\begin{aligned} &\lim_{n\to\infty} \omega(|z_n|)|g'(z_n)| \int_0^{|\varphi(z_n)|} \mu_*(s)ds \\ \lesssim\ & \lim_{n\to\infty} \|C_\varphi^g k_n\|_{\mathscr{Z}_\omega} + 2\lim_{n\to\infty} \left|\varphi'(z_n)g(z_n)\omega(|z_n|)\mu_*(|\varphi(z_n)|^2)\right| \\ \lesssim\ & \lim_{n\to\infty} \frac{\omega(|z_n|)|\varphi'(z_n)g(z_n)|}{\mu(|\varphi(z_n)|)} = 0, \end{aligned}$$

which implies that $\lim\limits_{|\varphi(z)|\to 1} \omega(|z|)|g'(z)|G_\mu(\varphi(z)) = 0$. The proof is completed. □

Theorem 3. *Suppose $g \in H(\mathbb{D}), \varphi \in S(\mathbb{D})$, ω and μ are normal on $[0, 1)$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is bounded.*

(ii) $\sup\limits_{z\in\mathbb{D}} \omega(|z|)|g(z)|G_\mu(\varphi(z)) < \infty$.

(iii) $\sup\limits_{z\in\mathbb{D}} \omega_+(|z|)|g'(z)|G_\mu(\varphi(z)) < \infty$ *and* $\sup\limits_{z\in\mathbb{D}} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \infty$.

Proof. (ii)⇒(i). Suppose that *(ii)* holds. For any $f \in \mathscr{Z}_\mu$, using Remark 1,

$$\|C_\varphi^g f\|_{\mathcal{B}_\omega} = \sup_{z\in\mathbb{D}} \omega(|z|)|g(z)f'(\varphi(z))| \le \sup_{z\in\mathbb{D}} \omega(|z|)|g(z)|G_\mu(\varphi(z))\|f\|_{\mathscr{Z}_\mu} \lesssim \|f\|_{\mathscr{Z}_\mu} < \infty.$$

So $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is bounded.

(ii)⇒(i). Suppose $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is bounded. Then

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g(z)| = \|C_\varphi^g z\|_{\mathcal{B}_\omega} < \infty. \tag{19}$$

For all $\eta \in \mathbb{D}$, let $u_a(z) = \int_0^{\overline{a}z}\int_0^t \mu_*(s)dsdt$, where $a = \varphi(\eta)$. By Lemma 2, $\sup_{\eta\in\mathbb{D}} \|u_a\|_{\mathscr{Z}_\mu} < \infty$. Thus $\sup_{\eta\in\mathbb{D}} \|C_\varphi^g u_a\|_{\mathcal{B}_\omega} < \infty$. When $|\varphi(\eta)| > \frac{1}{2}$,

$$\omega(|\eta|)|g(\eta)| \int_0^{|\varphi(\eta)|} \frac{1}{\mu(s)}ds \approx \omega(|\eta|)|(C_\varphi^g u_a)'(\eta)| \le \|C_\varphi^g u_a\|_{\mathcal{B}_\omega} < C. \tag{20}$$

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By (19) and (20),

$$\sup_{\eta\in\mathbb{D}} \omega(|\eta|)|g(\eta)|G_\mu(\varphi(\eta)) < \infty.$$

By Lemma 5 and Theorem 1, *(i)*⇔ *(iii)*. The proof is completed. □

Theorem 4. *Suppose* $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω *and* μ *are normal on* $[0,1)$ *such that* $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is bounded. Then the following statements hold.*

(i) *If* $\lim_{|z|\to 1} G_\mu(z) < \infty$, $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is compact.*

(ii) *if* $\lim_{|z|\to 1} G_\mu(z) = \infty$, *then the following statements are equivalent.*

- *(a)* $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is compact.*
- *(b)* $\lim_{|\varphi(z)|\to 1} \omega(|z|)|g(z)|G_\mu(\varphi(z)) = 0.$
- *(c)* $\lim_{|\varphi(z)|\to 1} \omega_+(|z|)|g'(z)|G_\mu(\varphi(z)) = 0$ *and* $\lim_{|\varphi(z)|\to 1} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0.$

Proof. Since $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is bounded, (19) holds.

(i). Suppose $\{f_n\}$ is bounded in $\mathscr{Z}_\mu$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Then $\{f'_n\}$ is also bounded in $\mathcal{B}_\mu$ and $f'_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. From Lemma 4, $\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f'_n(z)| = 0$. Using (19),

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} \omega(|z|)|(C^g_\varphi f_n)'(z)| = \lim_{n\to\infty} \sup_{z\in\mathbb{D}} \omega(|z|)|g(z) f'_n(\varphi(z))| \lesssim \lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f'_n(\varphi(z))| = 0.$$

Thus $\lim_{n\to\infty} \|C^g_\varphi f_n\|_{\mathcal{B}_\omega} = 0$. By Lemma 6, $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is compact.

(ii). *(b)*⇒*(a)*. Assume that $\lim_{|\varphi(z)|\to 1} \omega(|z|)|g(z)|G_\mu(\varphi(z)) = 0$. Then for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$, such that

$$\omega(|z|)|g(z)|G_\mu(\varphi(z)) < \varepsilon, \text{ when } \delta < |\varphi(z)| < 1.$$

Suppose that $\{f_n\}$ is bounded in $\mathscr{Z}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then $f'_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By (19) and Remark 1,

$$\begin{aligned} \|C^g_\varphi f_n\|_{\mathcal{B}_\omega} &= \sup_{z\in\mathbb{D}} \omega(|z|)|g(z) f'_n(\varphi(z))| \\ &\le \sup_{|\varphi(z)|\le\delta} \omega(|z|)|g(z) f'_n(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \omega(|z|)|g(z) f'_n(\varphi(z))| \\ &\lesssim \sup_{|\varphi(z)|\le\delta} |f'_n(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \omega(|z|)|g(z)|G_\mu(\varphi(z))\|f_n\|_{\mathscr{Z}_\mu} \\ &\lesssim \sup_{|\varphi(z)|\le\delta} |f'_n(\varphi(z))| + \varepsilon, \end{aligned}$$

which implies that $\lim_{n\to\infty} \|C^g_\varphi f_n\|_{\mathcal{B}_\omega} = 0$. By Lemma 6, $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is compact.

(a)⇒(b). Assume that $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is compact. Let $\{z_n\} \subset \mathbb{D}$ be a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. From the proof of Theorem 2, we see that $\{k_n\}$ is bounded in $\mathscr{Z}_\mu$ and $k_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemmas 1, 2 and 6,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\omega(|z_n|)|g(z_n)|\left(\int_0^{|a_n|^2} \frac{1}{\mu(s)}ds\right)^2}{\int_0^{|a_n|} \frac{1}{\mu(s)}ds} \\
\approx\ & \lim_{n\to\infty} \omega(|z_n|)\left|(C_\varphi^g k_n)'(z_n)\right| \le \lim_{n\to\infty} \|C_\varphi^g k_n\|_{\mathcal{B}_\omega} = 0,
\end{aligned}$$

which implies $\lim_{n\to\infty} \omega(|z_n|)|g(z_n)G_\mu(\varphi(z_n))| = 0$. So $\lim_{|\varphi(z)|\to 1} \omega(|z|)|g(z)|G_\mu(\varphi(z)) = 0$.

Using Lemma 5 and Theorem 2, we see that *(a)⇔(c)*. The proof is completed. □

4 The boundness and compactness of $C_\varphi^g : \mathcal{B}_\mu \to \mathscr{Z}_\omega(\mathcal{B}_\omega)$

From Lemma 5, Theorems 1 and 2, notice that $\lim_{|z|\to 1} G_{\mu_+}(z) = \infty$, we have the following theorems.

Theorem 5. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$. Then $C_\varphi^g : \mathcal{B}_\mu \to \mathscr{Z}_\omega$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) < \infty \quad \textit{and} \qquad \sup_{z\in\mathbb{D}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} < \infty.$$

Theorem 6. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$ such that $C_\varphi^g : \mathcal{B}_\mu \to \mathscr{Z}_\omega$ is bounded. Then C_φ^g is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \omega(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) = 0 \quad \textit{and} \qquad \lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} = 0.$$

Theorem 7. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ *is bounded.*

(ii) $\sup_{z\in\mathbb{D}} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} < \infty.$

(iii) $\sup_{z\in\mathbb{D}} \omega_+(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) < \infty$ *and* $\sup_{z\in\mathbb{D}} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} < \infty.$

(iv) $\sup_{z\in\mathbb{D}} \omega(|z|)|g(z)|G_{\mu_+}(\varphi(z)) < \infty.$

Proof. (ii)⇔(i). By Lemma 3 and taking the function $f(z) = \int_0^z \mu_*(\eta)d\eta \in \mathcal{B}_\mu$, we can get the desired result. Since the proof is similar to the proof of Theorem 1, we omit the details.

By Lemma 5, Theorems 1 and 3, we see that *(i)⇔ (iii)* and *(i)⇔ (iv)* hold. The proof is completed. □

Theorem 8. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$ such that $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is bounded. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ *is compact.*

(ii) $\lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} = 0.$

(iii) $\lim_{|\varphi(z)|\to 1} \omega_+(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) = 0, \ \lim_{|\varphi(z)|\to 1} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} = 0.$

(iv) $\lim_{|\varphi(z)|\to 1} \omega(|z|)|g(z)|G_{\mu_+}(\varphi(z)) = 0.$

Proof. Since $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is bounded, we get that $\sup_{z\in\mathbb{D}} \omega(|z|)|g(z)| = ||C_\varphi^g z||_{\mathcal{B}_\omega} < \infty$.

(ii)⇒(i). Suppose $\lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} = 0$. For any $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that

$$\frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} < \varepsilon, \text{ when } \delta < |\varphi(z)| < 1.$$

Let $\{f_n\}$ be bounded in $\mathcal{B}_\mu$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. From Cauchy estimate, $f_n' \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 3,

$$\begin{aligned} ||C_\varphi^g f_n||_{\mathcal{B}_\omega} &= \sup_{z\in\mathbb{D}} \omega(|z|)|f_n'(\varphi(z))g(z)| \\ &\leq \sup_{|\varphi(z)|\leq\delta} \omega(|z|)|f_n'(\varphi(z))g(z)| + \sup_{\delta<|\varphi(z)|<1} \omega(|z|)|f_n'(\varphi(z))g(z)| \\ &\lesssim \sup_{|\varphi(z)|\leq\delta} |f_n'(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} ||f_n||_{\mathcal{B}_\mu} \\ &\lesssim \sup_{|\varphi(z)|\leq\delta} |f_n'(\varphi(z))| + \varepsilon, \end{aligned}$$

which implies that $\lim_{n\to\infty} ||C_\varphi^g f_n||_{\mathcal{B}_\omega} = 0$. By Lemma 6, $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is compact.

(i)⇒(ii). Suppose that $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is compact. Let $\{z_n\} \subset \mathbb{D}$ be a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. Let $a_n = \varphi(z_n)$ and $f_n(z) = \mu(|a_n|) \int_0^{\overline{a_n}z} \mu_*^2(\eta)d\eta$. Then $\{f_n\}$ is bounded in $\mathcal{B}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 6, $\lim_{n\to\infty} ||C_\varphi^g f_n||_{\mathcal{B}_\omega} = 0$. Therefore

$$\lim_{n\to\infty} \frac{\omega(|z_n|)|g(z_n)|}{\mu(|\varphi(z_n)|)} = \lim_{n\to\infty} \omega(|z_n|)|(C_\varphi^g f_n)'(z_n))| = 0,$$

which implies that (ii) holds.

By Lemma 5, Theorems 2 and 4, *(i)⇔(iii)* and *(i)⇔(iv)* hold. The proof is completed. □

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Convergence and error estimates for the series solutions of higher-order differential equations

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January 30, 2016

Abstract

Little work on the convergence and error estimates of approximate series solutions exists in the literature. For general nth-order linear differential equations with initial conditions, a rigorous proof of convergence for the series solutions given by the homotopy analysis method is first presented in this paper. Furthermore, an upper bound for the absolute error of these approximations is obtained.

1 Introduction

Higher-order differential equations arise in various branches of science and engineering. However, unlike numerical solutions, little work on the convergence and error estimates of the approximate series solutions to these equations can be found in the literature.

Consider general nth-order linear differential equations with initial conditions

$$\begin{cases} L[u(x)] = f(x), \\ u^{(i)}(x_0) = A_i, \quad i = 0, \cdots, n-1 \end{cases} \tag{1}$$

where

$$L := \frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_1(x)\frac{d}{dx} + p_0(x),$$

$p_i(x)$, $i = 0, 1, \cdots, n-1$ and $f(x)$ are continuous in some neighborhood $[x_0 - \delta, x_0 + \delta]$ of x_0. The main purpose of this paper is to present a rigorous proof of convergence for the series solutions given by the homotopy analysis method and to establish an upper bound for the absolute error of these approximations.

The homotopy analysis method (or HAM) [1, 2] is a popular analytic approach for seeking series solutions to differential equations and related problems. It has been applied to solve many problems in different fields of science and engineering [3, 4, 5, 6, 7, 8, 9, 10].

For the sake of easy reference, the homotopy analysis method is briefly described as follows. Given a (usually nonlinear) problem

$$\mathcal{N}[u(x)] = 0, \; x \in \Omega, \tag{2}$$

one first constructs a zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(x;q) - u_0(x)] = q\, c_0\, \mathcal{N}[\phi(x;q)], \tag{3}$$

where $\mathcal{L}$ is an auxiliary linear operator, $u_0(x)$ an initial guess satisfying the given initial/boundary conditions, and $c_0 \neq 0$ the convergence-control parameter. At $q = 0$ and $q = 1$, one has

$$\phi(x;0) = u_0(x), \; \phi(x;1) = u(x), \tag{4}$$

respectively. To seek a series solution, one expands $\phi(x;q)$ into a Taylor series at $q = 0$

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x;c_0) q^m. \tag{5}$$

Assuming that c_0 is properly chosen so that the series (5) converges at $q = 1$, then

$$u(x) = \phi(x;1) = \sum_{m=0}^{+\infty} u_m(x;c_0) \tag{6}$$

must be one of the solutions to the given problem as shown in [1], where $u_m(x;c_0)$ is governed by the mth-order deformation equation

$$\mathcal{L}[u_m(x;c_0) - \chi_m u_{m-1}(x;c_0)] = \frac{c_0}{(m-1)!} \left. \frac{\partial^{m-1}\mathcal{N}[\phi(x;q)]}{\partial q^{m-1}} \right|_{q=0}, \tag{7}$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

In practice, one can only calculate an Nth-order approximation

$$\psi_N(x;c_0) = \sum_{m=0}^{N} u_m(x, c_0) \tag{8}$$

of which the higher-order terms $u_m(x;c_0), m \geq 1$ are calculated via (7).

To obtain an accurate approximation (8), the optimal value c_0 is determined by minimizing the average residual error

$$\mathbf{E}(c_0) = \frac{1}{M} \sum_{j=1}^{M} \left(\mathcal{N}[\psi_N(x_j;c_0)] \right)^2 , \tag{9}$$

where $x_1, x_2 \cdots, x_M \in \Omega$ are sample points.

For the problem (1), a rigorous proof of convergence for the series solutions given by the HAM is presented in Section 2. Moreover, an approach is also given for determining the valid region of c_0 that ensures the convergence of (6), and for obtaining an upper bound for the absolute error of the Nth-order approximation (8). In Section 3, two examples are given to illustrate the procedure and to demonstrate how accurate and convergent series solutions can be obtained. Some concluding remarks are given in the last section.

2 Convergence and error estimates

Let $F(x)$ be continuous on $[x_0 - \delta, x_0 + \delta]$. Denote

$$\|F(\cdot)\| := \max_{x \in [x_0 - \delta, x_0 + \delta]} |F(x)| .$$

For the initial value problem (1), it is assumed for the rest of the paper that the neighborhood $[x_0 - \delta, x_0 + \delta]$ of x_0 is sufficiently small so that

$$1 - \sum_{k=1}^{n} \frac{\delta^k}{(k-1)!} \|p_{n-k}(\cdot)\| > 0. \tag{10}$$

First, one constructs the zeroth-order deformation equation (3) with an initial guess $u_0(x)$ satisfying the initial conditions in (1)

$$u^{(i)}(x_0) = A_i, \qquad i = 0, \cdots, n-1. \tag{11}$$

The linear operator and nonlinear operator are set out below

$$\mathcal{L}[\phi(x;q)] = \frac{\partial^n \phi(x;q)}{\partial x^n}, \qquad \mathcal{N}[\phi(x;q)] = L[\phi(x;q)] - f(x).$$

Following the procedure outlined in Section 1, one obtains a series solution

$$u(x;c_0) = \sum_{m=0}^{+\infty} u_m(x;c_0) \tag{12}$$

and an Nth-order approximation

$$\psi_N(x;c_0) = \sum_{m=0}^{N} u_m(x;c_0) \tag{13}$$

to the initial value problem (1). For these solutions, one has the following convergence theorem:

Theorem 1. *For* $c_0 \in \left[-\frac{2-\epsilon}{1+\sum_{k=1}^{n} \frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}, -\frac{\epsilon}{1-\sum_{k=1}^{n} \frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|} \right]$*, the series solution (12) converges to the true solution* $u(x)$ *on* $[x_0-\delta, x_0+\delta]$*, where* ϵ *is a small number satisfying* $0 < \epsilon < 1 - \sum_{k=1}^{n} \frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|$*.* $K\,\|E_N(\cdot;c_0)\|$ *is an upper bound for the absolute error of the* N*th-order approximation (13), where* $K = \frac{\delta^n}{(n-1)!\left[1-\sum_{k=1}^{n} \frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\|\right]}$ *and* $E_N(x;c_0) = \frac{u_{N+1}^{(n)}(x;c_0)}{c_0}$*.*

Proof: Let

$$u(x) = \sum_{m=0}^{N} u_m(x;c_0) + R_N(x;c_0). \tag{14}$$

Then the series (12) converges to $u(x)$ on $[x_0-\delta, x_0+\delta]$ if and only if

$$\lim_{N\to\infty} \|R_N(\cdot;c_0)\| = 0. \tag{15}$$

To achieve this goal, we substitute (14) into the differential equation in (1), which gives a differential equation satisfied by $R_N(x;c_0)$,

$$L[R_N(x;c_0)] = -S_N(x;c_0) + f(x), \tag{16}$$

where

$$S_N(x;c_0) = \sum_{m=0}^{N} L[u_m(x;c_0)].$$

Since the initial guess $u_0(x)$ satisfies the initial conditions in (1), one sets

$$u_m(x_0;c_0) = 0, u_m'(x_0;c_0) = 0, \cdots, u_m^{(n-1)}(x_0;c_0) = 0 \quad \text{for } m \geq 1. \tag{17}$$

Consequently the initial conditions for the remainder $R_N(x;c_0)$ are

$$R_N(x_0;c_0) = R_N'(x_0;c_0) = \cdots = R_N^{(n-1)}(x_0;c_0) = 0. \tag{18}$$

Next, we want to simplify $S_N(x;c_0)$ in (16). Note that the mth-order deformation equation (7) becomes

$$u_m^{(n)}(x;c_0) - \chi_m u_{m-1}^{(n)}(x;c_0) = c_0[L(u_{m-1}(x;c_0)) - (1-\chi_m)f(x)]. \tag{19}$$

Adding both sides of (19) from $m = 1$ to $N + 1$ yields

$$\frac{1}{c_0} u_{N+1}^{(n)}(x; c_0) = S_N(x; c_0) - f(x). \tag{20}$$

Consequently (16) becomes

$$L[R_N(x; c_0)] = -\frac{1}{c_0} u_{N+1}^{(n)}(x; c_0). \tag{21}$$

Our next goal is to estimate $\|R_N(\cdot; c_0)\|$. Noticing (18), one has

$$\begin{aligned} R_N(x; c_0) &= \int_{x_0}^{x} R_N'(t_1; c_0)dt_1 = \int_{x_0}^{x}\int_{x_0}^{t_2} R_N''(t_1; c_0)dt_1 dt_2 = \cdots \\ &= \int_{x_0}^{x}\int_{x_0}^{t_{n-1}} \cdots \int_{x_0}^{t_2} R_N^{(n-1)}(t_1; c_0)dt_1 \cdots dt_{n-2}dt_{n-1}, \end{aligned}$$

which implies

$$\left\|R_N^{(k)}(\cdot; c_0)\right\| \le \frac{\delta^{n-1-k}}{(n-1-k)!}\left\|R_N^{(n-1)}(\cdot; c_0)\right\|, \quad k = 0, 1, \cdots, n-2. \tag{22}$$

Using the initial condition $R_N^{(n-1)}(x_0; c_0) = 0$ and integrating (21) from x_0 to x give

$$\begin{aligned} R_N^{(n-1)}(x; c_0) = -\Bigg(&\int_{x_0}^{x} p_{n-1}(t) R_N^{(n-1)}(t; c_0)dt + \cdots + \int_{x_0}^{x} p_0(t) R_N(t; c_0)dt \\ &+\frac{1}{c_0}\int_{x_0}^{x} u_{N+1}^{(n)}(t; c_0)dt\Bigg). \end{aligned}$$

Following a similar reasoning as above, one arrives at

$$\left\|R_N^{(n-1)}(\cdot; c_0)\right\| \le \sum_{k=1}^{n} \frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\| \cdot \left\|R_N^{(n-1)}(\cdot; c_0)\right\| + \frac{\delta\left\|u_{N+1}^{(n)}(\cdot; c_0)\right\|}{|c_0|}.$$

In view of (10), one has

$$\left\|R_N^{(n-1)}(\cdot; c_0)\right\| \le \frac{\delta}{1 - \sum_{k=1}^{n}\frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\|}\frac{\left\|u_{N+1}^{(n)}(\cdot; c_0)\right\|}{|c_0|}. \tag{23}$$

Combining (22) and (23), one finally achieves

$$\|R_N(\cdot;c_0)\| \le \frac{\delta^n}{(n-1)!\left[1-\sum_{k=1}^{n}\frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\|\right]}\frac{\left\|u_{N+1}^{(n)}(\cdot;c_0)\right\|}{|c_0|}.$$

Consequently

$$\|R_N(\cdot;c_0)\| \le K\,\|E_N(\cdot;c_0)\|, \tag{24}$$

where

$$K = \frac{\delta^n}{(n-1)!\left[1-\sum_{k=1}^{n}\frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\|\right]} \quad \text{and} \quad E_N(x;c_0) = \frac{u_{N+1}^{(n)}(x;c_0)}{c_0}.$$

Therefore, we have proved that $K\,\|E_N(\cdot;c_0)\|$ is an upper bound for the absolute error of the Nth-order approximation (13) on $[x_0-\delta, x_0+\delta]$. Our next goal is to prove that, if

$$c_0 \in \left[-\frac{2-\epsilon}{1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}, -\frac{\epsilon}{1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}\right],$$

then

$$\lim_{N\to\infty}\|R_N(\cdot;c_0)\| = 0. \tag{25}$$

First we figure out an expression for $E_N(x;c_0)$. The following lemma can be proved by mathematical induction.

Lemma 2. *Noticing* $E_N(x;c_0) = \frac{u_{N+1}^{(n)}(x;c_0)}{c_0}$, *one has*

$$E_N(x;c_0) = \sum_{k=0}^{N}\binom{N}{k}a_k(x)c_0^k, \tag{26}$$

where $a_0(x) = L[u_0(x;c_0)] - f(x)$, *and for* $0 \le k \le N-1$,

$$a_{k+1}(x) = a_k(x) + p_{n-1}(x)\int_{x_0}^{x}a_k(t_1)dt_1 + p_{n-2}(x)\int_{x_0}^{x}\int_{x_0}^{t_2}a_k(t_1)dt_1dt_2 + \cdots + p_0(x)\int_{x_0}^{x}\int_{x_0}^{t_n}\cdots\int_{x_0}^{t_2}a_k(t_1)dt_1\cdots dt_{n-1}dt_n. \tag{27}$$

Based on Lemma 2, one can determine the relation between $E_N(x;c_0)$ and $E_{N+1}(x;c_0)$. In view of (17), (26) and (27), one has

$$\begin{aligned}
&E_{N+1}(x;c_0)\\
=&(1+c_0)\frac{u_{N+1}^{(n)}(x;c_0)}{c_0}+p_{n-1}(x)u_{N+1}^{(n-1)}(x;c_0)+\cdots+p_0(x)u_{N+1}(x;c_0)\\
=&(1+c_0)E_N(x;c_0)+c_0p_{n-1}(x)\int_{x_0}^{x}E_N(t_1;c_0)dt_1+\cdots\\
&+c_0p_0(x)\int_{x_0}^{x}\int_{x_0}^{t_n}\cdots\int_{x_0}^{t_2}E_N(t_1;c_0)dt_1\cdots dt_{n-1}dt_n,
\end{aligned}$$

which implies

$$\|E_{N+1}(\cdot;c_0)\|\leq\left[|1+c_0|+|c_0|\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|\right]\|E_N(\cdot;c_0)\|.$$

Next, let

$$w(\delta,c_0)=|1+c_0|+|c_0|\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|. \tag{28}$$

Case I. If $c_0\in\left[-1,-\frac{\epsilon}{1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}\right]$, then (28) becomes

$$w(\delta,c_0)=1+c_0\left(1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|\right)\leq 1-\epsilon.$$

Case II. If $c_0\in\left[-\frac{2-\epsilon}{1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|},-1\right)$, then (28) becomes

$$w(\delta,c_0)=-1-c_0\left(1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|\right)\leq 1-\epsilon.$$

One thus concludes that if

$$c_0\in\left[-\frac{2-\epsilon}{1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|},-\frac{\epsilon}{1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}\right], \tag{29}$$

then

$$w(\delta,c_0)\leq 1-\epsilon<1.$$

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Thus $E_N(x;c_0)$ is a contraction mapping

$$\|E_{N+1}(\cdot;c_0)\| \leq w(\delta,c_0)\,\|E_N(\cdot;c_0)\| \leq (1-\epsilon)\,\|E_N(\cdot;c_0)\|\,.$$

Iteration then yields

$$\|E_N(\cdot;c_0)\| \leq (1-\epsilon)^N\,\|E_0(\cdot;c_0)\|\,,$$

Consequently

$$\lim_{N\to\infty}\|E_N(\cdot;c_0)\| = 0 \quad \text{and} \quad \lim_{N\to\infty}\|R_N(\cdot;c_0)\| = 0,$$

and the theorem has finally been proved. ♠

3 Examples

In this section, we apply the approach to investigate some initial value problems.

3.1 Buckling of a cantilever bar

A cantilever bar of length l is free at the upper end and is built-in at the bottom. The axial load P is supposed to be uniformly distributed along the bar axis. The deflection w satisfies the differential equation

$$\frac{d^3w}{dx^3} + \frac{P}{EI}(l-x)\frac{dw}{dx} = 0, \tag{30}$$

where EI is the bending rigidity. The initial conditions are

$$w(0) = 0, \qquad \left.\frac{dw}{dx}\right|_{x=0} = 0, \qquad \left.\frac{d^2w}{dx^2}\right|_{x=0} = 1. \tag{31}$$

A simple way to reduce the order of the equation is to set $u(x) = dw/dx$, but a more interesting way is as follows.

We make a change of variable

$$z = \frac{2}{3}\sqrt{\frac{P}{EI}}(l-x)^{\frac{3}{2}}. \tag{32}$$

Table 1: Upper bound for the absolute error of the Nth-order approximation.

N	2nd order	3rd order	4th order	5th order	6th order
$K \|E_N(\cdot;-1)\|$	1.190E-2	1.146E-3	8.188E-5	5.794E-6	3.800E-7

Step by step differentiation yields

$$\begin{aligned} \frac{dw}{dx} &= -\frac{dw}{dz}\sqrt[3]{\frac{3P}{2EI}}z, \\ \frac{d^2w}{dx^2} &= \left(\frac{3P}{2EI}\right)^{\frac{2}{3}}\left(\frac{1}{3}z^{-\frac{1}{3}}\frac{dw}{dz} + z^{\frac{2}{3}}\frac{d^2w}{dz^2}\right), \\ \frac{d^3w}{dx^3} &= \frac{3P}{2EI}\left(\frac{1}{9}z^{-1}\frac{dw}{dz} - \frac{d^2w}{dz^2} - z\frac{d^3w}{dz^3}\right). \end{aligned} \tag{33}$$

Introducing this in (30) and using the notation

$$\frac{dw}{dz} = u, \tag{34}$$

we get

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{1}{9z^2}\right)u = 0. \tag{35}$$

It turns out that (35) is a differential equation of Bessel type. For computational purposes, one sets $l = 1, EI = 1$, and $P = \left(\frac{1000\pi}{1122}\right)^2$, the buckling critical load (see [11]). Then the initial conditions become

$$u(z_0) = 0, \quad u'(z_0) = \frac{1}{P}, \tag{36}$$

where $z_0 = \frac{1000\pi}{1683}$. Now let us solve the initial value problem (35)-(36).

Following the procedure outlined at the beginning of Section 2, one obtains a series solution (12) and an Nth-order approximation (13) to the problem (35)-(36).

The radius $\delta = \frac{3}{5}$ of the neighborhood of $z_0 = \frac{1000\pi}{1683}$ is determined by the condition (10). Then a valid region $[-1.211828346 + 0.6059141731\,\epsilon, -2.860401756\,\epsilon]$ of c_0 is obtained by means of (29), where $0 < \epsilon < 0.3496012397$.

It follows from Theorem 1 that, for $c_0 \in [-1.211828346 + 0.6059141731\,\epsilon, -2.860401756\,\epsilon]$, the series solution (12) converges on $\left[\frac{1000\pi}{1683} - \frac{3}{5}, \frac{1000\pi}{1683} + \frac{3}{5}\right]$.

To get an accurate approximation, the optimal value of c_0 is determined by minimizing the averaged residual error (9) of the 5th-order approximation (13). It turns out that $c_0 = -1$. Notice that $-1 \in [-1.211828346 +$

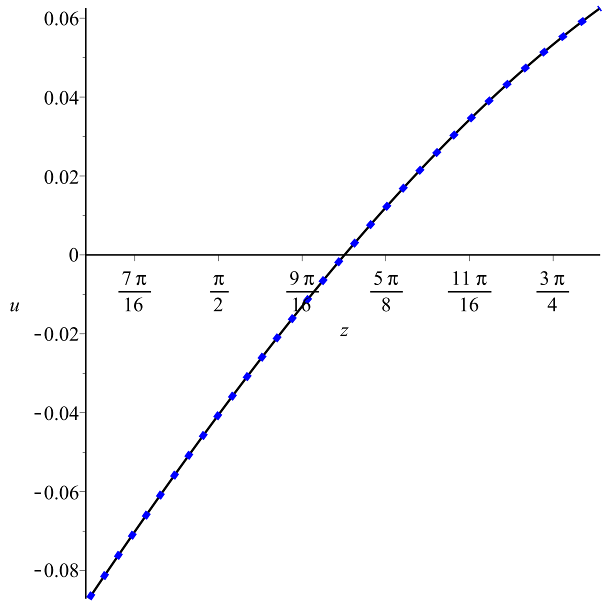

Figure 1: The solid line: numerical solution; the dot line: the 5th-order HAM approximation.

$0.6059141731\,\epsilon, -2.860401756\,\epsilon]$. So the corresponding series solution (12) does converge.

The upper bounds $K\,\|E_N(\cdot;-1)\|$ for the absolute error of the Nth-order approximation (13) when $N = 2, 3, 4, 5,$ and 6 on $\left[\frac{1000\pi}{1683} - \frac{3}{5}, \frac{1000\pi}{1683} + \frac{3}{5}\right]$ are calculated, as shown in Table 1. It is very accurate as shown in Figure 1.

3.2 Third-order equation with variable coefficients

Consider the third-order initial value problem (see [12])

$$\begin{aligned} &u^{(3)}(x) + xu''(x) + x^{\frac{2}{3}}u'(x) + x^{\frac{1}{3}}u(x) = 0, \\ &u(1) = 1,\ u'(1) = 0,\ u''(1) = 1. \end{aligned} \tag{37}$$

This initial value problem does not have a closed-form solution.

Following the procedure outlined at the beginning of Section 2, one obtains a series solution (12) and an Nth-order approximation (13) to the problem (37).

The radius $\delta = \frac{9}{20}$ of the neighborhood of $x_0 = 1$ is determined by the condition (10). Then a valid region $[-1.111481567+0.5557407833\,\epsilon, -4.985046404\,\epsilon]$ of c_0 is obtained by means of (29), where $0 < \epsilon < 0.2005999381$.

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Table 2: Upper bound for the absolute error of the Nth-order approximation.

N	5th order	10th order	15th order	20th order
$K\,\|E_N(\cdot;-1)\|$	4.000E-4	1.116E-8	1.347E-13	1.017E-18

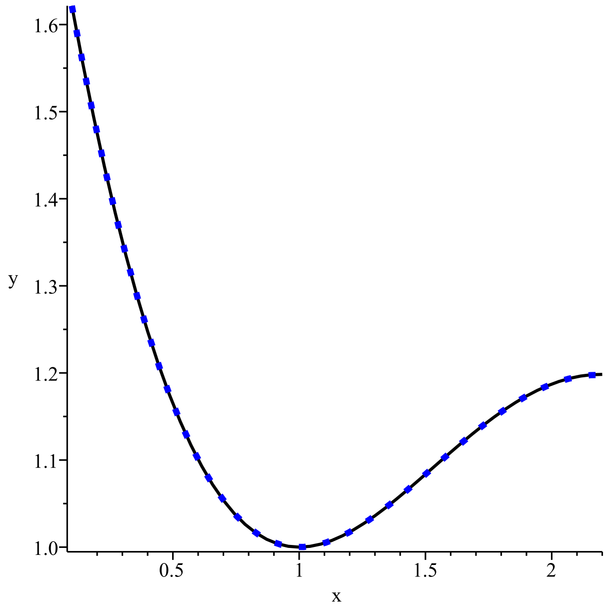

Figure 2: The solid line: numerical solution; the dot line: the 5th-order HAM approximation.

It follows from Theorem 1 that, for $c_0 \in [-1.111481567 + 0.5557407833\,\epsilon, -4.985046404\,\epsilon]$, the series solution (12) converges on $\left[\frac{11}{20}, \frac{29}{20}\right]$.

To get an accurate approximation, the optimal value of c_0 is determined by minimizing the averaged residual error (9) of the 5th-order approximation (13). It turns out that $c_0 = -1$. Notice that $-1 \in [-1.111481567 + 0.5557407833\,\epsilon, -4.985046404\,\epsilon]$. So the corresponding series solution (12) does converge.

The upper bounds $K\,\|E_N(\cdot;-1)\|$ for the absolute error of the Nth-order approximation (13) when $N = 5, 10, 15$ and 20 on $\left[\frac{11}{20}, \frac{29}{20}\right]$ are calculated, as shown in Table 2. It is very accurate as shown in Figure 2.

4 Conclusion

For general nth-order linear differential equations with initial conditions, we have presented a rigorous proof of convergence for the series solutions given

by the HAM for the first time. Furthermore, we have proposed an approach for seeking convergent series solutions to these problems. Some outstanding features of the approach include the determination of a valid region of the convergence-control parameter for ensuring convergence, and the calculation of an upper bound for the absolute error of an approximation.

Some issues still deserve further investigations. For example, how can one enlarge the neighborhood $[x_0 - \delta, x_0 + \delta]$ of x_0 so that Theorem 1 is still valid? How can one extend the techniques to investigate the convergence of nonlinear problems? It is believed that substantial work has to be done for dealing with these issues.

Acknowledgements The research was partially supported by the National Natural Science Foundation of China (51379033,51522902).

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ON THE GENERALIZED Z-ALGORITHM FOR THE NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. For any d-dimensional neutral stochastic functional differential equation with infinite delay and m-dimensional Brownian motion, we introduce a sequence of approximate equations and offer sufficient conditions such that the approximate solutions converge with probability one to the solution of the given equation. This iterative method called the generalized Z-algorithm is a generalization to many well-known analytic iterative method.

1. Introduction

The neutral stochastic functional differential equation, abbreviated as NSFDE, which was introduced by Kolmanovskii and Nosov [5], has been received much more attention in recent years. The existence and uniqueness theorems of the solution to the NSFDE with finite delay can be seen in Mao's books [6]. Recently, the existence of the solution to NSFDEs with infinite delay has been established in [1, 8, 9] by the classic Picard iteration argument. In 2010, S. Janković, M. Vasilova and M. Krstić [4] utilized successfully a general analytic method called the Z-algorithm to verify the existence of the solutions to NSFDEs with finite delay.

Actually, the Z-algorithm method could be backed to works [10, 11] in which Zuber posed an analytic iterative method for solving the Cauchy problem of the ordinary differential equation $X' = f(t, X)$ with $X(t_0) = X_0$. In fact, Zuber considered the approximate equations $X'_{n+1} = f_n(t; X_{n+1})$ with $X_{n+1}(t_0) = X_0$ for $n = 0, 1, \cdots$. It was showed in [10] that if $\sum_{n=1}^{\infty} \sup_{|t-t_0|<\varepsilon} |f(t, X_n(t)) - f_n(t, X_n(t))| < 1$, then the sequence of the solutions $\{X_n\}$ converges to the solution X of the initial equation, uniformly in an interval around t_0. Here we remark that, by the advantages of the Z-algorithm method, if we choose the functions $\{f_n\}$ good enough so that the approximate equations can be effectively solved, then the solution X of the initial equation can be effectively approximated. Later, Janković

Date: Jan 18, 2016.

Keywords. Neutral stochastic functional differential system; Infinite delay; Approximate solution; Z-algorithm.

This work was supported in part by the National Natural Science Foundation of China under grant #11171306 and #11071065, and sponsored by the Scientific Project of Zhejiang Provincial Science Technology Department under grant #2011C33012.

*Key words and phrases.*Neutral stochastic functional differential system; Infinite delay; Approximate solution; Z-algorithm.

[2, 3] applied an analogous analytic method to the following stochastic differential equation of the Itô type,

$$dX(t) = a(t,X(t))dt + b(t,X(t))dB(t), \quad t \in [0,T] \tag{1.1}$$
$$X(0) = X_0, \tag{1.2}$$

by comparing its solution with the solutions to the related equations

$$dX_{n+1}(t) = a_n(t,X_{n+1}(t))dt + b_n(t,X_{n+1}(t))dB(t), \quad t \in [0,T] \tag{1.3}$$
$$X_{n+1}(0) = X_0, \tag{1.4}$$

for $n = 0, 1, ...$ and some suitable functions $\{a_n\}$ and $\{b_n\}$. It was shown in [2] that, if

$$\sum_{n=1}^{\infty} \sup_{t,X}\{|a(t,X) - a_n(t,X)| + |b(t,X) - b_n(t,X)|\} < \infty,$$

then the solutions $X_{n+1}(t)$ of (1.3) converge to the solution $X(t)$ of (1.1) a.s. uniformly in $[0,T]$ as $n \to \infty$.

Motivated by the works mentioned above, we will discuss in this paper the existence of the Z-algorithm approximate solutions to NSFDEs with infinite delay. Our main theorem will be stated and shown in the next section, see Theorem 2.4; and some comments will be given in section 3. Here we need to give the notations and definitions which will be used in the paper.

Let $|\cdot|$ denote the Euclidean norm in $\mathbf{R}^d$. If A is a vector or a matrix, its transpose is denoted by A^T; The trace norm of a matrix A is represented by $|A| = \sqrt{trace(A^T A)}$. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq t_0}$ satisfying the usual conditions, i.e., it is right continuous and $\mathscr{F}_{t_0}$ contains all P-null sets. Assume that $B(t)$ is a m-dimensional Brownian motion defined on $(\Omega, \mathscr{F}, P)$, that is $B(t) = \big(B_1(t), B_2(t), ..., B_m(t)\big)^T$ and each $B_i(t)$ is a standard Brownian motion for $i = 1, 2, \cdots, n$. Let $BC((-\infty, 0]; \mathbf{R}^d)$ denote the family of bounded continuous $\mathbf{R}^d$-value functions φ defined on $(-\infty, 0]$ with the norm $||\varphi|| = \sup_{-\infty<\theta\leq 0} |\varphi(\theta)|$.

In this paper, we consider the following d-dimensional NSFDEs with infinite delay,

$$d\left[X(t) - D(X_t)\right] = f(t,X_t)dt + g(t,X_t)dB(t), \quad t_0 \leq t \leq T, \tag{1.5}$$

where

$$X_t = \big\{X(t+\theta) : -\infty < \theta \leq 0\big\}$$

can be regarded as a $BC((-\infty,0];\mathbf{R}^d)$-valued stochastic process; and we assume that D is a vector-value function from $BC((-\infty,0];\mathbf{R}^d)$ to $\mathbf{R}^d$, and f is a Borel measurable function from $[t_0,T] \times BC((-\infty,0];\mathbf{R}^d)$ to $\mathbf{R}^d$, and g is a matrix-value Borel measurable function from $[t_0,T] \times BC((-\infty,0];\mathbf{R}^d)$ to $\mathbf{R}^{md}$. Here one notes that the last term, $g(t,X_t)dB(t)$, in equation (1.5) can be rewritten as

$$g_1(t,X_t)dB_1(t) + g_2(t,X_t)dB_2(t) + \cdots + g_m(t,X_t)dB_m(t).$$

The initial value is assumed to be

$$X_{t_0} = \{\xi(\theta) : -\infty < \theta \leq 0\} \tag{1.6}$$

where $\xi(\theta)$ is an $\mathscr{F}_{t_0}$-measurable $BC((-\infty,0];\mathbf{R}^d)$-value random variable such that $\mathbf{E}||\xi||^2 < \infty$.

Definition 1.1. $\mathbf{R}^d$-value stochastic process $X(t)$ defined on $(-\infty, T]$ is called the solution of (1.5) with initial value (1.6), if $X(t)$ has the following properties:

(1) $X(t)$ is continuous and $\{X(t) : -\infty \leq t \leq T\}$ is $\mathscr{F}_t$-adapted;
(2) $f(t, X_t) \in \mathscr{L}^1([t_0, T]; \mathbf{R}^d)$ and $g(t, X_t) \in \mathscr{L}^2([t_0, T]; \mathbf{R}^{d\times m})$;
(3) $X_{t_0} = \xi$, and for each $t_0 \leq t \leq T$,

$$X(t) = \xi(0) + D(X_t) - D(\xi) + \int_{t_0}^{t} f(s, X_s)ds + \int_{t_0}^{t} g(s, X_s)dB(s), \quad a.s. \tag{1.7}$$

We say that the solution of (1.5) with initial value (1.6), $X(t)$, is the unique solution, if for any other solution $\bar{X}(t)$ distinguishable with $X(t)$ we have

$$P\left\{X(t) = \bar{X}(t) : -\infty < t \leq T\right\} = 1$$

Here we remarked that the existence and uniqueness of the solution to the equation (1.5) were established by the well-known Picard iteration in [1, 8, 9] under the following assumptions:

(M1) There exists a positive constant K such that, for all $\varphi, \psi \in BC((-\infty, 0]; \mathbf{R}^d)$ and $t \in [t_0, T]$,

$$|f(t, \varphi) - f(t, \psi)| + |g(t, \varphi) - g(t, \psi)| \leq K||\varphi - \psi||$$

(M2) There exists a positive constant $\bar{K}$ such that, for all $\varphi \in BC((-\infty, 0]; \mathbf{R}^d)$ and $t \in [t_0, T]$,

$$|f(t, \varphi)| + |g(t, \varphi)| \leq \bar{K}(1 + ||\varphi||)$$

(M3) There exists a constant $k \in (0, 1)$ such that, for all $\varphi, \psi \in BC((-\infty, 0]; \mathbf{R}^d)$,

$$|D(\varphi) - D(\psi)| \leq k||\varphi - \psi||$$

Recall the stochastic integral equation (1.7), we introduce the following sequence of the related equations:

$$\begin{aligned} X^{n+1}(t) - D_n(X_t^{n+1}) &= \xi^{n+1}(0) - D_n(\xi^{n+1}) + \int_{t_0}^{t} f_n(s, X_s^{n+1})ds \\ &\quad + \int_{t_0}^{t} g_n(s, X_s^{n+1})dB(s) \end{aligned} \tag{1.8}$$

for $t \in [t_0, T]$ and $n = 0, 1, \cdots$, where $X_t^{n+1} = \{X^{n+1}(t + \theta) : -\infty < \theta \leq 0\}$ are $BC((-\infty, 0]; \mathbf{R}^d)$-value stochastic processes, $\xi^{n+1} = X_{t_0}^{n+1}$ are the initial conditions. The functions D_n, f_n, g_n will be chosen late such that

$$D_n(X) \to D(X),\ f_n(t, X) \to f(t, X),\ g_n(t, X) \to g(t, X),\ \text{as } n \to \infty,$$

uniformly in $[t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d)$.

Let $X(t)$ be the unique $\mathscr{F}_t$-adapted solution to the equation (1.7) satisfying $\mathbf{E}\sup_{t\in(-\infty,T]} |X(t)|^p < \infty$. Let $X^{n+1}(t)$ be the unique $\mathscr{F}_t$-adapted solution to the equation (1.8) satisfying $\mathbf{E}\sup_{t\in(-\infty,T]} |X^{n+1}(t)|^p < \infty$ for $n = 0, 1, \cdots$.

We will use the following notations in this paper,

$$\begin{aligned} \gamma_n &= \mathbf{E}||\xi - \xi^n||^p, \\ \delta_n &= \mathbf{E}\sup_{t\in[t_0,T]} |D(X_t^n) - D_n(X_t^n)|^p, \\ \varepsilon_n &= \mathbf{E}\sup_{t\in[t_0,T]} \left[|f(t, X_t^n) - f_n(t, X_t^n)|^p + |g(t, X_t^n) - g_n(t, X_t^n)|^p\right]. \end{aligned}$$

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We take the initial iteration to be $X^0(t) = \xi(0)$ a.s., for $t \in [t_0, T]$, and $X^0_{t_0} = \xi$. We will use the following conditions:

$$\sum_{n=0}^{\infty} \gamma_n < \infty, \quad \sum_{n=0}^{\infty} \delta_n < \infty, \quad \sum_{n=0}^{\infty} \varepsilon_n < \infty \tag{1.9}$$

2. The main theorem and its proof

At first we give three lemmas, which are useful for our investigation.

Lemma 2.1. *For any $X, Y \in \mathbf{R}^d$ and $\theta \in (0,1)$, we have*

$$|X+Y|^p \leq \frac{|X|^p}{(1-\theta)^{p-1}} + \frac{|Y|^p}{\theta^{p-1}}, \quad p \geq 1$$

The proof of Lemma 2.1 can be found in [6]

Lemma 2.2. *Let $u, v : [a,b] \to \mathbf{R}_+$ be continuous functions and L be a positive constant, if*

$$u(t) \leq v(t) + L \int_a^t u(s)ds,$$

then for all $t \in [a,b]$ we have

$$u(t) \leq v(t) + L \int_a^t e^{L(t-s)} v(s)ds.$$

Especially, if $v(t) = M$ is a constant, then $u(t) \leq Me^{L(t-a)}$.

Lemma 2.2 is a special case of the Gronwall lemma, so we omit its proof here.

Lemma 2.3. *Let $p \geq 2$, $t \in [t_0, T]$, and let $X_t, X^n_t \in BC((-\infty, 0]; \mathbf{R}^d)$ be the stochastic process mentioned above, then for any $r \in [t_0, t]$ we have*

$$||X_r - X^n_r||^p \leq ||\xi - \xi^n||^p + \sup_{u \in [t_0, r]} |X(u) - X^n(u)|^p.$$

Proof. From the definition of the norm in $BC((-\infty, 0]; \mathbf{R}^d)$, we have

$$\begin{aligned}
||X_r - X^n_r||^p &= \sup_{\theta \in (-\infty, 0]} |X(r+\theta) - X^n(r+\theta)|^p \\
&= \sup_{u \in (-\infty, r]} |X(u) - X^n(u)|^p \\
&\leq \sup_{u \in (-\infty, t_0]} |X(u) - X^n(u)|^p + \sup_{u \in [t_0, r]} |X(u) - X^n(u)|^p \\
&= ||\xi - \xi^n||^p + \sup_{u \in [t_0, r]} |X(u) - X^n(u)|^p.
\end{aligned}$$

The lemma is proved. □

Now we state our main theorem.

Theorem 2.4. *Let $p \geq 2$ and $\mathbf{E}||\xi||^p < \infty$, $\mathbf{E}||\xi^n||^p < \infty$, $n = 0, 1, ...$ Assume that the functions D, f, g, D_n, f_n, g_n satisfy the Lipschitz conditions (M1) and (M3) with constants $K > 0$ and $k \in \left(0, 1/(3 \cdot 2^{4(p-1)})^{\frac{1}{p}}\right)$ for any $n = 0, 1, ...$ If the condition (1.9) is vilid. Then, the sequence of solutions $\left\{X^{n+1}(t) : t \in (-\infty, T], n = 0, 1, ...\right\}$ of the equations (1.8) converges uniformly in $[t_0, T]$, with probability one as $n \to \infty$, to the solution $\left\{X(t), t \in (-\infty, T]\right\}$ of equation (1.7).*

Proof. From (1.7) and (1.8), for all $r \in [t_0, T]$, we have

$$\begin{aligned} X(r) - X^{n+1}(r) &= \xi(0) - \xi^{n+1}(0) - D(\xi) + D_n(\xi^{n+1}) \\ &\quad + D(X_r) - D_n(X_r^{n+1}) \\ &\quad + \int_{t_0}^{r} [f(s, X_s) - f_n(s, X_s^{n+1})] ds \\ &\quad + \int_{t_0}^{r} [g(s, X_s) - g_n(s, X_s^{n+1})] dB(s). \end{aligned} \tag{2.1}$$

Since $k \in \left(0, 1/(3 \cdot 2^{4(p-1)})^{\frac{1}{p}}\right)$ and $p > 1$, we can choose θ such that

$$(3k^p)^{\frac{1}{4(p-1)}} \leq \theta < \frac{1}{2}. \tag{2.2}$$

Hence, for $t_0 \leq r \leq t \leq T$, we see from Lemma 2.1 that

$$\mathbf{E} \sup_{r \in [t_0, t]} |X(r) - X^{n+1}(r)|^p \leq \frac{I_1}{(1-\theta)^{p-1}} + \frac{J_1(t)}{\theta^{2(p-1)}} + \frac{J_2(t)}{(\theta(1-\theta))^{p-1}}, \tag{2.3}$$

where

$$\begin{aligned} I_1 &= \mathbf{E}|\xi(0) - \xi^{n+1}(0) - D(\xi) + D_n(\xi^{n+1})|^p \\ J_1(t) &= \mathbf{E} \sup_{r \in [t_0, t]} \left|D(X_r) - D_n(X_r^{n+1})\right|^p \\ J_2(t) &= \mathbf{E} \sup_{r \in [t_0, t]} \left| \int_{t_0}^{r} \left[f(s, X_s) - f_n(s, X_s^{n+1})\right] ds \right. \\ &\qquad \left. + \int_{t_0}^{r} \left[g(s, X_s) - g_n(s, X_s^{n+1})\right] dB(s) \right|^p \end{aligned}$$

Next we will give the estimates for I_1, J_1 and J_2, respectively. According to the condition (M3), we can deduce that

$$\begin{aligned} I_1 &\leq 2^{p-1}[\mathbf{E}|\xi(0) - \xi^{n+1}(0)|^p + \mathbf{E}|D(\xi) - D_n(\xi^{n+1})|^p] \\ &\leq 2^{p-1}\gamma_{n+1} + 8^{p-1}\mathbf{E}\big[|D(\xi) - D(\xi^n)|^p + |D(\xi^n) - D_n(\xi^n)|^p \\ &\qquad + |D_n(\xi^n) - D_n(\xi)|^p + |D_n(\xi) - D_n(\xi^{n+1})|^p\big] \\ &\leq 2^{p-1}\gamma_{n+1} + 8^{p-1}k^p\mathbf{E}||\xi - \xi^n||^p + 8^{p-1}\delta_n + 8^{p-1}k^p\mathbf{E}||\xi - \xi^n||^p \\ &\qquad + 8^{p-1}k^p\mathbf{E}||\xi - \xi^{n+1}||^p \\ &= 2^{p-1}\gamma_{n+1} + 8^{p-1}(2k^p\gamma_n + k^p\gamma_{n+1} + \delta_n). \end{aligned} \tag{2.4}$$

For $J_1(t)$, from Lemma 2.1 and the condition (M3), we get

$$\begin{aligned} &|D(X_r) - D_n(X_r^{n+1})|^p \\ &\leq \frac{1}{(1-\theta)^{p-1}}\big[\frac{1}{(1-\theta)^{p-1}}|D(X_r) - D(X_r^n)|^p + \frac{1}{\theta^{p-1}}|D(X_r^n) - D_n(X_r^n)|^p\big] \\ &\quad + \frac{1}{\theta^{p-1}}\big[\frac{1}{(1-\theta)^{p-1}}|D_n(X_r^n) - D_n(X_r)|^p + \frac{1}{\theta^{p-1}}|D_n(X_r) - D_n(X_r^{n+1})|^p\big] \\ &\leq k^p\big[\frac{1}{(1-\theta)^{2(p-1)}} + \frac{1}{(\theta(1-\theta))^{p-1}}\big]||X_r - X_r^n||^p + \frac{\delta_n}{(\theta(1-\theta))^{p-1}} \\ &\quad + \frac{k^p}{\theta^{2(p-1)}}||X_r - X_r^{n+1}||^p. \end{aligned}$$

Lemma 2.3 yields that

$$\mathbf{E}\sup_{r\in[t_0,t]}||X_r-X_r^n||^p\le\gamma_n+\mathbf{E}\sup_{r\in[t_0,t]}|X(r)-X^n(r)|^p$$

Therefore,

$$\begin{aligned}J_1(t)\quad&\le\quad k^p\Big[\frac{1}{(1-\theta)^{2(p-1)}}+\frac{1}{(\theta(1-\theta))^{p-1}}\Big]\Big[\gamma_n+\mathbf{E}\sup_{r\in[t_0,t]}|X(r)-X^n(r)|^p\Big]\\&+\quad\frac{k^p}{\theta^{2(p-1)}}\Big[\gamma_{n+1}+\mathbf{E}\sup_{r\in[t_0,t]}|X(r)-X^{n+1}(r)|^p\Big]+\frac{\delta_n}{(\theta(1-\theta))^{p-1}}.\end{aligned}\tag{2.5}$$

For $J_2(t)$, we can decompose it into two party

$$\begin{aligned}J_2(t)\quad&\le\quad 2^{p-1}\left[\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^r[f(s,X_s)-f_n(s,X_s^{n+1})]ds\Big|^p\right.\\&\qquad\left.+\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^r[g(s,X_s)-g_n(s,X_s^{n+1})]dB(s)\Big|^p\right]\\&=:\quad 2^{p-1}[J_{21}(t)+J_{22}(t)].\end{aligned}$$

To estimate $J_{21}(t)$, using the Hölder inequality and the condition (M1), we get that

$$\begin{aligned}&J_{21}(t)\\&\le 4^{p-1}\mathbf{E}\left[\sup_{r\in[t_0,t]}\Big|\int_{t_0}^r[f(s,X_s)-f(s,X_s^n)]ds\Big|^p+\sup_{r\in[t_0,t]}\Big|\int_{t_0}^r[f(s,X_s^n)-f_n(s,X_s^n)]ds\Big|^p\right.\\&\left.+\sup_{r\in[t_0,t]}\Big|\int_{t_0}^r[f_n(s,X_s^n)-f_n(s,X_s)]ds\Big|^p+\sup_{r\in[t_0,t]}\Big|\int_{t_0}^r[f_n(s,X_s)-f_n(s,X_s^{n+1})]ds\Big|^p\right]\\&\le 4^{p-1}\Big[K^p(t-t_0)^{p-1}\int_{t_0}^t\mathbf{E}||X_s-X_s^n||^pds+(t-t_0)^p\varepsilon_n\\&\quad+K^p(t-t_0)^{p-1}\int_{t_0}^t\mathbf{E}||X_s-X_s^n||^pds+K^p(t-t_0)^{p-1}\int_{t_0}^t\mathbf{E}||X_s-X_s^{n+1}||^pds\Big]\\&\le 4^{p-1}(T-t_0)^{p-1}\left[2K^p\int_{t_0}^t\mathbf{E}||X_s-X_s^n||^pds\right.\\&\qquad\qquad\left.+K^p\int_{t_0}^t\mathbf{E}||X_s-X_s^{n+1}||^pds+(t-t_0)\varepsilon_n\right].\end{aligned}$$

Hence by lemma 2.3 we obtain that

$$\begin{aligned}J_{21}(t)\quad&\le\quad 4^{p-1}(T-t_0)^{p-1}\Big[2K^p\int_{t_0}^t\big(\gamma_n+\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^p\big)ds\\&\qquad+K^p\int_{t_0}^t\big(\gamma_{n+1}+\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^p\big)ds+(t-t_0)\varepsilon_n\Big]\\&=\quad 4^{p-1}(T-t_0)^{p-1}\Big[2K^p\int_{t_0}^t\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^pds\\&\qquad+K^p\int_{t_0}^t\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^pds+(t-t_0)(2K^p\gamma_n+K^p\gamma_{n+1}+\varepsilon_n)\Big].\end{aligned}$$

In order to estimate $J_{22}(t)$, we use Cauchy inequality to get that

$$\begin{aligned} J_{22}(t) \quad \le \quad & 4^{p-1}\Big[\mathbf{E}\sup_{r\in[t_0,t]}\big|\int_{t_0}^{r}[g(s,X_s)-g(s,X_s^n)]dB(s)\big|^p \\ & +\mathbf{E}\sup_{r\in[t_0,t]}\big|\int_{t_0}^{r}[g(s,X_s^n)-g_n(s,X_s^n)]dB(s)\big|^p \\ & +\mathbf{E}\sup_{r\in[t_0,t]}\big|\int_{t_0}^{r}[g_n(s,X_s^n)-g_n(s,X_s)]dB(s)\big|^p \\ & +\mathbf{E}\sup_{r\in[t_0,t]}\big|\int_{t_0}^{r}[g_n(s,X_s)-g_n(s,X_s^{n+1})]dB(s)\big|^p\Big] \end{aligned}$$

Applying the Burkhölder-Davis-Gundy inequality and the Hölder inequality to the first Itô integral, we obtain that

$$\begin{aligned} & \mathbf{E}\sup_{r\in[t_0,t]}\big|\int_{t_0}^{r}[g(s,X_s)-g(s,X_s^n)]dB(s)\big|^p \le c_p\mathbf{E}\big[\int_{t_0}^{t}|g(s,X_s)-g(s,X_s^n)|^2ds\big]^{\frac{p}{2}} \\ & \le c_p(t-t_0)^{\frac{p}{2}-1}\int_{t_0}^{t}\mathbf{E}\big|g(s,X_s)-g(s,X_s^n)\big|^pds \end{aligned}$$

where c_p is a universal constant, more precisely, $c_p=[p^{p+1}/2(p-1)^{p-1}]^{\frac{p}{2}}$ for $p>2$ and $c_p=4$ for $p=2$. The other Itô integrals can be estimated analogously, Thus according to Lemma 2.3 and the condition (M1), we get that

$$\begin{aligned} & J_{22}(t) \\ & \le 4^{p-1}c_p(t-t_0)^{\frac{p}{2}-1}\Big[\int_{t_0}^{t}\mathbf{E}|g(s,X_s)-g(s,X_s^n)|^pds+\int_{t_0}^{t}\mathbf{E}|g(s,X_s^n)-g_n(s,X_s^n)|^pds \\ & \quad +\int_{t_0}^{t}\mathbf{E}|g_n(s,X_s^n)-g_n(s,X_s)|^pds+\int_{t_0}^{t}\mathbf{E}|g_n(s,X_s)-g_n(s,X_s^{n+1})|^pds\Big] \\ & \le 4^{p-1}c_p(t-t_0)^{\frac{p}{2}-1}\Big[2K^p\int_{t_0}^{t}\mathbf{E}||X_s-X_s^n||^pds+K^p\int_{t_0}^{t}\mathbf{E}||X_s-X_s^{n+1}||^pds+(t-t_0)\varepsilon_n\Big] \\ & \le 4^{p-1}c_p(T-t_0)^{\frac{p}{2}-1}\Big[2K^p\int_{t_0}^{t}(\gamma_n+\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^p)ds \\ & \quad +K^p\int_{t_0}^{t}(\gamma_{n+1}+\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^p)ds+(t-t_0)\varepsilon_n\Big] \\ & = 4^{p-1}c_p(T-t_0)^{\frac{p}{2}-1}\Big[2K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^pds \\ & \quad +K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^pds+(t-t_0)(2K^p\gamma_n+K^p\gamma_{n+1}+\varepsilon_n)\Big] \end{aligned}$$

Therefore,

$$\begin{aligned} J_2(t) \le \quad & C\Bigg[2K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^pds \\ & +K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^pds+(t-t_0)(2K^p\gamma_n+K^p\gamma_{n+1}+\varepsilon_n)\Bigg] \end{aligned} \tag{2.6}$$

with the positive constant $C = 8^{p-1}\Big[(T-t_0)^{p-1} + c_p(T-t_0)^{\frac{p}{2}-1}\Big]$.

Introduce the following notations

$$u_n(t) = \mathbf{E}\sup_{r\in[t_0,t]}|X(r)-X^n(r)|^p, \quad t\in[t_0,T], \quad n=0,1,\cdots$$

Then from the inequalities (2.4), (2.5) and (2.6), we can easily obtain

$$u_{n+1}(t) \le \frac{k^p}{\theta^{4(p-1)}}u_{n+1}(t)+\bar\alpha\int_{t_0}^t u_{n+1}(s)ds+2\bar\alpha\int_{t_0}^t u_n(s)ds+\bar\beta u_n(t)+(t-t_0)\mu_n+\nu_n$$

where $\bar\beta = \frac{k^p}{\theta^{2(p-1)}}[\frac{1}{(1-\theta)^{2(p-1)}} + \frac{1}{(\theta(1-\theta))^{p-1}}]$, $\bar\alpha$ is a positive constant, and $\mu_n = a_1\gamma_n + a_2\gamma_{n+1} + a_3\varepsilon_n$, $\nu_n = b_1\gamma_n + b_2\gamma_{n+1} + b_3\delta_n$, here a_i, b_i (i=1,2,3) are generic positive constants. Recall the inequality (2.2) we have $k^p < 3k^p \le \theta^{4(p-1)}$ and so $1-\frac{k^p}{\theta^{4(p-1)}} > 0$, then one can see that

$$u_{n+1}(t) \le \alpha\int_{t_0}^t u_{n+1}(s)ds + 2\alpha\int_{t_0}^t u_n(s)ds + \beta u_n(t) + \lambda(t-t_0)\mu_n + \lambda\nu_n$$

where α, λ are generic positive constants and $\beta = \frac{k^p}{\theta^{4(p-1)}-k^p}[(\frac{\theta}{1-\theta})^{2(p-1)}+(\frac{\theta}{1-\theta})^{(p-1)}]$. This and Lemma 2.2 yield that

$$\begin{aligned}
u_{n+1}(t) &\le 2\alpha\int_{t_0}^t u_n(s)ds + \beta u_n(t) + \lambda(t-t_0)\mu_n + \lambda\nu_n \\
&\quad +\alpha\int_{t_0}^t e^{\alpha(t-s)}\Big[2\alpha\int_{t_0}^s u_n(r)dr + \beta u_n(s) + \lambda(s-t_0)\mu_n + \lambda\nu_n\Big]ds \\
&= 2\alpha\int_{t_0}^t u_n(s)ds + 2\alpha^2\int_{t_0}^t e^{\alpha(t-s)}\int_{t_0}^s u_n(r)drds + \alpha\beta\int_{t_0}^t e^{\alpha(t-s)}u_n(s)ds \\
&\quad +\alpha\lambda\mu_n\int_{t_0}^t (s-t_0)e^{\alpha(t-s)}ds + \alpha\lambda\nu_n\int_{t_0}^t e^{\alpha(t-s)}ds + \beta u_n(t) \\
&\quad +\lambda(t-t_0)\mu_n + \lambda\nu_n.
\end{aligned}$$

By direct computation we note that

$$\begin{aligned}
&2\alpha^2\int_{t_0}^t e^{\alpha(t-s)}\int_{t_0}^s u_n(r)drds = 2\alpha\int_{t_0}^t (e^{\alpha(t-s)}-1)u_n(s)ds, \\
&\alpha\lambda\mu_n\int_{t_0}^t (s-t_0)e^{\alpha(t-s)}ds = -\lambda t\mu_n - \frac{\lambda}{\alpha}\mu_n + \frac{\lambda e^{\alpha(t-t_0)}}{\alpha}\mu_n + \lambda t_0\mu_n, \\
&\alpha\lambda\nu_n\int_{t_0}^t e^{\alpha(t-s)}ds = -\lambda\nu_n + \lambda e^{\alpha(t-t_0)}\nu_n.
\end{aligned}$$

Therefor we have

$$\begin{aligned}
u_{n+1}(t) &\le \alpha(2+\beta)\int_{t_0}^t e^{\alpha(t-s)}u_n(s)ds + \beta u_n(t) + \frac{\lambda e^{\alpha(t-t_0)}-\lambda}{\alpha}\mu_n + \lambda e^{\alpha(t-t_0)}\nu_n \\
&\le \alpha(2+\beta)e^{\alpha(T-t_0)}\int_{t_0}^t u_n(s)ds + \beta u_n(t) + \frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\mu_n + \lambda e^{\alpha(T-t_0)}\nu_n.
\end{aligned}$$

Thus, for every $m = 1, 2, ...$, we have

$$\sum_{n=0}^m u_n(t) - u_0(t) \quad \le \quad \alpha(2+\beta)e^{\alpha(T-t_0)}\int_{t_0}^t\sum_{n=0}^m u_n(s)ds + \beta\sum_{n=0}^m u_n(t)$$

$$+\frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\sum_{n=0}^{m}\mu_n+\lambda e^{\alpha(T-t_0)}\sum_{n=0}^{m}\nu_n.$$

Hence,

$$\begin{aligned}(1-\beta)\sum_{n=0}^{m}u_n(t) &\le \alpha(2+\beta)e^{\alpha(T-t_0)}\int_{t_0}^{t}\sum_{n=0}^{m}u_n(s)ds\\ &+\frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\sum_{n=0}^{m}\mu_n+\lambda e^{\alpha(T-t_0)}\sum_{n=0}^{m}\nu_n+u_0(T)\end{aligned}$$

Noting $0<\frac{\theta}{1-\theta}<1$ and $\theta^{4(p-1)}-k^p\ge 2k^p$, one has that $\beta<\frac{2k^p}{\theta^{4(p-1)}-k^p}\le 1$. From this and by Lemma 2.2, we get that

$$\begin{aligned}\sum_{n=0}^{m}u_n(t) \quad\le\quad & \frac{1}{1-\beta}\Big[\frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\sum_{n=0}^{m}\mu_n+\lambda e^{\alpha(T-t_0)}\sum_{n=0}^{m}\nu_n+u_0(T)\Big]\\ &\cdot e^{\frac{\alpha(2+\beta)e^{\alpha(T-t_0)}}{1-\beta}(t-t_0)}\end{aligned}$$

The condition (1.9) implies that $\sum_{n=0}^{\infty}\mu_n<\infty$ and $\sum_{n=0}^{\infty}\nu_n<\infty$. Therefore,

$$\sum_{n=0}^{\infty}u_n(T)<\infty.$$

Recall that

$$\begin{aligned}\mathbf{E}\sup_{t\in(-\infty,T]}|X(t)-X^n(t)|^p &\le \mathbf{E}||\xi-\xi^n||^p+\mathbf{E}\sup_{t\in[t_0,T]}|X(t)-X^n(t)|^p\\ &= \gamma_n+u_n(T),\end{aligned}$$

then from the condition (1.9) and Doob's martingale inequality [7], we find for an arbitrary $\varepsilon>0$ that,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbf{P}\Big\{\sup_{t\in(-\infty,T]}|X(t)-X^n(t)|\ge\varepsilon\Big\} &\le \frac{1}{\varepsilon^p}\sum_{n=0}^{\infty}\mathbf{E}|X(T)-X^n(T)|^p\\ &\le \frac{1}{\varepsilon^p}\Big[\sum_{n=0}^{\infty}\gamma_n+\sum_{n=0}^{\infty}u_n(T)\Big]\\ &< \infty.\end{aligned}$$

Hence, by applying the Borel-Cantelli lemma, we conclude that for an arbitrary small $\varepsilon>0$, there exists a positive integer $n_0=n_0(\omega)$ such that

$$\sup_{t\in(-\infty,T]}|X(t)-X^n(t)|\le\varepsilon,\qquad n\ge n_0.$$

This shows that the sequence $\{X^n(t),\ t\in(-\infty,T],\ n=0,1,...\}$ converges with probability one to the solution $\{X(t),\ t\in(-\infty,T]\}$. The proof is complete. □

3. Comments and examples

In [1, 8, 9], the proof of the existence of the solution to the equation (1.5) is based on the well-known Picard iteration, which establishes the iteration on the solution. However, the Z-algorithm method iterates for the whole equation. The Z-algorithm is a more general algorithm and can be applied to discuss more equations. Many historically and well-known analytic methods are its special cases, for example,

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Picard iteration, Newton-Kantorovich method and Chaplygin methods of chords and tangents.

Next, we give some examples to illustrate Theorem 2.4.

Example 1: Let $\{\xi^n, n = 0, 1, ...\}$ satisfy $\sum_{n=0}^{\infty} \mathbf{E}||\xi - \xi^n||^p < \infty$ and let D_n, f_n, g_n be defined in the following way: for $n = 0, 1, ...$, $X \in BC((-\infty, 0]; \mathbf{R}^d)$ and for every fixed $t \in [t_0, T]$,

$$D_n(X; X_t^n) := \varphi_n(X - X_t^n) + D(X_t^n) \tag{3.1}$$

$$f_n(t, X; X_t^n) := \psi_n(X - X_t^n) + f(t, X_t^n) \tag{3.2}$$

$$g_n(t, X; X_t^n) := \theta_n(X - X_t^n) + g(t, X_t^n), \tag{3.3}$$

where

$$\varphi_n : BC((-\infty, 0]; \mathbf{R}^d) \to \mathbf{R}^d,\ \varphi_n(0) = 0$$
$$\psi_n : [t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d) \to \mathbf{R}^d,\ \psi_n(t, 0) \equiv 0$$
$$\theta_n : [t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d) \to \mathbf{R}^{d+m},\ \theta_n(t, 0) \equiv 0.$$

The functions φ_n, ψ_n, θ_n satisfy the conditions (M1)-(M3) with constants $K > 0$ and $k \in \left(0, 1/(3 \cdot 2^{4(p-1)})^{\frac{1}{p}}\right)$. Obviously,

$$D_n(X_t^n; X_t^n) - D(X_t^n) \equiv 0$$
$$f_n(t, X_t^n; X_t^n) - f(t, X_t^n) \equiv 0$$
$$g_n(t, X_t^n; X_t^n) - g(t, X_t^n) \equiv 0.$$

This shows that the condition (1.9) is satisfied. Thus theorem 2.4 is obtained.

Example 2: In particular, we linearize the equation (3.1) by: for $n = 0, 1, ...$,

$$D_n(X; X_t^n) := (X - X_t^n) \cdot \varphi_n + D(X_t^n) \tag{3.4}$$

$$f_n(t, X; X_t^n) := (X - X_t^n) \cdot \psi_n + f(t, X_t^n) \tag{3.5}$$

$$g_n(t, X; X_t^n) := (X - X_t^n) \cdot \theta_n + g(t, X_t^n), \tag{3.6}$$

where $\theta_n = (\theta_{1n}, \theta_{2n}, ..., \theta_{mn})$ and φ_n, ψ_n, $\theta_{in}(i = 1, 2, ..., m)$ are scalar sequences. We can easily see that all conditions of Theorem 2.4 are satisfied. So Theorem 2.4 succeed.

Example 3: More specifically, we assume that $\xi^n = \xi$ a.s. and $\varphi_n = \psi_n = \theta_n = 0$ in equation (3.4) for all $n = 0, 1, ...$, then we obtain the Picard iteration. Of course, in this case, Theorem 2.4 is successful. Therefore, the Picard iteration is a special Z-algorithm.

Acknowledgements

This work was supported in part by the National Natural Science Foundation of China under grant #11171306 and #11071065, and sponsored by the Scientific Project of Zhejiang Provincial Science Technology Department under grant #2011C33012.

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Conflict of interest:

The authors declared that they have no conflicts of interest to this work. We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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An improved generalized parameterized inexact Uzawa method for singular saddle point problems *

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Abstract

In this paper, based on the generalized parameterized inexact Uzawa method (GPIU) presented by Zhang and Wang [*Applied Mathematics and Computation*, 219(2013) 4225-4231], we introduce and study an improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems. Moreover, theoretical analysis shows that the semi-convergence of the IGPIU method can be guaranteed by suitable choices of the iteration parameters. Finally, numerical experiments are carried out, which show that the improved generalized parameterized inexact Uzawa method (IGPIU) with appropriate parameters improve the convergence of iteration method efficiently when solving singular saddle point problems from the classic incompressible steady state Stokes problems.

Key words: Krylov subspace methods; Generalized saddle point matrices; Minimal polynomial; Preconditioners.

MSC: 65F10; 65F15

1 Introduction

Consider a singular saddle point problem

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} \equiv b, \tag{1}$$

*This research of this author is supported by NSFC Tianyuan Mathematics Youth Fund (11226337), NSFC(11501525,11471098,61203179,61202098,61170309,91130024,61272544, 61472462 and 11171039), Science Technology Innovation Talents in Universities of Henan Province(16HASTIT040), Aeronautical Science Foundation of China (2013ZD55006), Project of Youth Backbone Teachers of Colleges and Universities in Henan Province(2013GGJS-142,2015GGJS-179), ZZIA Innovation team fund (2014TD02), Major project of development foundation of science and technology of CAEP (2012A0202008), Defense Industrial Technology Development Program, China Postdoctoral Science Foundation (2014M552001), Basic and Advanced Technological Research Project of Henan Province (152300410126), Henan Province Postdoctoral Science Foundation (2013031), Natural Science Foundation of Zhengzhou City (141PQYJS560), Research on Innovation Ability Evaluation Index System and Evaluation Model (142400411268).

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where $A \in R^{m\times m}, B \in R^{m\times n}, m \geq n$. The matrix A is symmetric positive matrix and the matrix and B is a rank-deficient matrix. Systems of the form (1) arise in a variety of scientific and engineering applications and have attracted a lot of research, see [1-7] for a comprehensive survey. When A is symmetric positive definite and B is of full column rank, we refer the reader to [2,7-18] for many efficient iterative methods and [19] for a survey.

For large, sparse or structure matrices, iterative method is an attractive option. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \text{span}\left\{b, \mathcal{A}b, \mathcal{A}^2 b, ..., \mathcal{A}^{n-1}b, ...\right\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES) and generalized minimal residual method (GMRES) are common Krylov subspace methods. The CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices and GMRES for unsymmetric matrices [20].

Generally speaking, the matrix B is full column rank, but not always. If B is rank-deficient, how to effectively solve the singular saddle point problem (1) is important in both scientific computing and engineering applications. For solving the rank-deficient saddle point problems, Ma and Zheng et al. [17,21] presented the parameterized Uzawa method. Bai et al. [22-23] studied the PHSS iteration method. Fischer et al. [24] considered the preconditioned minimum residual (PMINRES) method. Wu et al. [7] discussed the preconditioned conjugate gradient (PCG) method. Zhang and Wang [17] introduced the generalized parameterized inexact Uzawa (GPIU) method.

In this paper, based on the generalized parameterized inexact Uzawa (GPIU) method presented by Zhang and Wang [17], we introduce and study an improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems (1). Similar to the proving process of section 3 in [17], theoretical analysis shows that the semi-convergence of IGPIU method can be guaranteed by suitable choices of the iteration parameters. Finally, one numerical example presented shows correctness and availability of our theory about the improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems.

This paper is organized as follows. In Section 2, we will present the improved generalized parameterized inexact uzawa method (IGPIU) for singular saddle point problems (1). The semi-convergence of the IGPIU method are discussed in Section 3. Moreover, our methods are the generalization of known literature. Some numerical examples are given to demonstrate the efficiency of the IGPIU method in Section 4. Finally, conclusions are made in Section 5.

2 An improved generalized parameterized inexact uzawa method (IGPIU)

Recently, for singular saddle point problems (1), Zhang and Wang [17] make the following splitting

$$\mathcal{A} := \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = \mathcal{M} - \mathcal{N},$$

where

$$\mathcal{M} = \begin{pmatrix} P & 0 \\ -B^T + Q_1 & Q_2 \end{pmatrix}, \mathcal{N} = \begin{pmatrix} P - A & -B \\ Q_1 & Q_2 \end{pmatrix}$$

$P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are prescribed symmetric positive definite matrices and $Q_1 \in R^{n\times m}$ is an arbitrary matrix.

To construct the improved generalized parameterized inexact Uzawa method (IGPIU), if we can add one parameter in the above splitting, then we may change the parameter to improve the performance of presented method. Hence, we propose the following splitting

$$\mathcal{A} := \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = \mathcal{L} - \mathcal{U},$$

where

$$\mathcal{L} = \begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix}, \mathcal{U} = \begin{pmatrix} P - A & -B \\ Q_1 & \omega Q_2 \end{pmatrix}$$

$P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are prescribed symmetric positive definite matrices and $Q_1 \in R^{n\times m}$ is an arbitrary matrix. Based the generalized parameterized inexact Uzawa (GPIU) iteration method presented by Zhang and Wang [17], we consider an improved generalized parameterized inexact uzawa method (IGPIU) for solving the singular saddle point (1).

$$\begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} P - A & -B \\ Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{2}$$

or equivalently,

$$\begin{cases} x^{(k+1)} = x^{(k)} + P^{-1}\left[p - Ax^{(k)} - By^{(k)}\right], \\ y^{(k+1)} = y^{(k)} + \frac{1}{\omega} Q_2^{-1}\left[B^T x^{(k+1)} - q\right] - \frac{1}{\omega} Q_2^{-1} Q_1 \left[x^{(k+1)} - x^{(k)}\right]. \end{cases} \tag{3}$$

The iteration matrix of the IGPIU method (2) or (3) is given by

$$\mathcal{T} = \begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix}^{-1} \begin{pmatrix} P - A & -B \\ Q_1 & \omega Q_2 \end{pmatrix} = I - \mathcal{L}^{-1}\mathcal{A}. \tag{4}$$

The IGPIU method: Let $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ be prescribed symmetric positive definite matrices and $Q_1 \in R^{n\times m}$ be an arbitrary matrix. Given initial vector $x^{(0)} \in R^m$ and $y^{(0)} \in R^n$ and the relaxation parameter ω with $\omega \neq 0$. For $k = 0, 1, 2, ...$ until the iteration sequence $\{(x^{(k)^T}, y^{(k)^T})^T\}$ is convergent, compute

$$\begin{cases} x^{(k+1)} = x^{(k)} + P^{-1}\left[p - Ax^{(k)} - By^{(k)}\right], \\ y^{(k+1)} = y^{(k)} + \frac{1}{\omega} Q_2^{-1}\left[B^T x^{(k+1)} - q\right] - \frac{1}{\omega} Q_2^{-1} Q_1 \left[x^{(k+1)} - x^{(k)}\right]. \end{cases} \tag{5}$$

Remark 2.1. It is obvious that when choosing $\omega = 1$, then the IGPIU method reduces to the GPIU method [17]. Hence, we may change the parameter to improve the performance of presented method.

3 The semi-convergence of IGPIU method

In this section, we discuss the semi-convergence of the IGPIU method for solving the singular saddle point problem (1). We first reveal some basic concepts and notations.

Denote $\sigma(\mathcal{A})$ and $\rho(\mathcal{A})$ as the spectrum and spectral radius of the matrix $\mathcal{A}$, respectively. The *rank* and *index* of the matrix $\mathcal{A}$ are denoted by *rank*$(\mathcal{A})$ and *index*$(\mathcal{A})$, respectively.

Assume that the matrix $\mathcal{A}$ can be split into $\mathcal{A} = \mathcal{M} - \mathcal{N}$ with $\mathcal{M}$ nonsingular. Then we can construct a splitting iteration method:

$$x^{(k+1)} = \mathcal{T}x^{(k)} + \mathcal{M}^{-1}c, k = 0, 1, 2, ... \tag{6}$$

where $\mathcal{T} = \mathcal{M}^{-1}\mathcal{N}$ is the iteration matrix.

It is well known that any of the following three conditions is necessary and sufficient for guaranteeing the semi-convergence of the iteration method (6) for the singular linear systems $\mathcal{A}X = c$ (see [17,22]):

(a) The spectral radius of the iteration matrix $\mathcal{T}$ is equal to one , i.e., $\rho(\mathcal{T}) = 1$;

(b) The elementary divisor associated with $\lambda \in \sigma(\mathcal{A})$ is linear when $\lambda = 1$, i.e., $rank((I-\mathcal{T})^2) = rank(I - \mathcal{T})$, or equivalently, $index(I - \mathcal{T}) = 1$;

(c) If $\lambda \in \sigma(\mathcal{T})$ with $|\lambda| = 1$, then $\lambda = 1$, i.e., $\mathcal{V}(\mathcal{T}) \equiv \max\{|\lambda| : \lambda \in \sigma(\mathcal{T}), \lambda \neq 1\} < 1$.

When iteration scheme (6) is semi-convergent, $\mathcal{V}(\mathcal{T})$ is said to be the semi-convergence factor. As usual, the splitting $\mathcal{A} = \mathcal{M} - \mathcal{N}$ and the corresponding iteration matrix $\mathcal{T}$ are called as semi-convergent if the iteration (6) is semi-convergent. Next we study the semi-convergence of the IGPIU iteration (5). To get the semi-convergence conditions, the following lemmas are used.

Lemma 3.1. **[25]** *Consider the quadratic equation* $x^2 - \delta x + \eta = 0$, *where* δ *and* η *are real numbers. Both roots of the equation are less than one in modulus if and only if* $|\eta| < 1$ *and* $|\delta| < 1 + \eta$.

Lemma 3.2. **[17]** *Let* $P \in R^{m\times m}$ *and* $Q_2 \in R^{n\times n}$ *be symmetric positive definite and* $B \in R^{m\times n}$ *be of column rank-deficient, with* $m \geq n$. *Suppose that* λ *is an eigenvalue of the iteration matrix* $\mathcal{T}$ *and* $(u^T, v^T)^T \in R^{m+n}$ *is the corresponding eigenvector. Then* $\lambda = 1$ *if and only if* $u = 0$.

Theorem 3.3. *Let* $P \in R^{m\times m}$ *and* $Q_2 \in R^{n\times n}$ *be symmetric positive definite and* $B \in R^{m\times n}$ *be of column rank-deficient, with* $m \geq n$. *Suppose that* $\lambda \neq 1$ *is an eigenvalue of the iteration matrix* $\mathcal{T}$ *and* $(u^T, v^T)^T \in R^{m+n}$ *is the corresponding eigenvector. Then* λ *satisfies the following quadratic equation:*

$$\lambda^2 + \frac{\beta + \gamma - 2\omega\alpha - \tau}{\alpha}\lambda + \frac{\alpha + \tau - \omega\beta}{\alpha} = 0,$$

where

$$\alpha = \frac{u^*Pu}{u^*u} > 0, \beta = \frac{u^*Au}{u^*u} > 0, \gamma = \frac{u^*BQ_2^{-1}B^Tu}{u^*u} \geq 0, \tau = \frac{u^*BQ_2^{-1}Q_1u}{u^*u}.$$

Proof. Firstly, since $\lambda \neq 1$, we know $u \neq 0$ from Lemma 3.2. By (4) we have

$$\begin{pmatrix} P - A & -B \\ Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{7}$$

or equivalently

$$\begin{cases} [(1-\lambda)P - A]\, u = Bv, \\ \left[(1-\lambda)Q_1 + \lambda B^T\right] u = \omega(\lambda - 1)Q_2 v. \end{cases} \tag{8}$$

Because Q_2 is symmetric positive definite and $\lambda \neq 1$, from the second equation in (8), we obtain that $v = \frac{1}{\omega}(-Q_2^{-1}Q_1 + \frac{\lambda}{\lambda-1}Q_2^{-1}B^T)u$, which together with the first equation in (8), result in

$$\omega\lambda^2 Pu + \lambda\omega(Au + BQ_2^{-1}B^Tu - 2\omega Pu - BQ_2^{-1}Q_1u) + \omega(Pu + BQ_2^{-1}Q_1u - \omega Au) = 0. \tag{9}$$

Since $u \neq 0$, by left multiplying u^* and with the positive definiteness of $P(u^*Pu \neq 0)$, we have

$$\lambda^2 + \frac{\beta+\gamma-2\omega\alpha-\tau}{\alpha}\lambda + \frac{\alpha+\tau-\omega\beta}{\alpha} = 0. \tag{10}$$

Thus, the proof is completed.

Theorem 3.4. *Assume that $A \in R^{m\times m}$ is symmetric positive definite, $B \in R^{m\times n}$ is rank-deficient, $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are symmetric positive definite and $Q_1 \in R^{n\times m}$ is an arbitrary matrix such that $BQ_2^{-1}Q_1$ is symmetric. Then $\sigma(\mathcal{T}) < 1$ holds if and only if one of the following conditions hold:*

$$\omega > 0, \tau < \omega\beta, 0 < \frac{\gamma}{2} < (1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta.$$

Proof. Since P is symmetric positive definite and $BQ_2^{-1}Q_1$ is symmetric. By Lemma 3.1 and Eq. (10), we know that the spectral radius of the IGPIU iteration matrix is less than one in modulus if and only if

$$\begin{cases} \left|\frac{\alpha+\tau-\omega\beta}{\alpha}\right| < 1, \\ \left|\frac{\beta+\gamma-2\omega\alpha-\tau}{\alpha}\right| < 1 + \frac{\alpha+\tau-\omega\beta}{\alpha}. \end{cases} \tag{11}$$

or equivalently,

$$\omega > 0, \tau < \omega\beta, 0 < \frac{\gamma}{2} < (1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta.$$

Thus, the proof is completed. $\diamond$

Theorem 3.5. **[17]** *Assume that $A \in R^{m\times m}$ is symmetric positive definite, $B \in R^{m\times n}$ is rank-deficient, $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are symmetric positive definite and $Q_1 \in R^{n\times m}$ is an arbitrary matrix such that $BQ_2^{-1}Q_1$ is symmetric and $\mathcal{N}(B) \cap \mathcal{R}(Q_2^{-1}B^TA^{-1}B) = \{0\}$, then index$(I-\mathcal{T}) = 1$. Here and in the sequel, $\mathcal{N}(\bullet)$ and $\mathcal{R}(\bullet)$ is used to represent the null space and range space of the corresponding matrix, respectively.*

Combining Theorem 3.4 and Theorem 3.5 , we immediately obtain the following sufficient conditions for the convergence result of the IGPIU method for solving singular saddle point problem (1).

Theorem 3.6. *Assume that $A \in R^{m\times m}$ is symmetric positive definite, $B \in R^{m\times n}$ is rank-deficient, $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are symmetric positive definite and $Q_1 \in R^{n\times m}$ is an arbitrary matrix such that $BQ_2^{-1}Q_1$ is symmetric and $\mathcal{N}(B) \cap \mathcal{R}(Q_2^{-1}B^TA^{-1}B) = \{0\}$. Then the IGPIU method for solving singular saddle point problem (1) is semi-convergent if $\omega, \tau, \beta, \gamma, \alpha$ satisfy one of the following conditions*

$$\omega > 0, \tau < \omega\beta, 0 < \frac{\gamma}{2} < (1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta.$$

Remark 3.1. From the Theorem 3.5, it is not difficult to find that when $\omega > 1, 2\alpha > \beta$ or $0 < \omega < 1, 2\alpha < \beta$, then $(1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta > 2\alpha + \tau - \beta$. Hence, under these conditions, the range of γ is wider and we will have more space of parameters range.

Remark 3.2. It is obvious that when choosing $\omega = 1, P = \frac{1}{\xi}A, Q_1 = 0,$ and $\mathrm{Q}_2 = \frac{1}{\zeta}\mathrm{Q}$, Q is an approximate matrix to the Schur complement $B^TA^{-1}B$, then the IGPIU method reduces to the PIU method in [10,17].

Remark 3.3. Some choices of the parameter matrices P, Q_1 and Q_2 are given in Table 1 [17]. When choosing different parameter matrices P, Q_1 and Q_2, we may immediately obtain a series of iterative methods for solving singular saddle problem (1).

Table 1: Some choices of the parameter matrices P, Q_1 and Q_2.

Case	P	Q_1	Q_2
I	$\frac{1}{\xi}A$	0	$\frac{1}{\zeta}I_n$
II	$\frac{1}{\xi}diag(A)$	0	$\frac{1}{\zeta}I_n$
III	$\frac{1}{\xi}tridiag(A)$	0	$\frac{1}{\zeta}I_n$
IV	$\frac{1}{\xi}A$	$-\frac{\theta}{\zeta}B^T$	$\frac{1}{\zeta}I_n$
V	$\frac{1}{\xi}A$	$-\frac{\theta}{\zeta}B^T$	$\frac{1}{\zeta}diag(\hat{B}^T P^{-1}\hat{B}, \tilde{B}^T\tilde{B})$
VI	$\frac{1}{\xi}A$	$-\theta Q_2 B^T$	$\frac{1}{\zeta}tridiag(\hat{B}^T P^{-1}\hat{B}, \tilde{B}^T\tilde{B})$

4 Numerical examples

In this section, we give numerical experiments to demonstrate the conclusions drawn above. The numerical experiments were done by using MATLAB 7.1 and the matrix of the numerical experiments were generated by IFISS software. In all our runs we used as a zero initial guess and stopped the iteration when the relative residual had been reduced by at least seven orders of magnitude (i.e, when $\|b - \mathcal{A}x^k\|_2 \leq 10^{-7}\|b\|_2$).

We consider the classic incompressible steady state Stokes problems:

$$\begin{cases} -\Delta u + gradp = f, \text{ in } \Omega, \\ -divu = 0, \text{ in } \Omega, \end{cases}$$

with suitable boundary condition on $\partial\Omega$. It is known that many discretization schemes for the above Stokes problems will lead to generalized saddle point problems of the form (1). Here, we get the test problem (leak-lid driven cavity) by using IFISS software written by David Silvester, Howard Elman and Alison Ramage. We take a finite element subdivision based on 32×32 uniform grids of square elements. The mixed finite element used is the bilinear-constant velocity-pressure: $Q_1 - P_0$ pair with stabilization. $Q_1 - P_0$ finite element subdivision is shown in Figure 1. The stabilization parameter is chosen to $\frac{1}{4}$. We get the (1,1) block A of the coefficient matrix corresponding to the discretization of the conservative term. Since the matrix B produced by the software is rank deficient, so $\mathcal{A}$ is singular matrix. In our experiment, we choose uniform grids 8×8, 16×16.

In Tables 2, when choosing different parameters, we show iteration counts, relative residual and computing time about the GPIU and the IGPIU methods for solving singular saddle problem (1), where IT, RES and CPU are the iteration numbers, relative residual and computing time about the GPIU and the IGPIU methods, respectively. Moreover, we also show the corresponding reduction of residual 2-norm and eigenvalues distributions about two methods for different parameters. Figures 2 $\sim$ 5 show the reduction of residual 2-norm with Case I, II, III and IV of Table 2. Figures 6 $\sim$ 9 show the eigenvalues distributions with Case I, II, III and IV of Table 2. Figures 10 and 11 show the reduction of residual 2-norm with uniform grids 16×16 and Cases I, II. Figures 12 and 13 show the eigenvalues distributions with uniform grids 16×16 and Cases I, II.

Remark 3.1. From Table 2, Figures $2 \sim 5$, 10 and 11 , it is very easy to get that the IGPIU method is in general better than the GPIU method when choosing suitable parameters. By numerical experiments for many times, we can find that, when $0.75 \leq \omega \leq 1.05$ the IGPIU method is very efficient. For Case II, when $\omega = 1.05$ the IGPIU method is little efficient. Hence, we suggest that, the selection range of the parameters may be $0.75 \leq \omega \leq 1$.

Remark 3.2. From Figures $6 \sim 9$,12 and 13, we may find that the eigenvalue distribution about the GPIU method has the same spectral clustering compared with the IGPIU method when choosing suitable parameters.

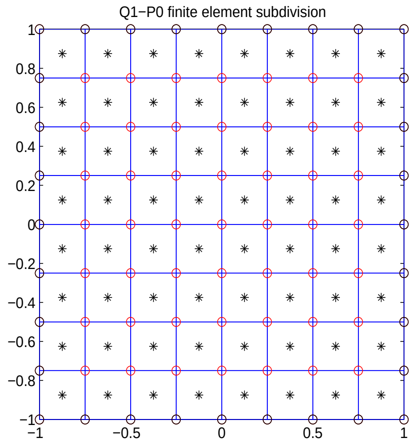

Figure 1: $Q_1 - P_0$ finite element subdivision

Table 2: numerical results of different parameters about GPIU and IGPIU methods for solving singular saddle problem (1). Here, uniform grids are 8×8.

Case	$(\xi, \theta, \zeta, \omega)$	IT	RES	CPU
Case I	$(0.8, 0, 10, 1)$	124	9.6146×10^{-8}	1.776
	$(0.8, 0, 10, 0.85)$	100	9.9265×10^{-8}	1.437
	$(0.8, 0, 10, 0.75)$	103	9.9106×10^{-8}	1.468
	$(0.8, 0, 10, 1.05)$	79	9.8442×10^{-8}	1.156
Case II	$(0.8, 0, 10, 1)$	451	8.8682×10^{-8}	0.813
	$(0.8, 0, 10, 0.85)$	435	9.6783×10^{-8}	0.812
	$(0.8, 0, 10, 0.75)$	439	9.6727×10^{-8}	0.797
	$(0.8, 0, 10, 1.05)$	462	7.7405×10^{-8}	0.844
Case III	$(0.8, 0, 10, 1)$	375	7.2718×10^{-8}	2.797
	$(0.8, 0, 10, 0.85)$	356	9.7272×10^{-8}	2.672
	$(0.8, 0, 10, 0.75)$	354	8.3278×10^{-8}	2.703
	$(0.8, 0, 10, 1.05)$	373	7.3365×10^{-8}	2.829
Case IV	$(0.8, 0, 10, 1)$	141	9.3439×10^{-8}	1.984
	$(0.8, 0, 10, 0.85)$	124	9.3478×10^{-8}	1.734
	$(0.8, 0, 10, 0.75)$	118	9.91×10^{-8}	1.625
	$(0.8, 0, 10, 1.05)$	139	9.913×10^{-8}	1.907

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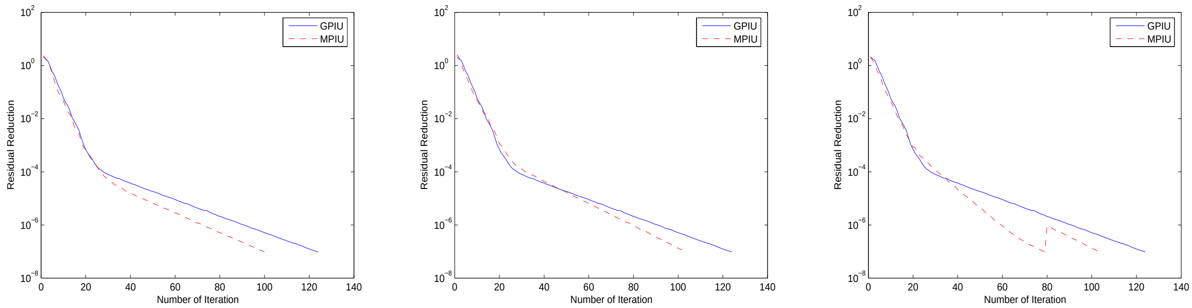

Figure 2: Reduction of residual 2-norm with Case I of Table 2. The left figure shows that the first line parameters (GPIU) of Case I compare with the second line parameters (IGPIU) of Case I; The middle figure shows that the first line parameters (GPIU) of Case I compare with the third line parameters (IGPIU) of Case I; The right figure shows that the first line parameters (GPIU) of Case I compare with the forth line parameters (IGPIU) of Case I.

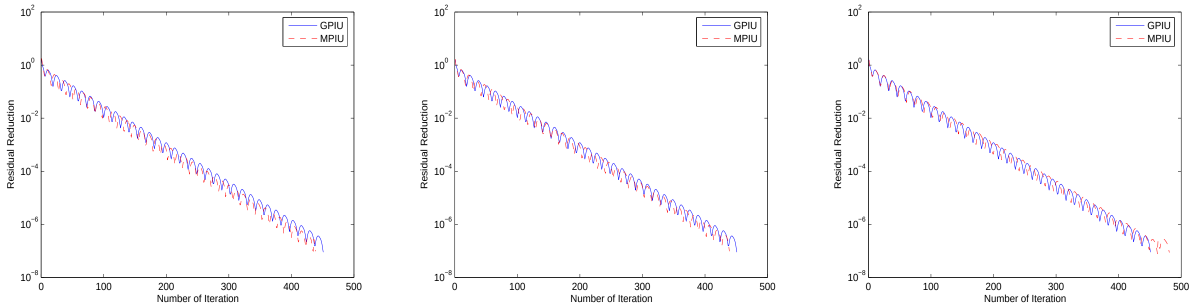

Figure 3: Reduction of residual 2-norm with Case II of Table 2. The left figure shows that the first line parameters (GPIU) of Case II compare with the second line parameters (IGPIU) of Case II; The middle figure shows that the first line parameters (GPIU) of Case II compare with the third line parameters (IGPIU) of Case II; The right figure shows that the first line parameters (GPIU) of Case II compare with the forth line parameters (IGPIU) of Case II.

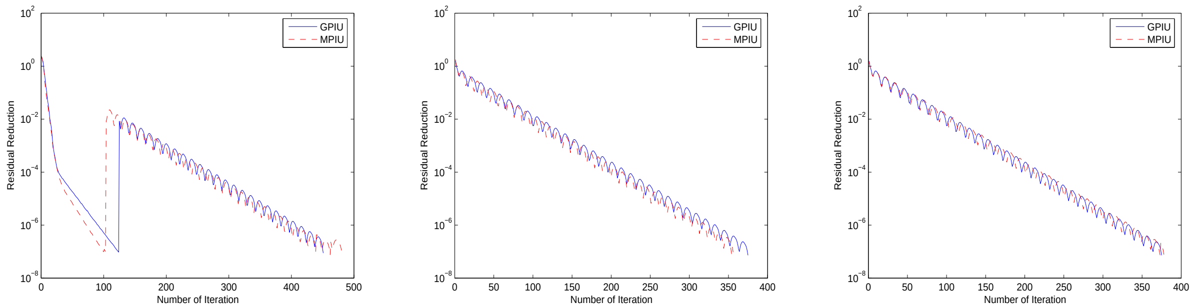

Figure 4: Reduction of residual 2-norm with Case III of Table 2. The left figure shows that the first line parameters (GPIU) of Case III compare with the second line parameters (IGPIU) of Case III; The middle figure shows that the first line parameters (GPIU) of Case III compare with the third line parameters (IGPIU) of Case III; The right figure shows that the first line parameters (GPIU) of Case III compare with the forth line parameters (IGPIU) of Case III.

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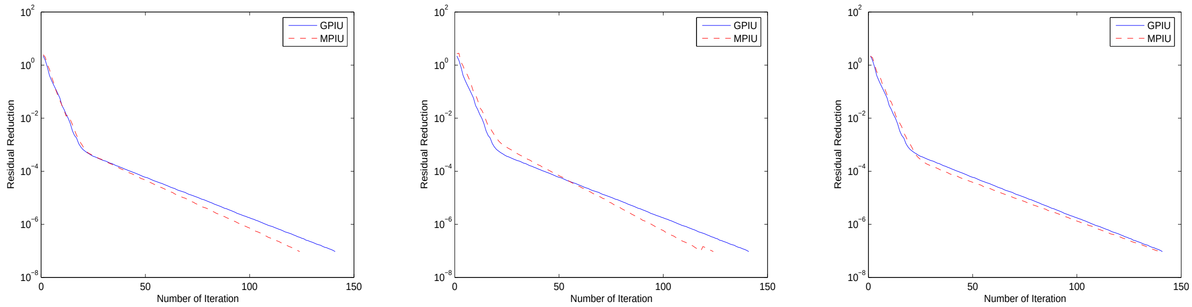

Figure 5: Reduction of residual 2-norm with Case IV of Table 2. The left figure shows that the first line parameters (GPIU) of Case IV compare with the second line parameters (IGPIU) of Case IV; The middle figure shows that the first line parameters (GPIU) of Case IV compare with the third line parameters (IGPIU) of Case IV; The right figure shows that the first line parameters (GPIU) of Case IV compare with the forth line parameters (IGPIU) of Case IV.

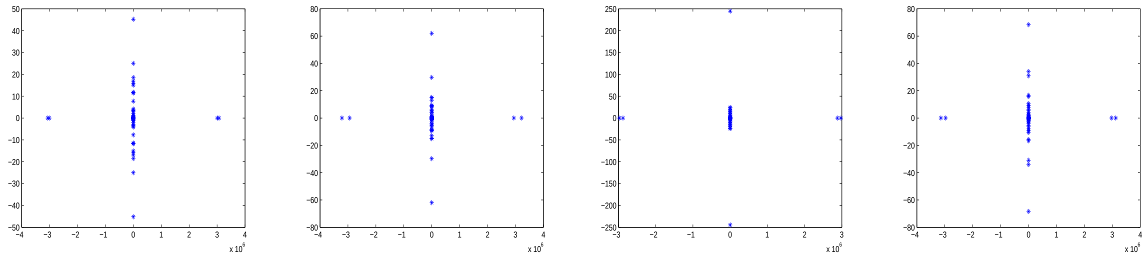

Figure 6: Eigenvalues distributions with Case I of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case I; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case I; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case I; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case I.

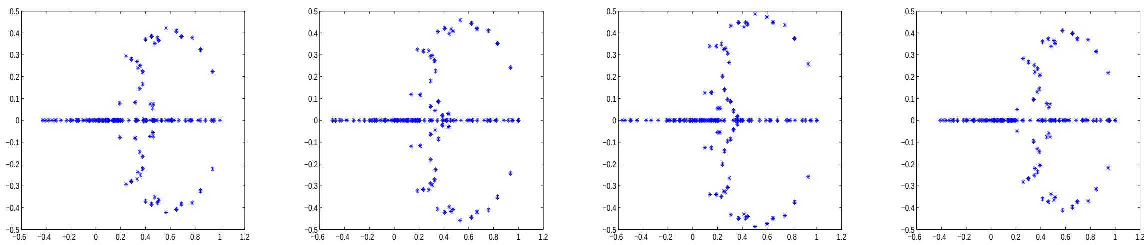

Figure 7: Eigenvalues distributions with Case II of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case II; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case II; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case II; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case II.

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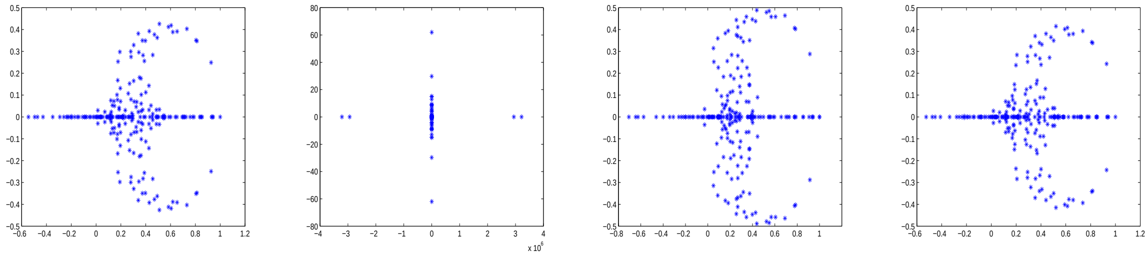

Figure 8: Eigenvalues distributions with Case III of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case III; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case III; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case III; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case III.

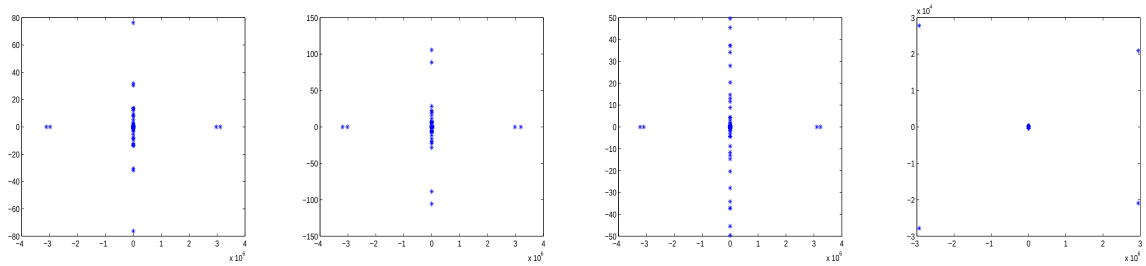

Figure 9: Eigenvalues distributions with Case IV of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case IV; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case IV; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case IV; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case IV.

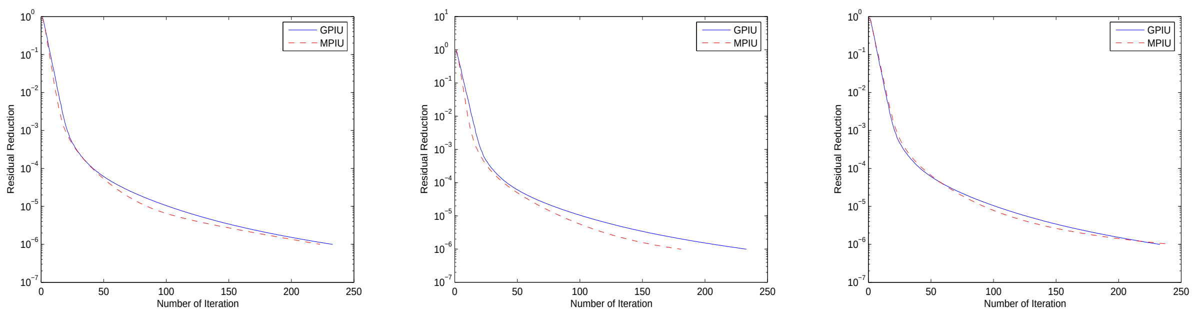

Figure 10: Reduction of residual 2-norm with uniform grids 16×16 and Case I. The left figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case I compare with the parameters $(1, 0, 10, 0.85)$ (IGPIU) of Case I; The middle figure shows that parameters $(1, 0, 10, 1)$ (GPIU) of Case I compare with the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case I; The right figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case I compare with the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case I.

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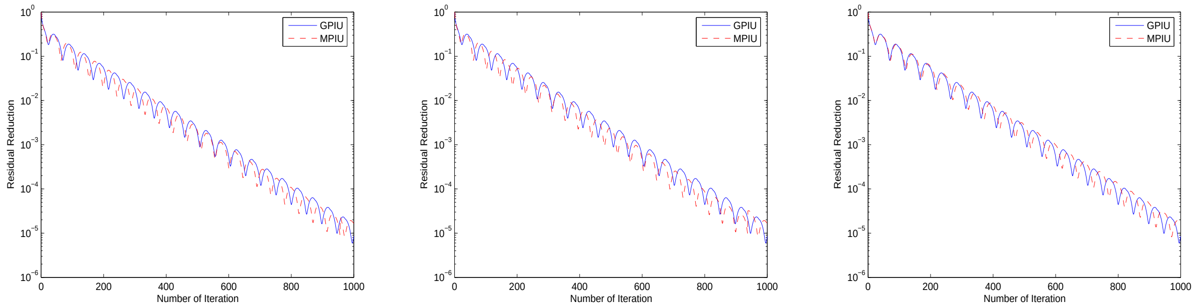

Figure 11: Reduction of residual 2-norm with uniform grids 16×16 and Case II. The left figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case II compare with the parameters $(1, 0, 10, 0.85)$ (IGPIU) of Case II; The middle figure shows that parameters $(1, 0, 10, 1)$ (GPIU) of Case II compare with the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case II; The right figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case II compare with the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case II.

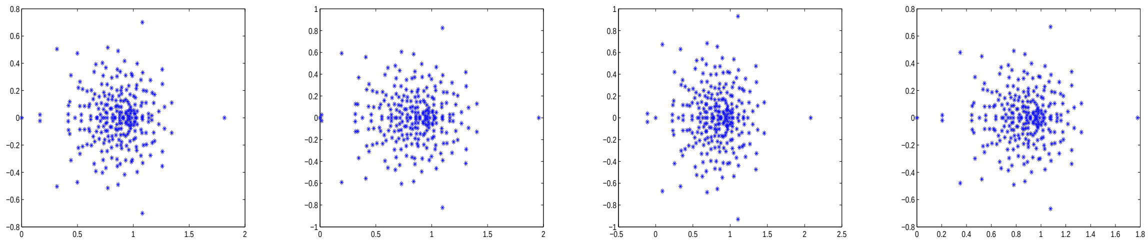

Figure 12: Eigenvalues distributions with uniform grids 16×16 and Case I. The first figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1)$ (GPIU) of Case I; The second figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.85)$ (IGPIU) if Case I; The third figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case I; The forth figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case I.

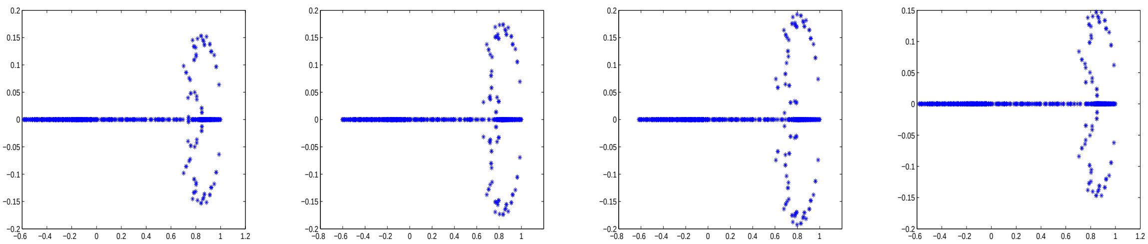

Figure 13: Eigenvalues distributions with uniform grids 16×16 and Case II. The first figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1)$ (GPIU) of Case II; The second figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.85)$ (IGPIU) of Case II; The third figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case II; The forth figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case II.

5 Conclusions

Based on the generalized parameterized inexact Uzawa method (GPIU) presented by Zhang and wang [17], we introduce and study an improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems (1). Moreover, theoretical analysis shows that the semi-convergence of IGPIU method can be guaranteed by suitable choices of the iteration parameters. Finally, numerical experiments are carried out, which show that the IGPIU method is in general better than the GPIU method when choosing suitable parameters. Moreover, we also may find that the eigenvalue distribution about GPIU method has the same spectral clustering compared with IGPIU method when choosing suitable parameters.

6 Acknowledgements

This research of this author is supported by NSFC Tianyuan Mathematics Youth Fund (11226337), NSFC(11501525,11471098,61203179,61202098,61170309,91130024,61272544, 61472462 and 11171039), Science Technology Innovation Talents in Universities of Henan Province(16HASTIT040), Aeronautical Science Foundation of China (2013ZD55006), Project of Youth Backbone Teachers of Colleges and Universities in Henan Province(2013GGJS-142,2015GGJS-179), ZZIA Innovation team fund (2014TD02), Major project of development foundation of science and technology of CAEP (2012A0202008), Defense Industrial Technology Development Program, China Postdoctoral Science Foundation (2014M552001), Basic and Advanced Technological Research Project of Henan Province (152300410126), Henan Province Postdoctoral Science Foundation (2013031), Natural Science Foundation of Zhengzhou City (141PQYJS560).

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IDENTITIES INVOLVING BESSEL POLYNOMIALS ARISING FROM LINEAR DIFFERENTIAL EQUATIONS

TAEKYUN KIM AND DAE SAN KIM

Abstract. In this paper, we study linear differential equations arising from Bessel polynomials and their applications. From these linear differential equations, we give some new and explicit identities for Bessel polynomials.

1. Introduction

As is well known, the Bessel differential equation is given by

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\alpha^2\right)y=0,\quad \text{(see [17])}. \tag{1.1}$$

for an arbitrary complex number α.

The Bessel functions of the first kind $J_\alpha(x)$ are defined by the solution of (1.1).

For $n\in\mathbb{Z}$, $J_n(x)$ are sometimes also called cylinder function or cylindrical harmonics.

It is known that

$$J_n(x)=\sum_{l=0}^{\infty}\frac{(-1)^l}{l!\,(n+l)!}\left(\frac{x}{2}\right)^{2l+n},\quad \text{(see [1, 16, 17])}. \tag{1.2}$$

The generating function of Bessel functions is given by

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}J_n(x)\,t^n, \tag{1.3}$$

and $J_n(x)$ can be also represented by the contour integral as

$$J_n(x)=\frac{1}{2\pi i}\oint e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}t^{-n-1}dt,\quad \text{(see [17])}, \tag{1.4}$$

where the contour encloses the origin and is traversed in a counterclockwise direction.

The Bessel polynomials are defined by the solution of the differential equation

$$x^2\frac{d^2y}{dx^2}+2(x+1)\frac{dy}{dx}-n(n+1)y=0,\quad \text{(see [1–6, 15, 16])}. \tag{1.5}$$

Indeed, the solutions of (1.5) are given by

$$\begin{aligned} y_n(x)&=\sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!k!}\left(\frac{x}{2}\right)^k \\ &=\sqrt{\frac{2}{\pi x}}e^{\frac{1}{x}}K_{-n-\frac{1}{2}}\left(\frac{1}{x}\right),\quad \text{(see [1, 15–17])}, \end{aligned} \tag{1.6}$$

2010 *Mathematics Subject Classification.* 05A19, 33C10, 34A30.
Key words and phrases. Bessel polynomials, linear differential equation.

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where

$$K_{\nu}(z) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)(2z)^{\nu}}{\sqrt{\pi}} \int_0^{\infty} \frac{\cos t}{\left(t^2+z^2\right)^{\nu+\frac{1}{2}}} dt.$$

We note that $y_n(x)$ are very similar to the modified spherical Bessel function of the second kind.

The first few are given as

$$\begin{aligned} &y_0(x) = 1, \quad y_1(x) = x+1, \quad y_2(x) = 3x^2+3x+1, \\ &y_3(x) = 15x^3+15x^2+6x+1, \\ &y_4(x) = 105x^4+105x^3+45x^2+10x+1, \quad \dots . \end{aligned}$$

Carlitz reverse Bessel polynomials are defined by

$$p_n(x) = x^n y_{n-1}\left(\frac{1}{x}\right), \quad (n \in \mathbb{N} \cup \{0\}), \quad (\text{see } [4, 15]). \tag{1.7}$$

These polynomials are also given by the generating function as

$$e^{x\left(1-\sqrt{1-2t}\right)} = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!}. \tag{1.8}$$

The explicit formulas for them are

$$\begin{aligned} p_n(x) &= \sum_{k=1}^{n} \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!} x^k \\ &= (2n-3)!! x \, {}_1F_1(1-n; 2-2n; 2x), \quad (\text{see } [1, 15, 16]), \end{aligned} \tag{1.9}$$

where

$$n!! = \begin{cases} n(n-2)\cdots 5\cdot 3\cdot 1 & \text{if } n>0 \text{ odd}, \\ n(n-2)\cdots 6\cdot 4\cdot 2 & \text{if } n>0 \text{ even}, \\ 1 & \text{if } n=-1,0, \end{cases}$$

and

$$\begin{aligned} {}_1F_1(a;b;z) &= 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)} \frac{z^k}{k!} \\ &= \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt. \end{aligned}$$

The first few polynomials are

$$\begin{aligned} &p_1(x) = x, \\ &p_2(x) = x^2 + x, \\ &p_3(x) = x^3 + 3x^2 + 3x, \\ &p_4(x) = x^4 + 6x^3 + 15x^2 + 15x, \cdots . \end{aligned}$$

Recently, several authors have studied non-linear differential equations related to special polynomials (see [7–14]).

The reverse Bessel polynomials are used in the design of Bessel electronic filters.

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In this paper, we consider linear differential equations arising from Carlitz reverse Bessel polynomials and give some new and explicit identities for Bessel polynomials.

2. Identities involving Bessel polynomials arising from linear differential equations

Let us put

$$F = F(t,x) = e^{x\left(1-\sqrt{1-2t}\right)}. \tag{2.1}$$

Thus, by (2.1), we get

$$F^{(1)} = \frac{d}{dt}F(t,x) = x(1-2t)^{-\frac{1}{2}}F, \tag{2.2}$$

$$\begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} \\ &= \left(x(1-2t)^{-\frac{3}{2}} + x^2(1-2t)^{-1}\right)F, \end{aligned} \tag{2.3}$$

$$\begin{aligned} F^{(3)} &= \frac{d}{dt}F^{(2)} \\ &= \left(3x(1-2t)^{-\frac{5}{2}} + 3x^2(1-2t)^{-2} + x^3(1-2t)^{-\frac{3}{2}}\right)F, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} F^{(4)} &= \frac{dF^{(3)}}{dt} \\ &= \left(15x(1-2t)^{-\frac{7}{2}} + 15x^2(1-2t)^{-3} + 6x^3(1-2t)^{-\frac{5}{2}} + x^4(1-2t)^{-2}\right)F. \end{aligned} \tag{2.5}$$

Continuing this process, we set

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t,x) \\ &= \left(\sum_{i=N}^{2N-1} a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}\right)F, \end{aligned} \tag{2.6}$$

where $N = 1,2,3,\ldots$.

From (2.6), we note that

$$\begin{aligned} &F^{(N+1)} \\ &= \frac{d}{dt}F^{(N)} \\ &= \left(\sum_{i=N}^{2N-1} a_{i-N}(N,x)\left(-\frac{i}{2}\right)(1-2t)^{-\frac{i}{2}-1}(-2)\right)F \\ &\quad + \sum_{i=N}^{2N-1} a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}F^{(1)} \end{aligned} \tag{2.7}$$

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$$
\begin{aligned}
&= \left(\sum_{i=N}^{2N-1} i a_{i-N}(N,x)(1-2t)^{-\frac{i+2}{2}} \right) F \\
&\quad + \left(\sum_{i=N}^{2N-1} a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}} \right) x(1-2t)^{-\frac{1}{2}} F \\
&= \left(\sum_{i=N}^{2N-1} i a_{i-N}(N,x)(1-2t)^{-\frac{i+2}{2}} \right) F + \left(\sum_{i=N}^{2N-1} x a_{i-N}(N,x)(1-2t)^{-\frac{i+1}{2}} \right) F \\
&= \Big\{ x a_0(N,x)(1-2t)^{-\frac{N+1}{2}} + (2N-1) a_{N-1}(N,x)(1-2t)^{-\frac{2N+1}{2}} \\
&\quad + \sum_{i=N+1}^{2N-1} \left((i-1) a_{i-N-1}(N,x) + x a_{i-N}(N,x) \right)(1-2t)^{-\frac{i+1}{2}} \Big\} F.
\end{aligned}
$$

By replacing N by $N+1$ in (2.6), we get

$$
\begin{aligned}
F^{(N+1)} &= \left(\sum_{i=N+1}^{2N+1} a_{i-N-1}(N+1,x)(1-2t)^{-\frac{i}{2}} \right) F \\
&= \left(\sum_{i=N}^{2N} a_{i-N}(N+1,x)(1-2t)^{-\frac{i+1}{2}} \right) F.
\end{aligned} \tag{2.8}
$$

By comparing the coefficients on both sides (2.7) and (2.8), we have

$$a_0(N+1,x) = x a_0(N,x), \tag{2.9}$$

$$a_N(N+1,x) = (2N-1) a_{N-1}(N,x), \tag{2.10}$$

and

$$a_{i-N}(N+1,x) = (i-1) a_{i-N-1}(N,x) + x a_{i-N}(N,x), \tag{2.11}$$

where $N+1 \le i \le 2N-1$.

From (2.2) and (2.6), we can derive the following equation (2.11):

$$x(1-2t)^{-\frac{1}{2}} F = F^{(1)} = a_0(1,x)(1-2t)^{-\frac{1}{2}} F. \tag{2.12}$$

Thus, by (2.12), we have

$$a_0(1,x) = x. \tag{2.13}$$

From (2.9), we note that

$$a_0(N+1,x) = x a_0(N,x) = x^2 a_0(N-1,x) = \cdots = x^N a_0(1,x) = x^{N+1}, \tag{2.14}$$

and, by (2.10), we see

$$
\begin{aligned}
a_N(N+1,x) &= (2N-1) a_{N-1}(N,x) \\
&= (2N-1)(2N-3) a_{N-2}(N-1,x) \\
&\ \ \vdots \\
&= (2N-1)(2N-3)\cdots 3 \cdot 1 a_0(1,x) \\
&= (2N-1)!! x.
\end{aligned} \tag{2.15}
$$

The matrix $(a_i(j,x))_{0 \le i \le N-1, 1 \le j \le N}$ is given by

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$$\begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & N \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \end{array} &
\left[\begin{array}{cccccc}
x & x^2 & x^3 & x^4 & \cdots & x^N \\
 & 1!!x & & & & \\
 & & 3!!x & & & \\
 & & & 5!!x & & \\
 & & & & \ddots & \\
 & & 0 & & & (2N-3)!!x
\end{array}\right]
\end{array}$$

From (2.11), we obtain

$$\begin{aligned}
&a_1(N+1,x) \\
&= Na_0(N,x) + xa_1(N,x) \\
&= Na_0(N,x) + x(N-1)a_0(N-1,x) + x^2a_1(N-1,x) \\
&\vdots \\
&= \sum_{i=0}^{N-2} x^i(N-i)a_0(N-i,x) + x^{N-1}a_1(2,x) \\
&= \sum_{i=0}^{N-2} x^i(N-i)a_0(N-i,x) + x^{N-1}x \\
&= \sum_{i=0}^{N-1} x^i(N-i)a_0(N-i,x),
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
&a_2(N+1,x) \\
&= (N+1)a_1(N,x) + xa_2(N,x) \\
&= (N+1)a_1(N,x) + xNa_1(N-1,x) + x^2a_2(N-1,x) \\
&\vdots \\
&= \sum_{i=0}^{N-3} x^i(N+1-i)a_1(N-i,x) + x^{N-2}a_2(3,x) \\
&= \sum_{i=0}^{N-3} x^i(N+1-i)a_1(N-i,x) + 3x^{N-2}a_1(2,x) \\
&= \sum_{i=0}^{N-2} x^i(N+1-i)a_1(N-i,x),
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
&a_3(N+1,x) \\
&= (N+2)a_2(N,x) + xa_3(N,x) \\
&= (N+2)a_2(N,x) + x(N+1)a_2(N-1,x) + x^2a_3(N-1,x)
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
&\vdots\\
&=\sum_{i=0}^{N-4}x^{i}\left(N-i+2\right)a_{2}\left(N-i,x\right)+5x^{N-3}a_{2}\left(3,x\right)\\
&=\sum_{i=0}^{N-3}x^{i}\left(N-i+2\right)a_{2}\left(N-i,x\right).
\end{aligned}$$

Continuing this process, we get

$$a_{j}\left(N+1,x\right)=\sum_{i=0}^{N-j}x^{i}\left(N-i+j-1\right)a_{j-1}\left(N-i,x\right), \tag{2.19}$$

where $j=1,2,\ldots,N-1$.

Now, we give explicit expressions for $a_{j}\left(N+1,x\right)$ $\left(j=1,2,\ldots,N-1\right)$. From (2.14) and (2.16), we can easily derive the following equation:

$$\begin{aligned}
a_{1}\left(N+1,x\right)&=\sum_{i_1=0}^{N-1}x^{i_1}\left(N-i_1\right)a_{0}\left(N-i_1,x\right)\\
&=x^{N}\sum_{i_1=0}^{N-1}\left(N-i_1\right).
\end{aligned} \tag{2.20}$$

By (2.17), (2.18) and (2.19), we get

$$\begin{aligned}
a_{2}\left(N+1,x\right)&=\sum_{i_2=0}^{N-2}x^{i_2}\left(N-i_2+1\right)a_{1}\left(N-i_2,x\right)\\
&=x^{N-1}\sum_{i_2=0}^{N-2}\sum_{i_1=0}^{N-2-i_2}\left(N-i_2+1\right)\left(N-i_2-i_1-1\right),
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
a_{3}\left(N+1,x\right)&=\sum_{i_3=0}^{N-3}x^{i_3}\left(N-i_3+2\right)a_{2}\left(N-i_3,x\right)\\
&=x^{N-2}\sum_{i_3=0}^{N-3}\sum_{i_2=0}^{N-3-i_3}\sum_{i_1=0}^{N-3-i_3-i_2}\left(N-i_3+2\right)\left(N-i_3-i_2\right)\\
&\quad\times\left(N-i_3-i_2-i_1-2\right),
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
a_{4}\left(N+1,x\right)&=\sum_{i_4=0}^{N-4}x^{i_4}\left(N-i_4+3\right)a_{3}\left(N-i_4,x\right)\\
&=x^{N-3}\sum_{i_4=0}^{N-4}\sum_{i_3=0}^{N-4-i_4}\\
&\quad\times\sum_{i_2=0}^{N-4-i_4-i_3}\sum_{i_1=0}^{N-4-i_4-i_3-i_2}\left(N-i_4+3\right)\left(N-i_4-i_3+1\right)\\
&\qquad\times\left(N-i_4-i_3-i_2-1\right)\left(N-i_4-i_3-i_2-i_1-3\right).
\end{aligned} \tag{2.23}$$

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Continuing this process, we get

(2.24)

$$a_j(N+1,x) = x^{N-j+1}\sum_{i_j=0}^{N-j}\sum_{i_{j-1}=0}^{N-j-i_j}\cdots\sum_{i_1=0}^{N-j-i_j-\cdots-i_2}\prod_{k=1}^{j}\left(N-i_j-\cdots-i_k-(j-(2k-1))\right).$$

Therefore, we obtain the following theorem.

Theorem 1. *For $N\in\mathbb{N}$, the linear differential equations*

$$F^{(N)}=\left(\frac{d}{dt}\right)^N F(t,x)=\left(\sum_{i=N}^{2N-1}a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}\right)F$$

has a solution $F=F(t,x)=e^{x\left(1-\sqrt{1-2t}\right)}$, where

$$a_0(N,x)=x^N,\quad a_{N-1}(N,x)=(2n-3)!!x,$$

$$a_j(N,x)=x^{N-j}\sum_{i_j=0}^{N-j-1}\sum_{i_{j-1}=0}^{N-j-1-i_j}\cdots\sum_{i_1=0}^{N-j-1-i_j-\cdots-i_2}\times\left(\prod_{k=1}^{j}\left(N-i_j-i_{j-1}-\cdots-i_k-(j-(2k-2))\right)\right).$$

Recall the the reverse Bessel polynomials $p_k(x)$ are given by the generating function as

$$F=F(t,x)=e^{x\left(1-\sqrt{1-2t}\right)}=\sum_{k=0}^{\infty}p_k(x)\frac{t^k}{k!}. \tag{2.25}$$

Thus, by (2.25), we get

$$\begin{aligned}F^{(N)}&=\left(\frac{d}{dt}\right)^N F(t,x)\\&=\sum_{k=N}^{\infty}p_k(x)(k)_N\frac{t^{k-N}}{k!}\\&=\sum_{k=0}^{\infty}p_{k+N}(x)(k+N)_N\frac{t^k}{(k+N)!}\\&=\sum_{k=0}^{\infty}p_{k+N}(x)\frac{t^k}{k!}.\end{aligned} \tag{2.26}$$

On the other hand, by Theorem 1, we get

$$\begin{aligned}F^{(N)}&=\left(\sum_{i=N}^{2N-1}a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}\right)F\\&=\sum_{i=N}^{2N-1}a_{i-N}(N,x)\left(\sum_{l=0}^{\infty}\left(-\frac{i}{2}\right)_l\frac{(-2t)^l}{l!}\right)\left(\sum_{m=0}^{\infty}p_m(x)\frac{t^m}{m!}\right)\end{aligned} \tag{2.27}$$

$$= \sum_{k=0}^{\infty} \left\{ \sum_{i=N}^{2N-1} a_{i-N}(N,x) \sum_{l=0}^{k} \binom{k}{l} 2^l \left(\frac{i}{2} + l - 1 \right)_l p_{k-l}(x) \right\} \frac{t^k}{k!}.$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2. *For $k \in \mathbb{N} \cup \{0\}$, and $N \in \mathbb{N}$, we have*

$$p_{k+N}(x) = \sum_{i=N}^{2N-1} a_{i-N}(N,x) \sum_{l=0}^{k} \binom{k}{l} 2^l \left(\frac{i}{2} + l - 1 \right)_l p_{k-l}(x),$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$, $(n \geq 1)$, and $(x)_0 = 1$.

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On existence and comparison results for solutions to stochastic functional differential equations in the G-framework

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October 5, 2016

Abstract

With the advancement in stochastic calculus, stochastic differential equations have now become very common in different fields such as engineering, population dynamics, physics, system sciences, ecological sciences, medicine and financial mathematics. In several stochastic dynamic systems, one assumes that the future state of the system does not depend on its past states. However, under close analysis, it becomes evident that most realistic models would contain some of the past states of the system, and one would require stochastic functional differential equations in order to study such systems. This paper presents the existence theory for stochastic functional differential equations in the G-framework (in short G-SFDEs). The comparison theorem has been developed in a bid to obtain the required results. It is ascertained that the G-SFDEs, whose coefficients may be discontinuous functions, have more than one continuous and bounded solutions.

Key words: Existence, G-Brownian motion, Stochastic functional differential equations, discontinuous coefficients.

1 Introduction

In the last twenty years, the greater requirement for tools and procedure of stochastic calculus has been recorded in different scientific fields. In the study of financial markets, it has acquired the state of an essential element, projected in dynamic phenomena of routine changes in share and stock prices. Stochastic calculus has its applications in engineering, as well as in filtering and control theory, and even in physics, when it deals with the effect of random changes on different physical phenomena. In Biology, its main usage is in modeling the achievement of stochastic changes in reproduction on populations processes. The idea of G-Brownian motion, which is a new

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stochastic process, was given by a Chinese mathematician Shige Peng in 2006 [12]. This theory opened a new era in stochastic calculus and financial mathematics. This type of motion has a newer construction as it does not depend on a specific probability space. This motion explains the ancient Brownian motion in an extraordinary way. In the framework of a sublinear expectation (called as G-expectation), he established the associated Itô's calculus. During his research on stochastic calculus, Peng set up the existence and uniqueness of solutions for stochastic differential equations driven by G-Brownian motion in short (G-SDEs) with Lipschitz continuous coefficients [12, 13]. Then F. Gao generalized the associated Itô's calculus and the existence theory of G-SDEs with Lipschitz continuity condition using the concept of G-capacity and quasi-sure analysis [6]. Y. Ren and L. Hu proved the existence and uniqueness of solutions for G-SDEs under the Carathéodory conditions, while later on, X. Bai and Y. Lin extended the theory for G-SDEs to the integral Lipschitz conditions [1]. In the G-frame, stochastic functional differential equations were introduced by Ren, Bi and Sakthivel [14]. Then studied by Faizullah[4]. He used the Cauchy-Maruyama approximation scheme to establish the existence-and-uniqueness theorem for SFDEs in the G-frame with linear growth condition as well as Lipschitz continuity condition [4]. In a different manner, this paper explores the existence theory for SFDEs in the G-frame, whose coefficients may not be continuous. This is the generalization of the previous work by Faizullah, Mukhtar and Rana [5]. We consider stochastic functional differential equations in the G-framework of the following type

$$dY(t) = \kappa(t, Y_t)dt + \lambda(t, Y_t)d\langle B, B\rangle(t) + \lambda(t, Y_t)dB(t),\ 0 \leq t \leq T. \tag{1.1}$$

Recall that $Y_t = \{Y(t+\theta) : -\delta \leq \theta \leq 0, \delta > 0\}$ is a bounded continuous stochastic process from $[-\tau, 0]$ to $\mathbb{R}$ where at time t, the value of stochastic process is denoted by $Y(t)$ [4]. Also, Y_t indicates the collection of continuous bounded real-valued functions ψ defined on $[-\delta, 0]$ with norm $\|\psi\| = \sup_{-\delta\leq\theta\leq 0} \mid \psi(\theta) \mid$. Let κ, λ and μ are Borel measurable functions from $[0, T] \times BC([-\tau, 0]; \mathbb{R})$ to $\mathbb{R}$. We define the initial data of equation (1.1) as follows;

$$\begin{aligned} Y_{t_0} =& \zeta = \{\zeta(\theta) : -\tau < \theta \leq 0\}\ is\ \mathcal{F}_0 - measurable,\ BC([-\tau, 0]; \mathbb{R}) -\ valued \\ &random\ variable\ so\ that\ \zeta \in M_G^2\left([-\tau, 0]; \mathbb{R}\right). \end{aligned} \tag{1.2}$$

The integral form of problem (1.1) is given as the following

$$Y(t) = \zeta(0) + \int_0^t \kappa(s, Y_s)ds + \int_0^t \lambda(s, Y_s)d\langle B, B\rangle(s) + \int_0^t \mu(s, Y_s)dB(s).$$

The G-SFDE (1.1) admit at most solution $Y(t) \in M_G^2\left([-\tau, T]; \mathbb{R}\right)$ if all its coefficients gratify the linear growth condition as well as Lipschitz condition. [4, 14]. On the other hand, in this article we assume that the coefficients κ and λ may be discontinuous functions. The solution to problem 1.1 with initial data 1.2 is a real valued stochastic process $Y(t)$, $t \in [-\tau, T]$ if it holds the following characteristics

(**a**) For every $t \in [0, T]$, $Y(t)$ is $\mathcal{F}_t$-adapted as well as path-wise continuous.

(**b**) $\kappa(t, Y_t), \lambda(t, Y_t) \in \mathcal{L}^1([o, T]; \mathbb{R})$ and $\mu(t, Y_t) \in \mathcal{L}^2([o, T]; \mathbb{R})$;

(**c**) $Y_0 = \zeta$ and $dY(t) = \kappa(t, Y_t)dt + \lambda(t, Y_t)d\langle B, B\rangle(t) + \mu(t, Y_t)dB(t)$ q.s. for each $t \in [0, T]$.

The rest of the paper is organized as follows. Some basic definitions and notions are given in the subsequent section. Section 3 presents an important results known as the comparison theorem. The final section develops the existence theorem with possible discontinuous coefficients.

2 Preliminary Concerns

This section presents some basic notions and results, which are used in forthcoming research work of this paper [2, 3, 6, 13].

2.1 Sublinear Expectation

Suppose that Ω (sample space) is a grand set and H be a family of linear and real valued functions described on Ω. Suppose that H fulfil $k \in H$, for any constant k and $|Y| \in H$ if $Y \in H$. H containing the stochastic variables.

Definition 2.1. A functional E, where $E : H \to R$, is known as a G-expectation or sublinear expectation if

(**1**) E is monotonic, that is, if $Y \geq Z$ for all $Y, Z \in H \Rightarrow E[Y] \geq E[Z]$.

(**2**) E is constant conserving, that is, $E[k] = k \;\; k \in H$.

(**3**) E is sub-additive, that is, if $E[Y + Z] \leq E[Y] + E[Z]$, for each $Y, Z \in H$.

(**4**) E is positive homogeneous, that is, $E[bY] = b[y]$ for $b \geq 0$.

the space given by triple (Ω, H, E) is said to be sublinear expectation space. And E is nonlinear expectation if it satisfies the above two conditions. Sublinear expectation is also able to state the supremum of linear expectation

Definition 2.2. G-Brownian motion A d-dimensional process $(B_t)_{t \geq 0}$, define on $(\Omega, C_{l,lip}(H), E)$, is known as G-Browmain motion, if the following conditions are hold.

(**1**) $B_0(w) = 0$.

(**2**) The increment $B_{t+r} - B_t$ is G-normally distributed for any $t, r \geq 0$.

(**3**) $B_{t+r} - B_t$ is independent from $B_{t_1}, B_{t_2}, ...B_{t_n}$ for any $n \in N,\ t, r \geq 0$ and $0 \leq t_1 \leq t_2 \leq , ... \leq t_n \leq t$.

2.2 Ito's integral of G-Brownian motion

Definition 2.3. If $T \in R^+$, a partition π_T of the interval $[0, T]$ is

$$\pi_T = \{t_0, t_1, ..., t_N\},$$

since

$$\rho(\pi_T) = max\{|t_{\epsilon+1} - t_\epsilon| : \epsilon = 0, 1, ...N - 1\},$$

where

$$0 = t_0 \leq t_1 \leq, ... \leq t_N = T,$$

we customize $\pi_T^N = \{t_0^N, t_1^N, ...t_N^N\}$ to represent a sequence of partition of $[0, T]$ since

$$\lim_{N\to\infty} \rho(\pi_T^N) = 0.$$

Let $p \geq 1$. Suppose the following sort of processes of a partition

$$\pi_T = \{t_0, t\pi_1, ..., t_N\}.$$

We take,

$$\eta_t(\omega) = \sum_{m=0}^{N-1} \xi_m(\omega) I_{[t_m, t_{m+1})}(t)$$

where $\xi_m \in L_G^p(\omega_{tm})$, for all $m = 0, 1, 2, ...N-1$. The group of these process is represented by $M_G^{p,0}(0, T)$.

Definition 2.4. Let $\eta \in M_G^{1,0}(0, T)$ with

$$\eta_t(\omega) = \sum_{m=0}^{N-1} \xi_m(\omega) I_{[t_m, t_{m+1})}(t)$$

it can be written as,

$$\int_0^T \eta_t(\omega) dt = \sum_{m=0}^{N-1} \xi_m(\omega)(t_{m+1} - t_m)$$

Definition 2.5. For every $p \geq 1$, we represent by $M_G^p(0, T)$ the completion of $M_G^{p,0}(0, T)$ under the norm

$$\|\eta\|_{M_G^p(0,T)} = \{E[\int_0^T |\eta_t|^p dt]\}^{1/p},$$

where for $1 \leq p \leq q$, $M_G^p(0, T) \supset M_G^q(0, T)$.

Definition 2.6. For every $\eta \in M_G^p(0, T)$ of the arrangement

$$\eta_t(w) = \sum_{\epsilon=0}^{N-1} \xi_\epsilon(w) I[t_\epsilon, t_{\epsilon+1})(t),$$

it can be written as,

$$I(\eta) = \int_0^T \eta_t dB_t = \sum_{\epsilon=0}^{N-1} \xi_\epsilon(B_{t_{\epsilon+1}} - B_{t_\epsilon}).$$

Lemma 2.7. *Let a function $I : M_G^{2,0}(0, T) \to L_G^2(\Omega_T)$, then it can be continuously extended to $I : M_G^2(0, T) \to L_G^2(\Omega_T)$. Moreover,*

$$E[\int_0^T \eta_t dB_t] = 0,$$

$$E[(\int_0^T \eta_t dB_t)^2] \leq \sigma^2 E[\int_0^T \eta_t^2 dt].$$

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2.3 (Peng's quadratic variation process $\langle B\rangle_t$)

Definition 2.8. A 1-dimensional G-quadratic variation process is introduced as follows. Let $\pi_t^N, N = 1, 2, ...$, be a sequence of the partition $[0, T]$ then

$$\begin{aligned}B_t^2 &= \sum_{\epsilon=0}^{N-1}(B^2_{tN_{\epsilon+1}} - B^2_{tN_\epsilon})\\ &= \sum_{\epsilon=0}^{N-1} 2B_{tN_\epsilon}(B_{tN_{i+\epsilon}} - B_{tN_\epsilon}) + \sum_{\epsilon=0}^{N-1}(B_{tN_{\epsilon+1}} - B_{tN_\epsilon})^2.\end{aligned}$$

Taking limit $\mu(\pi_t^N) \to 0$

$$\sum_{\epsilon=0}^{N-1} 2B_{tN_\epsilon}(B_{tN_{\epsilon+1}} - B_{tN_\epsilon}) \quad \textit{converges to} \quad 2\int_0^t B_s dB_s,$$

and we have

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_s dB_s.$$

Definition 2.9. Let $\mathcal{P}$ be a (weakly compact) collection of probability measures P defined on $(\Omega, \mathcal{B}(\Omega))$ then the capacity $\hat{c}(.)$ associated to $\mathcal{P}$ is defined by

$$\hat{c}(B) = \sup_{P\in\mathcal{P}} P(B), \quad B \in \mathcal{B}(\Omega),$$

where $\mathcal{B}(\Omega)$ is the Borel σ-algebra of Ω. A set B is said to be polar if its capacity is zero, that is, $\hat{c}(B) = 0$ and a statement holds quasi-surely in short (q.s.) if it holds except on a polar set.

3 An important result

In this section, we establish an important result known as comparison theorem. First, we assume two stochastic functional integral equations given as follows.

$$Y(t) = \zeta_1(0) + \int_{t_0}^t \kappa_1(s, Y_s)ds + \int_{t_0}^t \lambda_1(s, Y_s)d\langle B, B\rangle(s) + \int_{t_0}^t \mu(s, Y_s)dB(s), \qquad t \in [0, T], \quad (3.1)$$

$$Y(t) = \zeta_2(0) + \int_{t_0}^t \kappa_2(s, Y_s)ds + \int_{t_0}^t \lambda_2(s, Y_s)d\langle B, B\rangle(s) + \int_{t_0}^t \mu(s, Y_s)dB(s), \qquad t \in [0, T]. \quad (3.2)$$

Theorem 3.1. *Let Y^1 and Y^2 are the respective unique solutions of equations* (3.1) *and* (3.2)*. Suppose that $\kappa_1(s, Y_s) \le \kappa_2(s, Y_s)$ and $\lambda_1(s, Y_s) \le \lambda_2(s, Y_s)$ are componentwise for every $t \in [t_0, T]$, $y \in BC([-\tau, 0]; \mathbb{R}^d)$ and $\zeta^1 \le \zeta^2$. Also, let the coefficients κ_1, λ_1 or κ_2, λ_2 are increasing functions. Then for every $t > 0$, $Y^1 \le Y^2$ q.s.*

Proof. Suppose that κ_2 and λ_2 are increasing and consider the problem

$$\begin{aligned} Z(t) = \zeta_2(0) + \int_{t_0}^{t} \kappa_2(s, \max\{Y_s{}^1, Z_s\})ds + \int_{t_0}^{t} \lambda_2(s, \max\{Y_s{}^1, Y_s\})d\langle B, B\rangle(s) \\ + \int_{t_0}^{t} \mu(s, \max\{Y_s{}^1, Z_s\})dB(s), \ \ t_0 \leq t \leq T, \end{aligned} \tag{3.3}$$

where the function $x \to \max\{y, z\}$ satisfies the growth condition $|\max\{y, z\}| \leq |y| + |z|$ and the Lipschitz condition with constant one. It follows that all coefficients of the above equation 3.3 gratify the growth condition as well as Lipschitz condition. Thus problem 3.3 admit the only one solution say $Z(t)$. Now one has to show that $Z(t) \geq Y_s^1$ q.s. First define stopping times δ_1 and δ_2 as follows. More details on stopping times can be found in [9, 10, 11].

$$\begin{aligned} \delta_1 &= \inf\{t \in [t_0, T] : Y_s^1 - Z(t) > 0\} \ \ where \ \ \delta_1 < T, \\ \delta_2 &= \inf\{t \in [\tau_1, T] : Y_s^1 - Z(t) < 0\}. \end{aligned}$$

Contrary assume that $(\delta_1, \delta_2) \subset [t_0, T]$ be an arbitrary interval, such that $Z(\delta_1) = Y^1(\delta_1) = \zeta^*(0)$ and $Z(t) \leq Y^1(t)$ for every $t \in (\delta_1, \delta_2)$. Then,

$$\begin{aligned} Z(t) - Y^1(t) = \zeta^*(0) &+ \int_{\delta_1}^{t} \kappa_2(s, \max\{Y_s{}^1, Z_s\})ds + \int_{\delta_1}^{t} \lambda_2(s, \max\{Y_s{}^1, Z_s\})d\langle B, B\rangle(s) \\ &+ \int_{\delta_1}^{t} \mu(s, \max\{Y_s{}^1, Z_s\})dB(s) - \zeta^*(0) - \int_{\delta_1}^{t} \kappa_1(s, Y_s{}^1)ds \\ &- \int_{\delta_1}^{t} \lambda_1(s, Y_s{}^1)d\langle B, B\rangle(s) - \int_{\delta_1}^{t} \mu(s, Y_s{}^1)dB(s), \ \ t \in (\delta_1, \delta_2). \end{aligned}$$

$$\begin{aligned} Z(t) - Y^1(t) = &\int_{\delta_1}^{t} [\kappa_2(s, \max\{Y_s{}^1, Z_s\}) - \kappa_1(s, Y_s{}^1)]ds \\ &+ \int_{\delta_1}^{t} [\lambda_2(s, \max\{Y_s{}^1, Z_s\}) - \lambda_1(s, Y_s{}^1)]d\langle B, B\rangle(s) \\ &+ \int_{\delta_1}^{t} [\mu(s, \max\{Y_s{}^1, Z_s\}) - \mu(s, Y_s{}^1)]dB(s), \ \ t \in (\delta_1, \delta_2). \end{aligned}$$

But the assumption $Z(t) \leq Y^1(t)$ gives $\max[Y^1, Z] = Y^1$. So, we have

$$\begin{aligned} Z(t) - Y^1(t) = &\int_{\delta_1}^{t} [\kappa_2(s, Y_s{}^1) - \kappa_1(s, Y_s{}^1)]ds \\ &+ \int_{\delta_1}^{t} [\lambda_2(s, Y_s{}^1) - \lambda_1(s, Y_s{}^1)]d\langle B, B\rangle(s) \\ &+ \int_{\delta_1}^{t} [\mu(s, Y_s{}^1) - \mu(s, Y_s{}^1)]dB(s), \end{aligned}$$

which gives $Z(t) \geq Y^1(t)$ because $\kappa_2(t, y) \geq \kappa_1(t, y)$ and $\lambda_2(t, y) \geq \lambda_1(t, y)$. This gives contradiction. So, the supposition $Z(t) \leq Y^1(t)$ for every $t \in (\delta_1, \delta_2)$ is not true. Thus $Z(t) \geq Y^1(t)$ q.s. and hence $\max\{Y^1, Z\} = Z$. It follows that $Z = Y^2 \geq Y^1$ because problem (3.3) admit a single solution Y^2. The proof is complete. □

4 Existence of solutions to SFDEs in the G-framework

Next, we assume that the coefficients κ and λ are not continuous. However, they are increasing, left continuous and $\kappa(t,y) \geq 0$, $\lambda(t,y) \geq 0$ for every $(t,y) \in [0,T] \times BC([-\delta,0];\mathbb{R})$. Assume a sequence of problems given as follows.

$$Y^l(t) = \zeta(0) + \int_0^t \kappa(s, Y_s^{l-1})ds + \int_0^t \lambda(s, Y_s^l)d\langle B, B\rangle(s) + \int_0^t \mu(s, Y_s^l)dB(s), \ t \in [0,T], \tag{4.1}$$

where $Y^0 = L_t$, L_t is the unique solution of the equation given by

$$L_t = \zeta + \int_0^t \mu(s, L_s)dB(s), \tag{4.2}$$

where $t \in [0,T]$. By our supposition $\kappa(t,y) \geq 0$, $\lambda(t,y) \geq 0$ and comparison result we obtain $Y^1 \geq L_t$. Thus, one can see that the sequence $\{Y^l : l \geq 1\}$ is increasing. In the following lemma we show that Y^l is bounded.

Lemma 4.1. *Let $Y^l(t)$ denotes a solution of equation* (4.1)*. Then*

$$E\left(\sup_{-\delta\leq s\leq T} |Y^l(s)|^2\right) \leq K,$$

where $K = C_6 e^{C_5 T}$, $C_6 = E[\|\zeta\|] + C_4$, $C_5 = 4(C_1 + C_2 + C_3)$, $C_4 = 4[E|\zeta|^2 + C_1 T + C_2 T + C_3 T]$, C_1, C_2 and C_3 are positive constants.

Proof. Define the following stopping time, for any $l \geq 1$

$$\delta_m = T \wedge inf\{t \in [t_0, T] : \|Y_t^l\| \geq m\}.$$

We get $\delta_m \uparrow T$ and define $Y^{l,m}(t) = Y^l(t \wedge \delta_m)$ for $t \in (-\tau, T)$. Next we proceed as follows.

$$Y^{l,m}(t) = \zeta(0) + \int_0^t \kappa(s, Y_s^{l-1,m}) I_{[o,\delta_m]}ds + \int_0^t \lambda(s, Y_s^{l,m}) I_{[o,\delta_m]}d\langle B, B\rangle_s + \int_0^t \mu(s, Y_s^{l,m}) I_{[o,\delta_m]}dB_s.$$

$$\begin{aligned}
|Y^{l,m}(t)|^2 &= |\zeta(0) + \int_0^t \kappa(s, Y_s^{l-1,m}) I_{[0,\delta_m]}ds + \int_0^t \lambda(s, Y_s^{l,m}) I_{[0,\delta_m]}d\langle B, B\rangle_s \\
&+ \int_0^t \mu(s, Y_s^{l,m}) I_{[0,\delta_m]}dB_s|^2 \\
&\leq 4|\zeta(0)|^2 + 4|\int_0^t \kappa(s, Y_s^{l-1,m}) I_{[0,\delta_m]}ds|^2 + 4|\int_0^t \lambda(s, Y_s^{l,m}) I_{[0,\delta_m]}d\langle B, B\rangle_s|^2 \\
&+ 4|\int_0^t \mu(s, Y_s^{l,m}) I_{[0,\delta_m]}dB_s|^2
\end{aligned}$$

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By taking G-expectation on both sides, using the linear growth condition and Burkholder-Davis-Gundy inequalities [6, 13] we proceed as follows

$$\begin{aligned}
E[|Y^{l,m}(t)|^2] &\leq 4E|\zeta(0)|^2 + 4C_1\int_0^t [1+E|Y_s^{l-1,m}|^2]ds + 4C_2\int_0^t [1+E|Y_s^{l,m}|^2]ds| \\
&+ 4C_3\int_0^t [1+E|Y_s^{l,m}|^2]ds \\
&\leq 4E|\zeta(0)|^2 + 4C_1\int_0^t ds + 4C_1\int_0^t E|Y_s^{l-1,m}|^2 ds + 4C_2\int_0^t dt + 4C_2\int_0^t E|Y_s^{l,m}|^2 ds \\
&+ 4C_3\int_0^t ds + 4C_3\int_0^t E|Y_s^{l,m}|^2 ds \\
&= 4E|\xi(0)|^2 + 4C_1T + 4C_1\int_0^t E|Y_s^{l-1,m}|^2 ds + 4C_2T + 4C_2\int_0^t E|Y_s^{l,m}|^2 ds \\
&+ 4C_3T + 4C_3\int_0^t E|Y_s^{l,m}|^2 ds.
\end{aligned}$$

For any $j \in \mathbb{N}$ we get,

$$\max_{1\leq l\leq j} E[|Y^{l,m}(t)|^2] \leq C_4 + 4C_1\int_0^t \max_{1\leq l\leq j} E|Y_s^{l-1,m}|^2 ds + 4C_2\int_0^t \max_{1\leq l\leq j} E|Y_s^{l,m}|^2 ds + 4C_3\int_0^t \max_{1\leq l\leq j} E|Y_s^{l,m}|^2 ds,$$

where $C_4 = 4[E|\zeta|^2 + C_1T + C_2T + C_3T]$. Hence by Doob's martingale inequality we get for any $l, m \in \mathbb{N}$

$$E[\sup_{0\leq s\leq t} |Y^{l,m}(s)|^2] \leq C_4 + C_5\int_0^t E|Y_s^{l,m}|^2 ds, \tag{4.3}$$

where $C_5 = 4(C_1 + C_2 + C_3)$. One can observe the fact [11],

$$\sup_{-\delta\leq s\leq t} |Y^{l,m}(v)|^2 \leq \|\zeta\| + \sup_{0\leq s\leq t} |Y^{l,m}(s)|^2,$$

and hence 4.3 gives

$$\begin{aligned}
E[\sup_{-\delta\leq s\leq t} |Y^{l,m}(s)|^2] &\leq E[\|\zeta\|] + C_4 + C_5\int_0^t E|Y_s^{l,m}|^2 ds \\
&\leq C_6 + C_5\int_0^t E[\sup_{-\delta\leq q\leq s} |Y^{l,m}(q)|^2] ds,
\end{aligned}$$

where $C_6 = E[\|\zeta\|] + C_4$. Finally, taking $m \to \infty$ and by the Gronwall's inequality we get,

$$E[\sup_{-\delta\leq s\leq t} |Y^l(s)|^2] \leq C_6 e^{C_5 t}.$$

Letting $t = T$ we have

$$E[\sup_{-\delta\leq s\leq T} |Y^l(s)|^2] \leq K,$$

where $K = C_6 e^{C_5 T}$. Hence, the proof stands completed. □

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Theorem 4.2. *Let the coefficients $\kappa(t,y)$ and $\lambda(t,y)$ are increasing in the second variable y and left continuous. For all $(t,y) \in [0,T] \times BC([-\tau,0];\mathbb{R})$, $\kappa(t,y) \geq 0$ and $\lambda(t,y) \geq 0$. Then there exists at least one solution $Y(t) \in M_G^2([-\tau,T];\mathbb{R})$ to problem* (1.1).

Proof. Theorem 3.1 follows that the sequence $\{Y^l\}$ is increasing. On the other hand, Lemma 4.1 shows that $\{Y^l\}$ is a bounded sequence in the norm $\mathbb{L}^2$. Thus dominated convergence theorem yields that Y^n converges in $\mathbb{L}^2$. Let Y be the limit of Y^l. Then for almost all w, we have

$$\kappa(t,Y^l(t)) \to \kappa(t,Y(t)) \ as \ l \to \infty,$$
$$\lambda(t,Y^l(t)) \to \lambda(t,Y(t)) \ as \ l \to \infty.$$

Also

$$|\kappa(t,Y^l(t))| \leq K(1 + \sup_l |Y^l(t)|) \in L^1([t_0,T]),$$
$$|\lambda(t,Y^l(t))| \leq K(1 + \sup_l |Y^l(t)|) \in L^1([t_0,T]).$$

Since $\langle B \rangle$ is continuous, so, for uniformly in t and almost all w

$$\int_0^t \kappa(s,Y^l(s))ds \to \int_0^t \kappa(s,Y(s))ds, \ l \to \infty,$$
$$\int_0^t \lambda(s,Y^l(s))\langle B,B\rangle(s) \to \int_0^t \lambda(s,Y(s))\langle B,B\rangle(s), \ l \to \infty.$$

Since G-integral is continuous we get,

$$\sup_{0\leq t\leq T} \left| \int_0^t \mu(s,Y^l(s))dB(s) - \int_0^t \mu(s,Y(s))dB(s) \right| \to 0 \ (q.s), \ l \to \infty.$$

Obviously, the sequence Y^l converges uniformly to Y in t, hence Y is continuous. Taking limits $l \to \infty$ on both sides of equation (4.1), we obtain that Y is the solution to G-SFDE (1.1) with initial condition (1.2). □

5 Acknowledgment

The financial support of NUST research directorate for this research work is acknowledged and deeply appreciated.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.4, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Interval-valued intuitionistic fuzzy Choquet integral operators based on Archimedean t-norm and their calculations†

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Abstract: It is necessary to assume additivity and independent among decision making criteria for traditional multiple decision making (MDM) in which the weights given by decision makers based on a additive measure. However, most criteria have inter-dependent or interactive characteristics in the real decision making problems. Furthermore, with respect to multiple attribute group decision making (MAGDM) problems in which the attribute weights and the expert weights take the form of real numbers and the attribute values take the form of interval-valued intuitionistic sets, we propose interval-valued intuitionistic fuzzy Choquet integral operators based on Archimedean t-norm and discuss their calculations in this paper. First, we introduce some concepts of fuzzy measure, interval-valued intuitionistic sets and Archimedean t-norm. Then, the representations and transformations of Archimedean t-norm and Archimedean t-conorm are obtained, and the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are presented under intuitionistic fuzzy environment. Finally, as fuzzy Choquet integral operators, some aggregating of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are given.

Keywords: Intuitionistic sets; Fuzzy Choquet integral operators; Archimedean t-norm.

1. Introduction

Multiple attribute decision making (MADM) problem is an important research topic in decision theory. Because the objects are fuzzy and uncertain, the attributes involved in decision problems are not always expressed as real numbers, and some better suited to be denoted by fuzzy numbers, such as interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, linguistic numbers on uncertain linguistic variables, and intuitionistic fuzzy numbers. Because Zadeh initially proposed the basic model of fuzzy decision making based on the theory of fuzzy mathematics, fuzzy MADM has been receiving more and more attention. We also notice that the main technologies in multiple attribute decision making, whether the situation is certain or vague, are how to define and calculate the aggregation operators proposed in the practice.

The fuzzy set (FS) theory proposed by Zadeh [1] was a very good tool to research the fuzzy MADM problems, the fuzzy set is used to character the fuzziness just by membership degree. Different from fuzzy set, there is another parameter: non-membership degree in intuitionistic fuzzy set (IFS) which is proposed by Atanassov [2, 3]. Clearly, the IFS can describe and character the fuzzy essence of the objective world more accurately [2] than the fuzzy set, and has received more and more attention since its appearance. Later, Atanassov and Gargov [4, 5] further introduced the interval-valued intuitionistic fuzzy set (IVIFS), which is a generalization of the IFS. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are interval numbers rather real numbers.

Base on Archimedean t-conorm and t-norm [6, 7], and the aggregation functions for the classical fuzzy sets (FSs), Beliakov et al. gave some operations about intuitionistic fuzzy sets, proposed two general concepts for constructing other types of aggregation operators for intuitionistic fuzzy sets (IFSs)

† This work was supported by the Key Subjects Construction of Tianshui Normal University.

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extending the existing methods and showed that the operators obtained by using the Lukasiewicz t-norm are consistent with the ones on ordinary FSs. We can find above aggregation operators are all based on different relationships of the aggregated arguments, which can provide more choices for the decision makers.

As an aggregation function, it is well-known that Choquet integral [8] based on non-additive fuzzy measure, is a kind of non-additive and non-linear integral, and has been successfully used for handling information fusion and decision making problems (MCDM). The main characteristic of this aggregation function is that it is able to flexibly describe the relative importance of decision criteria as well as their interactions. There are many works on the Choquet integral of single-valued functions, set-valued functions and studied their mathematical properties. It is of interest to combine the Choquet integral and the IFS theory or MCDM under intuitionistic fuzzy environment, because, by doing this, we cannot only deals with the imprecise and uncertain decision information but also efficiently take into account the various interactions among the decision criteria. The intuitionistic fuzzy-valued Choquet integral, the combination of the Choquet integral and the IFS theory, can also act an aggregation tool employed in MCDM as well as other multicriteria analysis field. In this paper, we propose the interval-valued intuitionistic fuzzy Choquet integral operators based on Archimedean t-norm and discuss their calculations. First, we introduced some concepts of fuzzy measure and interval-valued intuitionistic sets based on Archimedean t-norm. Then, interval-valued intuitionistic weighted average(geometric) operator based on Archimedean t-norm, interval-valued intuitionistic ordered weighted average operator based on Archimedean t-norm are developed.

The rest of this study is organized as follows. In section 2, we recall the definitions of intuitionistic fuzzy set、Archimedean t-norm and Choquet integral. In section 3, the representations and transformations of Archimedean t-norm and Archimedean t-conorm are proposed and inveastigated, and some of its properties are investigated in detail by means of the representation theorem. In section 4, the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm is presented under intuitionistic fuzzy environment. In section 5, an aggregating of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are defined and discussed.

2. Definitions and preliminaries

A fuzzy measure on X is a set function $\mu : P(X) \to [0,1]$ such that
(i) $\mu(\emptyset) = 0, \mu(X) = 1$;
(ii) $A, B \subseteq X$, $A \subseteq B$ implies $\mu(A) \leqslant \mu(B)$.

Definition 2.1. Let $A, B \in P(X), A \cap B = \varnothing$. If fuzzy measure g satisfies the following conditions:

$$g(A \cup B) = g(A) + g(B) + \lambda g(A)g(B)$$

and $\lambda \in (-1, \infty)$.

Especially if $\lambda = 0$, then g is an additive measure, which means there is no interaction between coalitions A and B .

Let $X = \{x_1, x_2, ...x_n\}$ be a attribute index set, if $i, j = 1, 2, ..., n$ and $i \neq j, x_i \cap x_j = \varnothing, \bigcup_{i=1}^n x_i = X$, then

$$g(X) = \begin{cases} \frac{1}{\lambda}(\prod_{i=1}^n [1 + \lambda g(x_i)] - 1) & \lambda \neq 0, \\ \sum_{i=1}^n g(x_i) & \lambda = 0, \end{cases} \tag{1}$$

From Eq. (1), for the $A \in P(X)$, g can be expressed by

$$g(X) = \begin{cases} \frac{1}{\lambda}(\prod_{i \in A} [1 + \lambda g(x_i)] - 1) & \lambda \neq 0, \\ \sum_{i \in A} g(x_i) & \lambda = 0, \end{cases} \tag{2}$$

For x_i, $g(x_i)$ is called a fuzzy measure function, and it indicates the importance degree of x_i.

From $g(X) = 1$, we know λ is determined by $\lambda + 1 = \prod_{i=1}^n (1 + \lambda g(x_i))$.

Definition 2.2. Let f be a positive real-valued function on X, the discrete Choquet integral of f with respect to a fuzzy measure μ on X is defined as

$$C_\mu(f(x_{(1)}), ..., f(x_{(n)})) = \sum_{i=1}^n f(x_{(i)})[\mu(A_{(i)}) - \mu(A_{(i+1)})]$$

where $(\cdot)$ indicates a permutation on X such that $f(x_{(1)}) \leqslant \cdots \leqslant f(x_{(n)})$. $A_{(i)} = (i, \ldots, n)$, and $A_{(n+1)} = \emptyset$.

A function $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following four conditions [8,9]:

1) $T(1, x) = x$, for all x.
2) $T(x, y) = T(y, x)$, for all x and y.
3) $T(x, T(y, z)) = T(T(x, y), z)$, for all x, y and z.
4) $x \leqslant x', y \leqslant y'$ implies $T(x, y) \leqslant T(x', y')$, $x, y, x', y' \in [0,1]$.

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following four conditions[8,9]:

1) $S(0, x) = x$, for all x.
2) $S(x, y) = S(y, x)$, for all x and y.
3) $S(x, S(y, z)) = S(S(x, y), z)$, for all x, y and z.
4) $x \leqslant x', y \leqslant y'$ implies $S(x, y) \leqslant S(x', y')$, $x, y, x', y' \in [0,1]$.

Definition 2.3 [8,9]. A t-norm function $T(x, y)$ is called Archimedean t-norm if it is continuous and $T(x, x) < x$ for all $x \in [0,1]$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for $x, y \in (0,1)$.

A t-conorm function $S(x, y)$ is called Archimedean t-conorm if it is continuous and $S(x, x) > x$ for all $x \in [0,1]$. An Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for $x, y \in (0,1)$.

Definition 2.4. Let X be in a given domain. Then,

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$$

is called an interval-valued intuitionistic fuzzy set ($IVIFS$), where $\mu_A : X \to I \subset [0,1]$, $\nu_A : X \to J \subset [0,1]$ and I, J are closed intervals in $[0,1]$, the following condition is met: $\sup \mu_A(x) + \sup \nu_A(x) \leqslant 1$, $x \in X$. The intervals $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x on X.

Thus for each x, $\mu_A(x)$ and $\nu_A(x)$ are closed intervals and their lower and upper end points are, respectively, denoted by $\mu_A^L(x)$, $\mu_A^U(x)$,$\nu_A^L(x)$,$\nu_A^U(x)$. We can denote by

$$A = \{\langle x, [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]]\rangle | x \in X\},$$

where $0 \leqslant \mu_A^U(x) + \nu_A^U(x) \leqslant 1$, $x \in X$, $\mu_A^L(x) \geqslant 0$ and $\nu_A^L(x) \geqslant 0$.

Simply, we write $A = \langle [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]$.

For each element x, we can compute its hesitation interval of x as:

$$\pi_A(x) = [\pi_A^L(x), \pi_A^U(x)] = [1 - \nu_A^U(x) - \mu_A^U(x), 1 - \nu_A^L(x) - \mu_A^L(x)].$$

3. The representations and transformations of Archimedean t-norm and Archimedean t-conorm

Definition 3.1. A mapping $N : [0,1] \to [0,1]$ is called negation operator, if N is decreasing and $N(0) = 1$, $N(1) = 0$. Especially, we have

(i) If $N(x) = 1 - x$, it is called standard negation operator.

(ii) $\forall x \in [0,1]$, if $N(N(x)) = x$, then it is called cyclotron negation operator. Obviously, cyclotron negation operator is continuous and strictly increasing.

(iii) For each negation operator, T and S are dual with respect to $N(x)$ if and only if $T(N(x), N(y)) = N(S(x, y))$.

It is well known [9] that a strict Archimedean t-norm is expressed via its additive generator g as $T(x, y) = g^{-1}(g(x) + g(y))$, and similarly, applied to its dual t-conorm $S(x, y) = h^{-1}(h(x) + h(y))$ with $h(t) = g(N(t))$. We notice that an additive generator of a continuous Archimedean t-norm is a strictly decreasing function $g : [0,1] \to [0, +\infty)$ such that $g(1) = 0$. If we assign specific forms to the function g, then some well-known t-conorms and t-norms can be obtained. Let me emphasize that the results (1-4) were shown in [11], however, considering that the representation of the negation operator is always restricted by the policy mak- ers' historical knowledge, perceptual judgement and other factors in the game playing, benefit groups' voting or decision making process, we could define the negation operator by means of the fuzzy logic non-portal operators in this paper and calculate Archimedean t-norm and Archimedean t-conorm as results (5-8) as follows.

Theorem 3.1 Let $T(x,y)$ be Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm. Then we have the following statements:

If $N(x)=1-x$, i.e. $h(t)=g(1-t)$), then the following are valid:

(1) Let $g(t)=-\log t$, then $h(t)=-\log(1-t), g^{-1}(t)=\exp^{-t}, h^{-1}(t)=1-\exp^{-t}$, and Algebraic t-conorm and t-norm [10] are obtained as follows:

$$T^A(x,y)=x\cdot y,\ \ S^A(x,y)=x+y-xy.$$

(2) Let $g(t)=\log(\frac{2-t}{t})$, then $h(t)=\log(\frac{2-(1-t)}{1-t}), g^{-1}(t)=\frac{2}{\exp^t+1}, h^{-1}(t)=1-\frac{2}{\exp^t+1}$, and we get Einstein t-conorm and t-norm [10]:

$$T^E(x,y)=\frac{xy}{1+(1-x)(1-y)}\ ,\ S^E(x,y)=\frac{x+y}{1+xy}.$$

(3) Let $g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma>0$, then we have $h(t)=\log(\frac{\gamma+(1-\gamma)(1-t)}{1-t}), g^{-1}(t)=\frac{\gamma}{\exp^t+\gamma-1}$, $h^{-1}(t)=1-\frac{\gamma}{\exp^t+\gamma-1}$, and Hamacher t-conorm and t-norm [10] are obtained as follows:

$$T^H_\gamma(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}\ ,\ \gamma>0,$$

$$S^H_\gamma(x,y)=\frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy}\ ,\ \gamma>0.$$

Especially, if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Algebraic t-conorm and t-norm respectively; if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Einstein t-conorm and t-norm respectively.

(4) Let $g(t)=\log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma>1$, then $h(t)=\log(\frac{\gamma-1}{\gamma^{1-t}-1}), g^{-1}(t)=\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}, h^{-1}(t)=1-\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}$, and we have Frank t-conorm and t-norm [10] as follows:

$$T^F_\gamma(x,y)=\log_\gamma(1+\frac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1})\ ,\ \gamma>1,$$

$$S^F_\gamma(x,y)=1-\log_\gamma(1+\frac{(\gamma^{1-x}-1)(\gamma^{1-y}-1)}{\gamma-1})\ ,\ \gamma>1.$$

Especially, if $\gamma\to1$, then we have

$$\lim_{\gamma\to1}g(t)=\lim_{\gamma\to1}\log(\frac{\gamma-1}{\gamma^t-1})=\lim_{\gamma\to1}\log(\frac{1}{t\gamma^{t-1}-1})=-\log t.$$

which indicates that $\lim_{\gamma\to1}S^F_\gamma(x,y)=S^A_\gamma(x,y)$ and $\lim_{\gamma\to1}T^F_\gamma(x,y)=T^A_\gamma(x,y)$.

If $N(x)=1-x^2$, i.e. $h(t)=g(1-t^2)$ then the following are also valid:

(5) Let $g(t)=-\log t$, then $h(t)=-\log(1-t^2), g^{-1}(t)=\exp^{-t}, h^{-1}(t)=\sqrt{1-\exp^{-t}}$, and Algebraic t-conorm and t-norm [10] are obtained as follows:

$$T^A_2(x,y)=xy\ ,\ S^A_2(x,y)=\sqrt{1-(1-x^2)(1-y^2)}.$$

(6) Let $g(t)=\log(\frac{2-t}{t})$, then we have $h(t)=\log\frac{1+t^2}{1-t^2}, g^{-1}(t)=\frac{2}{\exp^t+1}, h^{-1}(t)=\sqrt{\frac{\exp^t-1}{\exp^t+1}}$, and we get Einstein t-conorm and t-norm [10] are obtained as follows:

$$T^E_2(x,y)=\frac{xy}{1+(1-x)(1-y)}\ ,\ S^E_2(x,y)=\sqrt{\frac{x^2+y^2}{1+x^2y^2}}/$$

(7) Let $g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma>0$, then we have $h(t)=\log(\frac{\gamma+(1-\gamma)(1-t^2)}{1-t^2}), g^{-1}(t)=\frac{\gamma}{\exp^t+\gamma-1}$, $h^{-1}(t)=\sqrt{1-\frac{\gamma}{\exp^t+\gamma-1}}$, and Hamacher t-conorm and t-norm [10] are obtained as follows:

$$T^H_{2\gamma}(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}\ ,\ \gamma>0,$$

$$S^H_{2\gamma}(x,y)=\sqrt{\frac{x^2+y^2-x^2y^2-(1-\gamma)x^2y^2}{1-(1-\gamma)x^2y^2}}\ ,\ \gamma>0.$$

Especially, if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Algebraic t-conorm and t-norm respectively; if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Einstein t-conorm and t-norm respectively.

(8) Let $g(t)=\log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma>1$, then $h(t)=\log(\frac{\gamma-1}{\gamma^{1-t^2}-1}), g^{-1}(t)=\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}$, $h^{-1}(t)=\sqrt{1-\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}}$,
and we have Frank t-conorm and t-norm [10] as follows:

$$T^F_{2\gamma}(x,y)=\log_\gamma(1+\frac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1})\ ,\ \gamma>1,$$

$$S^F_{2\gamma}(x,y)=\sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-x^2}-1)(\gamma^{1-y^2}-1)}{\gamma-1})}\ ,\ \gamma>1.$$

Especially, if $\gamma\to1$, then we have

$$\lim_{\gamma\to1}g(t)=\lim_{\gamma\to1}\log(\frac{\gamma-1}{\gamma^t-1})=\lim_{\gamma\to1}\log(\frac{1}{t\gamma^{t-1}-1})=-\log t.$$

which indicates that $\lim_{\gamma\to1}S^F_\gamma(x,y)=S^A_\gamma(x,y)$ and $\lim_{\gamma\to1}T^F_\gamma(x,y)=T^A_\gamma(x,y)$.

4. The operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm

Definition 4.1. Let $\widetilde{\alpha}_i=\langle[\mu^L(\alpha_i),\mu^U(\alpha_i)],[\nu^L(\alpha_i),\nu^U(\alpha_i)]\rangle$ $(i=1,2)$ be two interval-valued intuitionistic fuzzy sets, $T(x,y)$ Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm, and $\lambda\geqslant0$. We can define the operational rules about $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ based on Archimedean t-norm as follows

(1) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[S(\mu^L(\alpha_1),\mu^L(\alpha_2)),S(\mu^U(\alpha_1),\mu^U(\alpha_2))],[T(\nu^L(\alpha_1),\nu^L(\alpha_2)),T(\nu^U(\alpha_1),\nu^U(\alpha_2))]\rangle$
$=\langle[h^{-1}(h(\mu^L(\alpha_1))+h(\mu^L(\alpha_2))),h^{-1}(h(\mu^U(\alpha_1))+h(\mu^U(\alpha_2)))],$
$[g^{-1}(g(\nu^L(\alpha_1))+g(\nu^L(\alpha_2))),g^{-1}(g(\nu^U(\alpha_1))+g(\nu^U(\alpha_2)))]\rangle$;

(2) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[T(\mu^L(\alpha_1),\mu^L(\alpha_2)),T(\mu^U(\alpha_1),\mu^U(\alpha_2))],[S(\nu^L(\alpha_1),\nu^L(\alpha_2)),S(\nu^U(\alpha_1),\nu^U(\alpha_2))]\rangle$
$=\langle[g^{-1}(g(\mu^L(\alpha_1))+g(\mu^L(\alpha_2))),g^{-1}(g(\mu^U(\alpha_1))+g(\mu^U(\alpha_2)))],$
$[h^{-1}(h(\nu^L(\alpha_1))+h(\nu^L(\alpha_2))),h^{-1}(h(\nu^U(\alpha_1))+h(\nu^U(\alpha_2)))]\rangle$;

(3) $\lambda\widetilde{\alpha}_1=\langle[h^{-1}(\lambda h(\mu^L(\alpha_1))),h^{-1}(\lambda h(\mu^U(\alpha_1)))],[g^{-1}(\lambda g(\nu^L(\alpha_1))),g^{-1}(\lambda g(\nu^U(\alpha_1)))]\rangle$;

(4) $\widetilde{\alpha}_1^\lambda=\langle[g^{-1}(\lambda g(\mu^L(\alpha_1))),g^{-1}(\lambda g(\mu^U(\alpha_1)))],[h^{-1}(\lambda h(\nu^L(\alpha_1))),h^{-1}(\lambda h(\nu^U(\alpha_1)))]\rangle$.

Obviously, the above operational result is still an the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm. According Theorem 3.1 and Definition 4.1, we have Theorem 4.1 and Theorem 4.2, the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are obtained as follows.

Theorem 4.1. Let $\widetilde{\alpha}_i=\langle[\mu^L(\alpha_i),\mu^U(\alpha_i)],[\nu^L(\alpha_i),\nu^U(\alpha_i)]\rangle$ $(i=1,2)$ be two interval-valued fuzzy intuitionistic sets, $T(x,y)$ Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm, $N(x)=1-x$. Then the following operational rules based on Archimedean t-norm are hold:

(1) If $g(t)=-\log t$, then [9]

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2),\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2)],$
$[\nu^L(\alpha_1)\nu^L(\alpha_2),\nu^U(\alpha_1)\nu^U(\alpha_2)]\rangle$;

(ii) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[\mu^L(\alpha_1)\mu^L(\alpha_2),\mu^U(\alpha_1)\mu^U(\alpha_2)],$
$[\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2),\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2)]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1=\langle[s_{\lambda\times\theta(\alpha_1)},s_{\lambda\times\tau(\alpha_1)}],[1-(1-\mu^L(\alpha_1))^\lambda,1-(1-\mu^U(\alpha_1))^\lambda],[(\nu^L(\alpha_1))^\lambda,(\nu^U(\alpha_1))^\lambda]\rangle$;

(iv) $\widetilde{\alpha}_1^\lambda=\langle[s_{(\theta(\alpha_1))^\lambda},s_{(\tau(\alpha_1))^\lambda}],[(\mu^L(\alpha_1))^\lambda,(\mu^U(\alpha_1))^\lambda],[1-(1-\nu^L(\alpha_1))^\lambda,1-(1-\nu^U(\alpha_1))^\lambda]\rangle$.

(2) If $g(t)=\log(\frac{2-t}{t})$, then

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[\frac{\mu^L(\alpha_1)+\mu^L(\alpha_2)}{1+\mu^L(\alpha_1)\mu^L(\alpha_2)},\frac{\mu^U(\alpha_1)+\mu^U(\alpha_2)}{1+\mu^U(\alpha_1)\mu^U(\alpha_2)}],[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{1+(1-\nu^L(\alpha_1))(1-\nu^L(\alpha_2))},\frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{1+(1-\nu^U(\alpha_1))(1-\nu^U(\alpha_2))}]\rangle$;

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{1+(1-\mu^L(\alpha_1))(1-\mu^L(\alpha_2))}, \frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{1+(1-\mu^U(\alpha_1))(1-\mu^U(\alpha_2))}], [\frac{\nu^L(\alpha_1)+\nu^L(\alpha_2)}{1+\nu^L(\alpha_1)\nu^L(\alpha_2)}, \frac{\nu^U(\alpha_1)+\nu^U(\alpha_2)}{1+\nu^U(\alpha_1)\nu^U(\alpha_2)}]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1 = \langle [\frac{(1+\mu^L(\alpha_1))^\lambda-(1-\mu^L(\alpha_1))^\lambda}{(1+\mu^L(\alpha_1))^\lambda+(1-\mu^L(\alpha_1))^\lambda}, \frac{(1+\mu^U(\alpha_1))^\lambda-(1-\mu^U(\alpha_1))^\lambda}{(1+\mu^U(\alpha_1))^\lambda+(1-\mu^U(\alpha_1))^\lambda}], [\frac{2(\nu^L(\alpha_1))^\lambda}{(2-\nu^L(\alpha_1))^\lambda+\nu^L(\alpha_1)^\lambda}, \frac{2(\nu^U(\alpha_1))^\lambda}{(2-\nu^U(\alpha_1))^\lambda+\nu^U(\alpha_1)^\lambda}]\rangle$;

(iv) $\widetilde{\alpha}_1^\lambda = \langle [\frac{2(\mu^L(\alpha_1))^\lambda}{(2-\mu^L(\alpha_1))^\lambda+\mu^L(\alpha_1)^\lambda}, \frac{2(\mu^U(\alpha_1))^\lambda}{(2-\mu^U(\alpha_1))^\lambda+\mu^U(\alpha_1)^\lambda}], [\frac{(1+\nu^L(\alpha_1))^\lambda-(1-\nu^L(\alpha_1))^\lambda}{(1+\nu^L(\alpha_1))^\lambda+(1-\nu^L(\alpha_1))^\lambda}, \frac{(1+\nu^U(\alpha_1))^\lambda-(1-\nu^U(\alpha_1))^\lambda}{(1+\nu^U(\alpha_1))^\lambda+(1-\nu^U(\alpha_1))^\lambda}]\rangle$.

(3) If $g(t) = \log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma > 0$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \langle [\frac{\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2)-(1-\gamma)\mu^L(\alpha_1)\mu^L(\alpha_2)}{1-(1-\gamma)\mu^L(\alpha_1)\mu^L(\alpha_2)}, \frac{\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2)-(1-\gamma)\mu^U(\alpha_1)\mu^U(\alpha_2)}{1-(1-\gamma)\mu^U(\alpha_1)\mu^U(\alpha_2)}],$
$[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{\gamma+(1-\gamma)(\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2))}, \frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{\gamma+(1-\gamma)(\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2))}]\rangle$;

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{\gamma+(1-\gamma)(\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2))}, \frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{\gamma+(1-\gamma)(\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2))}],$
$[\frac{\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2)-(1-\gamma)\nu^L(\alpha_1)\nu^L(\alpha_2)}{1-(1-\gamma)\nu^L(\alpha_1)\nu^L(\alpha_2)}, \frac{\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2)-(1-\gamma)\nu^U(\alpha_1)\nu^U(\alpha_2)}{1-(1-\gamma)\nu^U(\alpha_1)\nu^U(\alpha_2)}]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1 = \langle [\frac{(1+(\gamma-1)\mu^L(\alpha_1))^\lambda-(1-\mu^L(\alpha_1))^\lambda}{(1+(\gamma-1)\mu^L(\alpha_1))^\lambda+(\gamma-1)(1-\mu^L(\alpha_1))^\lambda}, \frac{(1+(\gamma-1)\mu^U(\alpha_1))^\lambda-(1-\mu^U(\alpha_1))^\lambda}{(1+(\gamma-1)\mu^U(\alpha_1))^\lambda+(\gamma-1)(1-\mu^U(\alpha_1))^\lambda}],$
$[\frac{\gamma(\nu^L(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\nu^L(\alpha_1)))^\lambda+(\gamma-1)(\nu^L(\alpha_1))^\lambda}, \frac{\gamma(\nu^U(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\nu^U(\alpha_1)))^\lambda+(\gamma-1)(\nu^U(\alpha_1))^\lambda}]\rangle$;

(iv) $\widetilde{\alpha}_1^\lambda = \langle [\frac{\gamma(\mu^L(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\mu^L(\alpha_1)))^\lambda+(\gamma-1)(\mu^L(\alpha_1))^\lambda}, \frac{\gamma(\mu^U(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\mu^U(\alpha_1)))^\lambda+(\gamma-1)(\mu^U(\alpha_1))^\lambda}],$
$[\frac{(1+(\gamma-1)\nu^L(\alpha_1))^\lambda-(1-\nu^L(\alpha_1))^\lambda}{(1+(\gamma-1)\nu^L(\alpha_1))^\lambda+(\gamma-1)(1-\nu^L(\alpha_1))^\lambda}, \frac{(1+(\gamma-1)\nu^U(\alpha_1))^\lambda-(1-\nu^U(\alpha_1))^\lambda}{(1+(\gamma-1)\nu^U(\alpha_1))^\lambda+(\gamma-1)(1-\nu^U(\alpha_1))^\lambda}]\rangle$.

(4) If $g(t) = \log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma > 1$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \langle [1-\log_\gamma(1+\frac{(\gamma^{1-\mu^L(\alpha_1)}-1)(\gamma^{1-\mu^L(\alpha_2)}-1)}{\gamma-1}), 1-\log_\gamma(1+\frac{(\gamma^{1-\mu^U(\alpha_1)}-1)(\gamma^{1-\mu^U(\alpha_2)}-1)}{\gamma-1})],$
$[\log_\gamma(1+\frac{(\gamma^{\nu^L(\alpha_1)}-1)(\gamma^{\nu^L(\alpha_2)}-1)}{\gamma-1}), \log_\gamma(1+\frac{(\gamma^{\nu^U(\alpha_1)}-1)(\gamma^{\nu^U(\alpha_2)}-1)}{\gamma-1})]\rangle$;

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\log_\gamma(1+\frac{(\gamma^{\mu^L(\alpha_1)}-1)(\gamma^{\mu^L(\alpha_2)}-1)}{\gamma-1}), \log_\gamma(1+\frac{(\gamma^{\mu^U(\alpha_1)}-1)(\gamma^{\mu^U(\alpha_2)}-1)}{\gamma-1})],$
$[1-\log_\gamma(1+\frac{(\gamma^{1-\nu^L(\alpha_1)}-1)(\gamma^{1-\nu^L(\alpha_2)}-1)}{\gamma-1}), 1-\log_\gamma(1+\frac{(\gamma^{1-\nu^U(\alpha_1)}-1)(\gamma^{1-\nu^U(\alpha_2)}-1)}{\gamma-1})]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1 = \langle [1-\log_\gamma(1+\frac{(\gamma^{1-\mu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), 1-\log_\gamma(1+\frac{(\gamma^{1-\mu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})],$
$[\log_\gamma(1+\frac{(\gamma^{\nu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), \log_\gamma(1+\frac{(\gamma^{\nu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})]\rangle$;

(iv) $\widetilde{\alpha}_1^\lambda = \langle [\log_\gamma(1+\frac{(\gamma^{\mu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), \log_\gamma(1+\frac{(\gamma^{\mu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})],$
$[1-\log_\gamma(1+\frac{(\gamma^{1-\nu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), 1-\log_\gamma(1+\frac{(\gamma^{1-\nu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})]\rangle$.

Theorem 4.2. Let $\widetilde{\alpha}_i = \langle [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2)$ be be two interval-valued intuitionistic fuzzy sets , $T(x, y)$ Archimedean t-norm and $S(x, y)$ its dual Archimedean t-conorm, $N(x) = 1 - x$. Then the following operational rules based on Archimedean t-norm valid:

(1) If $g(t) = -\log t$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 =$
$\langle [\sqrt{(\mu^L(\alpha_1))^2 + (\mu^L(\alpha_2))^2 - (\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2}, \sqrt{(\mu^U(\alpha_1))^2 + (\mu^U(\alpha_2))^2 - (\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}],$
$[\nu^L(\alpha_1)\nu^L(\alpha_2), \nu^U(\alpha_1)\nu^U(\alpha_2)]\rangle$;

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\mu^L(\alpha_1)\mu^L(\alpha_2), \mu^U(\alpha_1)\mu^U(\alpha_2)],$
$[\sqrt{(\nu^L(\alpha_1))^2 + (\nu^L(\alpha_2))^2 - (\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2}, \sqrt{(\nu^U(\alpha_1))^2 + (\nu^U(\alpha_2))^2 - (\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1 = \langle [\sqrt{1-(1-(\mu^L(\alpha_1))^2)^\lambda}, \sqrt{1-(1-(\mu^U(\alpha_1))^2)^\lambda}], [(\nu^L(\alpha_1))^\lambda, (\nu^U(\alpha_1))^\lambda]\rangle$;

(iv) $\widetilde{\alpha}_1^\lambda = \langle [(\mu^L(\alpha_1))^\lambda, (\mu^U(\alpha_1))^\lambda], [\sqrt{1-(1-(\nu^L(\alpha_1))^2)^\lambda}, \sqrt{1-(1-(\nu^U(\alpha_1))^2)^\lambda}]\rangle$.

(2) If $g(t) = \log(\frac{2-t}{t})$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \langle [\sqrt{\frac{(\mu^L(\alpha_1))^2+(\mu^L(\alpha_2))^2}{1+(\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2}}, \sqrt{\frac{(\mu^U(\alpha_1))^2+(\mu^U(\alpha_2))^2}{1+(\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}}],$
$[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{1+(1-\nu^L(\alpha_1))(1-\nu^L(\alpha_2))}, \frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{1+(1-\nu^U(\alpha_1))(1-\nu^U(\alpha_2))}]\rangle$;

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{1+(1-\mu^L(\alpha_1))(1-\mu^L(\alpha_2))}, \frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{1+(1-\mu^U(\alpha_1))(1-\mu^U(\alpha_2))}],$
$[\sqrt{\frac{(\nu^L(\alpha_1))^2+(\nu^L(\alpha_2))^2}{1+(\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2}}, \sqrt{\frac{(\nu^U(\alpha_1))^2+(\nu^U(\alpha_2))^2}{1+(\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}}]\rangle$;

(iii) $\lambda\widetilde{\alpha}_1 = \langle [\sqrt{\frac{(1+(\mu^L(\alpha_1))^2)^\lambda-(1-(\mu^L(\alpha_1))^2)^\lambda}{(1+(\mu^L(\alpha_1))^2)^\lambda+(1-(\mu^L(\alpha_1))^2)^\lambda}}, \sqrt{\frac{(1+(\mu^U(\alpha_1))^2)^\lambda-(1-(\mu^U(\alpha_1))^2)^\lambda}{(1+(\mu^U(\alpha_1))^2)^\lambda+(1-(\mu^U(\alpha_1))^2)^\lambda}}],$
$[\frac{2(\nu^L(\alpha_1))^\lambda}{(2-\nu^L(\alpha_1))^\lambda+(\nu^L(\alpha_1))^\lambda}, \frac{2(\nu^U(\alpha_1))^\lambda}{(2-\nu^U(\alpha_1))^\lambda+(\nu^U(\alpha_1))^\lambda}]\rangle$;

(iv) $\widetilde{\alpha}_1^{\lambda}=\langle[\frac{2(\mu^L(\alpha_1))^{\lambda}}{(2-\mu^L(\alpha_1))^{\lambda}+(\mu^L(\alpha_1))^{\lambda}},\frac{2(\mu^U(\alpha_1))^{\lambda}}{(2-\mu^U(\alpha_1))^{\lambda}+(\mu^U(\alpha_1))^{\lambda}}],$
$[\sqrt{\frac{(1+(\nu^L(\alpha_1))^2)^{\lambda}-(1-(\nu^L(\alpha_1))^2)^{\lambda}}{(1+(\nu^L(\alpha_1))^2)^{\lambda}+(1-(\nu^L(\alpha_1))^2)^{\lambda}}},\sqrt{\frac{(1+(\nu^U(\alpha_1))^2)^{\lambda}-(1-(\nu^U(\alpha_1))^2)^{\lambda}}{(1+(\nu^U(\alpha_1))^2)^{\lambda}+(1-(\nu^U(\alpha_1))^2)^{\lambda}}}]\rangle.$

(3) If $g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma>0$, then

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[\sqrt{\frac{(\mu^L(\alpha_1))^2+(\mu^L(\alpha_2))^2-(\mu^L(\alpha_1))^2(\mu^L(\alpha_1))^2-(1-\gamma)(\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2}{1-(1-\gamma)(\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2}},$
$\sqrt{\frac{(\mu^U(\alpha_1))^2+(\mu^U(\alpha_2))^2-(\mu^U(\alpha_1))^2(\mu^U(\alpha_1))^2-(1-\gamma)(\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}{1-(1-\gamma)(\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}}],$
$[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{\gamma+(1-\gamma)(\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2))},\frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{\gamma+(1-\gamma)(\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2))}]\rangle;$

(ii) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{\gamma+(1-\gamma)(\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2))},\frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{\gamma+(1-\gamma)(\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2))}],$
$[\sqrt{\frac{(\nu^L(\alpha_1))^2+(\nu^L(\alpha_2))^2-(\nu^L(\alpha_1))^2(\nu^L(\alpha_1))^2-(1-\gamma)(\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2}{1-(1-\gamma)(\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2}},$
$\sqrt{\frac{(\nu^U(\alpha_1))^2+(\nu^U(\alpha_2))^2-(\nu^U(\alpha_1))^2(\nu^U(\alpha_1))^2-(1-\gamma)(\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}{1-(1-\gamma)(\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}}]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1=\langle[\sqrt{\frac{(1+(\gamma-1)(\mu^L(\alpha_1))^2)^{\lambda}-(1-(\mu^L(\alpha_1))^2)^{\lambda}}{(1+(\gamma-1)(\mu^L(\alpha_1))^2)^{\lambda}+(\gamma-1)(1-(\mu^L(\alpha_1))^2)^{\lambda}}},\sqrt{\frac{(1+(\gamma-1)(\mu^U(\alpha_1))^2)^{\lambda}-(1-(\mu^U(\alpha_1))^2)^{\lambda}}{(1+(\gamma-1)(\mu^U(\alpha_1))^2)^{\lambda}+(\gamma-1)(1-(\mu^U(\alpha_1))^2)^{\lambda}}}],$
$[\frac{\gamma(\nu^L(\alpha_1))^{\lambda}}{(1+(\gamma-1)(1-\nu^L(\alpha_1)))^{\lambda}+(\gamma-1)(\nu^L(\alpha_1))^{\lambda}},\frac{\gamma(\nu^U(\alpha_1))^{\lambda}}{(1+(\gamma-1)(1-\nu^U(\alpha_1)))^{\lambda}+(\gamma-1)(\nu^U(\alpha_1))^{\lambda}}]\rangle;$

(iv) $\widetilde{\alpha}_1^{\lambda}=\langle[\frac{\gamma(\mu^L(\alpha_1))^{\lambda}}{(1+(\gamma-1)(1-\mu^L(\alpha_1)))^{\lambda}+(\gamma-1)(\mu^L(\alpha_1))^{\lambda}},\frac{\gamma(\mu^U(\alpha_1))^{\lambda}}{(1+(\gamma-1)(1-\mu^U(\alpha_1)))^{\lambda}+(\gamma-1)(\mu^U(\alpha_1))^{\lambda}}],$
$[\sqrt{\frac{(1+(\gamma-1)(\nu^L(\alpha_1))^2)^{\lambda}-(1-(\nu^L(\alpha_1))^2)^{\lambda}}{(1+(\gamma-1)(\nu^L(\alpha_1))^2)^{\lambda}+(\gamma-1)(1-(\nu^L(\alpha_1))^2)^{\lambda}}},\sqrt{\frac{(1+(\gamma-1)(\nu^U(\alpha_1))^2)^{\lambda}-(1-(\nu^U(\alpha_1))^2)^{\lambda}}{(1+(\gamma-1)(\nu^U(\alpha_1))^2)^{\lambda}+(\gamma-1)(1-(\nu^U(\alpha_1))^2)^{\lambda}}}]\rangle.$

(4) If $g(t)=\log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma>1$, then

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=$
$\langle[\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\mu^L(\alpha_1))^2}-1)(\gamma^{1-(\mu^L(\alpha_2))^2}-1)}{\gamma-1})},\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\mu^U(\alpha_1))^2}-1)(\gamma^{1-(\mu^U(\alpha_2))^2}-1)}{\gamma-1})}],$
$[\log_{\gamma}(1+\frac{(\gamma^{1-\nu^L(\alpha_1)}-1)(\gamma^{1-\nu^L(\alpha_2)}-1)}{\gamma-1}),\log_{\gamma}(1+\frac{(\gamma^{1-\nu^U(\alpha_1)}-1)(\gamma^{1-\nu^U(\alpha_2)}-1)}{\gamma-1})]]\rangle;$

(ii) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[\log_{\gamma}(1+\frac{(\gamma^{1-\mu^L(\alpha_1)}-1)(\gamma^{1-\mu^L(\alpha_2)}-1)}{\gamma-1}),\log_{\gamma}(1+\frac{(\gamma^{1-\mu^U(\alpha_1)}-1)(\gamma^{1-\mu^U(\alpha_2)}-1)}{\gamma-1})],$
$[\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\nu^L(\alpha_1))^2}-1)(\gamma^{1-(\nu^L(\alpha_2))^2}-1)}{\gamma-1})},\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\nu^U(\alpha_1))^2}-1)(\gamma^{1-(\nu^U(\alpha_2))^2}-1)}{\gamma-1})}]]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1=\langle[\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\mu^L(\alpha_1))^2}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})},\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\mu^U(\alpha_1))^2}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})}],$
$[\log_{\gamma}(1+\frac{(\gamma^{\nu^L(\alpha_1)}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}}),\log_{\gamma}(1+\frac{(\gamma^{\nu^U(\alpha_1)}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})]]\rangle;$

(iv) $\widetilde{\alpha}_1^{\lambda}=\langle[\log_{\gamma}(1+\frac{(\gamma^{\mu^L(\alpha_1)}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}}),\log_{\gamma}(1+\frac{(\gamma^{\mu^U(\alpha_1)}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})],$
$[\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\nu^L(\alpha_1))^2}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})},\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\nu^U(\alpha_1))^2}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})}]\rangle.$

Theorem 4.3. Let $\widetilde{\alpha}_i$ $(i=1,2)$ be be two interval-valued intuitionistic fuzzy sets, $T(x,y)$ Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm, $N(x)=1-x$. We can easily prove the the following statements:

(1) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\widetilde{\alpha}_2\oplus\widetilde{\alpha}_1$;
(2) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\widetilde{\alpha}_2\otimes\widetilde{\alpha}_1$;
(3) $\lambda(\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2)=\lambda\widetilde{\alpha}_1\oplus\lambda\widetilde{\alpha}_2,\ \lambda>0$;
(4) $\lambda_1\widetilde{\alpha}_1\oplus\lambda_2\widetilde{\alpha}_1=(\lambda_1+\lambda_2)\widetilde{\alpha}_1,\ \lambda_1,\lambda_2>0$;
(5) $\widetilde{\alpha}_1^{\lambda_1}\otimes\widetilde{\alpha}_1^{\lambda_2}=(\widetilde{\alpha}_1)^{\lambda_1+\lambda_2},\ \lambda_1,\lambda_2>0$;
(6) $\widetilde{\alpha}_1^{\lambda}\otimes\widetilde{\alpha}_2^{\lambda}=(\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2)^{\lambda},\ \lambda>0$.

According Theorem 3.1 and Definition 4.1, Theorem 4.3 is easy to prove.

5. Aggregating of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm

Definition 5.1. Let $\widetilde{\alpha}_1=\langle[\mu^L(\alpha_1),\mu^U(\alpha_1)],[\nu^L(\alpha_1),\nu^U(\alpha_1)]\rangle$ be an interval-valued fuzzy intuitionistic sets. An expected value $E(\widetilde{\alpha}_1)$ of $\widetilde{\alpha}_1$ can be represented as follows

$$E(\widetilde{\alpha}_1)=\frac{1}{2}\times(\frac{\mu^L(\alpha_1)+\mu^U(\alpha_1)}{2}+1-\frac{\nu^L(\alpha_1)+\nu^U(\alpha_1)}{2})$$

$$= (\mu^L(\alpha_1) + \mu^U(\alpha_1) + 2 - \nu^L(\alpha_1) - \nu^U(\alpha_1))/4.$$

An accuracy function $H(\widetilde{\alpha}_1)$ can be represented as follows

$$H(\widetilde{\alpha}_1) = (\frac{\mu^L(\alpha_1) + \mu^U(\alpha_1)}{2} + \frac{\nu^L(\alpha_1) + \nu^U(\alpha_1)}{2})$$
$$= (\mu^L(\alpha_1) + \mu^U(\alpha_1) + \nu^L(\alpha_1) + \nu^U(\alpha_1))/4.$$

Let $\widetilde{\alpha}_i = \langle[\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2)$ be two interval-valued fuzzy intuitionistic sets. Then

(1) If $E(\widetilde{\alpha}_1) > E(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 \succ \widetilde{\alpha}_2$.

(2) If $E(\widetilde{\alpha}_1) = E(\widetilde{\alpha}_2)$, then:

If $H(\widetilde{\alpha}_1) > H(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 \succ \widetilde{\alpha}_2$.

If $H(\widetilde{\alpha}_1) = H(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$.

Based on the the above operational rules, we propose weighted average (geometric) operator, ordered weighted average (geometric) operator and hybrid average (geometric) operator for interval-valued intuitionistic fuzzy sets based on Archimedean t-norm in this part.

Definition 5.2. Let $\widetilde{\alpha}_i = \langle[\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic fuzzy sets, $T(x, y)$ Archimedean t-norm and $S(x, y)$ its dual Archimedean t-conorm, $N(x) = 1 - x$. We define interval-valued intuitionistic fuzzy weighted average operator based on Archimedean t-norm as follows: $ATS - IVIFWA : \Omega^n \to \Omega$,

$$ATS - IVIFWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \sum_{j=1}^{n} \mu_j \widetilde{\alpha}_j,$$

Specifically, if $\mu = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then $ATS - IVIFWA$ operator degenerates interval-valued intuitionistic fuzzy arithmetic average operator based on Archimedean t-norm $(ATS - IVIFAA)$:

$$ATS - IVIFAA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \frac{1}{n}(\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 \oplus \ldots \oplus \widetilde{\alpha}_n).$$

Similarly, we could define interval-valued intuitionistic fuzzy weighted geometric average operator based on Archimedean t-norm, $ATS - IVIFWGA : \Omega^n \to \Omega$, as follows

$$ATS - IVIFWGA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \prod_{j=1}^{n} (\widetilde{\alpha}_j)^{\mu_j},$$

Specifically, if $\mu = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then $ATS - IVIFWGA$ operator degenerates interval-valued intuitionistic fuzzy arithmetic geometric average operator based on Archimedean t-norm $(ATS - IVIFGA)$:

$$ATS - IVIFGA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = (\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 \otimes \ldots \otimes \widetilde{\alpha}_n)^{\frac{1}{n}}.$$

where Ω is the set of all interval-valued fuzzy intuitionistic sets, and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ is the weighted vector of $\widetilde{\alpha}_j (j = 1, 2, \ldots, n)$, μ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, and $\sum_{j=1}^{n} \mu_j = 1$, $A_{(j)} = (j, \ldots, n)$ with $A_{(n+1)} = \emptyset$.

Theorem 5.1. Let $\widetilde{\alpha}_i = \langle[\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic fuzzy sets, $T(x, y)$ Archimedean t-norm and $S(x, y)$ its dual Archimedean t-conorm, $N(x) = 1 - x$. Then, the result aggregated by Definition 5.1 is still an intuitionistic fuzzy set, and

$(i) ATS - IVIFWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) =$
$\langle[h^{-1}(\sum_{j=1}^{n} \mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{n} \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^{n} \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{n} \mu_j g(\nu^U(\alpha_j)))]\rangle$.

$(ii) ATS - IVIFWGA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) =$
$\langle[g^{-1}(\sum_{j=1}^{n} \mu_j g(\mu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{n} \mu_j g(\mu^U(\alpha_j)))], [h^{-1}(\sum_{j=1}^{n} \mu_j h(\nu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{n} \mu_j h(\nu^U(\alpha_j)))]\rangle$,
where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, and $\sum_{j=1}^{n} \mu_j = 1$, the parentheses used for indices represent a permutation on X such that $\widetilde{\alpha}_1 \leqslant \widetilde{\alpha}_2 \leqslant \cdots \leqslant \widetilde{\alpha}_n$, $A_{(j)} = (j, ..., n)$, $A_{(n+1)} = \varnothing$.

Theorem 5.1 can be proven by mathematical induction. The steps in the proof are as follows:

Proof. We only prove that (i) holds. the proof of (ii) is similar.

(1) When $n=1$, obviously, it is right.

(2) When $n=2$,

$\mu_1\widetilde{\alpha}_1=\langle[h^{-1}(\mu_1 h(\mu^L(\alpha_1))),h^{-1}(\mu_1 h(\mu^U(\alpha_1)))],[g^{-1}(\mu_1 g(\nu^L(\alpha_1))),g^{-1}(\nu_1 g(\nu^U(\alpha_1)))]\rangle.$

$\mu_2\widetilde{\alpha}_2=\langle[h^{-1}(\mu_2 h(\mu^L(\alpha_2))),h^{-1}(\mu_2 h(\mu^U(\alpha_2)))],[g^{-1}(\mu_2 g(\nu^L(\alpha_2))),g^{-1}(\nu_2 g(\nu^U(\alpha_2)))]\rangle.$

$ATS-ATS-IVIFWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2)=\mu_1\widetilde{\alpha}_1\oplus\mu_2\widetilde{\alpha}_2$
$=[h^{-1}(\mu_1 h(\mu^L(\alpha_1))),h^{-1}(\mu_1 h(\mu^U(\alpha_1)))],[g^{-1}(\mu_1 g(\nu^L(\alpha_1))),g^{-1}(\nu_1 g(\nu^U(\alpha_1)))]\rangle$
$\oplus\langle[h^{-1}(\mu_2 h(\mu^L(\alpha_2))),h^{-1}(\mu_2 h(\mu^U(\alpha_2)))],[g^{-1}(\mu_2 g(\nu^L(\alpha_2))),g^{-1}(\nu_2 g(\nu^U(\alpha_2)))]\rangle=$
$\langle[h^{-1}(h(h^{-1}(\mu_1 h(\mu^L(\alpha_1))))+h(h^{-1}(\mu_2 h(\mu^L(\alpha_2))))),h^{-1}(h(h^{-1}(\mu_1 h(\mu^U(\alpha_1))))+h(h^{-1}(\mu_2 h(\mu^U(\alpha_2)))))],$
$[g^{-1}(g(g^{-1}(\mu_1 g(\nu^L(\alpha_1))))+g(g^{-1}(\mu_2 g(\nu^L(\alpha_2))))),g^{-1}(g(g^{-1}(\mu_1 g(\nu^U(\alpha_1))))+g(g^{-1}(\mu_2 g(\nu^U(\alpha_2)))))]\rangle=$
$\langle[h^{-1}(\sum_{j=1}^2\mu_j h(\mu^L(\alpha_j))),h^{-1}(\sum_{j=1}^2\mu_j h(\mu^U(\alpha_j)))],[g^{-1}(\sum_{j=1}^2\mu_j g(\nu^L(\alpha_j))),g^{-1}(\sum_{j=1}^2\mu_j g(\nu^U(\alpha_j)))]\rangle.$

Therefore, when $n=2$, the conclusion is right.

(3) Suppose when $n=k$, the conclusion is right, i.e.

$ATS-IVIFWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_k)=$
$\langle[h^{-1}(\sum_{j=1}^k\mu_j h(\mu^L(\alpha_j))),h^{-1}(\sum_{j=1}^k\mu_j h(\mu^U(\alpha_j)))],[g^{-1}(\sum_{j=1}^k\mu_j g(\nu^L(\alpha_j))),g^{-1}(\sum_{j=1}^k\mu_j g(\nu^U(\alpha_j)))]\rangle.$

Then, when $n=k+1$,

$ATS-IVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_k,\widetilde{\alpha}_{k+1})=$
$\langle[h^{-1}(\sum_{j=1}^k\mu_j h(\mu^L(\alpha_j))),h^{-1}(\sum_{j=1}^k\mu_j h(\mu^U(\alpha_j)))],[g^{-1}(\sum_{j=1}^k\mu_j g(\nu^L(\alpha_j))),g^{-1}(\sum_{j=1}^k\mu_j g(\nu^U(\alpha_j)))]\rangle$
$\oplus\langle[h^{-1}(\mu_{k+1}h(\mu^L(\alpha_{k+1}))),h^{-1}(\mu_{k+1}h(\mu^U(\alpha_{k+1})))],[g^{-1}(\mu_{k+1}g(\nu^L(\alpha_{k+1}))),g^{-1}(\nu_{k+1}g(\nu^U(\alpha_{k+1})))]\rangle=$
$\langle[h^{-1}(h(h^{-1}(\sum_{j=1}^k\mu_j h(\mu^L(\alpha_j))))+h(h^{-1}(\mu_{k+1}h(\mu^L(\alpha_{k+1}))))),$
$h^{-1}(h(h^{-1}(\sum_{j=1}^k\mu_j h(\mu^U(\alpha_j))))+h(h^{-1}(\mu_{k+1}h(\mu^U(\alpha_{k+1})))))],$
$[g^{-1}(g(g^{-1}(\sum_{j=1}^k\mu_j g(\nu^L(\alpha_j))))+g(g^{-1}(\mu_{k+1}g(\nu^L(\alpha_{k+1}))))),$
$g^{-1}(g(g^{-1}(\sum_{j=1}^k\mu_j g(\nu^U(\alpha_j))))+g(g^{-1}(\mu_{k+1}g(\nu^U(\alpha_{k+1})))))]\rangle=$
$\langle[h^{-1}(\sum_{j=1}^{k+1}\mu_j h(\mu^L(\alpha_j))),h^{-1}(\sum_{j=1}^{k+1}\mu_j h(\mu^U(\alpha_j)))],[g^{-1}(\sum_{j=1}^{k+1}\mu_j g(\nu^L(\alpha_j))),g^{-1}(\sum_{j=1}^{k+1}\mu_j g(\nu^U(\alpha_j)))]\rangle.$

So, when $n=k+1$, the conclusion is right, too.

According to steps $(1),(2)$ and (3), we can conclude the conclusion is right for all n.

6. Conclusions

The main technologies in multiple attribute decision making, whether the situation is certain or vague, are how to define and calculate aggregation operators proposed in the practice. In this study we only discussed and investigated the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm, and the aggregating of interval-valued intuitionistic fuzzy sets based on Archimed -ean t-norm. In order to do this we also obtained the representations and transformations of Archimedean t-norm and Archimedean t-conorm. Based on these operators proposed in this note, we could make multiple attribute group decision making problems easily. Limited to the length of this paper it can not be discussed. However, it will be our main work in the future.

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Approximate bi-homomorphisms and bi-derivations in intuitionistic fuzzy ternary normed algebras

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Abstract. In this paper, we generalize the concept of homomorphisms and derivations in intuitionistic fuzzy normed algebras for 2-dimensional functional equations. Furthermore, we investigate the Hyers-Ulam stability bi-homomorphisms and bi-derivations in intuitionistic fuzzy ternary normed algebras concerning a 2-dimensional bi-additive functional equation.

1. Introduction and preliminaries

We say a functional equation (ζ) is stable if any function g satisfying the equation (ζ) approximately is near to true solution of (ζ). Also, we say that a functional equation is superstable if every approximately solution is an exact solution of it. The stability problem of functional equations originated from a question of Ulam [37] in 1940, concerning the stability of group homomorphisms. We are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. The case of approximately additive mappings was solved by Hyers [11] under the assumption that G_1 and G_2 are Banach spaces. In 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by Rassias [28]. In 1991, Gajda [8] answered the question for the case $p > 1$, which was raised by Rassias. For more information on functional equations, see [18, 25, 26, 27, 32, 34, 35].

Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. This new theory was introduced by Zadeh [38], in 1965 and since then a large number of research papers have appeared by using the concept of fuzzy set/numbers and fuzzification of many classical theories has also been made. It has also very useful application in various fields, e.g. population dynamics [5], chaos control [7], computer programming [9], nonlinear dynamical systems [10], fuzzy physics [12], fuzzy topology [31], fuzzy stability [13, 14, 15, 16, 24], nonlinear operators [20], statistical convergence [21, 23], etc. The concept of intuitionistic fuzzy normed spaces, initially has been introduced by Saadati and Park [29]. In [30], by modifying the separation condition and strengthening some conditions in the definition of Saadati and Park, Saadati et al. have obtained a modified case of intuitionistic fuzzy normed spaces. Many authors have considered the intuitionistic fuzzy normed linear spaces, and intuitionistic fuzzy 2-normed spaces(see [3, 4, 6, 19]).

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[0]**2010 Mathematics Subject Classification**: 39B52; 46S40; 26E50.

[0]**Keywords**: Hyers-Ulam stability; fuzzy ternary Banach space, intuitionistic fuzzy normed algebra; bi-additive functional equation.

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Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

$(N1)$ $N(x, c) = 0$ for $c \leqslant 0$;
$(N2)$ $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
$(N3)$ $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
$(N4)$ $N(x + y, s + t) \geqslant \min\{N(x, s), N(y, t)\}$;
$(N5)$ $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
$(N6)$ For $x \neq 0, N(x, .)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, t)$ as the truth value of the statement the norm of x is less than or equal to the real number t.

The stability problem for a 2-dimensional bi-additive functional equation was proved by Bae and Park [1] for mappings $f : X \times X \to Y$, where X is a real normed space and Y is a Banach space.

In this paper, we determine some stability results of bi-homomorphism and bi-derivation concerning the 2-dimensional bi-additive functional equation

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) - 2f(y, w) \tag{1.1}$$

in intuitionistic fuzzy ternary normed algebras. It has been discussed that $f(x, y) = ax^2 + by^2$ is a solution of (1.1) (see [2]).

We recall some notations and basic definitions used in this paper.

We use the definition of intuitionistic fuzzy normed spaces given in [17, 22, 29] to investigate some stability results for the functional equation (1.1) in the intuitionistic fuzzy normed vector space setting.

Definition 1.1. ([33]) *A binary operation* $* : [0, 1] \times [0, 1] \to [0, 1]$ *is said to be a continuous* t*-norm if it satisfies the following conditions:*

(a) *is commutative and associative;*
(b) *is continuous;*
(c) $a * 1 = a$ *for all* $a \in [0, 1]$*;*
(d) $a * b \leqslant c * d$ *whenever* $a \leqslant c$ *and* $b \leqslant d$ *for all* $a, b, c, d \in [0, 1]$.

Definition 1.2. ([33]) *A binary operation* $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ *is said to be a continuous* t*-conorm if it satisfies the following conditions:*

(a) *is commutative and associative;*
(b) *is continuous;*
(c) $a \diamond 0 = a$ *for all* $a \in [0, 1]$*;*
(d) $a \diamond b \leqslant c \diamond d$ *whenever* $a \leqslant c$ *and* $b \leqslant d$ *for all* $a, b, c, d \in [0, 1]$.

Using the continuous t-norm and t-conorm, Saadati and Park [29] have introduced the concept of intuitionistic fuzzy normed space.

Definition 1.3. ([22, 29]) *The five-tuple* $(X, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy normed space (for short, IFNS) if* X *is a vector space,* $*$ *is a continuous* t*-norm,* $\diamond$ *is a continuous* t*-conorm, and* μ, ν *are fuzzy sets on* $X \times (0, \infty)$ *satisfying the following conditions: for every* $x, y \in X$ *and* $s, t > 0$,

(i) $\mu(x, t) + \nu(x, t) \leqslant 1$, (ii) $\mu(x, t) > 0$, (iii) $\mu(x, t) = 1$ *if and only if* $x = 0$, (iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$, (v) $\mu(x, t) * \mu(y, s) \leqslant \mu(x + y, t + s)$, (vi) $\mu(x, .) : (0, \infty) \to [0, 1]$ *is continuous,* (vii) $\lim_{t\to\infty} \mu(x, t) = 1$ *and* $\lim_{t\to 0} \mu(x, t) = 0$, $(viii)$ $\nu(x, t) < 1$, (ix) $\nu(x, t) = 0$ *if and only if* $x = 0$, (x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$, (xi) $\nu(x, t) \diamond \nu(y, s) \geqslant \nu(x + y, t + s)$, (xii) $\nu(x, .) : (0, 1) \to [0, 1]$ *is continuous,* $(xiii)$ $\lim_{t\to\infty} \nu(x, t) = 0$ *and* $\lim_{t\to 0} \nu(x, t) = 1$.

Definition 1.4. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. A sequence $\{x_n\}$ is said to be* intuitionistic fuzzy convergent *to $L\in X$ if $\lim_{k\to\infty}\mu(x_k-L,t)=1$ and $\lim_{k\to\infty}\nu(x_k-L,t)=0$ for all $t>0$. In this case we write $x_k\to L$ as $k\to\infty$. A sequence $\{x_n\}$ is said to be* intuitionistic fuzzy Cauchy sequence *if $\lim_{k\to\infty}\mu(x_{k+p}-x_k,t)=1$ and $\lim_{k\to\infty}\nu(x_{k+p}-x_k,t)=0$ for all $p\in\mathbb{N}$ and all $t>0$. Then IFNS $(X,\mu,\nu,*,\diamond)$ is said to be* complete *if every intuitionistic fuzzy Cauchy sequence in $(X,\mu,\nu,*,\diamond)$ is intuitionistic fuzzy convergent in $(X,\mu,\nu,*,\diamond)$ and $(X,\mu,\nu,*,\diamond)$ is also called an* intuitionistic fuzzy Banach space*.*

The concepts of convergent sequence and Cauchy sequence in an intuitionistic fuzzy normed space are studied in [29].

Definition 1.5. *Let X be a ternary algebra with $[\cdot,\cdot,\cdot]$ and $(X,\mu,\nu,*,\diamond)$ be an IFNS.*
(1) *The intuitionistic fuzzy normed space $(X,\mu,\nu,*,\diamond)$ is called an intuitionistic fuzzy ternary normed algebra if*

$$\mu([x,y,z],stu)\geqslant \mu(x,s)*\mu(y,t)*\mu(z,u)$$
$$\nu([x,y,z],stu)\geqslant \nu(x,s)*\nu(y,t)*\nu(z,u)$$

for all $x,y,z\in X$ and $s,t,u>0$.

(2) *A complete intuitionistic fuzzy ternary normed algebra is called an intuitionistic fuzzy ternary Banach algebra.*

Definition 1.6. *Let X be a ternary normed (Banach) algebra and (Y,μ,ν) an intuitionistic fuzzy ternary Banach algebra.*

(1) *A bi-additive mapping $H:X\times X\to Y$ is called a ternary bi-homomorphism if*

$$H([x,y,z],[w,w,w])=[H(x,w),H(y,w),H(z,w)],$$
$$H([x,x,x],[y,z,w])=[H(x,y),H(x,z),H(x,w)]$$

for all $x,y,z,w\in X$.

(2) *A bi-additive mapping $\delta:X\times X\to X$ is called a ternary bi-derivation if*

$$\delta([x,y,z],w)=[\delta(x,w),y,z]+[x,\delta(y,w),z]+[x,y,\delta(z,w)],$$
$$\delta(x,[y,z,w])=[\delta(x,y),z,w]+[y,\delta(x,z),w]+[y,z,\delta(x,w)]$$

for all $x,y,z,w\in X$.

2. Bi-homomorphisms in intuitionistic fuzzy ternary normed algebras

We begin with a Hyers-Ulam type theorem in intuitionistic fuzzy ternary normed algebras to approximate bi-homomorphism associated to the functional equation (1.1). For notational convenience, given a function $f:X\times X\to Y$, we define the difference operator

$$D_qf(x,y,z,w)=f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)$$

Lemma 2.1. ([36, Theorem 3.1]) *Let X be a linear space and let (Z,μ',ν') be an IFNS. Let $\varphi:X^4\to Z$ be a mapping such that, for some $0<\alpha<4$.*

$$\begin{cases}\mu'(\varphi(2x,2y,2z,2w),t)\geqslant\mu'(\alpha\varphi(x,y,z,w),t),\\ \nu'(\varphi(2x,2y,2z,2w),t)\leqslant\nu'(\alpha\varphi(x,y,z,w),t),\end{cases}\tag{2.1}$$

for all $x,y,z,w\in X$ and all $t>0$. Let (Y,μ,ν) be an intuitionistic fuzzy Banach space and let $f:X\times X\to Y$ be a mapping satisfying $f(0,0)=0$ and

$$\begin{cases}\mu(D_qf(x,y,z,w),t)\geqslant\mu'(\varphi(x,y,z,w),t),\\ \nu(D_qf(x,y,z,w),t)\leqslant\nu'(\varphi(x,y,z,w),t)\end{cases}\tag{2.2}$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique bi-additive mapping $H : X \times X \to Y$ satisfying (1.1) *such that*

$$\begin{cases} \mu\Big(H(x,y) - f(x,y), t\Big) \\ \quad \geqslant *^{\infty}\mu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) *^{\infty} \mu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) *^{\infty} \mu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big), \\ \nu\Big(H(x,y) - f(x,y), t\Big) \\ \quad \leqslant \diamond^{\infty}\nu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) \diamond^{\infty} \nu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) \diamond^{\infty} \nu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big) \end{cases} \tag{2.3}$$

for all $x, y, z, w \in X$ and all $t > 0$, where $*^{\infty}a := a * a * \cdots$ and $\diamond^{\infty}a := a \diamond a \diamond \cdots$ for all $a \in [0, 1]$.

Theorem 2.2. *Let X be a ternary algebra and let (Z, μ', ν) be an IFNS. Let $\varphi : X^4 \to Z$ be a mapping satisfying* (2.1). *Let (Y, μ, ν) be an intuitionistic fuzzy ternary Banach algebra and let $f : X \times X \to Y$ be a mapping satisfying $f(0, 0) = 0$,* (2.2) *and*

$$\begin{cases} \mu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad +\mu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \geqslant \mu'(\varphi(x,y,z,w), t), \\ \nu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad +\nu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \leqslant \nu'(\varphi(x,y,z,w), t) \end{cases} \tag{2.4}$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique bi-homomorphism $H : X \times X \to Y$ satisfying (1.1) *and* (2.3).

Proof. In Lemma 2.1, the mapping $H : X \times X \to Y$ was defined by $H(x, y) = \lim_{n\to\infty} \frac{f(2^n x, 2^n y)}{4^n}$ for all $x, z \in X$.

From (2.4) and definition of H, it follows that

$$\begin{aligned} &\mu(H([x,y,z],[w,w,w]) - [H(x,y), H(y,w), H(z,w)], t) \\ &\quad + \mu(H([x,x,x],[y,z,w]) - [H(x,y), H(x,z), H(x,w)], t) \\ &\quad = \mu\Big(\frac{f([2^n x, 2^n y, 2^n z],[2^n w, 2^n w, 2^n w])}{64^n} - \Big[\frac{f(2^n x, 2^n w)}{4^n}, \frac{f(2^n y, 2^n w)}{4^n}, \frac{f(2^n z, 2^n w)}{4^n}\Big], t\Big) \\ &\quad + \mu\Big(\frac{f([2^n x, 2^n x, 2^n x],[2^n x, 2^n y, 2^n z])}{64^n} - \Big[\frac{f(2^n x, 2^n y)}{4^n}, \frac{f(2^n x, 2^n z)}{4^n}, \frac{f(2^n x, 2^n w)}{4^n}\Big], t\Big) \\ &\quad \geqslant \mu'(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^{3n} t) \geqslant \mu'(\varphi(x,y,z,w), \frac{4^{3n}}{\alpha^n} t) \to 1 \end{aligned}$$

as $n \to \infty$ for all $x, y, z, w \in X$ and all $t > 0$, and similarly

$$\begin{aligned} &\nu(H([x,y,z],[w,w,w]) - [H(x,y), H(y,w), H(z,w)], t) \\ &\quad + \nu(H([x,x,x],[y,z,w]) - [H(x,y), H(x,z), H(x,w)], t) \leqslant 0 \end{aligned}$$

for all $x, y, z, w \in X$ and all $t > 0$. So we conclude that

$$\begin{aligned} H([x,y,z],[w,w,w]) &= [H(x,w), H(y,w), H(z,w)], \\ H([x,x,x],[y,z,w]) &= [H(x,y), H(x,z), H(x,w)] \end{aligned}$$

for all $x, y, z, w \in X$. □

Corollary 2.3. *Let p be a nonnegative real number with $p < 2$, X be a ternary normed algebra with norm $\|.\|$, (Z, μ', ν') be an intuitionistic fuzzy ternary normed algebra, (Y, μ, ν) be a complete intuitionistic fuzzy ternary normed algebra, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping satisfying $f(0,0) = 0$ and*

$$\begin{cases} \mu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad +\mu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \\ \quad \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) z_0, t) \\ \nu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad +\mu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \\ \quad \leqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) z_0, t) \end{cases} \tag{2.5}$$

and

$$\begin{cases} \mu(D_q f(x,y), t) \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) z_0, t) \\ \nu(D_q f(x,y), t) \leqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) z_0, t) \end{cases} \tag{2.6}$$

for all $x, y, z, w \in X$ and $t > 0$, then there exists a unique bi-homomorphism $H : X \times X \to Y$ such that

$$\begin{cases} \mu(H(x,y) - f(x,y), t) \geqslant *^2 \mu'\left((\|x\| + \|y\|) z_0, \frac{(4-2^p)t}{16}\right) * \mu'\left((\|x\| + \|y\|) z_0, \frac{(4-2^p)t}{8}\right) \\ \nu(H(x,y) - f(x,y), t) \leqslant *^2 \nu'\left((\|x\| + \|y\|) z_0, \frac{(4-2^p)t}{16}\right) * \nu'\left((\|x\| + \|y\|) z_0, \frac{(4-2^p)t}{8}\right) \end{cases}$$

for all $x, y \in X$ and $t > 0$.

Lemma 2.4. ([36, Theorem 3.3]) *Let X be a linear space and let (Z, μ', ν') be an IFNS. Let $\varphi : X \times X \times X \times X \to Z$ be a mapping such that, for some $\alpha > 4$,*

$$\begin{cases} \mu'\left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t\right) \geqslant \mu'(\varphi(x,y,z,w), \alpha t), \\ \nu'\left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t\right) \leqslant \nu'(\varphi(x,y,z,w), \alpha t), \end{cases} \tag{2.7}$$

for all $x, y, z, w \in X$ and all $t > 0$. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and let $f : X \times X \to Y$ be a φ-approximately bi-additive mapping in the sense of (2.2) and (2.4) with $f(0,0) = 0$. Then there exists a unique bi-additive mapping $H : X \times X \to Y$ such that

$$\begin{aligned} \mu(H(x,y) - f(x,y), t) \geqslant *^\infty \, \mu'\left(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8} t\right) *^\infty \, \mu'\left(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8} t\right) \\ *^\infty \, \mu'\left(\varphi(0,x,0,y), \frac{(\alpha-4)}{8} t\right) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \mu(H(x,y) - f(x,y), t) \leqslant \diamond^\infty \, \nu'\left(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8} t\right) \diamond^\infty \, \nu'\left(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8} t\right) \\ \diamond^\infty \, \nu'\left(\varphi(0,x,0,y), \frac{(\alpha-4)}{8} t\right) \end{aligned} \tag{2.9}$$

for all $x, y \in X$ and all $t > 0$.

Theorem 2.5. *Let X be a ternary algebra and let (Z, μ', ν') be an IFNS. Let $\varphi : X \times X \times X \times X \to Z$ be a mapping satisfying (2.7). Let (Y, μ, ν) be an intuitionistic fuzzy ternary Banach algebra and let $f : X \times X \to Y$ be a φ-approximately bi-additive mapping in the sense of (2.2) and (2.4) with $f(0,0) = 0$. Then there exists a unique bi-homomorphism $H : X \times X \to Y$ satisfying (2.8) and (2.9).*

Proof. The proof is similar to the proof of Theorem 2.2. □

Corollary 2.6. *Let p be a nonnegative real number with $p > 2$, X be a ternary normed algebra with norm $\|.\|$, (Z, μ', ν') be an intuitionistic fuzzy ternary normed algebra, (Y, μ, ν) be a complete intuitionistic fuzzy ternary normed algebra, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping satisfying $f(0,0) = 0$, (2.5) and (2.6). then there exists a unique bi-homomorphism $H : X \times X \to Y$ such that*

$$\begin{cases} \mu(H(x,y) - f(x,y), t) \geqslant *^2\mu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \mu'\Big((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{8}\Big) \\ \nu(H(x,y) - f(x,y), t) \leqslant *^2\nu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \nu'\Big((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{8}\Big) \end{cases}$$

for all $x, y \in X$ and $t > 0$.

3. Bi-derivations on intuitionistic fuzzy ternary normed algebras

In this section, we investigate generalized Hyers-Ulam stability of bi-derivations on intuitionistic fuzzy ternary normed algebrasfor the functional equation (1.1).

Theorem 3.1. *Let X be an intuitionistic fuzzy ternary Banach algebra and let (Z, μ', ν') be an IFNS. Let $f : X \times X \to X$ be a mapping with $f(0,0) = 0$ for which there exists a mapping $\varphi : X \times X \times X \times X \to Z$ such that, for some $0 < \alpha < 4$ satisfying (2.1), (2.2) and*

$$\begin{cases} \mu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\ \qquad +\mu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\ \quad \geqslant \mu'(\varphi(x,y,z,w),t), \\ \nu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\ \qquad +\nu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\ \quad \leqslant \nu'(\varphi(x,y,z,w),t) \end{cases} \tag{3.1}$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique bi-derivation $\delta : X \times X \to X$ satisfying (1.1) such that

$$\begin{cases} \mu\Big(\delta(x,y) - f(x,y), t\Big) \\ \geqslant *^\infty\mu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) *^\infty \mu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) *^\infty \mu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big), \\ \nu\Big(\delta(x,y) - f(x,y), t\Big) \\ \leqslant \diamond^\infty\nu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) \diamond^\infty \nu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) \diamond^\infty \nu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big) \end{cases} \tag{3.2}$$

for all $x, y, z, w \in X$ and all $t > 0$, where $*^\infty a := a * a * \cdots$ and $\diamond^\infty a := a \diamond a \diamond \cdots$ for all $a \in [0,1]$.

Proof. By the same argument as in the proof of Theorem 2.2, there exists a unique bi-additive mapping $\delta : X \times X \to X$ satisfying (3.2). The mapping δ is given by

$$\delta(x,y) = \lim_{n\to\infty} \frac{1}{4} f(2^n x, 2^n y)$$

for all $x, y \in X$. It follows from (3.1) that

$$\begin{aligned}
&\mu(\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w),z] - [x,y,\delta(z,w)],t) \\
&\quad + \mu(\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x,z),w] - [y,z,\delta(x,w)],t) \\
&\quad = \mu\Big(\frac{1}{4^{3n}}f(2^{3n}[x,y,z],2^{3n}w) - \Big[\frac{1}{4^n}f(2^nx,2^nw),y,z\Big] \\
&\qquad - \Big[x,\frac{1}{4^n}f(2^nx,2^nw),z\Big] - \Big[x,y,\frac{1}{4^n}f(2^nz,2^nw)\Big],t\Big) \\
&\qquad + \mu\Big(\frac{1}{4^{3n}}f(2^{3n}x,2^{3n}[y,z,w]) - \Big[\frac{1}{4^n}f(2^nx,2^ny),z,w\Big] \\
&\qquad - \Big[y,\frac{1}{4^n}f(2^nx,2^nz),w\Big] - \Big[y,z,\frac{1}{4^n}f(2^nx,2^nw)\Big],t\Big) \\
&\quad = \mu\Big(\frac{1}{4^{3n}}f([2^nx,2^ny,2^nz],2^{3n}w) - \frac{1}{4^{3n}}[f(2^nx,2^{3n}w),2^ny,2^nz] \\
&\qquad - \frac{1}{4^{3n}}[2^nx,f(2^ny,2^{3n}w),2^nz] - \frac{1}{4^{3n}}[2^nx,2^ny,f(2^nz,2^{3n}w)],t\Big) \\
&\qquad + \mu\Big(\frac{1}{4^{3n}}f(2^{3n}x,[2^ny,2^nz,2^nw]) - \frac{1}{4^{3n}}[f(2^{3n}x,2^ny),2^nz,2^nw] \\
&\qquad - \frac{1}{4^{3n}}[2^ny,f(2^{3n}x,2^nz),2^nw] - \frac{1}{4^{3n}}[2^ny,2^nz,f(2^{3n}x,2^nw)],t\Big) \\
&\qquad \leqslant \mu'(\varphi(2^nx,2^ny,2^nz,2^{3n}w),4^{3n}t) + \mu'(\varphi(2^{3n}x,2^ny,2^nz,2^nw),4^{3n}t)) \\
&\qquad \leqslant 2\mu'\Big(\varphi(x,y,z,w),\frac{4^{3n}t}{\alpha^{3n}}\Big) \longrightarrow 1
\end{aligned}$$

as $n \to \infty$ for all $x, y, z, w \in \mathcal{A}$. Similarly, we obtain

$$\begin{aligned}
&\nu(\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w),z] - [x,y,\delta(z,w)],t) \\
&\quad + \nu(\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x,z),w] - [y,z,\delta(x,w)],t) = 0
\end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$. Thus

$$\begin{aligned}
\delta([x,y,z],w) &= [\delta(x,w),y,z] + [x,\delta(y,w),z] + [x,y,\delta(z,w)], \\
\delta(x,[y,z,w]) &= [\delta(x,y),z,w] + [y,\delta(x,z),w] + [y,z,\delta(x,w)]
\end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$. So we conclude that δ is a unique bi-derivation satisfying (3.2). $\square$

Corollary 3.2. *Let p be a nonnegative real number with $p < 2$, (Z, μ', ν') be an intuitionistic fuzzy ternary normed algebra, (X, μ, ν) be a complete intuitionistic fuzzy ternary Banach algebra, and let $z_0 \in Z$. If $f : X \to X$ is a mapping with $f(0,0) = 0$ such that*

$$\begin{cases}
\mu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\
\qquad +\mu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\
\qquad \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0,t) \\
\nu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\
\qquad +\nu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\
\qquad \geqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0,t)
\end{cases} \tag{3.3}$$

and

$$\begin{cases} \mu(D_q f(x,y),t) \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \\ \nu(D_q f(x,y),t) \leqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \end{cases} \tag{3.4}$$

for all $x, y, z, w \in X$ and $t > 0$, then there exists a unique bi-derivation $\delta : X \times X \to X$ such that

$$\begin{cases} \mu(\delta(x,y) - f(x,y),t) \geqslant *^2\mu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{16}\right) * \mu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{8}\right) \\ \nu(\delta(x,y) - f(x,y),t) \leqslant *^2\nu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{16}\right) * \nu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{8}\right) \end{cases}$$

for all $x, y \in X$ and $t > 0$.

Theorem 3.3. *Let X be an intuitionistic fuzzy ternary Banach algebra and let $(Z < \mu', \nu,)$ be an IFNS. Let $f : X \times X \to Y$ be a mapping with $f(0,0) = 0$ for which there exists a mapping $\varphi : X \times X \times X \times X \to Z$ satisfying (2.1), (2.7) and (3.1) for some $\alpha > 4$. Then there exists a unique bi-derivation $\delta : X \times X \to X$ such that*

$$\mu(\delta(x,y) - f(x,y),t) \geqslant *^\infty \mu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) *^\infty \mu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big)$$
$$*^\infty \mu\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big)$$

and

$$\mu(\delta(x,y) - f(x,y),t) \leqslant \diamond^\infty \nu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) \diamond^\infty \nu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big)$$
$$\diamond^\infty \nu'\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big)$$

for all $x, y \in X$.

Proof. The proof is similar to the proof of Theorems 2.5 and 3.1. □

Corollary 3.4. *Let p be a nonnegative real number with $p > 2$, (Z, μ', ν') be an intuitionistic fuzzy ternary normed algebra, (X, μ, ν) be a complete intuitionistic fuzzy ternary Banach algebra and let $z_0 \in Z$. If $f : X \to Y$ is a mapping satisfying $f(0,0) = 0$, (3.3) and (3.4), then there exists a unique bi-derivation $\delta : X \times X \to X$ such that*

$$\begin{cases} \mu(\delta(x,y) - f(x,y),t) \geqslant *^2\mu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \mu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{8}\right) \\ \nu(\delta(x,y) - f(x,y),t) \leqslant *^2\nu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \nu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{8}\right) \end{cases}$$

for all $x, y \in X$ and $t > 0$.

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ON NEW REFINEMENTS AND APPLICATIONS OF EFFICIENT QUADRATURE RULES USING N-TIMES DIFFERENTIABLE MAPPINGS

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Abstract. In this paper, new efficient quadrature rules are established using a newly developed special type of kernel for n-times differentiable mappings, having five steps. Some previous inequalities are also recaptured as special cases of our main inequalities. At the end, efficiency of the newly developed quadrature rules are discussed.

1. Introduction

In 1938, Ostrowski [13] first announced his inequality for different differentiable mappings, which is given below:

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° (I° is the interior of I) and let $a, b \in I^\circ$ with $a < b$. If $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b) i.e. $\|f'\|_\infty = \sup\limits_{t\in[a,b]} |f''(t)| < \infty$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty , \tag{1.1}$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In 1976, Milovanovic et. al [11], proved a generalization of Ostrowski's inequality for n-time differentiable mappings. Up till now, a large number of research papers and books have been written on inequalities and their applications (see for instance [2]-[5], [8] and [14]-[16]). In many practical problems, it is important to bound one quantity by another quantity. The classical inequalities like Ostrowski are very helpful for this purpose. Ostrowski type inequalities have immediate applications in numerical integration, optimization theory, statistics, and integral operator theory.

We indicate another inequality called Grüss inequality [11] which is stated as the integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals, which is given below.

Theorem 2. *Let $f,\ g : [a, b] \to \mathbb{R}$ be integrable functions such that $\varphi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$, for some constants $\varphi,\ \Phi,\ \gamma,\ \Gamma$ and $x \in [a, b]$. Then*

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx.\frac{1}{b-a}\int_a^b g(x)dx \right| \tag{1.2}$$
$$\leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).$$

2000 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.
Key words and phrases. Ostrowski inequality, Grüss inequality, Quadrature formula,Numerical Integration, peano kerenel.

Dragomir et. al [4] combined Ostrowski and Grüss inequality to give a new inequality which they named Ostrowski-Grüss type inequality. Dragomir [3], Liu [6], Alomari [1] and Liu et. al [8] established some companions of ostrowski type integral inequalities.

Recently, Liu [7] proved the following companions of ostrowski type inequalities for 3-step kernels.

Theorem 3. *Let* $f : [a,b] \to \mathbb{R}$ *be differentiable in* (a,b). *If* $f' \in L^1[a,b]$, *and* $\gamma \leq f'(x) \leq \Gamma$, *for all* $x \in [a,b]$, *then for all* $x \in \left[a, \frac{a+b}{2}\right]$, *we have*

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{b-a}{4} + \left|x - \frac{3a+b}{4}\right|\right](S-\gamma) \quad (1.3)$$

and

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \quad (1.4)$$
$$\leq \left[\frac{b-a}{4} + \left|x - \frac{3a+b}{4}\right|\right](\Gamma - S).$$

More recently, Qayyum et. al [9]-[10] proved companions of Ostrowski inequality for 5-step linear and quadratic kernels but in this paper, we establish our results for 5-step kernel for n-times differentiable mappings. In this paper, new ontrowski inequalities are extended. Using these inequalities, some efficient quadrature rules are established. Some previous inequalities are also recaptured as special cases of our main inequalities. At the end, efficiency of the newly developed quadrature rules are discussed.

2. Derivation of Ostrowski inequalities using 5-step kernel

We will start our work by introducing a new Peano kernel defined by $P(x,.) : [a,b] \to \mathbb{R}$

$$P_n(x,t) = \begin{cases} \frac{1}{n!}(t-a)^n, & t \in \left(a, \frac{a+x}{2}\right], \\ \frac{1}{n!}\left(t - \frac{3a+b}{4}\right)^n, & t \in \left(\frac{a+x}{2}, x\right], \\ \frac{1}{n!}\left(t - \frac{a+b}{2}\right)^n, & t \in (x, a+b-x], \\ \frac{1}{n!}\left(t - \frac{a+3b}{4}\right)^n, & t \in \left(a+b-x, \frac{a+2b-x}{2}\right], \\ \frac{1}{n!}(t-b)^n, & t \in \left(\frac{a+2b-x}{2}, b\right], \end{cases} \quad (2.1)$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

The following lemma is the main tool to prove the main results.

Lemma 1. *Let* $f : [a,b] \to \mathbb{R}$ *be an n-times differentiable function such that* $f^{(n-1)}(x)$ *for* $n \in N$ *is absolutely continuous on* $[a,b]$ *then*

$$\frac{1}{b-a}\int_a^b P_n(x,t) f^{(n)}(t)dt \quad (2.2)$$
$$= \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x - \frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right.$$

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$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\Bigg]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt,$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. The proof of (2.2) is established using mathematical induction.
Take $n=1$,

$$L.H.S \text{ of } (2.2)=\int_a^b P_1(x,t)f'(t)dt. \tag{2.3}$$

After integrating by parts, we get

$$\frac{1}{b-a}\int_a^b P_1(x,t)f'(t)dt \tag{2.4}$$
$$=\frac{1}{4}\left[f\left(\frac{a+x}{2}\right)+f(x)+f(a+b-x)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt.$$

We have

$$L.H.S=\int_a^b P_1(x,t)f'(t)dt.$$

Equation (2.3), is identical to the $R.H.S$ of (2.2).
Assume that (2.2) is true for n.

$$\int_a^b P_{n+1}(x,t)f^{(n+1)}(t)dt$$
$$=\sum_{k=0}^{n}\frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right.$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+(-1)^{n+1}\int_a^b f(t)dt,$$

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where

$$P_{n+1}(x,t)=\begin{cases}\frac{1}{(n+1)!}(t-a)^{n+1}, & t\in\left(a,\frac{a+x}{2}\right],\\ \frac{1}{(n+1)!}\left(t-\frac{3a+b}{4}\right)^{n+1}, & t\in\left(\frac{a+x}{2},x\right],\\ \frac{1}{(n+1)!}\left(t-\frac{a+b}{2}\right)^{n+1}, & t\in(x,a+b-x],\\ \frac{1}{(n+1)!}\left(t-\frac{a+3b}{4}\right)^{n+1}, & t\in\left(a+b-x,\frac{a+2b-x}{2}\right],\\ \frac{1}{(n+1)!}(t-b)^{n+1}, & t\in\left(\frac{a+2b-x}{2},b\right].\end{cases}$$

After integeration by parts, we get

$$\begin{aligned}&\int_a^b P_{n+1}(x,t)f^{(n+1)}(t)dt\\ &=\frac{1}{(n+1)!}\left[\frac{1}{2^{n+1}}(x-a)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(x)\right.\\ &-\frac{1}{2^{n+1}}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+(-1)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(a+b-x)\\ &-\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(x)+\left(\frac{-1}{2}\right)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\\ &\left.+(-1)^n\left\{\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(a+b-x)+\left(\frac{x-a}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\right\}\right]\end{aligned}$$

$$\begin{aligned}&-\frac{1}{n!}\left[\int_a^{\frac{a+x}{2}}(t-a)^n f^{(n)}(t)dt+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^n f^{(n)}(t)dt\right.\\ &+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^n f^{(n)}(t)dt+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{3a+b}{4}\right)^n f^{(n)}(t)dt\\ &\left.+\int_{\frac{a+2b-x}{2}}^{b}(t-b)^n f^{(n)}(t)dt\right]\end{aligned}$$

$$\begin{aligned}&=\frac{1}{(n+1)!}\left[\frac{1}{2^{n+1}}(x-a)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(x)\right.\\ &-\frac{1}{2^{n+1}}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+(-1)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(a+b-x)\\ &-\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(x)+\left(\frac{-1}{2}\right)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\\ &-(-1)^{n+1}\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(a+b-x)\\ &\left.-\left(\frac{-1}{2}\right)^{n+1}(x-a)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\right]-\int_a^b P_n(x,t)f^{(n)}(t)dt\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{(n+1)!}\left[\left\{(x-a)^{n+1}-\left(x-\frac{a+b}{2}\right)^{n+1}\right\}\frac{1}{2^{n+1}}f^{(n)}\left(\frac{a+x}{2}\right)\right.\\
&+\left\{\left(x-\frac{3a+b}{4}\right)^{n+1}-\left(x-\frac{a+b}{2}\right)^{n+1}\right\}f^{(n)}(x)\\
&+\left\{\left(x-\frac{a+b}{2}\right)^{n+1}-\left(x-\frac{3a+b}{4}\right)^{n+1}\right\}(-1)^{n+1}f^{(n)}(a+b-x)\\
&\left.+\left\{\left(x-\frac{a+b}{2}\right)^{n+1}-(x-a)^{n+1}\right\}\left(\frac{-1}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=0}^{n-1}\frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right.\\
&+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)\\
&+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)\\
&\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]\\
&+(-1)^{n+1}\int_a^b f(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{k=0}^{n}\frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right.\\
&+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)\\
&+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)\\
&\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]\\
&+(-1)^{n+1}\int_a^b f(t)dt.
\end{aligned}
$$

This completes the proof of lemma 1. □

Now we will present our results by imposing three different conditions on $f^{(n)}$ and $f^{(n+1)}$.

3. Case A: When $f^{(n)}\in L^1[a,b]$

Theorem 4. *Let $f\colon [a,b]\to\mathbb{R}$ be an n-times differentiable function on (a,b), $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $\gamma\le f^{(n)}(t)\le\Gamma$, $\forall\ t\in[a,b]$, then for*

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all $x \in \left[a, \frac{a+b}{2}\right]$, *we have*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right.\right. \tag{3.1}$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\left[\times\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1}+(1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$
$$\leq \delta(x)(b-a)(S-\gamma)$$

and

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right.\right. \tag{3.2}$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\times\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1}+(1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$
$$\leq \delta(x)(b-a)(\Gamma-S),$$

where

$$S=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}, \tag{3.3}$$

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$$\delta(x) = \max\left\{ \left| \frac{1}{n!}\left(\frac{x-a}{2}\right)^n - \frac{\lambda(x)}{b-a} \right|, \left| \frac{1}{n!}\left(x - \frac{3a+b}{4}\right)^2 - \frac{\lambda(x)}{b-a} \right|, \right.$$
$$\left. \left| \frac{1}{n!}\left(x - \frac{a+b}{2}\right)^n - \frac{\lambda(x)}{b-a} \right|, \left| \frac{1}{4n!}\left(x - \frac{a+b}{2}\right)^2 - \frac{\lambda(x)}{b-a} \right|, \frac{\lambda(x)}{b-a} \right\}$$

and

$$\lambda(x) = \frac{1}{(n+1)!}\left[(1+(-1)^n)\left\{ \frac{1}{2^{n+1}}(x-a)^{n+1} + \left(x - \frac{3a+b}{4}\right)^{n+1} \right\} \right.$$
$$\left. + \left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1 \right)\left(x - \frac{a+b}{2}\right)^{n+1} \right].$$

Proof. Let

$$\frac{1}{b-a}\int_a^b P_n(x,t)dt \tag{3.4}$$
$$= \frac{1}{(b-a)(n+1)!}\left[(1+(-1)^n)\left\{ \frac{1}{2^{n+1}}(x-a)^{n+1} + \left(x - \frac{3a+b}{4}\right)^{n+1} \right\} \right.$$
$$\left. + \left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1 \right)\left(x - \frac{a+b}{2}\right)^{n+1} \right].$$

Using (3.4), we get

$$\frac{1}{b-a}\int_a^b P_n(x,t)f^{(n)}(t)dt - \frac{1}{(b-a)^2}\int_a^b P_n(x,t)dt\int_a^b f^{(n)}(t)dt \tag{3.5}$$
$$= \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{ (x-a)^{k+1} - \left(x - \frac{a+b}{2}\right)^{k+1} \right\} f^{(k)}\left(\frac{a+x}{2}\right) \right.$$
$$+ \left\{ \left(x - \frac{3a+b}{4}\right)^{k+1} - \left(x - \frac{a+b}{2}\right)^{k+1} \right\} f^{(k)}(x)$$
$$+ (-1)^{k+1}\left\{ \left(x - \frac{a+b}{2}\right)^{k+1} - \left(x - \frac{3a+b}{4}\right)^{k+1} \right\} f^{(k)}(a+b-x)$$
$$\left. + \left(\frac{-1}{2}\right)^{k+1}\left\{ \left(x - \frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1} \right\} f^{(k)}\left(\frac{a+2b-x}{2}\right) \right]$$
$$+ \frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\left[\times \frac{1}{2^{n+1}}(1+(-1)^n)(x-a)^{n+1} + (1+(-1)^n)\left(x - \frac{3a+b}{4}\right)^{n+1} \right.$$
$$\left. + \left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1 \right)\left(x - \frac{a+b}{2}\right)^{n+1} \right].$$

Denote the L.H.S of (3.5) by $R_n(x)$. If $C \in R$ is an arbitrary constant, then we have

$$R_n(x) = \frac{1}{b-a}\int_a^b \left(f^{(n)}(t) - C\right)\left[P_n(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right]dt. \tag{3.6}$$

Furthermore, we have

$$|R_n(x)| \le \frac{1}{b-a}\max_{t\in[a,b]}\left|P_n(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right|\int_a^b \left|f^{(n)}(t) - C\right|dt. \tag{3.7}$$

Now

$$\left|P_n(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right| \tag{3.8}$$

$$= \max\left\{\begin{array}{c} \left|\frac{1}{n!}\left(\frac{x-a}{2}\right)^n - \frac{\lambda(x)}{b-a}\right|, \left|\frac{1}{n!}\left(x-\frac{3a+b}{4}\right)^2 - \frac{\lambda(x)}{b-a}\right|, \\ \left|\frac{1}{n!}\left(x-\frac{a+b}{2}\right)^n - \frac{\lambda(x)}{b-a}\right|, \left|\frac{1}{4n!}\left(x-\frac{a+b}{2}\right)^2 - \frac{\lambda(x)}{b-a}\right|, \frac{\lambda(x)}{b-a} \end{array}\right\} = \delta(x),$$

where

$$\lambda(x) = \frac{1}{(n+1)!}\left[(1+(-1)^n)\left\{\frac{1}{2^{n+1}}(x-a)^{n+1} + \left(x-\frac{3a+b}{4}\right)^{n+1}\right\}\right.$$
$$\left. + \left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right].$$

We also have

$$\int_a^b \left|f^{(n)}(t) - \gamma\right|dt = (S-\gamma)(b-a), \tag{3.9}$$

$$\int_a^b \left|f^{(n)}(t) - \Gamma\right|dt = (\Gamma - S)(b-a). \tag{3.10}$$

Using (3.4) to (3.10) and using $C=\gamma$ and $C=\Gamma$ in (3.7), we can obtain (3.1) and (3.2). □

Remark 1. *If we substitute $n=2$ in (3.1) and (3.2), we get Qayyum et. al result proved in* [9].

Corollary 1. *Substitution of $x=a$ in (3.1) and (3.2) gives*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left\{(-1)^k f^{(k)}(a) + \left(1+\frac{1}{2^{k+1}}+\frac{(-1)^k}{4^{k+1}}\right)f^{(k)}(b)\right\}\right. \tag{3.11}$$

$$\left.+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)(1+(-1)^n)\right|$$
$$\le \delta(a)(b-a)(S-\gamma)$$

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and

$$\Bigg|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left\{(-1)^k f^{(k)}(a)+\left(1+\frac{1}{2^{k+1}}+\frac{(-1)^k}{4^{k+1}}\right)f^{(k)}(b)\right\} \tag{3.12}$$

$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)(1+(-1)^n)\Bigg|$$

$$\le \delta(a)(b-a)(\Gamma-S).$$

Corollary 2. *Substitution of* $x=\frac{a+b}{2}$ *in (3.1) and (3.2) gives*

$$\Bigg|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left\{f^{(k)}\left(\frac{3a+b}{4}\right)+\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+b}{2}\right)\right. \tag{3.13}$$

$$\left.+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\}+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$

$$-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\times\frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\Bigg|$$

$$\le \delta\left(\frac{a+b}{2}\right)(b-a)(S-\gamma)$$

and

$$\Bigg|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left\{f^{(k)}\left(\frac{3a+b}{4}\right)+\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+b}{2}\right)\right. \tag{3.14}$$

$$\left.+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\}+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$

$$-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\times\frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\Bigg|$$

$$\le \delta\left(\frac{a+b}{2}\right)(b-a)(\Gamma-S).$$

Corollary 3. *Substitution of* $x=\frac{3a+b}{4}$ *in (3.1) and (3.2) gives*

$$\Bigg|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left[\frac{\left(1+(-1)^k\right)}{2^{k+1}}f^{(k)}\left(\frac{7a+b}{8}\right)+(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)\right. \tag{3.15}$$

$$\left.+f^{(k)}\left(\frac{a+3b}{4}\right)+\frac{1}{2^{k+1}}\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+7b}{8}\right)\right]+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$

$$-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\frac{1}{(n+1)!}\frac{(b-a)^{n-1}}{4^{n+1}}\left\{1+(-1)^n+\frac{1}{2^n}+\frac{(-1)^n}{2^n}\right\}\Bigg|$$

$$\le \delta\left(\frac{3a+b}{4}\right)(b-a)(S-\gamma)$$

and

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left[\frac{1}{2^{k+1}}\left\{1+(-1)^k\right\}f^{(k)}\left(\frac{7a+b}{8}\right)\right.\right. \tag{3.16}$$
$$+(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)+f^{(k)}\left(\frac{a+3b}{4}\right)$$
$$\left.+\frac{1}{2^{k+1}}\left\{1+(-1)^k\right\}f^{(k)}\left(\frac{a+7b}{8}\right)\right]+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$
$$\left.-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}\frac{(b-a)^{n+1}}{4^{n+1}}\left\{1+(-1)^n+\frac{1}{2^n}+\frac{(-1)^n}{2^n}\right\}\right|$$
$$\le\delta\left(\frac{3a+b}{4}\right)(b-a)(\Gamma-S).$$

4. Case B: When $f^{(n+1)}\in L^2[a,b]$

Theorem 5. *Let* $f:[a,b]\to\mathbb{R}$ *be an n-times differentiable function on* (a,b), $f^{(n+1)}\in L^2[a,b]$, *then for all* $x\in\left[a,\frac{a+b}{2}\right]$, *we have*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{f^{(k)}\left(\frac{a+x}{2}\right)}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}\right.\right. \tag{4.1}$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\times\left[(1+(-1)^n)\left\{\frac{1}{2^{n+1}}(x-a)^{n+1}+\left(x-\frac{3a+b}{4}\right)^{n+1}\right\}\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$
$$\le\frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\left[\frac{1}{(n!)^2(2n+1)}\left\{\frac{(x-a)^{2n+1}}{2^{2n}}+2\left(x-\frac{3a+b}{4}\right)^{2n+1}\right.\right.$$
$$\left.-\left(\frac{1}{2^{2n}}+2\right)\left(x-\frac{a+b}{2}\right)^{2n+1}\right\}$$
$$-\frac{1}{(b-a)(n+1)!}\left\{(1+(-1)^n)\left(\frac{(x-a)^{n+1}}{2^{n+1}}+\left(x-\frac{3a+b}{4}\right)^{n+1}\right)\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}-\frac{(-1)^n}{2^{n+1}}-(-1)^n-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right\}^2\right]^{\frac{1}{2}}.$$

Proof. Substitute $C = f^{(n)}\left(\frac{a+b}{2}\right)$, in $R_n(x)$ given in (3.5) and use the Cauchy Inequality, then we get

$$
\begin{aligned}
&|R_n(x)| \qquad (4.2)\\
&\leq \frac{1}{b-a}\int_a^b \left|f^{(n)}(t)-f^{(n)}\left(\frac{a+b}{2}\right)\right|\left|P^{(n)}(x,t)-\frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right|dt\\
&\leq \frac{1}{b-a}\left[\int_a^b \left(f^{(n)}(t)-f^{(n)}\left(\frac{a+b}{2}\right)\right)^2 dt\right]^{\frac{1}{2}}\\
&\times\left[\int_a^b \left(P^{(n)}(x,t)-\frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right)^2 dt\right]^{\frac{1}{2}}.
\end{aligned}
$$

Use the Diaz-Metcalf inequality [12] or [17], to get

$$
\int_a^b \left(f^{(n)}(t)-f^{(n)}\left(\frac{a+b}{2}\right)\right)^2 dt \leq \frac{(b-a)^2}{\pi^2}\left\|f^{(n+1)}\right\|_2^2.
$$

Therefore, using the above relations, we obtain (4.1). □

Corollary 4. *Substitution of $x=a$ in (4.1) gives*

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left\{(-1)^k f^{(k)}(a)+\left(1+\frac{1}{2^{k+1}}+\frac{(-1)^k}{4^{k+1}}\right)f^{(k)}(b)\right\}\right.\\
&\left.+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)(1+(-1)^n)\right|\\
&\leq \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\left[\frac{(b-a)^{2n+1}}{2^{2n}(n!)^2(2n+1)}-\frac{(1+(-1)^n)^2(b-a)^{2n+1}}{2^{2n+2}(n+1)!}\right]^{\frac{1}{2}}.
\end{aligned}
$$

Corollary 5. *Substitution of $x=\frac{a+b}{2}$ in (4.1) gives*

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left\{f^{(k)}\left(\frac{3a+b}{4}\right)\right.\right. \qquad (4.3)\\
&\left.+\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+b}{2}\right)+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\}\\
&\left.+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\right|\\
&\leq \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\left[\frac{1}{(n!)^2(2n+1)}\frac{4}{4^{2n+1}}(b-a)^{2n+1}\right.\\
&\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{2(b-a)^{n+1}}{4^{n+1}}(1+(-1)^n)\right\}^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

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Corollary 6. *Substitution of* $x = \frac{3a+b}{4}$ *in (4.1) gives*

$$\Bigg|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left[\frac{\left\{1+(-1)^k\right\}}{2^{k+1}} f^{(k)}\left(\frac{7a+b}{8}\right) \right. \tag{4.4}$$
$$+ (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) + f^{(k)}\left(\frac{a+3b}{4}\right)$$
$$\left. + \frac{1}{2^{k+1}}\left(1+(-1)^k\right) f^{(k)}\left(\frac{a+7b}{8}\right)\right] + \frac{(-1)^n}{b-a}\int_a^b f(t)dt - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)$$
$$\times \frac{1}{(n+1)!} \frac{(b-a)^{n-1}}{4^{n+1}} \left\{1+(-1)^n + \frac{1}{2^n} + \frac{(-1)^n}{2^n}\right\}\Bigg|$$
$$\leq \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2 \left[\frac{1}{(n!)^2(2n+1)} \frac{(b-a)^{2n+1}}{4^{2n+1}} \left\{\frac{4}{2^{2n+1}} - 2\right\}\right.$$
$$\left. - \frac{1}{b-a}\frac{1}{(n+1)!}\frac{(b-a)^{n+1}}{4^{n+1}}\left\{(1+(-1)^n)\left(2+\frac{1}{2^{n+1}}\right)\right\}^2\right]^{\frac{1}{2}}.$$

5. Case C: When $f^{(n)} \in L^2[a,b]$.

Theorem 6. *Let* $f : [a;b] \to R$ *be an n-times differentiable function on* (a,b)*, with* $f^{(n)} \in L^2[a,b]$*. Then, we have*

$$\Bigg|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right. \tag{5.1}$$
$$+ \left\{\left(x-\frac{3a+b}{4}\right)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)$$
$$+ (-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - \left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)$$
$$\left. + \left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+ \frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\times\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1} + \left(1-(-1)^{n+1}\right)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left. + \left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\Bigg|$$
$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\left\{\frac{(x-a)^{2n+1}}{2^{2n}} + 2\left(x-\frac{3a+b}{4}\right)^{2n+1}\right.\right.$$
$$\left. - \left(\frac{1}{2^{2n}}+2\right)\left(x-\frac{a+b}{2}\right)^{2n+1}\right\}$$

$$-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(x-a)^{n+1}}{2^{n+1}}(1+(-1)^n)+(1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}-\frac{(-1)^n}{2^{n+1}}-(-1)^n-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right\}^2\right]^{\frac{1}{2}},$$

for all $x\in\left[a,\frac{a+b}{2}\right]$, *where*

$$\sigma\left(f^{(n)}\right) \tag{5.2}$$
$$=\left\|f^{(n)}\right\|_2^2-\frac{\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)^2}{b-a}=\left\|f^{(n)}\right\|_2^2-k^2(b-a),$$

where S *is as defined in Theorem 4.*

Proof. Let $R_n(x)$ is defined as in (3.5). If we choose $C=\frac{1}{b-a}\int_a^b f^{(n)}(s)ds$ in (3.6) and use the Cauchy inequality and (3.5), then we get

$$|R_n(x)|$$
$$\le\frac{1}{b-a}\int_a^b\left|f^{(n)}(t)-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right|\left|P_n(x,t)-\frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right|dt$$
$$\le\frac{1}{b-a}\left[\int_a^b\left(f^{(n)}(t)-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right)^2dt\right]^{\frac{1}{2}}$$
$$\times\left[\int_a^b\left(P_n(x,t)-\frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right)^2dt\right]^{\frac{1}{2}}$$
$$=\frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\int_a^b(P_n(x,t))^2-\frac{1}{b-a}\left(\int_a^b P^{(n)}(x,t)dt\right)^2\right]^{\frac{1}{2}}$$

$$\le\frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\left\{\frac{1}{2^{2n}}(x-a)^{2n+1}+2\left(x-\frac{3a+b}{4}\right)^{2n+1}\right.\right.$$
$$\left.-\left(\frac{1}{2^{2n}}+2\right)\left(x-\frac{a+b}{2}\right)^{2n+1}\right\}$$
$$-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{1}{2^{n+1}}(1+(-1)^n)(x-a)^{n+1}\right.$$
$$+(1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)$$
$$\left.\left.\times\left(x-\frac{a+b}{2}\right)^{n+1}\right\}^2\right]^{\frac{1}{2}}.$$

Hence theorem is completed. □

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Corollary 7. *Substitution of $x = a$ in (5.1) gives*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left\{ (-1)^k f^{(k)}(a) + \left(1 + \frac{1}{2^{k+1}} + \frac{(-1)^k}{4^{k+1}}\right) f^{(k)}(b) \right\}\right. \tag{5.3}$$

$$\left. + \frac{(-1)^n}{b-a} \int_a^b f(t)dt - \frac{(b-a)^{n-1}}{2^{n+1}(n+1)!} \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)(1 + (-1)^n) \right|$$

$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a} \left[\frac{-1}{(n!)^2 (2n+1)} \frac{(b-a)^{2n+1}}{4^{3n+1}} \right.$$

$$\left. - \frac{1}{b-a} \frac{1}{(n+1)!} \left\{ \frac{(b-a)^{n+1}}{2^{n+1}} (1 + (-1)^n) \right\}^2 \right]^{\frac{1}{2}}.$$

Corollary 8. *Substitution of $x = \frac{a+b}{2}$ in (5.1) gives*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left\{ f^{(k)}\left(\frac{3a+b}{4}\right) + \left(1 + (-1)^k\right) f^{(k)}\left(\frac{a+b}{2}\right) \right.\right. \tag{5.4}$$

$$\left. + (-1)^k f^{(k)}\left(\frac{a+3b}{4}\right) \right\} + \frac{(-1)^n}{b-a} \int_a^b f(t)dt$$

$$\left. - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) \frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!} (1 + (-1)^n) \right|$$

$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a} \left[\frac{(b-a)^{2n+1}}{2^{2n}(n!)^2(2n+1)} \left(1 + \frac{1}{2^{2n+1}}\right) - \frac{(b-a)^{2n+1}(1+(-1)^n)^2}{4^{2n+1}(n+1)!} \right]^{\frac{1}{2}}.$$

Corollary 9. *Substitution of $x = \frac{3a+b}{4}$ in (5.1) gives*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left[\frac{\left(1 + (-1)^k\right)}{2^{k+1}} f^{(k)}\left(\frac{7a+b}{8}\right) \right.\right. \tag{5.5}$$

$$+ (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) + f^{(k)}\left(\frac{a+3b}{4}\right)$$

$$\left. + \frac{1}{2^{k+1}} \left(1 + (-1)^k\right) f^{(k)}\left(\frac{a+7b}{8}\right) \right] + \frac{(-1)^n}{b-a} \int_a^b f(t)dt$$

$$- \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) . \frac{1}{(n+1)!} \frac{(b-a)^{n-1}}{4^{n+1}}$$

$$\left. \times \left(1 + (-1)^n + \frac{1}{2^n} + \frac{(-1)^n}{2^n}\right) \right|$$

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$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\frac{2(b-a)^{2n+1}}{4^{2n+1}}\left(\frac{1}{2^{2n}}+1\right)\right.$$
$$\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(b-a)^{n+1}}{4^{n+1}}\left(1+(-1)^n\right)\left(1+\frac{1}{2^n}\right)\right\}^2\right]^{\frac{1}{2}}.$$

Remark 2. *By choosing $n=1$ in case A, B and C, we get all results obtained in* [10].

Remark 3. *By choosing $n=2$ in case A, B and C, we get all results obtained in* [9].

6. Derivation of Numerical Quadrature Rules

We propose some new quadrature rules involving higher order derivatives of the function f. In fact, the following new quadrature rules can be obtained while investigating error bounds using theorem 5.

$$Q_{n,1}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n-1}\frac{(b-a)^{k+2}}{2^{k+1}(k+1)!}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]$$
$$+\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]\frac{(b-a)^n}{2^{n+1}(n+1)!}\left(1+(-1)^n\right),$$

$$Q_{n,2}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n-1}\frac{(b-a)^{k+2}(-1)^k}{4^{k+1}(k+1)!}\left[f^{(k)}\left(\frac{3a+b}{4}\right)\right.$$
$$\left.+\left\{1+(-1)^k\right\}f^{(k)}\left(\frac{a+b}{2}\right)+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right]$$
$$+\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\times\frac{2}{4^{n+1}}\frac{(b-a)^n}{(n+1)!}\left((-1)^n+1\right),$$

$$Q_{n,3}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}\frac{(b-a)^{k+2}}{4^{k+1}}\left[\frac{1}{2^{k+1}}\left(1+(-1)^k\right)\left(f^{(k)}\left(\frac{7a+b}{8}\right)+f^{(k)}\left(\frac{a+7b}{8}\right)\right)\right.$$
$$\left.+\left\{(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)+f^{(k)}\left(\frac{a+3b}{4}\right)\right\}\right]$$
$$+\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]\times\frac{(b-a)^n}{4^{n+1}(n+1)!}\left((-1)^n+1\right)\left(\frac{1}{2^n}+1\right).$$

Performance of the efficient quadrature rules

	Method	$n:\ Q_{n,1}(f)$	$n:\ Q_{n,2}(f)$	$n:\ Q_{n,3}(f)$	Exact Value
1.	$\int_0^1 f_1(x)dx$	**2:** 2.83333	**2:** 2.83333	**2:** 2.83333	2.83333
	Error:	0	0	0	
2.	$\int_0^1 f_2(x)dx$	**6:** 0.30117	**4:** 0.301172	**4:** 0.30117	0.301169
	Error:	1.1381×10^{-6}	3.08726×10^{-6}	1.38925×10^{-6}	
3.	$\int_0^1 f_3(x)dx$	**6:** 0.909328	**4:** 0.909324	**4:** 0.909327	0.909331
	Error:	2.33999×10^{-6}	7.13925×10^{-6}	3.21638×10^{-6}	
4.	$\int_0^1 f_4(x)dx$	**5:** 0.793022	**4:** 0.793031	**4:** 0.793031	0.793031
	Error:	8.63182×10^{-6}	2.9641×10^{-7}	1.33626×10^{-7}	
5.	$\int_0^1 f_5(x)dx$	**11:** 1.46266	**7:** 1.46265	**6:** 1.46266	1.46265
	Error:	5.8789×10^{-6}	2.29707×10^{-6}	5.20247×10^{-6}	
6.	$\int_0^1 f_6(x)dx$	**11:** 1.31384	**6:** 1.31383	**6:** 1.31383	1.31383
	Error	7.37624×10^{-6}	2.13363×10^{-6}	1.73918×10^{-6}	
7.	$\int_0^1 f_7(x)dx$	**6:** 1.34146	**4:** 1.34137	**4:** 1.34147	1.34147
	Error:	1.4808×10^{-7}	5.42574×10^{-7}	2.44601×10^{-7}	
8.	$\int_0^1 f_8(x)dx$	**9:** 0.62977	**5:** 0.629762	**4:** 0.629774	0.629769
	Error:	1.18074×10^{-6}	6.3567×10^{-6}	5.647×10^{-6}	

Table:

$$\begin{aligned}
&f_1(x) = x^2 + x + 2, && f_2(x) = x \sin x, \\
&f_3(x) = e^x \sin x\ , && f_4(x) = x^2 + \sin x, \\
&f_5(x) = e^{x^2}, && f_6(x) = e^x \cos\left(e^x - 2x\right), \\
&f_7(x) = x + \cos x, && f_8(x) = \log\left(x^2+2\right)\sin\left[\log\left(x^2+2\right)\right].
\end{aligned} \tag{6.1}$$

From the above table, we observe that all three quadrature rules show exact value of the integral of f_1 for $n = 2$. For any polynomial of degree k, $n = k + 1$ will give exact value of the integral f_1. Acceptable error estimates can be obtained for smaller values of n to save computational time.

The integral of f_5 , $Q_{n,3}(f)$ report an error of the order of 10^{-6} for $n = 6$ while the other two quadrature rules give a similar error for $n = 7$ and $n = 11$. Similarly for all other functions $Q_{n,3}(f)$ report errors of the order of 10^{-6} or 10^{-7} for relatively smaller values of n as compared to the other two quadrature rules. Specifically, $Q_{n,3}(f)$ give an excellent estimate for the integrals of f_5 and f_8 at $n = 6$ and $n = 4$ respectively. In general $Q_{n,3}(f)$ gave better results as compared to the rest of the quadrature rules for much smaller values of n. Therefore we can conclude that overall $Q_{n,3}(f)$ is computationally more efficient both in terms of error approximation, simplicity, and time. As a rough estimate we integrated $\log\left(x^2+2\right)\sin\left[\log\left(x^2+2\right)\right]$ using the built in algorithms of Mathematica 10.0 which took 26.30 seconds to give its approximate answer. To obtain similar approximation for the integral of f_8, $Q_{n,3}(f)$ took less than a second.

Based on this analysis, we can conjecture that $Q_{n,3}(f)$ is the most efficient quadrature rule, while $Q_{n,2}(f)$ comes second in terms of performance. It should be noted that if desired the value of n can be adjusted to improve the error bounds or decrease computational time.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.4, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Duality in multiobjective nonlinear programming under generalized second order $(F, b, \phi, \rho, \theta)-$ univex functions

Falleh R. Al-Solamy, Meraj Ali Khan

Abstract

In the present paper, second order duality for multiobjective nonlinear programming are investigated under the second order generalized $(F, b, \phi, \rho, \theta)-$ univex functions. The weak, strong and converse duality theorems are proved. Further, we also illustrated an example of $(F, b, \phi, \rho, \theta)-$ univex functions. Results obtained in this paper extend some previously known results of multiobjective nonlinear programming in the literature.

Keywords: Duality, Multiobjective programming, Univex functions

Mathematics Subject Classification (2000): 90C32, 49K35, 49N15

1 Introduction

In recent years, the concept of convexity and generalized convexity is well known in optimization theory and plays a central role in mathematical economics, management science and optimization theory. Therefore, the research on convexity and generalized convexity is one of the most important aspects in mathematical programming. In particular, the concept of generalized $(F, \rho)-$ convexity introduced by Preda [8]. In [9, 13], the concept of $V-\rho$-invexity and $(F, \alpha, \rho, d)-$ convexity were introduced respectively. Zhang and Mond [12] extended the class of $(F, \rho)-$ convex functions to second oder $(F, \rho)-$ convex functions and obtained the duality results for Mangasarian type, Mond-Weir type and general Mond-Weir type multiobjective dual problems. Motivated by Liang et al. [13] and Aghezzaf [2], I. Ahmad and Z. Husain [5] introduced second order $(F, \alpha, \rho, d)-$ convex functions and their generalization and they investigate weak, strong and strict converse duality theorems for second order Mond Weir type Multiobjective dual. Bector et al. [15] generalized the notion of convex function to univex functions. Rueda et al. [16] obtained optimality and duality results for several mathematical programs by combining the concepts of type I and univex functions. Mishra [8] obtained optimality results and saddle point results for multiobjective programs under generalized type I univex functions. Recently, Zalmai [14] introduced the notion of second order $(F, b, \phi, \rho, \theta)-$ univex functions and he called these functions $(F, b, \phi, \rho, \theta)-$sounivex functions, these function generalize the second order $(F, \alpha, \rho, d)-$convex functions defined by Ahmad and Husain [5].

The concept of second order duality in nonlinear programming problems was first introduced by Mangasarian [11]. One significant practical application of second order dual over first order is that it may provide tighter bounds for value of objective function because there ae more parameters involved, several researchers [1, 4, 7, 21] considered second order dual models for multiobjective

programming. In this paper , we formulate second order dual model and investigate weak, strong and strict converse duality theorems under $(F, b, \phi, \rho, \theta)-$ sounivexity assumptions. Further, an example have been constructed, which shows the existence of $(F, b, \phi, \rho, \theta)-$ sounivex functions.

2 Notations and Preliminaries

We consider the following multiobjective nonlinear programming problem:

$$(\mathbf{P}) \qquad \text{Minimize} \qquad f(x),$$
$$\text{subject to} \qquad g(x) \leqq 0, \quad x \in X, \tag{1}$$

where $f = (f_1, f_2, \ldots, f_k) : X \to R^k$, $g = (g_1, g_2, \ldots g_m) : X \to R^m$ are assumed to be twice differentiable function over X, an open subset of R^n.

Definition 2.1. A function $\mathcal{F} : X \times X \times R^n \to R$, where $X \subseteq R^n$ is said to be sublinear in its third argument, if $\forall \ x, \bar{x} \in X$,
(i) $\mathcal{F}(x, \bar{x}; a_1 + a_2) \leq \mathcal{F}(x, \bar{x}; a_1) + \mathcal{F}(x, \bar{x}; a_2), \ \forall \ a_1, a_2 \in R^n$,

(ii) $\mathcal{F}(x, \bar{x}; \alpha a) = \alpha \mathcal{F}(x, \bar{x}; a), \ \forall \ \alpha \in R_+, a \in R^n$.

Definition 2.2. A point $\bar{x} \in S$ is said to efficient solution of (P), if there exists no other feasible point x such that $f(x) \leq f(\bar{x})$ for each $x, \bar{x} \in X$.

Let $u \in R^n$ and assume that the function $f : X \to R$ is twice differentiable at u.

Definition 2.3. [14] The function f is said to be (strictly) $(F, b, \phi, \rho, \theta)-$ sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X (x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2} p^t \nabla^2 f(u) p)(>) \geqq F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u) p])$$
$$+\rho(x, u) \|\theta(x, u)\|^2,$$

where $\|.\|^2$ is a norm on R^n.

A twice differentiable vector function $f : X \to R^k$ is said to be $(F, b, \phi, \rho, \theta)-$ sounivex at u, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$sounivex at u. Now we define generalized $(F, b, \phi, \rho, \theta)-$ sounivex functions

Definition 2.4. A twice differentiable function f, over X is said to be $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X (x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2} p^t \nabla^2 f(u) p) < 0$$
$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u) p]) < -\rho(x, u) \|\theta(x, u)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$pseudo sounivex at u.

Definition 2.5. A twice differentiable function f, over X is said to be strictly $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X(x \neq u)$ and $p \in R^n$,

$$F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) \geqq -\rho(x, u)\|\theta(x, u)\|^2$$

$$\Rightarrow \quad \phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) > 0,$$

or equivalently

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leqq 0$$

$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) < -\rho(x, u)\|\theta(x, u)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be strictly $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components is strictly f_i is $(F, b, \phi, \rho, \theta)-$pseudo sounivex at u.

Definition 2.6. A twice differentiable function f, over X is said to be $(F, b, \phi, \rho, \theta)-$ quasi sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X(x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leqq 0$$

$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) \leqq -\rho(x, u)\|\theta(x, u)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$quasi sounivex at u.

Definition 2.7. A twice differentiable function f, over X is said to be strong $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X(x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leq 0$$

$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) \leq -\rho(x, u)\|\theta(x, u)\|^2$$

A twice differentiable vector function $f : X \to R^k$ is said to be strong $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components f_i is strong $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u.

Every $(F, b, \phi, \rho, \theta)-$ sounivex function need not to be second order $(F, \alpha, \rho, d)-$ convex, definded in [5]. To show this, consider the following example.

Example 2.1. Let $f : X = (0, \infty) \to R$ be defined as $f(x) = -x^2 - x$. Let $\phi(t) = -t$, $b(x, u) = x - u$, $\rho = -10$, $\theta(x, u) = \frac{u+2}{2}$ and sublinear function is defined as $F(x, u, a) = a(x - u) + x$

$$F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) = -(x^2 - u^2)(2u + 1 + 2p) + x - 10\|\frac{u+2}{2}\|^2,$$

at $u = 0$,

$$F(x, 0; b(x, 0)[\nabla f(0) + \nabla^2 f(0)p]) = -x^2(1 + 2p) + x - 10$$

and

$$f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p = -x^2 - x + u^2 + u - p^2,$$

at $u = 0$

$$\phi(f(x) - f(0) + \frac{1}{2}p^t\nabla^2 f(0)p) = x^2 + x + p^2,$$

and it is easy to see that

$$\phi(f(x) - f(0) + \frac{1}{2}p^t\nabla^2 f(0)p) - F(x, 0; b(x, 0)[\nabla f(0) + \nabla^2 f(0)p])$$

$$= x^2 + p^2 + x^2(1 + 2p) + 10 \geq 0$$

for all $x \in R$ and $-1 \leq p < \infty$, so the function is $(F, b, \rho, \phi, \theta)-$ sounivex at $x = 0$, but at $p = -1, x = 10$

$$(f(x) - f(0) + \frac{1}{2}p^t\nabla^2 f(0)p) - F(x, 0; b(x, 0)[\nabla f(0) + \nabla^2 f(0)p]) < 0$$

Hence, the function is not $(F, \alpha, \rho, d)-$convex at $x = 0$.

Now we have following Kuhn-Tucker type necessary conditions, which will be useful to prove the strong duality theorem.

Theorem 2.1. (*Kuhn-Tucker type necessary conditions*) Assume that x^* is an efficient solution for (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\lambda^* \in R^k$ and $y^* \in R^m$, such that

$$\lambda^{*t}\nabla f(x^*) + y^{*t}\nabla g(x^*) = 0,$$

$$y^{*t}\nabla g(x^*) = 0,$$

$$y^* \geqq 0,$$

$$\lambda^* \geq 0.$$

3 Second order Mond-Weir type duality

In this section, we consider the following Mond-Weir second order dual associated with multiobjective problem (P) and establish weak, strong and strict converse duality theorems under generalized $(F, b, \rho, \phi, \theta)-$ sounivexity

(**MD**) Maximize

$$f(u) - \frac{1}{2}p^t\nabla^2 f(u)p$$

Subject to

$$\nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p + \nabla\lambda^t g(u) + \nabla^2\lambda^t g(u)p = 0, \tag{2}$$

$$y^t g(u) - \frac{1}{2}p^t\nabla^2 y^t g(u)p \geqq 0, \tag{3}$$

$$y \geqq 0, \tag{4}$$

$$\lambda \geq 0, \tag{5}$$

where λ is a $k-$dimensional vector, and y is an $m-$dimensional vector.

Theorem 2 (*weak duality*) Suppose that for all feasible solutions x in (P) and all feasible solutions (u, y, λ, p) in MD

(i) $y^t g(0)$ is $(F, b, \phi, \rho, \theta)-$quasi sounivex at u,

(ii) $\lambda > 0$, and $f(.)$ is strong $(F, b_1, \phi, \rho_1, \theta)-$ pseudo sounivex at u with $b^{-1}\rho + b_1^{-1}\rho_1\lambda \geqq 0$,

(iii) $u \leq 0 \Rightarrow \phi(u) \leq 0$ and $v \leqq 0 \Rightarrow \phi(v) \leqq 0$, for all $u, v \in R^n$.

Then the following can not hold

$$f(x) \leq f(u) - \frac{1}{2}p^t\nabla^2 f(u)p. \tag{6}$$

Proof. Now suppose contrary to the result that (6) holds, i.e.,

$$f(x) \leq f(u) - \frac{1}{2}p^t\nabla^2 f(u)p,$$

or

$$f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p \leq 0,$$

then by assumption (iii)

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leq 0, \tag{7}$$

which by virtue of assumption (ii) leads

$$F(x, u, b_1(x, u)\{\nabla f(u) + \nabla^2 f(u)p\}) \leq -\rho_1\|\theta(x, u)\|^2. \tag{8}$$

On multiplying (8) by $\lambda > 0$ and using sublinearity of F with $b_1(x, u) > 0$, we have

$$F(x, u, \nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p) < -b_1^{-1}(x, u)\rho_1\lambda\|\theta(x, u)\|^2. \tag{9}$$

The first dual constraint and sublinearity of F give

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u)p) \geqq -F(x, u; \nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p).$$

Applying (9) in above inequality, we have

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u)) > b_1^{-1}(x, u)\rho_1\lambda\|\theta(x, u)\|^2. \tag{10}$$

Let x be any feasible solution in (P) and (u, y, λ, p) be any feasible solution in (MD). Then we have

$$y^t g(x) \leqq 0 \leqq y^t g(u) - \frac{1}{2} p^t \nabla^2 y^t g(u) p), \tag{11}$$

by assumption (iii), (11) yields

$$\phi(y^t g(x) - y^t g(u) + \frac{1}{2} p^t \nabla^2 y^t g(u) p) \leqq 0. \tag{12}$$

Using $(F, b, \phi, \rho, \theta)$-quasi sounivexity of $y^t g(.)$, we have

$$F(x, u; b(x, u)\{\nabla y^t g(u) + \nabla^2 y^t g(u) p)\}) \leqq -\rho \|\theta(x, u)\|^2. \tag{13}$$

Since $b(x, u) > 0$, the above inequality with the sublinearity of F give

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u) p) \leqq -b^{-1} \rho \|\theta(x, u)\|^2. \tag{14}$$

Now using the assumption $b^{-1}\rho + b_1^{-1}\rho_1 \lambda \geqq 0$, the above inequality yields

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u) p) \leqq b_1^{-1} \rho_1 \lambda \|\theta(x, u)\|^2. \tag{15}$$

Which contradict (10), hence (6) can not hold.

Theorem 3 (*Strong duality*). Let $\bar{x}$ be an efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\bar{y} \in R^m$ and $\bar{\lambda} \in R^k$, such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is a feasible for (MD) and the corresponding values of (P) and (MD) are equal. If in addition, the assumptions of weak duality (Theorem 2) hold for all feasible solutions of (P) and (MD), then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is an efficient solution of (MD).

Proof. Since $\bar{x}$ is an efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied, then by Theorem 1, there exist $\bar{y} \in R^m$ and $\bar{\lambda} \in R^k$, such that

$$\bar{\lambda}^t \nabla f(\bar{x}) + \bar{y}^t \nabla g(\bar{x}) = 0,$$

$$\bar{y}^t \nabla g(\bar{x}) = 0,$$

$$\bar{y} \geqq 0,$$

$$\bar{\lambda} \geq 0.$$

Therefore $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is feasible for (MD) and the corresponding values of (P) and (MD) are equal. The efficiency of this feasible solution for (MD) thus follows from weak duality (Theorem 2).

Theorem 4 (*Strict converse duality*) Let $\bar{x}$ and $(\bar{u}, \bar{y}, \bar{\lambda}, \bar{p})$ be the efficient solution of (P) and (MD), respectively such that

$$\bar{\lambda}^t f(\bar{x}) = \bar{\lambda}^t f(\bar{u}) - \frac{1}{2} \bar{p}^t \nabla^2 \bar{\lambda}^t f(\bar{u}) \bar{p}. \tag{16}$$

Suppose

(i) $y^t g(.)$ is $(F, b, \phi, \rho, \theta)-$quasi sounivex at $\bar{u}$,

(ii) $\bar{\lambda}^t f(.)$ be $(F, b_1, \phi, \rho_1, \theta)-$ pseudo sounivex at $\bar{u}$ with $b^{-1}\rho + b_1^{-1}\rho_1\lambda \geqq 0$,

(iii) $u \leqq 0 \Rightarrow \phi(u) \leqq 0$ and $v < 0 \Rightarrow \phi(v) < 0$, for all $u, v \in R^n$.

Then $\bar{x} = \bar{u}$, that is $\bar{u}$ is an efficient solution.

Proof. We assume that $\bar{x} \neq \bar{u}$ and reach a contradiction, since $\bar{x}$ and $(\bar{u}, \bar{y}, \bar{\lambda}, p)$ are respectively the feasible solution of (P) and (MD), we have

$$\bar{y}^t g(\bar{x}) - \bar{y}^t g(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 \bar{y}^t g(\bar{u})\bar{p} \leqq 0. \tag{17}$$

Using the assumption (iii), we have

$$\phi(\bar{y}^t g(\bar{x}) - \bar{y}^t g(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 \bar{y}^t g(\bar{u})\bar{p}) \leqq 0. \tag{18}$$

By $(F, b, \phi, \rho, \theta)-$quasi sounivexity of $\bar{y}^t g(.)$ at $\bar{u}$, we get

$$F(\bar{x}, \bar{u}; b(\bar{x}, \bar{u})\{\nabla \bar{y}^t g(\bar{u}) + \nabla^2 \bar{y}^t g(\bar{u})\bar{p})\}) \leqq -\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{19}$$

Since $b(\bar{x}, \bar{u}) > 0$, the inequality (19) along with the sublinearity of F, imply

$$F(\bar{x}, \bar{u}; \nabla \bar{y}^t g(\bar{u}) + \nabla^2 \bar{y}^t g(\bar{u})\bar{p}) \leqq -b^{-1}(x, u)\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{20}$$

The first dual constraint and sublinearity of F imply

$$F(\bar{x}, \bar{u}; \nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p}) \geqq -F(\bar{x}, \bar{u}, \nabla \bar{y}^t g(\bar{u}) + \nabla^2 \bar{y}^t g(\bar{u})\bar{p}).$$

Applying (20) and $b^{-1}\rho + b_1^{-1}\rho \geqq 0$ in above inequality, we get

$$F(\bar{x}, \bar{u}; \nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p}) \geqq -b_1^{-1}(\bar{x}, \bar{u})\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{21}$$

Suppose (16) does not holds, then we have

$$\bar{\lambda}^t f(\bar{x}) < \bar{\lambda}^t f(\bar{u}) - \frac{1}{2}\bar{p}^t \nabla^2 \bar{\lambda}^t f(\bar{u})\bar{p},$$

now using assumption (iii)

$$\phi(\bar{\lambda}^t f(\bar{x}) - \bar{\lambda}^t f(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 \bar{\lambda}^t f(\bar{u})\bar{p}) < 0.$$

Now by the assumption (ii), the above inequality gives

$$F(\bar{x}, \bar{u}; b_1(\bar{x}, \bar{u})(\nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p})) < -\rho\|\theta(\bar{x}, \bar{u})\|^2,$$

or

$$F(\bar{x}, \bar{u}; \nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p}) < -b_1^{-1}(\bar{x}, \bar{u})\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{22}$$

Which contradict (21). Hence result.

4 Conclusion

In this paper anew concept of generalized invex functions is introduced. Under this generalized invexity we establish weak, strong and converse duality theorems. These duality relations lead to duality in nonlinear fractional programming problems.

5 Authors contributions

Both the authors contributed equally to writing of this paper and the final manuscript is read and approved by the authors.

6 Competing interests

The author declare that they have no competing interests.

7 Acknowledgments

This project was funded by the Deanship of Scientific Research(DSR), King Abdulaziz University, Jeddah, under grant No. (G-1436-130-242). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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STABILITY OF FRACTIONAL DIFFERENTIAL EQUATION WITH BOUNDARY CONDITIONS

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Abstract. In this paper, we prove the Hyers-Ulam stability of a fractional differential equation of order $\alpha \in (1, 2)$ with certain boundary conditions.

1. Introduction

The recent concentric area in the research world of mathematics is fractional differential equations. The concept of fractional derivative is not new and is very much as old as classical differential equations. In recent years, many authors disscussed and proved the existence results of fractional differential equations using various methods. For example, one can refer the monographs of Kilbas et al. [10], Miller and Ross [14], Podulbny [20], Diethelm et al. [4, 5], Benchora [2] and so on. Obviously, the differential equations of fractional order has been proved to be a valuable tool in the modeling of many phenomena in various fields of science and engineering. Indeed, one can find many applications in electromagnetic, control, electrochemistry, etc. (see [6, 7]).

At the same instance, the stability concept is more devoloped in the research world of mathematics, particularly in functional equations. But the analysis of stability concepts of fractional differential equations has been very slow and there are only countable number of works. In 2009, Li [12], first proposed the Mittag-Leffler stability and in 2010 [13], the fractional Lyapunovs second method. In the next year, Li and Zhang [11] have been given a brief overview on the stability of the fractional differential equations. However, there are only few works available on the local stability and Mittag-Leffler stability for fractional differential equations and very rare works on the Ulam stability of fractional differential equations.

In 2011, Wang [24] carried out a pioneering work on the Hyers-Ulam stability and data dependence for fractional differential equations with Caputo derivative. Wang [25] proved the Hyers-Ulam stability of fractional differential equation of order $0 < \alpha < 1$ via a generalized fixed point approach, by adopting some part idea of Wang et al. [24], Cadariu and Radu [3] and Jung [9] in the next year. Particularly, there are very rare works on the Hyers-Ulam stability of fractional differential equations with boundary conditions. Recently, Rabha [8], Muniyappan and Rajan [16] had given Ulam stabilities with boundary conditions in the interval $(0, 1)$. For more information on functional equations and their stability problems, see [15, 17, 18, 19, 21, 22, 23].

2010 *Mathematics Subject Classification.* Primary 34A08, 34K10, 34K20.

Key words and phrases. Hyers-Ulam stability; fractional differential equation; boundary condition.

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In this paper, the Hyers-Ulam stability of the following fractional boundary value problem is proved.

$$^{C}D^{\alpha}y(t) = F(t, y(t)), \quad 1 < \alpha < 2 \tag{1.1}$$

$$y(0) = y_0 \quad y(T) = y_T \tag{1.2}$$

This paper is organized as follows: In Section 2, basic definitions and notations are given. In Section 3, the Hyers-Ulam stability of the above fractional boundary value problem is proved.

2. Preliminaries

Throughout this paper, we assume that Y is a normed space and $I = [0, T]$ is a given interval.

Definition 2.1. ([2]) The fractional order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where Γ is the gamma function.

Definition 2.2. ([2]) For a function h given on the interval $[a, b]$, the Caputo fractional order derivative of h, is defined by

$$({}^{c}D_{a+}^{\alpha} h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3. ([1]) A function $y \in C^2(I, \mathbb{R})$ is said to be a solution of (1.1)-(1.2) if y satisfies the equation ${}^{c}D^{\alpha}y(t) = F(t, y(t))$ on I, and the condition $y(0) = y_0$ and $y(T) = y_T$

Lemma 2.4. ([1]) *Let $1 < \alpha < 2$ and let $F : I \to \mathbb{R}$ be continuous. A function $y \in C^2(I, \mathbb{R})$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T$$

if and only if y is a solution of the fractional boundary value problem

$$^{c}D^{\alpha}y(t) = F(t, y(t)), t \in [0, T]$$

$$y(0) = y_0 \quad y(T) = y_T$$

Definition 2.5. ([25]) The fractional differential equation (1.1) is Hyers-Ulam stable if there exists a continuously differentiable function $f : I \to Y$ satisfying the inequality

$$\|{}^{c}D^{\alpha}y(t) - F(t, y(t))\| \leq \epsilon$$

for all $t \in I$ and for some $\epsilon > 0$, there exists a solution $f_0 : I \to Y$ of the fractional differential equation 1.1 such that

$$\|f(t) - f_0(t)\| \leq K\epsilon$$

for all $t \in I$.

Definition 2.6. ([25]) The fractional differential equation (1.1) is Hyers-Ulam-Rassias stable if there exists a continuously differentiable function $f : I \to Y$ satisfying the inequality

$$\|{}^c D^\alpha y(t) - F(t, y(t))\| \leq \varphi(t)$$

for all $t \in I$, there exists a solution $f_0 : I \to Y$ of the fractional differential equation (1.1) such that

$$\|f(t) - f_0(t)\| \leq \Phi(t)$$

for all $t \in I$. where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on f and f_0 explicitly.

Definition 2.7. ([25]) For a nonempty set X, a function $d : X \times X \to [0, \infty]$ is called generalized metric on X if and only if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Such a space is called a generalized complete metric space.

Theorem 2.8. ([3]) *Let (X,d) be a generalised complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$, If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k) < \infty$ for some $x \in X$, then the following are true:*

(1) *The sequence $\{\Lambda^n x\}$ converges to a fixed end point x^* of Λ*
(2) *x^* is the unique fixed point of Λ in $X^* = \{y \in X | d(\Lambda^k x, y) < \infty\}$;*
(3) *If $y \in X^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$*

3. Hyers-Ulam stability

In this section, we first investigate the Hyers-Ulam stability of the fractional differential equation (1.1) with boundary condition (1.2) via Theorem 2.8.

Theorem 3.1. *Let $I = [0, T]$ be a closed interval. Let $K, P,$ and L be positive constants with $0 < KPL < 1$. Assume that $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies the standard Lipschitz condition*

$$|F(t, y) - F(t, z)| \leq L\,|y - z| \tag{3.1}$$

for all $t \in I$ and $y, z \in \mathbb{R}$. If a continuously differential function $y : I \to \mathbb{R}$ satisfies

$$|{}^c D^\alpha y(t) - F(t, y(t))| \leq \varphi(t) \tag{3.2}$$

for all $t \in I$, where $\varphi : I \to (0, \infty)$ is a continuous function with

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \varphi(\tau) d\tau \right| \leq K\varphi(t) \tag{3.3}$$

for all $t \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$y_0(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} F(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T - s)^{\alpha - 1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T \tag{3.4}$$

and

$$|y(t) - y_0(t)| \leq \frac{K}{1 - KPL} \varphi(t) \tag{3.5}$$

for all $t \in I$.

Proof. Let us define a set X of all continuous functions $f : I \to \mathbb{R}$ by

$$X = \{f : I \to \mathbb{R} | f \text{ is continuous}\}. \tag{3.6}$$

Similar to [9, Theorem 3.1], one can introduce a generalised complete metric on X as follows

$$d(f,g) = \inf\{C \in [0,\infty] | \, |f(t) - g(t)| \le C\varphi(t) \quad \text{for all} \quad t \in I\}. \tag{3.7}$$

Define an operator $\Lambda : X \to X$ by

$$(\Lambda f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T \tag{3.8}$$

for all $f \in X$.
It is easy to see that Λ is well defined, since F and f are continuous functions.
To acheive our aim, we need to prove that Λ is strictly contractive on X.
For any $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f,g) \le C_{fg}$, that is, by (3.7), we have

$$|f(t) - g(t)| \le C_{fg}\varphi(t) \tag{3.9}$$

for all $t \in I$. It then follows from (3.1), (3.3), (3.7), (3.8) and (3.9) that

$$\begin{aligned}
|(\Lambda f)t - (\Lambda g)t| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds \\
&\quad + \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds \\
&\le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s) - g(s)| \, ds + \frac{tL}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s) - g(s)| \, ds \\
&\le \frac{L}{\Gamma(\alpha)} C_{fg} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds + \frac{tL}{T\Gamma(\alpha)} C_{fg} \int_0^T (T-s)^{\alpha-1} \varphi(s) ds \\
&\le KPLC_{fg}\varphi(t)
\end{aligned}$$

for all $t \in I$, where $P = \left(1 + \frac{t}{T}\right)$. That is,

$$d(\Lambda f, \Lambda g) \le KPLC_{fg}.$$

Hence we can conclude that

$$d(\Lambda f, \Lambda g) \le KPLd(f, g)$$

for all $f, g \in X$, where we note that $0 < KPL < 1$.
It follows from (3.6) and (3.8) that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < \infty$ with

$$\begin{aligned}
|(\Lambda g_0)(t) - g_0(t)| &= \Big| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds \\
&\quad - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T - g_0(t) \Big| \\
&\le C\varphi(t)
\end{aligned}$$

for all $t \in I$, since $f(t, g_0(t))$ and $g_0(t)$ are bounded on I and $min_{t\in I}\varphi(t) > 0$.
Thus (3.7) implies that

$$d(\Lambda g_0, g_0) < \infty.$$

Therefore, according to Theorem 2.8, there exists a continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda^n g_0 \to y_0$ in (X, d) and $\Lambda y_0 = y_0$, that is, y_0 satisfies (3.4) for every $t \in I$.
we will now verify that $\{g \in X/d(g_0, g) < \infty\} = X$.
For any $g \in X$, since g and g_0 are bounded on I and $min_{t\in I}\varphi(t) > 0$, there exists a constant $0 < C_g < \infty$ such that $|g_0(t) - g(t)| \leq C_g\varphi(t)$
Hence, we have $d(g_0, g) < \infty$ for all $g \in X$, that is $\{g \in X/d(g_0, g) < \infty\} = X$.
Hence in view of Theorem 2.8, we conclude that y_0 is the unique continuous function with the property (3.4). On the other hand, it follows from (3.2) that

$$-\varphi(t) \leq^c D^\alpha_{a+}y(t) - F(t, y(t)) \leq \varphi(t)$$

for all $t \in I$.
If we integrate each term in the above inequality and substitute the boundary conditions, then we obtain

$$|y(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, y(s))ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s, y(s))ds - \left(\frac{t}{T} - 1\right)y_0 + \frac{t}{T}y_T|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\varphi(t)ds$$

for all $t \in I$.
Thus by (3.3) and (3.8) we get

$$|y(t) - (\Lambda y)(t)| \leq K\varphi(t)$$

for each $t \in I$, which implies that

$$d(y, \Lambda y) \leq K. \tag{3.10}$$

Finally, Theorem 2.8 and (3.10) imply that

$$d(y, y_0) \leq \frac{1}{1 - KPL}d(y, \Lambda y) \leq \frac{K}{1 - KPL}.$$

□

Now, we will prove the Hyers-Ulam stability of the (1.1) with boundary condition (1.2)

Theorem 3.2. *Let $I = [0, T]$ be a closed interval. Let $r > 0$ be a positive constant with $0 \leq t, T \leq r$ and let $F : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies a Lipschitz condition (3.1) for all $t \in I$ and $y, z \in \mathbb{R}$, where L is a constant with $0 < \frac{LPr^\alpha}{\Gamma(\alpha+1)} < 1$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfying the differential inequality*

$$\left|{}^cD^\alpha_{a+}y(t) - F(t, y(t))\right| \leq \epsilon \tag{3.11}$$

for all $t \in I$ and for some $\epsilon \geq 0$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ satisfying (3.4) and

$$|y(t) - y_0(t)| \leq \frac{r^\alpha}{\Gamma(\alpha+1) - LPr^\alpha}\epsilon \tag{3.12}$$

for all $t \in I$.

Proof. First, we define a set X of all continuous functions $f : I \to \mathbb{R}$ by

$$X = \{f : I \to \mathbb{R} | f \text{ is continuous}\}$$

and introduce a generalized complete metric on X as follows

$$d(f, g) = \inf\{C \in [0, \infty] | \, |f(t) - g(t)| \leq C \quad \text{for all} \quad t \in I\}$$

Define an operator $\Lambda : X \to X$ by

$$(\Lambda f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T$$

for all $f \in X$.

We now assert that Λ is strictly contractive on X.

For all $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f, g) \leq C_{fg}$, that is, let us assume that

$$|f(t) - g(t)| \leq C_{fg} \tag{3.13}$$

for any $t \in I$. Moreover, it follows from (3.1), (3.8) and (3.13) that

$$\begin{aligned}
|(\Lambda f)t - (\Lambda g)t| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds \\
&\quad + \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |F(s, f(s)) - F(s, g(s))| \, ds \\
&\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s) - g(s)| \, ds \\
&\quad + \frac{tL}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s) - g(s)| \, ds \\
&\leq LC_{fg} \left[\frac{r^\alpha}{\alpha\Gamma(\alpha)} + \frac{tr^\alpha}{T\alpha\Gamma(\alpha)}\right] \\
&\leq \frac{LC_{fg} r^\alpha}{\Gamma(\alpha+1)} \left[\frac{t+T}{T}\right] \\
&\leq \frac{LPC_{fg} r^\alpha}{\Gamma(\alpha+1)}
\end{aligned}$$

for all $t \in I$, where $P = \left(1 + \frac{t}{T}\right)$, that is

$$d(\Lambda f, \Lambda g) \leq \frac{LPr^\alpha}{\Gamma(\alpha+1)} C_{fg}.$$

Thus it follows that

$$d(\Lambda f, \Lambda g) \leq \frac{LPr^\alpha}{\Gamma(\alpha+1)} d(f, g)$$

for all $f, g \in X$, and we note that $0 < \frac{LPr^\alpha}{\Gamma(\alpha+1)} < 1$.

Analogously to the proof of Theorem 3.1, we can show that each $g_0 \in X$ satisfies the property $d(\Lambda g_0, g_0) < \infty$.

Therefore, Theorem 2.8 implies that there exists a continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda^n g_0 \to y_0$ in (X, d) as $n \to \infty$, and such that $y_0 = \Lambda y_0$, that is, y_0 satisfies the equation (3.4) for all $t \in I$.

If $g \in X$, then g_0 and g are continuous functions defined on a compact interval I. Hence, there exists a constant $C > 0$ with $|g_0(t) - g(t)| \leq C$ for all $t \in I$. This implies that $d(g_0, g) < \infty$ for every $g \in X$, or equivalently, $\{g \in X | d(g_0, g) < \infty\} = X$. Therefore, according to Theorem 2.8, y_0 is a unique continuous function with property (3.4). Furtheremore, it follows from (3.11) that

$$-\epsilon \leq^c D^{\alpha}_{a+} y(t) - F(t, y(t)) \leq \epsilon$$

for all $t \in I$. If we integrate each term of the above inequality and appling the boundary conditions, then we have

$$|(\Lambda y)(t) - y(t)| \leq \frac{r^{\alpha}}{\Gamma(\alpha+1)}\epsilon$$

for all $t \in I$, that is, it holds that $d(\Lambda y, y) \leq \frac{r^{\alpha}}{\Gamma(\alpha+1)}\epsilon$.
It now follows from Theorem 2.8 that

$$d(y, y_0) \leq \frac{1}{1 - \frac{LPr^{\alpha}}{\Gamma(\alpha+1)}} d(\Lambda y, y) \leq \frac{r^{\alpha}}{\Gamma(\alpha+1) - LPr^{\alpha}}\epsilon, \tag{3.14}$$

which implies the validity of (3.12) for each $t \in I$ □

Acknowledgments

S. Yun was supported by Hanshin University Research Grant.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.4, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Bernstein-Stancu type operators which preserve polynomials

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Abstract

In the last years there is an increasing interest in modifying linear operators so that the new versions reproduce some basic functions. This idea motivated us to modify the sequence of linear Bernstein Stancu type operators. Using numerical examples we show that these operators present a better degree of approximation than the original ones. In this note the modified Bernstein Stancu operators are studied in regard to uniform convergence and global smoothness preservation.

2010 Mathematics Subject Classification: Primary 41A36; Secondary 41A25

Key words and phrases: Bernstein-Stancu operator, rate of convergence, moduli of continuity

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1 Introduction

In 1912 in Bernstein's constructive proof of the Weierstrass approximation theorem [3] were introduced the classical Bernstein operators $B_n : C[0,1] \to C[0,1]$, defined by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad \text{where } p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Lemma 1.1. *The Bernstein operators verify the following identities*

(i) $B_n(e_0;x) = 1$,

(ii) $B_n(e_1;x) = x$,

(iii) $B_n(e_2;x) = \frac{x}{n}(1+xn-x)$, *where* $e_i(t) = t^i$, $i = 0, 1, \ldots$.

In the last years there is an increasing interest in modifying linear operators so that the new versions reproduce some basic functions. King [12] consider for the first time this kind of modification for the Bernstein operators and proved that the modified operators reproduce the functions $e_i(x) = x^i$ for $i = 0, 2$ and approximate each continuous function on $[0,1]$ with an order of approximation at least as good as that of the classic Bernstein whenever $0 \le x < \frac{1}{3}$. Using the same type of technique introduced by King or new methods many authors published new results in regard with this subject. Cárdenas-Morales et al. [4] extended this result considering a family of sequences of operators $B_{n,\alpha}$ that preserve e_0 and $e_2+\alpha e_1$ with $\alpha \in [0,\infty)$. Gonska et al. [11] studied the sequence $V_n^\tau : C[0,1] \to C[0,1]$ defined by

$$V_n^\tau f := (B_n f) \circ (B_n \tau)^{-1} \circ \tau,$$

where τ is a continuous strictly increasing function defined on $[0,1]$ with $\tau(0) = 0$ and $\tau(1) = 1$. Note that if $\tau = \frac{e_2+\alpha e_1}{1+\alpha}$, then $V_n^\tau = B_{n,\alpha}$ and the operators V_n^τ preserve e_0 and τ. In [5], the authors inspired by the above ideas consider the sequence of linear Bernstein-type operators defined for $f \in C[0,1]$ by $B_n(f \circ \tau^{-1}) \circ \tau$, τ being any function that is continuously differentiable ∞ times on $[0,1]$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$. Note that the Korovkin set $\{1, e_1, e_2\}$ is generalized to $\{1, \tau, \tau^2\}$ and these operators present a better degree of approximation than B_n.

Since the modified operators present a better degree of approximation than the original ones leads to an interesting area of research, so that generalized Bernstein-Durrmeyer operators and their approximation properties were studied in [1] and [6]. Also, the modified Szasz operators were considered recently in [2].

2 Bernstein-Stancu operators

In 1968, Stancu [15] proposed the sequence of positive linear operators $S_n^{<\alpha>} : C[0,1] \to C[0,1]$ depending on a non-negative parameter α given by

$$S_n^{<\alpha>}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{<\alpha>}(x),\ x \in [0,1], \tag{2.1}$$

where

$$p_{n,k}^{<\alpha>}(x) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}$$

and $t^{[n,h]} := t(t-h)\cdots(t-\overline{n-1}h)$ is the n^{th} factorial power of t with increment h.

For $\alpha = 0$ these operators reduce to the classical Bernstein operators.

The values of the test function by Bernstein-Stancu operators were given by Stancu [15] as follows

Lemma 2.1. *If* $x \in [0,1]$, *then*
(i) $S_n^{<\alpha>}(e_0;x) = 1$,
(ii)$S_n^{<\alpha>}(e_1;x) = x$,
(iii)$S_n^{<\alpha>}(e_2;x) = \frac{1}{1+\alpha}\left(\frac{x(1-x)}{n} + x(x+\alpha)\right)$.

Recently, in [13] Miclăuş proposed a new technique to obtain the values of the test function, without using properties of Bernstein operators.

It is well known the following form of Bernstein operators using the divided difference

$$B_n(f;x) = \sum_{k=0}^{n} \frac{k!}{n^k}\binom{n}{k}\left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right] x^k. \tag{2.2}$$

Starting with the form (2.2) of the Bernstein operators, the following Stancu type operators are constructed in [7]-[8]:

$$C_n : C[0,1] \to \Pi_n,$$

$$C_n(f;x) = \sum_{k=0}^{n} \frac{k!}{n^k}\binom{n}{k}\mathbf{m}_{k,n}\left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k, \quad f \in C[0,1], \tag{2.3}$$

where the real numbers $(m_{k,n})_{k=0}^{\infty}$ are selected in order to preserve some important properties of Bernstein operators and Π_n is the linear space of all real polynomials of degree $\le n$.

Let $\mathbf{m}_{0,n} = 1$, $\lim_{n\to\infty}\mathbf{m}_{1,n} = 1$ and $\mathbf{m}_{k,n} = \frac{(a_n)_k}{k!}$, $a_n \in (0,1]$. For this special case of real sequence $(m_{k,n})_{k=0}^{\infty}$ the Bernstein-Stancu operators C_n were written in the Bernstein basis as follows (see [7], Theorem 10):

$$C_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) C_{k,n}[f], \tag{2.4}$$

where

$$C_{k,n}[f] = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)(a_n)_j(1-a_n)_{k-j}.$$

We remark that $a_n \in (0,1]$ leads to C_n linear positive operators.

The coefficients $C_{k,n}[f]$ can be written as follows

$$C_{k,n}[f] = \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n) f\left(\frac{j}{n}\right).$$

Therefore,

$$C_{k,n}[f] = S_k^{<1>}(\tilde{f}; a_n), \quad \text{where } \tilde{f}(t) = f\left(t\frac{k}{n}\right).$$

Lemma 2.2. ([7]) *The Bernstein-Stancu operators C_n verify the following identities*

(i) $C_n(e_0; x) = 1$,

(ii) $C_n(e_1; x) = a_n x$,

(iii) $C_n(e_2; x) = x^2 + \frac{x(1-x)}{n} a_n + \frac{1-a_n}{2}\left(\frac{a_n}{n} - (2+a_n)\right) x^2$.

Let

$$\begin{aligned}\mu_{n,m}(x) &= C_n\left((t-x)^m; x\right)\\ &= \sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n)\left(\frac{j}{n} - x\right)^m, \quad n, m \in \mathbb{N},\end{aligned}$$

be the central moment operators.

Lemma 2.3. ([7]) *The central moment operators verify*

(i) $\mu_{n,2}(x) = \frac{x(1-x)}{n} a_n + x^2(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right)$,

(ii) $\mu_{n,4}(x) = x^4 + \left[\frac{6(a_n)_3}{n^4}\binom{n}{3} - \frac{12(a_n)_2}{n^3}\binom{n}{2} + \frac{6a_n}{n}\right] x^3 + \left[\frac{7(a_n)_2}{n^4}\binom{n}{2} - \frac{4a_n}{n^2}\right] x^2$
$+ \frac{a_n}{n^3} x + \frac{(a_n)_4}{n^4}\binom{n}{4} - \frac{4(a_n)_3}{n^3}\binom{n}{3} + \frac{6(a_n)_2}{n^2}\binom{n}{2} - 4a_n$.

In [7], Cleciu obtained the following Voronovskaya type theorem:

Theorem 2.4. ([7]) *Suppose that $x_0 \in [0,1]$ and $f''(x_0)$ exists. If $a_n \in (0,1)$, $\lim_{n\to\infty} a_n = 1$ and $L := \lim_{n\to\infty} n(1-a_n)$ exists, then*

$$\lim_{n\to\infty} n\left[f(x_0) - C_n(f; x_0)\right] = -\frac{x_0(1-x_0)}{2} f''(x_0) + \left[x_0 f'(x_0) - \frac{x_0^2}{4} f''(x_0)\right] L.$$

3 Modified Bernstein-Stancu operators

In this section we deal with Bernstein-Stancu type generalization of (2.4). We investigate its sharp preserving and convergence properties.

We define the modified Bernstein-Stancu operators as follows:

$$C_n^{\tau}(f; x) = \sum_{k=0}^{n} p_{n,k}^{\tau}(x) \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n)\left(f \circ \tau^{-1}\right)\left(\frac{j}{n}\right), \tag{3.1}$$

where $p_{n,k}^{\tau}(x) = \binom{n}{k}\tau(x)^k\left(1-\tau(x)\right)^{n-k}$ and τ is any function that is continuously differentiable ∞ times on $[0,1]$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$.

Note that these operators are positive and linear and for the case $\tau(x) = x$, these operators (3.1) reduce to the Bernstein-Stancu operators defined by Cleciu [7]-[8].

Lemma 3.1. *The modified operators C_n^τ verify*
(i) $C_n^\tau e_0 = 1$,
(ii) $C_n^\tau \tau = a_n \tau$,
(iii) $C_n^\tau \tau^2 = \tau^2 + \frac{\tau(1-\tau)}{n} a_n + \frac{1-a_n}{2}\left(\frac{a_n}{n} - (2+a_n)\right)\tau^2$.

Let

$$\begin{aligned}\mu_{n,m}^\tau(x) &= C_n^\tau\left((\tau(t)-\tau(x))^m; x\right)\\ &= \sum_{k=0}^n p_{n,k}^\tau(x) \sum_{j=0}^k p_{k,j}^{<1>}(a_n)\left(\frac{j}{n} - \tau(x)\right)^m, \quad n, m \in \mathbb{N},\end{aligned}$$

be the central moment operators.

Lemma 3.2. *The central moment operators verify*
(i) $\mu_{n,0}^\tau(x) = 1$,
(ii) $\mu_{n,1}^\tau(x) = (a_n - 1)\tau(x)$,
(iii) $\mu_{n,2}^\tau(x) = \frac{\tau(x)(1-\tau(x))}{n} a_n + \tau(x)^2(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right)$,
(iv) $\mu_{n,4}(x) = \tau(x)^4 + \left[\frac{6(a_n)_3}{n^4}\binom{n}{3} - \frac{12(a_n)_2}{n^3}\binom{n}{2} + \frac{6a_n}{n}\right]\tau(x)^3$
$+\left[\frac{7(a_n)_2}{n^4}\binom{n}{2} - \frac{4a_n}{n^2}\right]\tau(x)^2 + \frac{a_n}{n^3}\tau(x) + \frac{(a_n)_4}{n^4}\binom{n}{4}$
$-\frac{4(a_n)_3}{n^3}\binom{n}{3} + \frac{6(a_n)_2}{n^2}\binom{n}{2} - 4a_n$.

Lemma 3.3. *For all $n \in \mathbb{N}$ we have*

$$\mu_{n,2}^\tau(x) \le \delta_{n,\tau}^2(x) \quad \textit{for all } x \in [0,1],$$

where $\delta_{n,\tau}^2(x) := \frac{a_n}{n}\varphi_\tau^2(x) + (1-a_n)$ and $\varphi_\tau^2(x) := \tau(x)(1-\tau(x))$.

Proof. We have

$$\begin{aligned}|\mu_{n,2}^\tau(x)| &= \left|\frac{\tau(x)(1-\tau(x))a_n}{n}\right| + \left|\tau^2(x)(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right)\right|\\ &\le \varphi_\tau^2(x)\frac{a_n}{n} + (1-a_n) = \delta_{n,\tau}^2(x).\end{aligned}$$

□

Lemma 3.4. *If $f \in C[0,1]$, then $\|C_n^\tau f\| \le \|f\|$, where $\|\cdot\|$ is the uniform norm on $C[0,1]$.*

Proof. From the definition of the operator C_n^τ and using Lemma 3.1 it follows

$$\begin{aligned}|C_n^\tau(f;x)| &\le \sum_{k=0}^n p_{n,k}^\tau(x)\sum_{j=0}^k p_{k,j}^{<1>}(a_n)\left|\left(f\circ\tau^{-1}\right)\left(\frac{j}{n}\right)\right|\\ &\le \|f\circ\tau^{-1}\| C_n^\tau(e_0;x) = \|f\|.\end{aligned}$$

□

Theorem 3.5. *Let $f \in C[0,1]$, $a_n \in (0,1]$ and $\lim_{n\to\infty} a_n = 1$. Then $C_n^\tau f$ converges to f as n tends to infinity, uniformly on $[0,1]$.*

Proof. Using the well known Korovkin theorem and Lemma 3.1 and the fact that $\{e_0, \tau, \tau^2\}$ is an extended complete Tchebychev system on $[0,1]$ it follows the uniform convergence of the operators C_n^τ. □

Let ω be the usual modulus of continuity of $f \in C[0,1]$ which is defined as

$$\omega(f;\delta) = \sup_{|h|\le\delta} \sup_{x,x+h\in[0,1]} |f(x+h) - f(x)|.$$

Proposition 3.6. *Let $f \in C[0,1]$ with modulus of continuity $\omega(f,\cdot)$. Then*

$$|C_n^\tau(f;x) - f(x)| \le \left(1 + \frac{\mu_{n,2}^\tau(x)}{\delta^2}\right)\omega(f,\delta)$$

for $\delta > 0$ and $x \in [0,1]$.

Example 3.7. If we choose $\tau(x) = \frac{x^2+x}{2}$, we have $\tau(x)(1-\tau(x)) \le x(1-x)$ for all $x \in [0,1/2]$ and this inequality leads to $\mu_{n,2}^\tau(x) \le \mu_{n,2}(x)$. Therefore, the modified operators C_n^τ presents an order of approximation better than C_n in that interval.

Example 3.8. Now using a graphical example we try to illustrate these approximation processes. Let $f(x) = \sin(9x)$, $\tau(x) = \frac{x^2+x}{2}$ and $a_n = 1/2$. For $n = 20$, the approximation to the function f by C_n and C_n^τ is shown in the Figure 1.

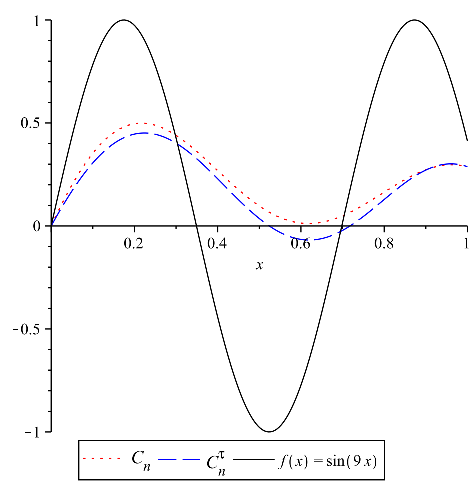

Figure 1. Approximation process by C_n and C_n^τ

Example 3.9. Let us take $f(x) = \log(x+1)$, $\tau(x) = \frac{x^2+x}{2}$ and $a_n = \frac{1}{2}$. In the Table 1 we computed the error of approximation for C_n and C_n^τ at the point $x_0 = 0.8$.

Table 1. Error of approximation for C_n and C_n^τ

n	$\lvert C_n(f;x_0)-f(x_0)\rvert$	$\lvert C_n^\tau(f;x_0)-f(x_0)\rvert$
5	0.2800807097	0.2613318434
10	0.2762200954	0.2502212648
15	0.2749367804	0.2465367167
20	0.2742959594	0.2447029941
25	0.2739117553	0.2436063038
30	0.2736557443	0.2428768564
35	0.2734729425	0.2423567117
40	0.2733358757	0.2419671158
45	0.2732292893	0.2416644116
50	0.2731440335	0.2414224523

From the above results it follows that C_n^τ converge faster than C_n to the function $f(x)=\log(x+1)$ at the point $x_0=0.8$. Also, the approximation to the function f by C_n and C_n^τ is shown in the Figure 2.

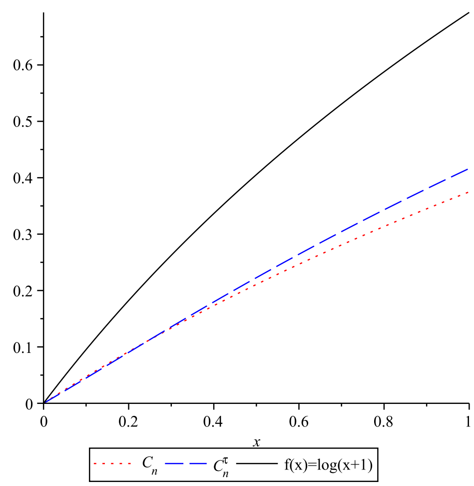

Figure 2. Approximation process by C_n and C_n^τ

4 Voronovskaya type theorem

Let $L_n : C[0,1] \to C[0,1]$, $n \geq 1$, be a positive linear operator and $L_n e_0 = e_0$. Acar et al. [1] defined a general operator $K_n : C[0,1] \to C[0,1]$ by

$$K_n g := \left(L_n(g \circ \tau^{-1})\right) \circ \tau,\ n \geq 1.$$

The authors obtained the following Voronovskaya type formula for the modified operators K_n.

Theorem 4.1. ([1]) *Let $f \in C[0,1]$ with $f''(x)$ finite for $x \in [0,1]$. If there exists $\alpha, \beta \in C[0,1]$ such that*

$$\lim_{n\to\infty} n(L_n(f,x)-f(x)) = \alpha(x)f''(x)+\beta(x)f'(x),$$

then we have

$$\lim_{n\to\infty} n\left(K_n(g,t)-g(t)\right) = \frac{\alpha(\tau(t))}{\tau'(t)^2} g''(t) + \left(\frac{\beta(\tau(t))}{\tau'(t)} - \frac{\alpha(\tau(t))\tau''(t)}{\tau'(t)^3}\right) g'(t)$$

for $g \in C[0,1]$ *with* $g''(x)$ *finite for* $x \in [0,1]$.

Using Theorem 2.4 and Theorem 4.1 we obtain a Voronovskaya type theorem for C_n^τ.

Theorem 4.2. *Let* $f \in C^2[0,1]$. *If* $a_n \in (0,1)$, $\lim_{n\to\infty} a_n = 1$ *and* $L := \lim_{n\to\infty} n(1-a_n)$ *exists, then*

$$\lim_{n\to\infty} n\left(C_n^\tau(f,x)-f(x)\right) = \frac{\alpha(\tau(x))}{\tau'(x)^2} f''(x) + \left(\frac{\beta(\tau(x))}{\tau'(x)} - \frac{\alpha(\tau(x))\tau''(x)}{\tau'(x)^3}\right) f'(x)$$

uniformly on $[0,1]$ *with* $\alpha(x) = -\frac{x(1-x)}{2} - \frac{x^2}{4}L$ *and* $\beta(x) = xL$.

5 Local Approximation

Let

$$W^2[0,1] = \left\{g \in C[0,1] : g' \in C[0,1]\right\}.$$

For $f \in C[0,1]$ and $\delta > 0$, the Peetre's K-functional [14] is defined by

$$K_2(f;\delta) = \inf_{g\in W^2[0,1]} \left\{\|f-g\| + \delta\|g\|_{W^2[0,1]}\right\},$$

where

$$\|f\|_{W^2[0,1]} = \|f\| + \|f'\| + \|f''\|.$$

Throughout this paper we assume that $\inf_{x\in[0,1]} \tau'(x) \geq a$, $a \in \mathbb{R}^+$.

Theorem 5.1. *Let* $a_n \in (0,1)$ *and* $\lim_{n\to\infty} a_n = 1$ *and* $f \in C[0,1]$. *For the operator* $C_n^\tau(f;\cdot)$, *there exists absolute constant* $C > 0$ *such that*

$$|C_n^\tau(f;x) - f(x)| \leq CK_2\left(f;\delta_{n,\tau}^2(x)\right) + \omega\left(f;\frac{1}{a}(1-a_n)\tau(x)\right).$$

Proof. Let $g \in W^2[0,1]$ and $t \in [0,1]$. Then by Taylor's expansion, we get

$$\begin{aligned} g(t) &= \left(g\circ\tau^{-1}\right)(\tau(t)) \\ &= \left(g\circ\tau^{-1}\right)(\tau(x)) + \left(g\circ\tau^{-1}\right)'(\tau(x))\,(\tau(t)-\tau(x)) \\ &\quad + \int_{\tau(x)}^{\tau(t)} (\tau(t)-u)\left(g\circ\tau^{-1}\right)''(u)du. \end{aligned} \tag{5.1}$$

If we consider the change of variable $u = \tau(y)$, it follows

$$\int_{\tau(x)}^{\tau(t)} (\tau(t)-u)\left(g\circ\tau^{-1}\right)''(u)du = \int_x^t (\tau(t)-\tau(y))\left(g\circ\tau^{-1}\right)''(\tau(y))\,\tau'(y)dy,$$

but

$$\left(g\circ\tau^{-1}\right)''(\tau(y))=\frac{1}{\tau'(y)}\cdot\frac{g''(y)\tau'(y)-g'(y)\tau''(y)}{(\tau'(y))^2},$$

therefore

$$\begin{aligned}&\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(g\circ\tau^{-1}\right)''(u)du\\&=\int_x^t(\tau(t)-\tau(y))\frac{g''(y)}{\tau'(y)}dy-\int_x^t(\tau(t)-\tau(y))\frac{g'(y)\tau''(y)}{(\tau'(y))^2}dy\\&=\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}du-\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}du.\end{aligned}\tag{5.2}$$

From (5.1) and (5.2) we can write

$$\begin{aligned}g(t)=g(x)+\left(g\circ\tau^{-1}\right)'(\tau(x))(\tau(t)-\tau(x))+\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}du\\-\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}du.\end{aligned}\tag{5.3}$$

We define

$$\tilde{C_n^\tau}(f;x)=C_n^\tau(f;x)+f(x)-\left(f\circ\tau^{-1}\right)(a_n\tau(x)).$$

From Lemma 3.1 it follows

$$\tilde{C_n^\tau}(e_0;x)=1 \text{ and } \tilde{C_n^\tau}(\tau;x)=C_n^\tau(\tau;x)+\tau(x)-a_n\tau(x)=\tau(x).$$

Now applying $\tilde{C_n^\tau}$ to both side of the relation (5.3) we can write

$$\begin{aligned}\tilde{C_n^\tau}(g;x)=g(x)&+C_n^\tau\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}du\right)\\&-\int_{\tau(x)}^{a_n\tau(x)}(a_n\tau(x)-u)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}du\\&-C_n^\tau\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}du\right)\\&+\int_{\tau(x)}^{a_n\tau(x)}(a_n\tau(x)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}du.\end{aligned}$$

Since $\inf_{x\in[0,1]}\tau'(x)\geq a, a\in\mathbb{R}^+$ and τ is strictly increasing on the interval $(0,1)$, we obtain

$$\begin{aligned}\left|\tilde{C_n^\tau}(\tau;x)-g(x)\right|&\leq\frac{1}{2}\mu_{n,2}^\tau(x)\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right]\\&\quad+\frac{1}{2}\left(a_n\tau(x)-\tau(x)\right)^2\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right]\\&\leq\frac{1}{2}\left[\delta_{n,\tau}^2(x)+\tau^2(x)(1-a_n)^2\right]\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right]\\&\leq\delta_{n,\tau}^2(x)\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right].\end{aligned}$$

By Lemma 3.4, it follows

$$|\tilde{C}_n^\tau(g;x)| \leq |C_n^\tau(g;x)| + |g(x)| + \left|\left(g\circ\tau^{-1}\right)(a_n\tau(x))\right| \leq 3\|g\|.$$

For $f\in C[0,1]$ and $g\in W_2[0,1]$, we can write

$$\begin{aligned}
&|C_n^\tau(f;x)-f(x)|\\
&= \left|\tilde{C}_n^\tau(f;x)-f(x)+\left(f\circ\tau^{-1}\right)(a_n\tau(x))-f(x)\right|\\
&\leq |\tilde{C}_n^\tau(f-g;x)| + |\tilde{C}_n^\tau(g;x)-g(x)| + |g(x)-f(x)|\\
&\quad + \left|\left(f\circ\tau^{-1}\right)(a_n\tau(x)) - \left(f\circ\tau^{-1}\right)(\tau(x))\right|\\
&\leq 4\|f-g\| + \frac{\delta_{n,\tau}^2(x)}{a^2}\|g''\| + \frac{\delta_{n,\tau}^2(x)}{a^3}\|\tau''\|\,\|g'\| + \omega\left(f\circ\tau^{-1};(1-a_n)\tau(x)\right).
\end{aligned}$$

Let $C:=\max\left\{4,\frac{1}{a^2},\frac{\|\tau''\|}{a^3}\right\}$. Then

$$|C_n^\tau(f;x)-f(x)| \leq C\left\{\|f-g\| + \delta_{n,\tau}^2(x)\|g\|_{W^2[0,1]}\right\} + \omega\left(f\circ\tau^{-1};(1-a_n)\tau(x)\right).$$

Using the following result (see [1]) $\omega\left(f\circ\tau^{-1};t\right)\leq\omega\left(f;\frac{t}{a}\right)$, the theorem is proved. □

To describe our next result, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional [9]. Let $\varphi_\tau(x):=\sqrt{\tau(x)(1-\tau(x))}$ and $f\in C[0,1]$. The first order modulus of smoothness is given by

$$\omega_{\varphi_\tau}(f;t) = \sup_{0<h\leq t}\left\{\left|f\left(x+\frac{h\varphi_\tau(x)}{2}\right)-f\left(x-\frac{h\varphi_\tau(x)}{2}\right)\right|, x\pm\frac{h\varphi_\tau(x)}{2}\in[0,1]\right\}. \quad (5.4)$$

Further, the corresponding K-functional to (5.4) is defined by

$$K_{\varphi_\tau}(f;t) = \inf_{g\in W_{\varphi_\tau}[0,1]}\{||f-g|| + t||\varphi_\tau g'||\}\ (t>0), \quad (5.5)$$

where $W_{\varphi_\tau}[0,1]=\{g: g\in AC[0,1], \|\varphi_\tau g'\|<\infty\}$ and $AC[0,1]$ is the class of all absolutely continuous functions on $[0,1]$. It is well known ([9], p.11) that there exists a constant $C>0$ such that

$$K_{\varphi_\tau}(f;t)\leq C\omega_{\varphi_\tau}(f;t). \quad (5.6)$$

Now, we establish a direct approximation theorem by means of Ditzian-Totik modulus of smoothness.

Theorem 5.2. *Let $f\in C[0,1]$ and $\varphi_\tau(x)=\sqrt{\tau(x)(1-\tau(x))}$, then for every $x\in(0,1)$, we have*

$$|C_n^\tau(f;x)-f(x)|\leq\tilde{C}\omega_{\varphi_\tau}\left(f;\frac{\delta_{n,\tau}(x)}{\varphi_\tau(x)}\right),$$

where $\tilde{C}$ is a constant independent of n and x.

Proof. Using the representation

$$g(t)=\left(g\circ\tau^{-1}\right)(\tau(t))=\left(g\circ\tau^{-1}\right)(\tau(x))+\int_{\tau(x)}^{\tau(t)}\left(g\circ\tau^{-1}\right)'(u)du,$$

we get

$$|C_n^\tau(g;x)-g(x)|=\left|C_n^\tau\left(\int_{\tau(x)}^{\tau(t)}\left(g\circ\tau^{-1}\right)'(u)du\right)\right|. \tag{5.7}$$

But,

$$\begin{aligned}\left|\int_{\tau(x)}^{\tau(t)}\left(g\circ\tau^{-1}\right)'(u)du\right|&=\left|\int_x^t\frac{g'(y)}{\tau'(y)}\tau'(y)dy\right|=\left|\int_x^t\frac{\varphi_\tau(y)}{\varphi_\tau(y)}\cdot\frac{g'(y)}{\tau'(y)}\tau'(y)dy\right|\\&\leq\frac{\|\varphi_\tau g'\|}{a}\left|\int_x^t\frac{\tau'(y)}{\varphi_\tau(y)}dy\right|,\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}\left|\int_x^t\frac{\tau'(y)}{\varphi_\tau(y)}dy\right|&\leq\left|\int_x^t\left(\frac{1}{\sqrt{\tau(y)}}+\frac{1}{\sqrt{1-\tau(y)}}\right)\tau'(y)dy\right|\\&\leq2\left(\left|\sqrt{\tau(t)}-\sqrt{\tau(x)}\right|+\left|\sqrt{1-\tau(t)}-\sqrt{1-\tau(x)}\right|\right)\\&=2\left|\tau(t)-\tau(x)\right|\left(\frac{1}{\sqrt{\tau(t)}+\sqrt{\tau(x)}}+\frac{1}{\sqrt{1-\tau(t)}+\sqrt{1-\tau(x)}}\right)\\&<2|\tau(t)-\tau(x)|\left(\frac{1}{\sqrt{\tau(x)}}+\frac{1}{\sqrt{1-\tau(x)}}\right)\\&\leq\frac{2\sqrt{2}|\tau(t)-\tau(x)|}{\varphi_\tau(x)}.\end{aligned} \tag{5.9}$$

From relations (5.7)-(5.9) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|C_n^\tau(g;x)-g(x)|&\leq2\sqrt{2}\frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)}C_n^\tau\left(|\tau(t)-\tau(x)|;x\right)\\&\leq2\sqrt{2}\frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)}\left[C_n^\tau\left((\tau(t)-\tau(x))^2;x\right)\right]^{1/2}\\&\leq2\sqrt{2}\frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)}\delta_{n,\tau}(x).\end{aligned} \tag{5.10}$$

Using Lemma 3.4 and (5.10) it follows

$$\begin{aligned}|C_n^\tau(f;x)-f(x)|&\leq|C_n^\tau(f-g;x)|+|f(x)-g(x)|+|C_n^\tau(g;x)-g(x)|\\&\leq C\left\{\|f-g\|+\frac{\delta_{n,\tau}(x)}{\varphi_\tau(x)}\|\varphi_\tau g'\|\right\},\end{aligned}$$

where $C=\max\left\{2,\frac{2\sqrt{2}}{a}\right\}$.

Taking infimum on the right hand side of the above inequality over all $g\in W_{\varphi_\tau}[0,1]$, we get

$$|C_n^\tau(f;x)-f(x)|\leq CK_{\varphi_\tau}\left(f;\frac{\delta_{n,\tau}(x)}{\varphi_\tau(x)}\right).$$

Using the relation (5.6) this theorem is proven. □

Acknowledgment

This work was supported by the Dong-A University research fund.

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A NOTE ON HERMITE POLYNOMIALS

TAEKYUN KIM AND DAE SAN KIM

Abstract. In this paper, we consider linear differential equations satisfied by the generating function for Hermite polynomials and derive some new identities involving those polynomials.

1. Introduction

The Hermite polynomials form a Sheffer sequence and are given by the generating function

$$e^{2xt-t^2}=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n, \quad (\text{see } [1\text{–}8,\ 10,\ 13,\ 14]). \tag{1.1}$$

By using Taylor series, we get

$$\begin{aligned}H_n(x)&=\left[\left(\frac{\partial}{\partial t}\right)^n e^{\left(2xt-t^2\right)}\right]_{t=0}\\&=\left[e^{x^2}\left(\frac{\partial}{\partial t}\right)^n e^{-(x-t)^2}\right]_{t=0}\\&=(-1)^n e^{x^2}\left[\left(\frac{\partial}{\partial x}\right)^n e^{-(x-t)^2}\right]_{t=0}\\&=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}, \quad (n\geq 0), \quad (\text{see } [1\text{–}15,\ 18]).\end{aligned}$$

The Hermite polynomials can be represented by the Contour integral as follows:

$$H_n(z)=\frac{n!}{2\pi i}\oint e^{-t^2+2tz}t^{-n-1}dt, \tag{1.2}$$

where the Contour encloses the origin and is traversed in a counterclockwise direction (see [2, 8, 11, 13]).

The probabilists' Hermite polynomials are given by the generating function

$$\begin{aligned}H_n^*(x)&=(-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}}\\&=\left(x-\frac{d}{dx}\right)^n\cdot 1, \quad (\text{see } [10]).\end{aligned} \tag{1.3}$$

The physicists' Hermite polynomials are also given by

$$\begin{aligned}H_n(x)&=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}\\&=\left(2x-\frac{d}{dx}\right)^n\cdot 1 \quad (\text{see } [20]).\end{aligned} \tag{1.4}$$

2010 *Mathematics Subject Classification.* 05A19, 11B83, 33C45, 34A30.

Key words and phrases. Hermite polynomials, linear differential equation.

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Thus, by (1.3) and (1.4), we get

$$H_n(x) = 2^{\frac{n}{2}} H_n^*\left(\sqrt{2}x\right), \quad H_n^*(x) = 2^{-\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2}}\right), \tag{1.5}$$

where $n \geq 0$ (see [9, 11, 12, 15, 18]).

The first several Hermite polynomials are $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2-2$, $H_3(x) = 8x^2-12x$, $H_4(x) = 16x^4-48x^2+12$, $H_5(x) = 32x^5-160x^3+120x$, $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$, ...

The probabilists' Hermite polynomials are solutions of the differential equation:

$$\left(e^{-\frac{x^2}{2}}u'\right)' + \lambda e^{-\frac{1}{2}x^2}u = 0,$$

where λ is a constant, with the boundary conditions that u should be polynomially bounded at infinity.

The generating function of the probabilists' Hermite polynomials is given by

$$e^{xt-\frac{t^2}{2}} = \sum_{n=0}^{\infty} H_n^*(x)\frac{t^n}{n!}, \quad (\text{see } [12, 15, 18]). \tag{1.6}$$

The Hermite polynomials $H_n^{(\nu)}(x)$ of variance ν form an Appell sequence and are defined by the generating function

$$\sum_{k=0}^{\infty} \frac{H_k^{(\nu)}(x)}{k!}t^k = e^{xt-\frac{\nu t^2}{2}}, \quad (\text{see } [12]). \tag{1.7}$$

Thus, by (1.7), we get

$$x^{2m+1} = \sum_{l=0}^{m} \binom{2m+1}{2l+1} \frac{(2m-2l)!}{(m-l)!} \left(\frac{\nu}{2}\right)^{m-l} H_{2l+1}^{(\nu)}(x), \tag{1.8}$$

and

$$x^{2m} = \sum_{l=0}^{m} \binom{2m}{2l} \frac{(2m-2l)!}{(m-l)!} \left(\frac{\nu}{2}\right)^{m-l} H_{2l}^{(\nu)}(x), \quad (\text{see } [12]). \tag{1.9}$$

The Hermite polynomials have been studied in probability, combinatorics, numerical analysis, finite element methods, physics and system theory (see [1–15, 18]).

Recently, Kim has studied nonlinear differential equations arising from Frobenius-Euler numbers and polynomials.

In this paper, we consider linear differential equations arising from Hermite polynomials of variance ν and give some new and explicit identities for those polynomials.

2. Hermite polynomials of variance ν

Let

$$F = F(t : x, \nu) = e^{xt-\frac{\nu t^2}{2}}. \tag{2.1}$$

From (2.1), we note that

$$\begin{aligned} F^{(1)} &= \frac{d}{dt}F(t : x, \nu) \\ &= (x-\nu t)\, e^{xt-\frac{\nu t^2}{2}} \\ &= (x-\nu t)\, F, \end{aligned} \tag{2.2}$$

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$$F^{(2)} = \frac{d}{dt}F^{(1)} = \left(-\nu + (x - t\nu)^2\right) F, \tag{2.3}$$

$$F^{(3)} = \frac{d}{dt}F^{(2)} = \left(-3\nu (x - \nu t) + (x - \nu t)^3\right) F, \tag{2.4}$$

and

$$F^{(4)} = \frac{d}{dt}F^{(3)} = \left(3\nu^2 - 6\nu (x - \nu t)^2 + (x - \nu t)^4\right) F. \tag{2.5}$$

Continuing this process, we set

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t : x, \nu) \\ &= \left(\sum_{i=0}^{N} a_i (N, \nu) (x - \nu t)^i\right) F, \end{aligned} \tag{2.6}$$

where $N \in \mathbb{N} \cup \{0\}$.

From (2.6), we have

$$\begin{aligned} F^{(N+1)} &= \frac{d}{dt}F^{(N)} \\ &= \sum_{i=0}^{N} a_i (N, \nu)\, i\, (x - \nu t)^{i-1} (-\nu) F \\ &\quad + \sum_{i=0}^{N} a_i (N, \nu) (x - \nu t)^i F^{(1)}. \end{aligned} \tag{2.7}$$

By (2.2) and (2.7), we easily get

$$\begin{aligned} F^{(N+1)} = \Big\{ &-\nu a_1 (N, \nu) + a_N (N, \nu) (x - \nu t)^{N+1} + a_{N-1} (N, \nu) (x - \nu t)^N \\ &+ \sum_{i=1}^{N-1} \left(-(i+1)\nu a_{i+1}(N, \nu) + a_{i-1}(N, \nu)\right) (x - \nu t)^i \Big\} F. \end{aligned} \tag{2.8}$$

By replacing N by $(N + 1)$ in (2.6), we get

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i (N + 1, \nu) (x - \nu t)^i\right) F. \tag{2.9}$$

From (2.8) and (2.9), we can derive the following equations:

$$a_0 (N + 1, \nu) = -\nu a_1 (N, \nu), \tag{2.10}$$

$$a_N (N + 1, \nu) = a_{N-1} (N, \nu), \tag{2.11}$$

$$a_{N+1} (N + 1, \nu) = a_N (N, \nu) \tag{2.12}$$

and

$$a_i (N + 1, \nu) = -(i + 1)\nu a_{i+1} (N, \nu) + a_{i-1} (N, \nu), \tag{2.13}$$

where $1 \le i \le N - 1$.

It is not difficult to show that

$$F = F^{(0)} = a_0 (0, \nu) F. \tag{2.14}$$

Thus, by (2.14), we get

$$a_0 (0, \nu) = 1. \tag{2.15}$$

From (2.2) and (2.6), we note that

$$(x-\nu t)\,F = F^{(1)} = \left(a_0\,(1,\nu) + a_1\,(1,\nu)\,(x-\nu t)\right)F. \tag{2.16}$$

Thus, by comparing the coefficients on both sides of (2.16), we get

$$a_0\,(1,\nu) = 0, \quad a_1\,(1,\nu) = 1. \tag{2.17}$$

From (2.11), (2.12), (2.15) and (2.17), we have

$$a_N\,(N+1,\nu) = a_{N-1}\,(N,\nu) = \cdots = a_0\,(1,\nu) = 0, \tag{2.18}$$

and

$$a_{N+1}\,(N+1,\nu) = a_N\,(N,\nu) = \cdots = a_1\,(1,\nu) = 1. \tag{2.19}$$

Therefore, we obtain the following theorem.

Theorem 1. *The linear differential equations*

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F\,(t:x,\nu) \\ &= \left(\sum_{i=0}^{N} a_i\,(N,\nu)\,(x-\nu t)^i\right) F, \quad (N \in \mathbb{N}\cup\{0\}) \end{aligned}$$

has a solution $F = F\,(t:x,\nu) = e^{xt-\frac{\nu t^2}{2}}$, *where*

$$\begin{aligned} a_0\,(N,\nu) &= -\nu a_1\,(N-1,\nu)\,, \\ a_{N-1}\,(N,\nu) &= a_{N-2}\,(N-1,\nu) = \cdots = a_1\,(2,\nu) = a_0\,(1,\nu) = 0, \\ a_N\,(N,\nu) &= a_{N-1}\,(N-1,\nu) = \cdots = a_1\,(1,\nu) = a_0\,(0,\nu) = 1, \end{aligned}$$

and

$$a_i\,(N,\nu) = -\,(i+1)\,\nu a_{i+1}\,(N-1,\nu) + a_{i-1}\,(N-1,\nu)\,, \quad (1 \le i \le N-2)\,.$$

Example.

(1) $N=3,\ i=1$. By (2.13), we get

$$\begin{aligned} a_1\,(3,\nu) &= -2\nu a_2\,(2,\nu) + a_0\,(2,\nu) \\ &= -2\nu - \nu = -3\nu. \end{aligned}$$

(2) $N=4,\ 1 \le i \le 2$. By (2.13), we have

$$a_1\,(4,\nu) = 0, \quad a_2\,(4,\nu) = -6\nu.$$

(3) $N=5,\ 1 \le i \le 3$. By (2.13), we get

$$a_1\,(5,\nu) = 15\nu^2, \quad a_2\,(5,\nu) = 0, \quad a_3\,(5,\nu) = -10\nu.$$

(4) $N=6,\ 1 \le i \le 4$. From (2.13), we have

$$a_1\,(6,\nu) = 0, \quad a_2\,(6,\nu) = 45\nu^2, \quad a_3\,(6,\nu) = 0, \quad a_4\,(6,\nu) = -15\nu.$$

Thus, we obtain the following result.

Remark. The matrix $(a_i(j,\nu))_{0\leq i,j\leq 6}$ is given by

$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & -\nu & 0 & 3\nu^2 & 0 & -15\nu^3 \\ & 1 & 0 & -3\nu & 0 & 15\nu^2 & 0 \\ & & 1 & 0 & -6\nu & 0 & 45\nu^2 \\ & & & 1 & 0 & -10\nu & 0 \\ & & & & 1 & 0 & -15\nu \\ & & 0 & & & 1 & 0 \\ & & & & & & 1 \end{bmatrix} \end{array}.$$

From (1.7), we note that

$$F=F(t:x,\nu)=e^{xt-\frac{\nu t^2}{2}} = \sum_{k=0}^{\infty} H_k^{(\nu)}(x)\frac{t^k}{k!}. \tag{2.20}$$

Thus, by (2.20), we get

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t:x,\nu) \\ &= \sum_{k=N}^{\infty} H_k^{(\nu)}(x)(k)_N \frac{t^{k-N}}{k!} \\ &= \sum_{k=0}^{\infty} H_{k+N}^{(\nu)}(x)(k+N)_N \frac{t^k}{(n+k)!} \\ &= \sum_{k=0}^{\infty} H_{k+N}^{(\nu)}(x)\frac{t^k}{k!}. \end{aligned} \tag{2.21}$$

By Theorem 1, we easily get

$$\begin{aligned} &F^{(N)} \\ &= \left(\sum_{i=0}^{N} a_i(N,\nu)(x-\nu t)^i\right)F \\ &= \sum_{i=0}^{N} a_i(N,\nu)\sum_{m=0}^{\infty}(i)_m x^{i-m}(-\nu)^m\frac{t^m}{m!}\sum_{l=0}^{\infty}H_l^{(\nu)}(x)\frac{t^l}{l!} \\ &= \sum_{k=0}^{\infty}\left\{\sum_{i=0}^{N} a_i(N,\nu)\sum_{l=0}^{k}\binom{k}{l}(i)_{k-l}(-\nu)^{k-l}x^{i+l-k}H_l^{(\nu)}(x)\right\}\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left\{\sum_{i=0}^{N} a_i(N,\nu)\sum_{l=\max\{0,k-i\}}^{k}\binom{k}{l}(i)_{k-l}(-\nu)^{k-l}x^{i+l-k}H_l^{(\nu)}(x)\right\}\frac{t^k}{k!}. \end{aligned} \tag{2.22}$$

Therefore, by (2.21) and (2.22), we obtain the following theorem.

Theorem 2. *For $k, N \in \mathbb{N} \cup \{0\}$, we have*

$$H_{k+N}^{(\nu)}(x) = \sum_{i=0}^{N} a_i(N,\nu) \sum_{l=\max\{0,k-i\}}^{k} \binom{k}{l} (i)_{k-l} (-\nu)^{k-l} x^{i+l-k} H_l^{(\nu)}(x).$$

It is easy to show that

$$H_{k+1}^{(\nu)}(x) = \left(x - \nu \frac{\partial}{\partial x}\right) H_k^{(\nu)}(x). \tag{2.23}$$

Thus, by (2.23), we have

$$H_{k+N}^{(\nu)}(x) = \left(x - \nu \frac{\partial}{\partial x}\right)^N H_k^{(\nu)}(x), \quad (N \in \mathbb{N} \cup \{0\}). \tag{2.24}$$

From Theorem 2, we note that

$$\begin{aligned} &\left(x - \nu \frac{\partial}{\partial x}\right)^N H_k^{(\nu)}(x) \\ &= \sum_{i=0}^{N} a_i(N,\nu) \sum_{l=\max\{0,k-i\}}^{k} \binom{k}{l} (i)_{k-l} (-\nu)^{k-l} x^{i+l-k} H_l^{(\nu)}(x), \end{aligned} \tag{2.25}$$

where $\frac{\partial}{\partial x} x - x \frac{\partial}{\partial x} = \text{identity}$.

Now, we observe explicit determination of $a_i(j,\nu)$.

From (2.12) and (2.13), we can derive the following equations:

$$a_N(N,\nu) = 1, \tag{2.26}$$

$$\begin{aligned} a_{N-2}(N,\nu) &= -(N-1)\nu a_{N-1}(N-1,\nu) + a_{N-3}(N-1,\nu) \\ &= -(N-1)\nu a_{N-1}(N-1,\nu) - (N-2)\nu a_{N-2}(N-2,\nu) \\ &\quad + a_{N-4}(N-2,\nu) \\ &\;\;\vdots \\ &= -(N-1)\nu a_{N-1}(N-1,\nu) - (N-2)\nu a_{N-2}(N-2,\nu) \\ &\quad - \cdots - 2\nu a_2(2,\nu) + a_0(2,\nu) \\ &= -(N-1)\nu a_{N-1}(N-1,\nu) - (N-2)\nu a_{N-2}(N-2,\nu) \\ &\quad - \cdots - 2\nu a_2(2,\nu) - \nu a_1(1,\nu) \\ &= -\nu \sum_{i=1}^{N-1} i a_i(i,\nu), \end{aligned} \tag{2.27}$$

$$\begin{aligned} a_{N-4}(N,\nu) &= -(N-3)\nu a_{N-3}(N-1,\nu) + a_{N-5}(N-1,\nu) \\ &= -(N-3)\nu a_{N-3}(N-1,\nu) - (N-4)\nu a_{N-4}(N-2,\nu) \\ &\quad + a_{N-6}(N-2,\nu) \\ &\;\;\vdots \\ &= -(N-3)\nu a_{N-3}(N-1,\nu) - (N-4)\nu a_{N-4}(N-2,\nu) \end{aligned} \tag{2.28}$$

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$$
\begin{aligned}
&-\cdots-2\nu a_2(4,\nu)+a_0(4,\nu)\\
&=-(N-3)\nu a_{N-3}(N-1,\nu)-(N-4)\nu a_{N-4}(N-2,\nu)\\
&\quad-\cdots-2\nu a_2(4,\nu)-\nu a_1(3,\nu)\\
&=-\nu\sum_{i=0}^{N-3} i a_i(i+2,\nu),
\end{aligned}
$$

and

$$
\begin{aligned}
a_{N-6}(N,\nu)&=-(N-5)\nu a_{N-5}(N-1,\nu)+a_{N-7}(N-1,\nu)\\
&=-(N-5)\nu a_{N-5}(N-1,\nu)-(N-6)\nu a_{N-6}(N-2,\nu)\\
&\quad+a_{N-8}(N-2,\nu)\\
&\vdots\\
&=-(N-5)\nu a_{N-5}(N-1,\nu)-(N-6)\nu a_{N-6}(N-2,\nu)\\
&\quad-\cdots-2\nu a_2(6,\nu)-\nu a_1(5,\nu)\\
&=-\nu\sum_{i=1}^{N-5} i a_i(i+4,\nu).
\end{aligned}
\tag{2.29}
$$

Continuing in this fashion, for l with $1\le l\le\left[\frac{N-1}{2}\right]$,

$$
a_{N-2l}(N,\nu)=-\nu\sum_{i=1}^{N-2l+1} i a_i(i+2l-2,\nu).
\tag{2.30}
$$

By (2.26), (2.27), (2.28), (2.29) and (2.30), we get

$$
a_{N-2}(N,\nu)=-\nu\sum_{i_1=1}^{N-1} i_1,
\tag{2.31}
$$

$$
\begin{aligned}
a_{N-4}(N,\nu)&=-\nu\sum_{i_2=1}^{N-3} i_2 a_{i_2}(i_2+2,\nu)\\
&=(-\nu)^2\sum_{i_2=1}^{N-3}\sum_{i_1=1}^{i_2+1} i_2 i_1,
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
a_{N-6}(N,\nu)&=-\nu\sum_{i_3=1}^{N-5} i_3 a_{i_3}(i_3+4,\nu)\\
&=(-\nu)^3\sum_{i_3=1}^{N-5}\sum_{i_2=1}^{i_3+1}\sum_{i_1=1}^{i_2+1} i_3 i_2 i_1,
\end{aligned}
\tag{2.33}
$$

and

$$
a_{N-2l}(N,\nu)=(-\nu)^l\sum_{i_l=1}^{N-2l+1}\sum_{i_{l-1}=1}^{i_l+1}\cdots\sum_{i_1=1}^{i_2+1} i_l\cdot i_{l-1}\cdots i_1,
\tag{2.34}
$$

where $1\le l\le\left[\frac{N-1}{2}\right]$.

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By (2.11) and (2.13), we easily get

(2.35)
$$a_{N-1}(N,\nu)=a_{N-2}(N-1,\nu)=a_{N-3}(N-2,\nu)=\cdots=a_0(1,\nu)=0,$$

(2.36)
$$\begin{aligned}a_{N-3}(N,\nu)&=-(N-2)\nu a_{N-2}(N-1,\nu)+a_{N-4}(N-1,\nu)\\&=a_{N-4}(N-1,\nu)\\&\vdots\\&=a_0(3,\nu)=-\nu a_1(2,\nu)=-\nu a_0(1,\nu)=0,\end{aligned}$$

(2.37)
$$\begin{aligned}a_{N-5}(N,\nu)&=-(N-4)\nu a_{N-4}(N-1,\nu)+a_{N-6}(N-1,\nu)=a_{N-6}(N-1,\nu)\\&\vdots\\&=a_0(5,\nu)=-\nu a_1(4,\nu)=0,\end{aligned}$$

(2.38)
$$\begin{aligned}a_{N-7}(N,\nu)&=-(N-6)\nu a_{N-6}(N-1,\nu)+a_{N-8}(N-1,\nu)\\&\vdots\\&=a_0(7,\nu)=-\nu a_1(6,\nu)=0,\end{aligned}$$

and

$$a_{N-(2l-1)}(N,\nu)=0,\quad \left(1\le l\le\left[\frac{N}{2}\right]\right). \tag{2.39}$$

Therefore, we obtain the following theorem.

Theorem 3. *For $N\in\mathbb{N}\cup\{0\}$, we have*

$$a_{N-2l}(N,\nu)=(-\nu)^l\sum_{i_l=1}^{N-2l+1}\sum_{i_{l-1}=1}^{i_l+1}\cdots\sum_{i_1=1}^{i_2+1}i_li_{l-1}\cdots i_1,$$

where $1\le l\le\left[\frac{N-1}{2}\right]$.

Also,

$$a_{N-(2l-1)}(N,\nu)=0,\quad \textit{if } 1\le l\le\left[\frac{N}{2}\right].$$

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Volume 23, Number 5 October 30, 2017
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

Editor in Chief: George Anastassiou
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JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.5, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

On new λ^2-convergent difference BK-spaces

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In this paper, we introduce the spaces $c(\lambda^2, \Delta)$ and $c_0(\lambda^2, \Delta)$, which are BK-spaces of non-absolute type and we prove that these spaces are linearly isomorphic to the spaces c and c_0, respectively. Moreover, we give some inclusion relations and compute the $\alpha-$, $\beta-$ and $\gamma-$duals of these spaces. We also determine the Schauder basis of the $c(\lambda^2, \Delta)$ and $c_0(\lambda^2, \Delta)$. Lastly we give some matrix transformations between of these spaces and others.

2010 Mathematics Subject Classification: 46A45, 46B20

Key words: λ^2-convergence, BK-spaces, $\alpha-, \beta-$ and $\gamma-$ duals, matrix mappings, difference sequence spaces

1 Introduction

A sequence space is defined to be a linear space of real or complex sequences. Let w denote the spaces of all complex sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$.

Let X be a sequence space. If X is a Banach space and

$$\tau_k : X \to C, \quad \tau_k(x) = x_k \quad (k = 1, 2, ...)$$

is a continuous for all k, X is called a $BK-$space.

We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively, which are $BK-$spaces with the norm given by $\|x\|_\infty = \sup_k |x_k|$ for all $k \in \mathbb{N}$.

For a sequence space X, the matrix domain X_A of an infinite matrix A defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1}$$

which is a sequence space. We denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

M. Mursaleen and A. K. Noman [9] introduced the sequence spaces ℓ_∞^λ, c^λ and c_0^λ as the sets of all $\lambda-$*bounded*, $\lambda-$*convergent* and $\lambda-$*null* sequences as follows;

$$\begin{aligned} \ell_\infty^\lambda &= \{x \in w : \sup_n |\Lambda_n(x)| < \infty\} \\ c^\lambda &= \{x \in w : \lim_{n\to\infty} \Lambda_n(x) \ exists\} \\ c_0^\lambda &= \{x \in w : \lim_{n\to\infty} \Lambda_n(x) = 0\} \end{aligned}$$

where $\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k,\ k \in \mathbb{N}$. Also they generalized c^λ and c_0^λ spaces defining $c^\lambda(\Delta)$, $c_0^\lambda(\Delta)$ spaces using the difference operator. They studied some properties of these spaces in [8]. N. L. Braha and F. Başar introduced the infinite matrix $A(\lambda) = \{a_{nk}(\lambda)\}_{n,k=0}^{\infty}$ such as;

$$a_{nk}(\lambda) = \begin{cases} \frac{\Delta^2\lambda_k}{\Delta\lambda_k}, & 0 \le k \le n; \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$ and they defined $A_\lambda(\ell_\infty)$, $A_\lambda(c)$ and $A_\lambda(c_0)$ spaces in [11] as follows;

$$\begin{aligned} A_\lambda(\ell_\infty) &= \left\{x \in w : \sup_n |(A_\lambda x)_n| < \infty\right\}, \\ A_\lambda(c) &= \left\{x \in w : \exists l \in \mathbb{C} \ni \lim_n (A_\lambda x)_n = l\right\}, \\ A_\lambda(c_0) &= \left\{x \in w : \lim_n (A_\lambda x)_n = 0\right\} \end{aligned}$$

where $(A_\lambda x)_n = \frac{1}{\Delta\lambda_n} \sum_{k=0}^{n} (\Delta^2\lambda_k) x_k$. They examined some properties of these spaces. In literature, some authors have constructed new sequence spaces by using matrix domain of infinite matrix and have introduced some topological properties. (see [2], [4], [12])

2 The sequence spaces $c(\lambda^2,\Delta)$ and $c_0(\lambda^2,\Delta)$

In this section, we define the sequence spaces $c(\lambda^2,\Delta)$ and $c_0(\lambda^2,\Delta)$ as follows;

$$c(\lambda^2,\Delta)=\left\{x\in w:\lim_{n\to\infty}\Lambda_n^2(x)\ exists\right\}$$

$$c_0(\lambda^2,\Delta)=\left\{x\in w:\lim_{n\to\infty}\Lambda_n^2(x)=0\right\}$$

where $\Lambda_n^2(x)=\frac{1}{\Delta\lambda_n}\sum\limits_{k=0}^{n}\left(\Delta^2\lambda_k\right)(x_k-x_{k-1})$ for all $k,n\in\mathbb{N}$. Δ denotes the difference operator. i.e., $\Delta^0\lambda_n=\lambda_n$, $\Delta\lambda_n=\lambda_n-\lambda_{n-1}$, $\Delta^2\lambda_n=\lambda_n-2\lambda_{n-1}+\lambda_{n-2}$ and $\Delta x_k=x_k-x_{k-1}$. $\lambda=(\lambda_k)_{k=0}^{\infty}$ is a strictly increasing sequence of positive reals tending to infinity, that is $0<\lambda_0<\lambda_1<...$and $\lambda_k\to\infty$ as $k\to\infty$ and $\lambda_{n+1}\geq 2\lambda_n$ for all $n\in\mathbb{N}$. Here and in sequel, we use the convention that any term with a negative subscript is equal to naught. e.g. $\lambda_{-1}=\lambda_{-2}=0$ and $x_{-1}=0$. On the other hand, we define the matrix $\Lambda^2=\left(\lambda_{nk}^2\right)$ for all $k,n\in\mathbb{N}$ by

$$\lambda_{nk}^2=\begin{cases}\frac{\Delta^2(\lambda_k-\lambda_{k+1})}{\Delta\lambda_k}; & k<n,\\ \frac{\Delta^2\lambda_n}{\Delta\lambda_n}; & n=k,\\ 0; & n>k.\end{cases} \tag{2}$$

The equality can be eaisly seen from

$$\Lambda_n^2(x)=\frac{1}{\Delta\lambda_n}\sum_{k=0}^{n}\left(\Delta^2\lambda_k\right)(x_k-x_{k-1}) \tag{3}$$

for all $m,n\in\mathbb{N}$ and every $x=(x_k)\in w$. Then it leads us together with (1) to the fact that

$$c_0\left(\lambda^2,\Delta\right)=(c_0)_{\Lambda^2}\ \text{ and } c\left(\lambda^2,\Delta\right)=(c)_{\Lambda^2}\,. \tag{4}$$

The matrix $\Lambda^2=\lambda_{nk}^2$ is a triangle, i.e., $\lambda_{nn}^2\neq 0$ and $\lambda_{nk}^2=0$ $(k>n)$ for all $n,k\in\mathbb{N}$. Further, for any sequence $x=(x_k)$ we define the sequence $y\left(\lambda^2\right)=\left\{y_k\left(\lambda^2\right)\right\}$ as the Λ^2-transform of x, i.e., $y\left(\lambda^2\right)=\Lambda^2(x)$ and so we have that

$$y_k\left(\lambda^2\right)=\sum_{j=0}^{k-1}\frac{\Delta^2\left(\lambda_j-\lambda_{j+1}\right)}{\Delta\lambda_k}x_j+\frac{\Delta^2\lambda_k}{\Delta\lambda_k}x_k \tag{5}$$

for $k\in\mathbb{N}$. Here and in what follows, the summation running from 0 to $k-1$ is equal to zero when $k=0$. Also it can be written from (3) with (5) for $k\in\mathbb{N}$ such as;

$$y_k\left(\lambda^2\right)=\sum_{j=0}^{k}\frac{\Delta^2\lambda_j}{\Delta\lambda_k}\left(x_j-x_{j-1}\right).$$

Theorem 1 *$c_0(\lambda^2,\Delta)$ and $c(\lambda^2,\Delta)$ are BK-spaces with the norm*

$$\|x\|_{(c_0)_{\Lambda^2}}=\|x\|_{(c)_{\Lambda^2}}=\sup_n\left|\Lambda_n^2(x)\right|.$$

Proof. We know that c and c_0 are $BK-$spaces with their natural norms from [6]. (4) holds and $\Lambda^2=\lambda_{nk}^2$ is a triangle matrix and from Theorem 4.3.12 of Wilansky [1], we derive that $c_0(\lambda^2,\Delta)$ and $c(\lambda^2,\Delta)$ are $BK-$spaces. This completes the proof. ■

Remark 2 *The absolute property does not hold on the $c_0(\lambda^2,\Delta)$ and $c(\lambda^2,\Delta)$ spaces. For instance, if we take $|x|=(|x_k|)$ we hold $\|x\|_{(c)_{\Lambda^2}}\neq\||x|\|_{(c)_{\Lambda^2}}$ .Thus, the space $c_0(\lambda^2,\Delta)$ and $c(\lambda^2,\Delta)$ are BK-space of non-absolute type.*

Theorem 3 *The sequence spaces $c_0(\lambda^2,\Delta)$ and $c(\lambda^2,\Delta)$ of non-absolute type are linearly isomorphic to the spaces c_0 and c, respectively, that is $c_0(\lambda^2,\Delta)\cong c_0$ and $c(\lambda^2,\Delta)\cong c$.*

Proof: We only consider $c_0(\lambda^2,\Delta)\cong c_0$ and others will prove similarly. To prove the theorem we must show the existence of linear bijection operator between $c_0(\lambda^2,\Delta)$ and c_0. Hence, let define the linear operator with the notation (5), from $c_0(\lambda^2,\Delta)$ and c_0 by $x\to y\left(\lambda^2\right)=Tx$.

Then $Tx=y\left(\lambda^2\right)=\Lambda^2\left(x\right)\in c_0$ for every $x\in c_0(\lambda^2,\Delta)$. Also, the linearity of T is clear. Further, it is trivial that $x=0$ whenever $Tx=0$. Hence T is injective.

Let $y=(y_k)\in c_0$ and define the sequence $x=\left\{x\left(\lambda^2\right)\right\}$ by

$$x_k\left(\lambda^2\right)=\sum_{j=0}^{k}\sum_{i=j-1}^{j}\left(-1\right)^{j-i}\frac{\Delta\lambda_i}{\Delta^2\lambda_j}y_i. \tag{6}$$

and we have

$$x_k\left(\lambda^2\right)-x_{k-1}\left(\lambda^2\right)=\sum_{i=k-1}^{k}\left(-1\right)^{k-i}\frac{\Delta\lambda_i}{\Delta^2\lambda_k}y_i.$$

Thus, for every $k\in\mathbb{N}$, we have by (5) that

$$\Lambda_n^2(x)=\frac{1}{\Delta\lambda_n}\sum_{k=0}^{n}\left[\Delta\left(\lambda_k y_k-\lambda_{k-1}y_{k-1}\right)\right]=y_n$$

This shows that $\Lambda^2(x)=y$ and since $y\in c_0$, we obtain that $\Lambda^2(x)\in c_0$. Thus we deduce that $x\in c_0(\lambda^2,\Delta)$ and $Tx=y$. Hence T is surjective.

Further, we have for every $x\in c_0(\lambda^2,\Delta)$ that

$$\left\|Tx\right\|_{c_0}=\left\|Tx\right\|_{\ell_\infty}=\left\|y\left(\lambda^2\right)\right\|_{\ell_\infty}=\left\|\Lambda^2(x)\right\|_{\ell_\infty}=\left\|x\right\|_{(c_0)_{\Lambda^2}}$$

which means that $c_0(\lambda^2,\Delta)$ and c_0 are linearly isomorphic.

3 Some inclusion relations

Theorem 4 *The inclusion $c_0\left(\lambda^2,\Delta\right)\subset c\left(\lambda^2,\Delta\right)$ strictly holds.*

Proof. $c_0\left(\lambda^2,\Delta\right)\subset c\left(\lambda^2,\Delta\right)$ is clear. To show strict, consider the sequence $x=(x_k)$ defined by $x_k=k+1$ for all $k\in\mathbb{N}$. Then we obtain that

$$\Lambda_n^2(x)=\frac{1}{\Delta\lambda_n}\sum_{k=0}^{n}\left(\Delta^2\lambda_k\right)\left(x_k-x_{k-1}\right)=1;\quad(n\in\mathbb{N})$$

for $n\in\mathbb{N}$ which shows that $\Lambda^2(x)\in c-c_0$. Thus, the sequence x is in $c\left(\lambda^2,\Delta\right)$ but not in $c_0\left(\lambda^2,\Delta\right)$. Hence the inclusion $c_0\left(\lambda^2,\Delta\right)\subset c\left(\lambda^2,\Delta\right)$ is strict and this completes the proof. ■

Theorem 5 *The inclusion $c\subset c_0\left(\lambda^2,\Delta\right)$ strictly holds.*

Proof. Let $x\in c$. Then, $\Lambda^2(x)\in c_0$. This shows that $x\in c_0\left(\lambda^2,\Delta\right)$. Hence, the inclusion $c\subset c_0\left(\lambda^2,\Delta\right)$ holds. Then, consider the sequence $y=(y_k)$ defined by $y_k=\sqrt{k+1}$ for $k\in\mathbb{N}$. It is trivial that $y\notin c$. On the other hand, it can easily be seen that $\Lambda^2(y)\in c_0$ and $y\in c_0\left(\lambda^2,\Delta\right)$ .Consequently, the sequence y is in $c_0\left(\lambda^2,\Delta\right)$ but not in c. We therefore deduce that the inclusion $c\subset c_0\left(\lambda^2,\Delta\right)$ is strict. ■

Corollary 6 *$c_0\subset c_0\left(\lambda^2,\Delta\right)$ and $c\subset c\left(\lambda^2,\Delta\right)$ strictly hold.*

Theorem 7 *Although the spaces ℓ_∞ and $c_0\left(\lambda^2,\Delta\right)$ overlap, the space ℓ_∞ does not include the space $c_0\left(\lambda^2,\Delta\right)$.*

Proof. It can be seen from the sequence y, which was defined in Theorem 5, is in $c_0\left(\lambda^2,\Delta\right)$ but not in ℓ_∞. ■

Lemma 8 *$A \in (\ell_\infty : c_0)$ if and only if* $\lim_n \sum\limits_k |a_{nk}| = 0$.

Theorem 9 *The inclusion $\ell_\infty \subset c_0(\lambda^2, \Delta)$ strictly holds if and only if $z \in A_\lambda(c_0)$ where the sequence $z = (z_k)$ is defined by*

$$z_k = \left|1 - \frac{\Delta^2 \lambda_{k+1}}{\Delta^2 \lambda_{k-1}}\right|; \quad (k \in \mathbb{N}).$$

Proof. Let $\ell_\infty \subset c_0(\lambda^2, \Delta)$. Then, we obtain that $\Lambda^2(x) \in c_0$ for every $x \in \ell_\infty$ and the matrix $\Lambda^2 = (\lambda^2_{nk})$ is in the class $(\ell_\infty : c_0)$. It follows by Lemma 8

$$\lim_n \sum_k |\lambda^2_{nk}| = 0. \tag{7}$$

From definition of $\Lambda^2 = (\lambda^2_{nk})$ given in (2) we have

$$\sum_k |\lambda^2_{nk}| = \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \left|(\Delta^2 \lambda_k - \Delta^2 \lambda_{k-1})\right| + \frac{\Delta^2 \lambda_n}{\Delta \lambda_n}. \tag{8}$$

From (7)

$$\lim_n \frac{\Delta^2 \lambda_n}{\Delta \lambda_n} = 0 \tag{9}$$

and

$$\lim_n \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \left|\Delta^2 (\lambda_k - \lambda_{k+1})\right| = 0. \tag{10}$$

We have

$$\frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \left|\Delta^2 (\lambda_k - \lambda_{k+1})\right| = \frac{\Delta \lambda_{n-1}}{\Delta \lambda_n} \left[\frac{1}{\Delta \lambda_{n-1}} \sum_{k=0}^{n-1} (\Delta^2 \lambda_k) z_k\right]$$

and since $\lim_n \frac{\Delta \lambda_{n-1}}{\Delta \lambda_n} = 1$ by (9); we have from (10) that

$$\lim_n \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} (\Delta^2 \lambda_k) z_k = 0 \tag{11}$$

which shows that $z = (z_k) \in A_\lambda(c_0)$. ∎

Conversely, let $z = (z_k) \in A_\lambda(c_0)$. Then we have that (11) holds. Also we obtain that

$$\begin{aligned} \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n} \left|\Delta^2 (\lambda_k - \lambda_{k+1})\right| &= \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \Delta^2 \lambda_k z_k \\ &\leq \frac{1}{\Delta \lambda_{n-1}} \sum_{k=0}^{n-1} \Delta^2 \lambda_k z_k. \end{aligned}$$

This and (11) provides (10). On the other hand, we have that

$$\begin{aligned} \left|\frac{\Delta^2 \lambda_n - \lambda_0}{\Delta \lambda_n}\right| &= \left|\frac{2\lambda_{n-1} - (\lambda_n + \lambda_{n-2} - \lambda_0)}{\Delta \lambda_n}\right| \\ &= \left|\frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \Delta^2 (\lambda_k - \lambda_{k+1})\right| \\ &\leq \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \left|\Delta^2 (\lambda_k - \lambda_{k+1})\right|. \end{aligned}$$

From (10), we derive that

$$\lim_n \frac{\Delta^2 \lambda_n}{\Delta \lambda_n} = \lim_n \frac{\Delta^2 \lambda_n - \lambda_0}{\Delta \lambda_n} = 0.$$

This provides (9). Hence, we obtain from (8) that (7) holds. From Lemma 8 $\Lambda^2 \in (\ell_\infty : c_0)$. Hence, the inclusion $\ell_\infty \subset c_0(\lambda^2, \Delta)$ holds. This inclusion is strict from Theorem 7. The proof is completed.

Corollary 10 *If* $\lim_n \frac{\Delta^2\lambda_{n+1}}{\Delta^2\lambda_n} = 1$, *then the inclusion* $\ell_\infty \subset c_0\left(\lambda^2, \Delta\right)$ *is strict.*

4 The bases for the spaces $c\left(\lambda^2, \Delta\right)$ and $c_0\left(\lambda^2, \Delta\right)$

If a normed sequence space X contains a sequence (b_n) with the property that for every $x \in X$ there is a unique sequence (α_n) of scalars such that

$$\lim_n \|x - (\alpha_0 b_0 + \alpha_1 b_1 + ... + \alpha_n b_n)\| = 0.$$

Then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_k \alpha_k b_k$.

Theorem 11 *Define the sequence* $b^{(k)}\left(\lambda^2\right) \in c_0\left(\lambda^2, \Delta\right)$ *for every fixed* $k \in \mathbb{N}$ *and by*

$$b_n^{(k)}(\lambda^2)=\begin{cases} \frac{\Delta\lambda_k}{\Delta^2\lambda_k} - \frac{\Delta\lambda_k}{\Delta^2\lambda_{k+1}}; & n > k, \\ \frac{\Delta\lambda_k}{\Delta^2\lambda_k}; & n = k, \\ 0; & n < k. \end{cases}$$

(i) The sequence $\left\{b_n^{(k)}\left(\lambda^2\right)\right\}_{k=0}^{\infty}$ is a Schauder basis for the space $c_0\left(\lambda^2, \Delta\right)$ and every $x \in c_0\left(\lambda^2, \Delta\right)$ has a unique representation of the form

$$x = \sum_k \alpha_k\left(\lambda^2\right) b^{(k)}\left(\lambda^2\right)$$

(ii) The sequence $\left\{b, b_n^{(0)}\left(\lambda^2\right), b_n^{(1)}\left(\lambda^2\right), ...\right\}$ is a Schauder basis for the space $c\left(\lambda^2, \Delta\right)$ and every $x \in c\left(\lambda^2, \Delta\right)$ has a unique representation of the form

$$x = lb + \sum_k \left[\alpha_k\left(\lambda^2\right) - l\right] b_n^{(k)}\left(\lambda^2\right)$$

where $\alpha_k\left(\lambda^2\right) = \Lambda^2(x)$ for all $k \in \mathbb{N}$ and the sequence $b = (b_k)$ is defined by $b_k = k + 1$.

Corollary 12 *The difference sequence spaces* $c\left(\lambda^2, \Delta\right)$ *and* $c_0\left(\lambda^2, \Delta\right)$ *are seperable.*

5 The $\alpha-, \beta-$ and $\gamma-$duals of the spaces $c\left(\lambda^2, \Delta\right)$ and $c_0\left(\lambda^2, \Delta\right)$

In this section, we introduce and prove the theorems determining the $\alpha-, \beta-$ and $\gamma-$ duals of the difference sequence spaces $c\left(\lambda^2, \Delta\right)$ and $c_0\left(\lambda^2, \Delta\right)$ of non-absolute type. For arbitrary sequence spaces X and Y ,the set $M(X, Y)$ defined by

$$M(X, Y) = \{a = (a_k) \in w : ax = (a_k x_k) \in Y \text{ for all } x = (x_k) \in X\} \tag{12}$$

is called the multipier space of X and Y. With the notation of (12); the $\alpha-, \beta-$ and $\gamma-$duals of a sequence space X, which are respectively denoted by X^α, X^β and X^γ are defined by

$$X^\alpha = M(X, \ell_1),\ X^\beta = M(X, cs) \text{ and } X^\gamma = M(X, bs).$$

Now, we may begin with lemmas which are given in [10]. We are needed them in proving theorems.

Lemma 13 $A \in (c_0 : \ell_1) = (c : \ell_1)$ *if and only if*

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty.$$

Lemma 14 *$A \in (c_0 : c)$ if and only if*

$$\lim_n a_{nk} \text{ exists for each } k \in \mathbb{N}, \tag{13}$$

$$\sup_n \sum_k |a_{nk}| < \infty. \tag{14}$$

Lemma 15 *$A \in (c : c)$ if and only if (13) and (14) hold, and*

$$\lim_n \sum_k a_{nk} \text{ exists.} \tag{15}$$

Lemma 16 *$A \in (c_0 : \ell_\infty) = (c : \ell_\infty)$ if and only if (14) holds.*

Lemma 17 *$A \in (\ell_\infty : c)$ if and only if (13) holds and*

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k |\alpha_k|.$$

Theorem 18 *The $\alpha-$dual of the space $c\left(\lambda^2, \Delta\right)$ and $c_0\left(\lambda^2, \Delta\right)$ is the set*

$$h_1 = \left\{ a = (a_k) \in w : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} b_{nk}\left(\lambda^2\right) \right| < \infty \right\};$$

where the matrix $B^{\lambda^2} = \left(b_{nk}^{\lambda^2}\right)$ is defined via the sequence $a = (a_k)$ by

$$b_{nk}^{\lambda^2} = \begin{cases} \left(\frac{\Delta\lambda_k}{\Delta^2\lambda_k} - \frac{\Delta\lambda_k}{\Delta^2\lambda_{k+1}}\right) a_n; & n > k, \\ \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n; & n = k, \\ 0; & n < k. \end{cases}$$

Proof. We prove the theorem for the space $c_0\left(\lambda^2, \Delta\right)$. Let $a = (a_k) \in w$. Then, we obtain the equality

$$a_k x_k = \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\Delta\lambda_j}{\Delta^2\lambda_k} y_j a_n = B_n^{\lambda^2}(y); \quad (n \in \mathbb{N}). \tag{16}$$

Thus, we observe by (16) that $ax = (a_k x_k) \in \ell_1$ whenever $x = (x_k) \in c_0\left(\lambda^2, \Delta\right)$ or $c\left(\lambda^2, \Delta\right)$ if and only if $B^{\lambda^2} y \in \ell_1$ whenever $y = (y_k) \in c_0$ or c. This means that the sequence $a = (a_k)$ is in the $\alpha-$dual of the spaces $c_0\left(\lambda^2, \Delta\right)$ or $c\left(\lambda^2, \Delta\right)$ if and only if $B^\lambda \in (c_0 : \ell_1) = (c : \ell_1)$. We therefore obtain by Lemma 13 with B^λ instead of A that $a \in \left\{c_0\left(\lambda^2, \Delta\right)\right\}^\alpha = \left\{c\left(\lambda^2, \Delta\right)\right\}^\alpha$ if and only if

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} b_{nk}^{\lambda^2} \right| < \infty.$$

Which leads us to the consequence that $\left\{c_0\left(\lambda^2, \Delta\right)\right\}^\alpha = \left\{c\left(\lambda^2, \Delta\right)\right\}^\alpha = h_1$. This concludes proof. ∎

Theorem 19 *Define the sets*

$$h_2 = \left\{ a = (a_k) \in w : \sum_{j=k}^{\infty} a_j \text{ exists for each } k \in \mathbb{N}. \right\}$$

$$h_3 = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n-1} |g_k(n)| < \infty. \right\}$$

$$h_4 = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n \right| < \infty. \right\}$$

$$h_5 = \left\{ a = (a_k) \in w : \sum_k (k+1)\, a_k \text{ converges.} \right\}$$

where

$$g_k(n) = \Delta\lambda_k \left(\frac{a_k}{\Delta\lambda_k} + \left(\frac{1}{\Delta\lambda_k} - \frac{1}{\Delta\lambda_{k+1}}\right) \sum_{j=k+1}^{n} a_j \right)$$

for $k < n$. Then $\left\{c\left(\lambda^2, \Delta\right)\right\}^\beta = h_3 \cap h_4 \cap h_5$ and $\left\{c_0\left(\lambda^2, \Delta\right)\right\}^\beta = h_2 \cap h_3 \cap h_4$.

Proof. We have from (6) that

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \left(\sum_{i=j-1}^{j} (-1)^{j-i} \frac{\Delta\lambda_i}{\Delta^2\lambda_j} y_i \right) \right] a_k \\
&= \sum_{k=0}^{n-1} \Delta\lambda_k \left[\frac{a_k}{\Delta^2\lambda_k} + \left(\frac{1}{\Delta^2\lambda_k} - \frac{1}{\Delta^2\lambda_{k+1}} \right) \sum_{j=k+1}^{n} a_j \right] y_k + \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n y_n \qquad (17) \\
&= \sum_{k=0}^{n-1} g_k(n)\, y_k + \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n y_n \\
&= T_n(y); \quad (n \in \mathbb{N})
\end{aligned}$$

where the matrix $T = (t_{nk})$

$$t_{nk} = \begin{cases} g_k(n); & k < n, \\ \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n; & k = n, \\ 0; & k > n. \end{cases} \qquad (k, n \in \mathbb{N}).$$

Then we derive that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0\left(\lambda^2, \Delta\right)$ if and only if $Ty \in c$ whenever $y = (y_k) \in c_0$. This means that $a = (a_k) \in \left\{c_0\left(\lambda^2, \Delta\right)\right\}^\beta$ if and only if $T \in (c_0 : c)$. Therefore, by using Lemma 14, we obtain from (13) and (14) that

$$\sum_{j=k}^{\infty} a_j \text{ exists for each } k \in \mathbb{N}, \qquad (18)$$

$$\sup_n \sum_{k=0}^{n-1} |g_k(n)| < \infty, \qquad (19)$$

$$\sup_k \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_k < \infty. \qquad (20)$$

Hence we conclude that $\left\{c_0\left(\lambda^2, \Delta\right)\right\}^\beta = h_2 \cap h_3 \cap h_4$. We can derive from Lemma 15 and 16 that $a = (a_k) \in \left\{c\left(\lambda^2, \Delta\right)\right\}^\beta$ if and only if $T \in (c : c)$. Therefore, we have from (13) and (14) that (18), (19) and (20) hold. It can be seen that the equality

$$\sum_{k=0}^{n} (k+1)\, a_k = \sum_{k=0}^{n-1} g_k(n) + \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n; \ (n \in \mathbb{N})$$

holds, which can be written as follows;

$$\sum_{k=0}^{n} (k+1)\, a_k = \sum_k t_{nk}; \quad (n \in \mathbb{N}).$$

Consequently, we have from (15) that

$$\{(k+1)\, a_k\} \in cs.$$

Hence (18) is redundant. We conclude that $\left\{c\left(\lambda^2, \Delta\right)\right\}^\beta = h_3 \cap h_4 \cap h_5$. ∎

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Theorem 20 $\{c_0(\lambda^2,\Delta)\}^{\gamma} = \{c(\lambda^2,\Delta)\}^{\gamma} = h_3 \cap h_4$.

Proof. It can be proved similarly as the proof of the Theorem 19 with Lemma 16 instead of Lemma 14. ■

6 Some matrix transformations

In this section, we state some matrix classes of matrix mappings on the $c_0(\lambda^2,\Delta)$ and $c(\lambda^2,\Delta)$. Let $x, y \in w$ be connected by the relation $y = \Lambda^2(x)$ like given in (5). For an infinite matrix $A = (a_{nk})$, we have by (17)

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m-1} g_{nk}(m)\, y_k + \frac{\Delta\lambda_m}{\Delta^2\lambda_m} a_{nm} y_m \tag{21}$$

where

$$g_{nk}(m) = \Delta\lambda_k \left[\frac{a_{nk}}{\Delta\lambda_k} + \left(\frac{1}{\Delta\lambda_k} - \frac{1}{\Delta\lambda_{k+1}} \right) \sum_{j=k+1}^{m} a_{nj} \right].$$

Let $x \in c(\lambda^2,\Delta)$ and $A_n = (a_{nk})_{k=0}^{\infty} \in (c(\lambda^2,\Delta))^{\beta}$ for all $n \in \mathbb{N}$. By passing limits in (21) as $m \to \infty$

$$\begin{aligned} \sum_k a_{nk}x_k &= \sum_k g_{nk}y_k + l a_n \\ &= \sum_k g_{nk}(y_k - l) + l\left(\sum_k g_{nk} + a_n \right) \end{aligned} \tag{22}$$

where $l = \lim_{k\to\infty} y_k$ and $a_n = \lim_{k\to\infty}\left(\frac{\Delta\lambda_k}{\Delta^2\lambda_k} a_{nk} \right)$ for all $n \in \mathbb{N}$. Let consider following conditions;

$$\sup_{F\in\mathcal{F}} \sum_n \left| \sum_{k\in F} g_{nk} \right|^p < \infty, \tag{23}$$

$$\sup_m \sum_{k=0}^{m-1} |g_{nk}(m)| < \infty, \tag{24}$$

$$\{(k+1)\, a_{nk}\}_{k=0}^{\infty} \in cs, \tag{25}$$

$$\lim_k \frac{\Delta\lambda_k}{\Delta^2\lambda_k} a_{nk} = a_n, \tag{26}$$

$$\sum_n |a_n|^p < \infty, \tag{27}$$

$$\sup_n \sum_k |g_{nk}| < \infty, \tag{28}$$

$$\sup_n |a_n| < \infty, \tag{29}$$

$$\sum_{j=k}^{\infty} a_{nj}\ exists, \tag{30}$$

$$\left\{ \frac{\Delta\lambda_k}{\Delta^2\lambda_k} a_{nk} \right\}_{k=0}^{\infty} \in \ell_\infty, \tag{31}$$

$$\lim_n a_n = a, \tag{32}$$

$$\lim_n g_{nk} = \alpha_k, \tag{33}$$

$$\lim_n \sum_k g_{nk} = \alpha, \tag{34}$$

$$\lim_n a_n = 0, \tag{35}$$

$$\lim_n g_{nk} = 0, \tag{36}$$

$$\lim_n \sum_k g_{nk} = 0. \tag{37}$$

Using Theorem 19 and the results given in [10] with (21) and (22), we derive the following result:

Theorem 21

(a) Let $1 \leq p < \infty$. Then $A \in \left(c\left(\lambda^2, \Delta\right) : \ell_p\right)$ if and only if (23), (24), (25), (26) and (27).
(b) $A \in \left(c\left(\lambda^2, \Delta\right) : \ell_p\right)$ if and only if (25), (26), (28), (29).
(c) Let $1 \leq p < \infty$. Then $A \in \left(c_0\left(\lambda^2, \Delta\right) : \ell_p\right)$ if and only if (23), (24), (30) and (31).
(d) $A \in \left(c_0\left(\lambda^2, \Delta\right) : \ell_\infty\right)$ if and only if (28), (30) and (31).
(e) $A \in \left(c\left(\lambda^2, \Delta\right) : c\right)$ if and only if (25), (26), (28), (32), (33) and (34).
(f) $A \in \left(c\left(\lambda^2, \Delta\right) : c_0\right)$ if and only if (25), (26), (28), (35), (36) and (37).
(g) $A \in \left(c_0\left(\lambda^2, \Delta\right) : c\right)$ if and only if (28), (30), (31) and (33).
(h) $A \in \left(c_0\left(\lambda^2, \Delta\right) : c_0\right)$ if and only if (28), (30), (31) and (36).

Acknowledgements
We thank the reviewer for his/her careful reading and useful comments which improved the presentation of the paper.
Disclosure Statement
The authors declares to have no competing interests.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.5, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Stable cubic sets

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Abstract. The notions of (almost) stable cubic set, stable element, evaluative set and stable degree are introduced, and related properties are investigated. Regarding internal (external) cubic sets and the complement of cubic set, their (almost) stableness and unstableness are discussed. Regarding the P-union, R-union, P-intersection and R-intersection of cubic sets, their (almost) stableness and unstableness are investigated.

1. Introduction

Fuzzy sets are initiated by Zadeh [14]. In [15], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. In traditional fuzzy logic, to represent, e.g., the expert's degree of certainty in different statements, numbers from the interval $[0, 1]$ are used. It is often difficult for an expert to exactly quantify his or her certainty; therefore, instead of a real number, it is more adequate to represent this degree of certainty by an interval or even by a fuzzy set. In the first case, we get an interval-valued fuzzy set. In the second case, we get a second-order fuzzy set. Interval-valued fuzzy sets have been actively used in real-life applications. For example, Sambuc [8] in Medical diagnosis in thyroidian pathology, Kohout [7] also in Medicine, in a system CLINAID, Gorzalczany [10] in Approximate reasoning, Turksen [10, 11] in Interval-valued logic, in preferences modelling [12], etc. These works and others show the importance of these sets. Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [4] introduced a new notion, called a (internal, external) cubic set, and investigated several properties. They dealt with P-union, P-intersection, R-union and R-intersection of cubic sets, and investigated several related properties. Cubic set theory is applied to CI-algebras (see [1]), B-algebras (see [9]), BCK/BCI-algebras (see [5, 6]), KU-Algebras (see [2, 13]), and semigroups (see [3]).

In this paper, we introduce the notions of (almost) stable cubic set, stable element, evaluative set and stable degree. We investigate related properties. Regarding internal (external) cubic sets and the complement of cubic set, we investigate their (almost) stableness and unstableness.

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[0] **2010 Mathematics Subject Classification**: 03E72, 08A72.

[0] **Keywords**: (almost) stable cubic set, stable element, evaluate set, stable degree.

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Regarding the P-union, R-union, P-intersection and R-intersection of cubic sets, we deal with their (almost) stableness and unstableness.

2. **Preliminaries**

A *fuzzy set* in a set X is defined to be a function $\lambda : X \to [0,1]$. Denote by I^X the collection of all fuzzy sets in a set X. Define a relation $\leq$ on I^X as follows:

$$(\forall \lambda, \mu \in I^X)\,(\lambda \leq \mu \iff (\forall x \in X)(\lambda(x) \leq \mu(x))).$$

The join ($\vee$) and meet ($\wedge$) of λ and μ are defined by $(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}$, and $(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}$, respectively, for all $x \in X$. The complement of λ, denoted by λ^c, is defined by $(\forall x \in X)\,(\lambda^c(x) = 1 - \lambda(x))$. For a family $\{\lambda_i \mid i \in \Lambda\}$ of fuzzy sets in X, we define the join ($\vee$) and meet ($\wedge$) operations as follows: $\left(\bigvee_{i\in\Lambda} \lambda_i\right)(x) = \sup\{\lambda_i(x) \mid i \in \Lambda\}$, $\left(\bigwedge_{i\in\Lambda} \lambda_i\right)(x) = \inf\{\lambda_i(x) \mid i \in \Lambda\}$, respectively, for all $x \in X$.

Let $D[0,1]$ be the set of all closed subintervals of the unit interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^-, M^+]$, where M^- and M^+ are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$, and $\mathbf{a} = [a,a]$ for every $a \in (0,1)$. We also note that

(i) $(\forall M, N \in D[0,1])\ (M = N \Leftrightarrow M^- = N^-,\ M^+ = N^+)$.
(ii) $(\forall M, N \in D[0,1])\ (M \leq N \Leftrightarrow M^- \leq N^-,\ M^+ \leq N^+)$.

For every $M \in D[0,1]$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^+, 1 - M^-]$.

Let X be a nonempty set. A function $A : X \to D[0,1]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $A(x)^-$ and $A(x)^+$, respectively. For any $[a,b] \in D[0,1]$, the IVF set whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $\widetilde{[a,b]}$. Denote by D^X the collection of all interval-valued fuzzy sets in a set X. In particular, for any $a \in [0,1]$, the IVF set whose value is $\mathbf{a} = [a,a]$ for all $x \in X$ is denoted by simply $\tilde{a}$.

For every $A, B \in D^X$, we define

$$A = B \Leftrightarrow (\forall x \in X)\,(A(x)^- = B(x)^-,\ A(x)^+ = B(x)^+),$$

$$A \subseteq B \Leftrightarrow (\forall x \in X)\,(A(x)^- \leq B(x)^-,\ A(x)^+ \leq B(x)^+).$$

The *complement* A^c of A is defined by $(\forall x \in X)\,(A^c(x)^- = 1 - A(x)^+,\ A^c(x)^+ = 1 - A(x)^-)$. For a family $\{A_i \mid i \in \Lambda\}$ of IVF sets where Λ is an index set, the *union* $G = \bigcup_{i\in\Lambda} A_i$ and the

intersection $F = \bigcap_{i \in \Lambda} A_i$ are defined by

$$(\forall x \in X)\ \big(G(x)^- = \sup_{i\in\Lambda} A_i(x)^-,\ G(x)^+ = \sup_{i\in\Lambda} A_i(x)^+\big),$$

$$(\forall x \in X)\ \big(F(x)^- = \inf_{i\in\Lambda} A_i(x)^-,\ F(x)^+ = \inf_{i\in\Lambda} A_i(x)^+\big),$$

respectively.

Definition 2.1 ([4])**.** Let X be a nonempty set. By a *cubic set* in X we mean a structure

$$\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in X\}$$

in which A is an IVF set in X and λ is a fuzzy set in X.

A cubic set $\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in X\}$ is simply denoted by $\mathscr{A} = \langle A, \lambda\rangle$. Note that a cubic set is a generalization of an intuitionistic fuzzy set.

Definition 2.2 ([4])**.** Let X be a nonempty set. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in X is said to be an *internal cubic set* (briefly, ICS) if $A(x)^- \le \lambda(x) \le A(x)^+$ for all $x \in X$.

Definition 2.3 ([4])**.** Let X be a nonempty set. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in X is said to be an *external cubic set* (briefly, ECS) if $\lambda(x) \notin (A(x)^-, A(x)^+)$ for all $x \in X$.

Theorem 2.4 ([4])**.** *Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in X. If $\mathscr{A}$ is both an ICS and an ECS, then* $(\forall x \in X)$ $(\lambda(x) \in U(A) \cup L(A))$ *where* $U(A) = \{A(x)^+ \mid x \in X\}$ *and* $L(A) = \{A(x)^- \mid x \in X\}$.

Definition 2.5 ([4])**.** Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be cubic sets in X. Then we define

(a) (Equality) $\mathscr{A} = \mathscr{B} \Leftrightarrow A = B$ and $\lambda = \nu$.
(b) (P-order) $\mathscr{A} \sqsubseteq \mathscr{B} \Leftrightarrow A \subseteq B$ and $\lambda \le \nu$.
(c) (R-order) $\mathscr{A} \Subset \mathscr{B} \Leftrightarrow A \subseteq B$ and $\lambda \ge \nu$.

Definition 2.6 ([4])**.** Let $\mathscr{A} = \langle A, \lambda\rangle$, $\mathscr{B} = \langle B, \mu\rangle$ and $\mathscr{A}_i = \{\langle x, A_i(x), \lambda_i(x)\rangle \mid x \in X\}$, $i \in \Lambda$, be cubic sets in X for $i \in \Lambda$. The *complement, P-union, P-intersection, R-union and R-intersection* are defined as follows;

(a) (Complement) $\mathscr{A}^c = \{\langle x, A^c(x), 1 - \lambda(x)\rangle \mid x \in X\}$.
(b) (P-union) $\mathscr{A} \sqcup \mathscr{B} = \{\langle x, (A \cup B)(x), (\lambda \vee \nu)(x)\rangle \mid x \in X\}$ and
$\sqcup \mathscr{A}_i = \{\langle x, (\bigcup A_i)(x), (\bigvee \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.
(c) (P-intersection) $\mathscr{A} \sqcap \mathscr{B} = \{\langle x, (A \cap B)(x), (\lambda \wedge \nu)(x)\rangle \mid x \in X\}$ and
$\sqcap \mathscr{A}_i = \{\langle x, (\bigcap A_i)(x), (\bigwedge \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.
(d) (R-union) $\mathscr{A} \Cup \mathscr{B} = \{\langle x, (A \cup B)(x), (\lambda \wedge \nu)(x)\rangle \mid x \in X\}$ and
$\Cup \mathscr{A}_i = \{\langle x, (\bigcup A_i)(x), (\bigwedge \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.
(e) (R-intersection) $\mathscr{A} \Cap \mathscr{B} = \{\langle x, (A \cap B)(x), (\lambda \vee \nu)(x)\rangle \mid x \in X\}$ and
$\Cap \mathscr{A}_i = \{\langle x, (\bigcap A_i)(x), (\bigvee \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.

3. (Almost) stable cubic sets

In what follows, let X denote a nonempty set unless otherwise specified.

Definition 3.1. Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in X. Then the *evaluative set* of $\mathscr{A} = \langle A, \lambda\rangle$ is defined to be a structure

$$\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\} \tag{3.1}$$

where $E_{\mathscr{A}}(x) = \langle l(E_{\mathscr{A}}(x)), r(E_{\mathscr{A}}(x))\rangle$ with $l(E_{\mathscr{A}}(x)) = \lambda(x) - A(x)^-$ and $r(E_{\mathscr{A}}(x)) = A(x)^+ - \lambda(x)$ which are called the *left evaluative point* and the *right evaluative point*, respectively, of $\mathscr{A} = \langle A, \lambda\rangle$ at $x \in X$. We say that $E_{\mathscr{A}}(x)$ is the *evaluative point* of $\mathscr{A} = \langle A, \lambda\rangle$ at $x \in X$.

Example 3.2. Let $\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in I\}$ be a cubic set in $I = [0, 1]$.

(1) If $A(x) = [0.3, 0.7]$ and $\lambda(x) = 0.4$ for all $x \in I$, then $\mathbf{E}_{\mathscr{A}} = \{(x, \langle 0.1, 0.3\rangle) \mid x \in I\}$.
(2) If $A(x) = [0.3, 0.7]$ and $\lambda(x) = 0.2$ for all $x \in I$, then $\mathbf{E}_{\mathscr{A}} = \{(x, \langle -0.1, 0.5\rangle) \mid x \in I\}$.
(3) If $A(x) = [0.3, 0.7]$ and $\lambda(x) = 0.8$ for all $x \in I$, then $\mathbf{E}_{\mathscr{A}} = \{(x, \langle 0.5, -0.1\rangle) \mid x \in I\}$.

Example 3.3. Let $\mathscr{B} = \{\langle x, B(x), \mu(x)\rangle \mid x \in I\}$ be a cubic set in $I = [0, 1]$ with $B(x) = [\frac{x}{4}, 1-\frac{x}{4}]$ and $\mu(x) = \frac{x}{3}$. Then $\mathbf{E}_{\mathscr{B}} = \left\{\left(x, \langle \frac{x}{12}, 1 - \frac{7x}{12}\rangle\right) \mid x \in I\right\}$, and so the evaluative point of $\mathscr{B}$ at $\frac{1}{2} \in I$ is $E_{\mathscr{B}}(\frac{1}{2}) = \langle \frac{1}{24}, \frac{17}{24}\rangle$.

Example 3.4. Let $\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in I\}$ be a cubic set in $X = \{0, a, b, c\}$ which is defined by Table 1.

Table 1. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
0	$[\frac{1}{8}, \frac{7}{8}]$	$\frac{7}{8} = 0.875$
a	$[\frac{1}{4}, \frac{3}{4}]$	$\frac{3}{8} = 0.375$
b	$[\frac{3}{8}, \frac{5}{8}]$	$\frac{1}{4} = 0.250$
c	$[\frac{1}{2}, \frac{1}{2}]$	$\frac{5}{8} = 0.625$

Then every evaluative point of $\mathscr{A}$ at each $x \in X$ is $E_{\mathscr{A}}(0) = \langle \frac{3}{4}, 0\rangle$, $E_{\mathscr{A}}(a) = \langle \frac{1}{8}, \frac{3}{8}\rangle$, $E_{\mathscr{A}}(b) = \langle -\frac{1}{8}, \frac{3}{8}\rangle$, and $E_{\mathscr{A}}(c) = \langle \frac{1}{8}, -\frac{1}{8}\rangle$, respectively. Hence the evaluative set of $\mathscr{A}$ is

$$\mathbf{E}_{\mathscr{A}} = \{(0, \langle \tfrac{3}{4}, 0\rangle), (a, \langle \tfrac{1}{8}, \tfrac{3}{8}\rangle), (b, \langle -\tfrac{1}{8}, \tfrac{3}{8}\rangle), (c, \langle \tfrac{1}{8}, -\tfrac{1}{8}\rangle)\}.$$

Definition 3.5. Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in X with the evaluative set

$$\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\}.$$

An element $a \in X$ is called a *stable element* of $\mathscr{A} = \langle A, \lambda\rangle$ in X if it satisfies: $l(E_{\mathscr{A}}(a)) = \lambda(a) - A(a)^- \geq 0,\ r(E_{\mathscr{A}}(a)) = A(a)^+ - \lambda(a) \geq 0$. Otherwise, we say that a is an *unstable element* of $\mathscr{A} = \langle A, \lambda\rangle$ in X. The set of all stable elements of $\mathscr{A} = \langle A, \lambda\rangle$ in X is called the *stable cut* of

$\mathscr{A} = \langle A, \lambda \rangle$ in X and is denoted by $S_{\mathscr{A}}$. The set of all unstable elements of $\mathscr{A} = \langle A, \lambda \rangle$ in X is called the *unstable cut* of $\mathscr{A} = \langle A, \lambda \rangle$ in X and is denoted by $U_{\mathscr{A}}$. We say that $\mathscr{A} = \langle A, \lambda \rangle$ is a *stable* cubic set if $S_{\mathscr{A}} = X$. Otherwise, $\mathscr{A} = \langle A, \lambda \rangle$ is called an *unstable* cubic set.

It is clear that $X = S_{\mathscr{A}} \cup U_{\mathscr{A}}$, $S_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) \geq 0,\ r(E_{\mathscr{A}}(x)) \geq 0\}$ and $U_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) < 0\} \cup \{x \in X \mid r(E_{\mathscr{A}}(x)) < 0\}$.

Example 3.6. Let $\mathscr{A} = \langle A, \lambda \rangle$ be a cubic set in $X = \{0, a, b, c\}$ given by Table 2.

TABLE 2. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
0	$[0.2, 0.3]$	0.10
a	$[0.2, 0.3]$	0.25
b	$[0.7, 0.8]$	0.75
c	$[0.3, 0.7]$	0.80

Then a and b are stable elements of $\mathscr{A}$ in X, and 0 and c are unstable elements of $\mathscr{A}$ in X. Hence $S_{\mathscr{A}} = \{a, b\}$ and $U_{\mathscr{A}} = \{0, c\}$.

Example 3.7. (1) Let $\mathscr{A} = \langle A, \lambda \rangle$ be a cubic set in $X = \{a, b, c\}$ defined by Table 3.

TABLE 3. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.1, 0.6]$	0.5
b	$[0.6, 0.9]$	0.7
c	$[0.1, 0.9]$	0.6

It is routine to verify that $\mathscr{A} = \langle A, \lambda \rangle$ is a stable cubic set.
(2) Let $\mathscr{B} = \langle B, \mu \rangle$ be a cubic set in $X = \{a, b, c\}$ defined by Table 4.

TABLE 4. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.1, 0.3]$	0.5
b	$[0.6, 0.9]$	0.7
c	$[0.1, 0.9]$	0.6

Then $\mathscr{B}$ is an unstable cubic set since $E_{\mathscr{B}}(a) = (0.5 - 0.1, 0.3 - 05) = (0.4, -0.2)$.

Theorem 3.8. *Every ICS is a stable cubic set.*

Proof. Straightforward. □

The following example shows that every ECS would be stable or unstable.

Example 3.9. (1) Let $\mathscr{A} = \langle A, \lambda\rangle$ be an ECS in $X = \{a, b, c\}$ given by Table 5.

TABLE 5. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.1, 0.6]$	0.6
b	$[0.6, 0.9]$	0.5
c	$[0.1, 0.9]$	0.1

Then $\mathscr{A}$ is unstable because $E_{\mathscr{A}}(b) = (0.5 - 0.6, 0.9 - 0.5) = (-0.1, 0.4)$.
(2) Let $\mathscr{B} = \langle B, \mu\rangle$ be an ECS in $X = \{a, b, c\}$ defined by Table 6.

TABLE 6. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.1, 0.3]$	0.1
b	$[0.6, 0.9]$	0.9
c	$[0.1, 0.9]$	0.1

Then $\mathscr{B}$ is stable since $E_{\mathscr{B}}(a) = (0, 0.2)$, $E_{\mathscr{B}}(b) = (0.3, 0)$, and $E_{\mathscr{B}}(c) = (0, 0.8)$.

We provide a condition for an ECS to be a stable cubic set.

Theorem 3.10. *If an ECS $\mathscr{A} = \langle A, \lambda\rangle$ in X satisfies the following condition*

$$(\forall x \in X)\left(\mathscr{A}^-(x) = \lambda(x) \text{ or } \mathscr{A}^+(x) = \lambda(x)\right), \tag{3.2}$$

then $\mathscr{A} = \langle A, \lambda\rangle$ is a stable cubic set.

Proof. Straightforward. □

Corollary 3.11. *Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in X. If $\mathscr{A}$ is both an ICS and an ECS, then $\mathscr{A}$ is stable.*

Proof. Straightforward. □

Theorem 3.12. *The complement of a stable cubic set is also stable.*

Proof. Let $\mathscr{A} = \langle A, \lambda\rangle$ be a stable cubic set in X. Then $X = S_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) \geq 0,\ r(E_{\mathscr{A}}(x)) \geq 0\}$. Hence $\lambda(x) - A(x)^- \geq 0$ and $A(x)^+ - \lambda(x) \geq 0$ for all $x \in X$. It follows that $l(E_{\mathscr{A}^c}(x)) = (1 - \lambda(x)) - (1 - A(x)^+) = A(x)^+ - \lambda(x)) \geq 0$ and $r(E_{\mathscr{A}^c}(x)) = (1 - A(x)^-) - (1 - \lambda(x)) = \lambda(x) - A(x)^- \geq 0$. Therefore $\mathscr{A}^c = \langle A^c, \lambda^c\rangle$ is a stable cubic set. □

Theorem 3.13. *The complement of an unstable cubic set is also unstable.*

Proof. Let $\mathscr{A} = \langle A, \lambda\rangle$ be an unstable cubic set in X. Then $U_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) < 0\} \cup \{x \in X \mid r(E_{\mathscr{A}}(x)) < 0\} \neq \emptyset$, and so there exist $x \in X$ such that $\lambda(x) - A(x)^- < 0$ or $A(x)^+ - \lambda(x) < 0$. It follows that $l(E_{\mathscr{A}^c}(x)) = (1 - \lambda(x)) - (1 - A(x)^+) = A(x)^+ - \lambda(x)) < 0$ or $r(E_{\mathscr{A}^c}(x)) = (1 - A(x)^-) - (1 - \lambda(x)) = \lambda(x) - A(x)^- < 0$. Hence $U_{\mathscr{A}^c} \neq \emptyset$, and therefore $\mathscr{A}^c = \langle A^c, \lambda^c\rangle$ is an unstable cubic set in X. □

The following example illustrates Theorem 3.13.

Example 3.14. Note that the cubic set $\mathscr{B} = \langle B, \mu\rangle$ in Example 3.7(2) is unstable, and its complement is represented by Table 7.

TABLE 7. Tabular representation of the cubic set $\mathscr{B}^c$

X	$B^c(x)$	$\mu^c(x)$
a	$[0.7, 0.9]$	0.5
b	$[0.1, 0.4]$	0.3
c	$[0.1, 0.9]$	0.4

Then $\mathscr{B}^c = \langle B^c, \mu^c\rangle$ is unstable since $a \in U_{\mathscr{B}^c}$.

Theorem 3.15. *The P-union and P-intersection of two stable cubic sets in X are stable cubic sets in X.*

Proof. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be stable cubic sets in X. Then $S_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) \geq 0,\ r(E_{\mathscr{A}}(x)) \geq 0\} = X$ and $S_{\mathscr{B}} = \{x \in X \mid l(E_{\mathscr{B}}(x)) \geq 0,\ r(E_{\mathscr{B}}(x)) \geq 0\} = X$. It follows that $\lambda(x) - A(x)^- \geq 0$, $A(x)^+ - \lambda(x) \geq 0$ for all $x \in X$ and $\mu(x) - B(x)^- \geq 0$, $B(x)^+ - \mu(x) \geq 0$ for all $x \in X$. Assume that $\lambda(x) \geq \mu(x)$ and consider four cases:

(i) $A(x)^- \geq B(x)^-$ and $A(x)^+ \geq B(x)^+$,
(ii) $A(x)^- \geq B(x)^-$ and $A(x)^+ \leq B(x)^+$,
(iii) $A(x)^- \leq B(x)^-$ and $A(x)^+ \geq B(x)^+$,
(iv) $A(x)^- \leq B(x)^-$ and $A(x)^+ \leq B(x)^+$.

The first case implies that $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq A(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} = \lambda(x) \leq A(x)^+ = \max\{A(x)^+, B(x)^+\}$. It follows that $\lambda(x) - A(x)^- \geq 0$ and $A(x)^+ - \lambda(x) \geq 0$. From the second case, we have $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq A(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} = \lambda(x) \leq B(x)^+ = \max\{A(x)^+, B(x)^+\}$. Hence $\lambda(x) - A(x)^- \geq 0$ and $B(x)^+ - \lambda(x) \geq A(x)^+ - \lambda(x) \geq 0$. The third case induces $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq \mu(x) \geq B(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} = \lambda(x) \leq A(x)^+ = \max\{A(x)^+, B(x)^+\}$, and so $\lambda(x) - B(x)^- \geq \mu(x) - B(x)^- \geq 0$ and $A(x)^+ - \lambda(x) \geq 0$. For the final case, we get $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq \mu(x) \geq B(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} =$

$\lambda(x) \leq A(x)^+ \leq B(x) = \max\{A(x)^+, B(x)^+\}$. Thus $\lambda(x) - B(x)^- \geq \mu(x) - B(x)^- \geq 0$ and $B(x)^+ - \lambda(x) \geq 0$. In the case of $\mu(x) \geq \lambda(x)$, we can obtain the same results in a similar way. Therefore $\mathscr{A} \sqcup \mathscr{B}$ is a stable cubic set in X. By the similar method, we know that $\mathscr{A} \sqcap \mathscr{B}$ is a stable cubic set in X. □

The following example shows that the R-union and the R-intersection of two stable cubic sets in X may not be stable in X.

Example 3.16. Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 8 and 9, respectively.

TABLE 8. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.2, 0.3]$	0.20
b	$[0.7, 0.8]$	0.75
c	$[0.3, 0.7]$	0.60

TABLE 9. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.1, 0.3]$	0.15
b	$[0.6, 0.9]$	0.70
c	$[0.1, 0.9]$	0.80

Then

$$\mathscr{A} \Cup \mathscr{B} = \{\langle a, [0.2, 0.3], 0.15\rangle, \langle b, [0.7, 0.9], 0.7\rangle, \langle c, [0.3, 0.9], 0.6\rangle\}$$

and

$$\mathscr{A} \Cap \mathscr{B} = \{\langle a, [0.1, 0.3], 0.2\rangle, \langle b, [0.6, 0.8], 0.75\rangle, \langle c, [0.1, 0.7], 0.8\rangle\}.$$

Hence we know that $E_{\mathscr{A} \Cup \mathscr{B}}(a) = \langle -0.05, 0.15\rangle$ and $E_{\mathscr{A} \Cap \mathscr{B}}(c) = \langle 0.7, -0.1\rangle$. Thus $\mathscr{A} \Cup \mathscr{B}$ and $\mathscr{A} \Cap \mathscr{B}$ are unstable.

Now, we provide conditions for the R-union (resp. R-intersection) of two ICSs to be stable.

Theorem 3.17. *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be ICSs in X such that*

$$(\forall x \in X) \left(\max\{A(x)^-, B(x)^-\} \leq (\lambda \wedge \mu)(x) \right). \tag{3.3}$$

Then the R-union of $\mathscr{A}$ and $\mathscr{B}$ is a stable cubic set in X.

Proof. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in X. Then $A(x)^- \leq \lambda(x) \leq A(x)^+$ and $B(x)^- \leq \mu(x) \leq B(x)^+$ for all $x \in X$. It follows from (3.3) that $\max\{A(x)^-, B(x)^-\} \leq (\lambda \wedge \mu)(x) \leq \max\{A(x)^+, B(x)^+\}$ for all $x \in X$. Hence the R-union of $\mathscr{A}$ and $\mathscr{B}$ is an ICS, and so it is stable by Theorem 3.8. □

Theorem 3.18. *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ICSs in X such that*

$$(\forall x \in X)\left(\max\{A(x)^+, B(x)^+\} \leq (\lambda \vee \mu)(x)\right). \tag{3.4}$$

Then the R-intersection of $\mathscr{A}$ and $\mathscr{B}$ is a stable cubic set in X.

Proof. The proof is by the similar method to Theorem 3.17. □

Theorem 3.19. *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be ECSs in X such that $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$ are ICSs in X. Then the P-union $\mathscr{A} \sqcup \mathscr{B}$ and the P-intersection $\mathscr{A} \sqcap \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are stable in X.*

Proof. It is straightforward by Theorems 3.20 and 3.21 in [4] and Theorem 3.8. □

Definition 3.20. Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set with the evaluative set $\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\}$ in X. Then the *stable degree* of $\mathscr{A}$ in X is denoted by $SD_{\mathscr{A}}$ and is defined by

$$SD_{\mathscr{A}} = \left(\sum_{x\in X} l(E_{\mathscr{A}}(x)),\ \sum_{x\in X} r(E_{\mathscr{A}}(x))\right). \tag{3.5}$$

Definition 3.21. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ with the evaluative set $\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\}$ in X is said to be *almost stable* if there exists the stable degree $SD_{\mathscr{A}}$ in which $\sum\limits_{x\in X} l(E_{\mathscr{A}}(x)) \geq 0$ and $\sum\limits_{x\in X} r(E_{\mathscr{A}}(x)) \geq 0$.

Example 3.22. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 10 and 11, respectively.

TABLE 10. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.2, 0.3]$	0.2
b	$[0.7, 0.8]$	0.9
c	$[0.3, 0.7]$	0.6

Then

$$\mathbf{E}_{\mathscr{A}} = \{(a, \langle 0, 0.1\rangle),\ (b, \langle 0.2, -0.1\rangle),\ (c, \langle 0.3, 0.1\rangle)\}$$

and

TABLE 11. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.2, 0.3]$	0.9
b	$[0.6, 0.9]$	0.7
c	$[0.1, 0.9]$	1

$$\mathbf{E}_{\mathscr{B}} = \{(a, \langle 0.7, -0.6\rangle), (b, \langle 0.1, 0.2\rangle), (c, \langle 0.9, -0.1\rangle)\}.$$

Thus $SD_{\mathscr{A}} = (0 + 0.2 + 0.3,\ 0.1 - 0.1 + 0.1) = (0.5, 0.1)$ and so $\mathscr{A}$ is almost stable. But $\mathscr{B}$ is not almost stable since $SD_{\mathscr{B}} = (0.7 + 0.1 + 0.9,\ -0.6 + 0.2 - 0.1) = (1.7, -0.5)$.

Theorem 3.23. *Every stable cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in X is almost stable.*

Proof. Straightforward. □

In Example 3.22, the almost stable cubic set $\mathscr{A} = \langle A, \lambda\rangle$ is not stable. This shows that the converse of Theorem 3.23 is not true in general.

Combining Theorems 3.8, 3.10, 3.15, 3.19 and 3.23, we know that

(1) Every ICS is almost stable.
(2) Every ESC satisfying the condition (3.2) is almost stable.
(3) The P-union and P-intersection of two stable cubic sets is almost stable.
(4) If $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are ECSs in X such that $\mathscr{A}^* = \langle A, \mu\rangle$ and $\mathscr{B}^* = \langle B, \lambda\rangle$ are ICSs in X, then the P-union and the P-intersection of $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are almost stable in X.

Proposition 3.24. *If $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are cubic sets in X, then either*

$$(\forall x \in X)\left(\max\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\} \leq \lambda(x) - A(x)^-\right) \tag{3.6}$$

or

$$(\forall x \in X)\left(\max\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\} \leq \mu(x) - B(x)^-\right). \tag{3.7}$$

Proof. For each $x \in X$, we consider the four cases as follows:

(1) $\max\{\lambda(x), \mu(x)\} = \lambda(x)$ and $\max\{A(x)^-, B(x)^-\} = A(x)^-$.
(2) $\max\{\lambda(x), \mu(x)\} = \lambda(x)$ and $\max\{A(x)^-, B(x)^-\} = B(x)^-$.
(3) $\max\{\lambda(x), \mu(x)\} = \mu(x)$ and $\max\{A(x)^-, B(x)^-\} = A(x)^-$.
(4) $\max\{\lambda(x), \mu(x)\} = \mu(x)$ and $\max\{A(x)^-, B(x)^-\} = B(x)^-$.

First two cases induce the inequality (3.6), and the inequality (3.7) is induced by the last two cases. □

Proposition 3.25. *If $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are cubic sets in X, then either*

$$(\forall x \in X)\left(\max\{A(x)^+, B(x)^+\} - \max\{\lambda(x), \mu(x)\} \leq A(x)^+ - \lambda(x)\right) \tag{3.8}$$

or

$$(\forall x \in X)\left(\max\{A(x)^+, B(x)^+\} - \max\{\lambda(x), \mu(x)\} \leq B(x)^+ - \mu(x)\right). \tag{3.9}$$

Proof. It is similar to the proof of Proposition 3.24. □

In the following example, we know that the P-union and the R-union of almost stable cubic sets may not be almost stable.

Example 3.26. Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 12 and 13, respectively.

Table 12. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[1.0, 1.0]$	0.7
b	$[0.5, 1.0]$	0.7
c	$[0.6, 1.0]$	0.7

Table 13. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.5, 1.0]$	0.7
b	$[1.0, 1.0]$	0.7
c	$[0.6, 1.0]$	0.7

Then $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are almost stable cubic sets in X because

$$\sum_{x\in X} l(E_{\mathscr{A}}(x)) = 0,\ \sum_{x\in X} r(E_{\mathscr{A}}(x)) = 0.9,\ \sum_{x\in X} l(E_{\mathscr{B}}(x)) = 0, \text{ and } \sum_{x\in X} r(E_{\mathscr{B}}(x)) = 0.9.$$

But the P-union $\mathscr{A} \sqcup \mathscr{B}$ and the R-union $\mathscr{A} \Cup \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are not almost stable because $\sum_{x\in X} l(E_{\mathscr{A}\sqcup\mathscr{B}}(x)) = \sum_{x\in X} (\max\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\}) = -0.5 \ngeq 0$ and $\sum_{x\in X} l(E_{\mathscr{A}\Cup\mathscr{B}}(x)) = \sum_{x\in X} (\min\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\}) = -0.5 \ngeq 0$.

We now provide conditions for the P-union of almost stable cubic sets to be almost stable.

Theorem 3.27. *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be almost stable cubic sets in X such that*

$$(\forall x \in X)\left(\sum_{x\in X} (|\lambda(x) - \mu(x)| - A(x)^-) \geq 0, \sum_{x\in X} (|A(x)^+ - B(x)^+| - \lambda(x)) \geq 0\right). \tag{3.10}$$

Then the P-union $\mathscr{A} \sqcup \mathscr{B} = \langle A \cup B, \lambda \vee \mu\rangle$ of $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ is almost stable in X.

Proof. Assume that $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are almost stable in X. Then there exist stable degrees $SD_{\mathscr{A}}$ and $SD_{\mathscr{B}}$, respectively, such that

$$\sum_{x\in X} l(E_{\mathscr{A}}(x)) = \sum_{x\in X}(\lambda(x) - A(x)^-) \geq 0, \sum_{x\in X} r(E_{\mathscr{A}}(x)) = \sum_{x\in X}(A(x)^+ - \lambda(x)) \geq 0,$$

$$\sum_{x\in X} l(E_{\mathscr{B}}(x)) = \sum_{x\in X}(\mu(x) - B(x)^-) \geq 0, \text{ and } \sum_{x\in X} r(E_{\mathscr{B}}(x)) = \sum_{x\in X}(B(x)^+ - \mu(x)) \geq 0.$$

Now, we have to show that $\sum_{x\in X} l(E_{\mathscr{A}\sqcup\mathscr{B}}(x)) \geq 0$ and $\sum_{x\in X} r(E_{\mathscr{A}\sqcup\mathscr{B}}(x)) \geq 0$ in the stable degree $SD_{\mathscr{A}\sqcup\mathscr{B}}$ of $\mathscr{A} \sqcup \mathscr{B}$. Using (3.10), we have

$$\begin{aligned}
&\sum_{x\in X} l(E_{\mathscr{A}\sqcup\mathscr{B}}(x)) = \sum_{x\in X}\big((\lambda\vee\mu)(x) - (A\cup B)(x)^-\big)\\
&= \sum_{x\in X}\big(\max\{\lambda(x),\mu(x)\} - \max\{A(x)^-, B(x)^-\}\big)\\
&= \sum_{x\in X}\Big(\tfrac{|\lambda(x)-\mu(x)|+\lambda(x)+\mu(x)}{2} - \tfrac{|A(x)^- - B(x)^-|+A(x)^-+B(x)^-}{2}\Big)\\
&= \sum_{x\in X}\Big(\tfrac{|\lambda(x)-\mu(x)|-|A(x)^- - B(x)^-|+\lambda(x)-A(x)^-+\mu(x)-B(x)^-}{2}\Big)\\
&= \tfrac{1}{2}\sum_{x\in X}\big(|\lambda(x)-\mu(x)| - |A(x)^- - B(x)^-| + \lambda(x) - A(x)^- + \mu(x) - B(x)^-\big)\\
&= \tfrac{1}{2}\sum_{x\in X}\big(|\lambda(x)-\mu(x)| - |A(x)^- - B(x)^-|\big)\\
&\qquad\qquad + \tfrac{1}{2}\sum_{x\in X}\big(\lambda(x) - A(x)^-\big) + \tfrac{1}{2}\sum_{x\in X}\big(\mu(x) - B(x)^-\big)\\
&\geq \tfrac{1}{2}\sum_{x\in X}\big(|\lambda(x)-\mu(x)| - A(x)^-\big) + \tfrac{1}{2}\sum_{x\in X}\big(\lambda(x) - A(x)^-\big) + \tfrac{1}{2}\sum_{x\in X}\big(\mu(x) - B(x)^-\big)\\
&\geq 0.
\end{aligned}$$

Similarly, we have $\sum_{x\in X} r(E_{\mathscr{A}\sqcup\mathscr{B}}(x)) \geq 0$. Therefore $\mathscr{A} \sqcup \mathscr{B} = \langle A\cup B, \lambda\vee\mu\rangle$ is almost stable in X. □

Theorem 3.28. *The complement of an almost stable cubic set is also almost stable.*

Proof. Let $\mathscr{A} = \langle A, \lambda\rangle$ be an almost stable cubic set in X. Then there exists a stable degree $SD_{\mathscr{A}}$ such that

$$\sum_{x\in X} l(E_{\mathscr{A}}(x)) = \sum_{x\in X}(\lambda(x) - A(x)^-) \geq 0, \text{ and } \sum_{x\in X} r(E_{\mathscr{A}}(x)) = \sum_{x\in X}(A(x)^+ - \lambda(x)) \geq 0.$$

It follows that $\sum_{x\in X} l(E_{\mathscr{A}^c}(x)) = \sum_{x\in X} ((1-\lambda(x)) - (1-A(x)^+)) = \sum_{x\in X} (A(x)^+ - \lambda(x)) \geq 0$ and $\sum_{x\in X} r(E_{\mathscr{A}^c}(x)) = \sum_{x\in X} ((1-A(x)^-) - (1-\lambda(x))) = \sum_{x\in X} (\lambda(x) - A(x)^-) \geq 0$. Therefore $\mathscr{A}^c = \langle A^c, \lambda^c\rangle$ is almost stable. □

We now provide conditions for the R-union of almost stable cubic sets to be almost stable.

Theorem 3.29. *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be almost stable cubic sets in X such that*

$$\sum_{x\in X} (|\lambda(x)-\mu(x)| + |A(x)^- - B(x)^-|) \leq \left(\sum_{x\in X} \lambda(x) - A(x)^-\right) + \sum_{x\in X} (\mu(x) - B(x)^-) \tag{3.11}$$

and

$$\sum_{x\in X} (|\lambda(x)-\mu(x)| + |A(x)^+ - B(x)^+|) \geq \sum_{x\in X} (\lambda(x) - A(x)^+) + \sum_{x\in X} (\mu(x) - B(x)^+) \tag{3.12}$$

for all $x \in X$. Then the R-union $\mathscr{A} \Cup \mathscr{B} = \langle A\cup B, \lambda\wedge\mu\rangle$ is almost stable in X.

Proof. Assume that $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are almost stable in X. Then there exist stable degrees $SD_{\mathscr{A}}$ and $SD_{\mathscr{B}}$, respectively, such that

$$\sum_{x\in X} l(E_{\mathscr{A}}(x)) = \sum_{x\in X}(\lambda(x) - A(x)^-) \geq 0, \sum_{x\in X} r(E_{\mathscr{A}}(x)) = \sum_{x\in X}(A(x)^+ - \lambda(x)) \geq 0,$$
$$\sum_{x\in X} l(E_{\mathscr{B}}(x)) = \sum_{x\in X}(\mu(x) - B(x)^-) \geq 0, \text{ and } \sum_{x\in X} r(E_{\mathscr{B}}(x)) = \sum_{x\in X}(B(x)^+ - \mu(x)) \geq 0.$$

It follows from (3.11) that

$$\begin{aligned}
&\sum_{x\in X} l(E_{\mathscr{A}\Cup\mathscr{B}}(x)) = \sum_{x\in X} \big((\lambda\wedge\mu)(x) - (A\cup B)(x)^-\big)\\
&= \sum_{x\in X} \big(\min\{\lambda(x),\mu(x)\} - \max\{A(x)^-, B(x)^-\}\big)\\
&= \sum_{x\in X} \left(\tfrac{-|\lambda(x)-\mu(x)|+\lambda(x)+\mu(x)}{2} - \tfrac{|A(x)^- - B(x)^-|+A(x)^- + B(x)^-}{2}\right)\\
&= \sum_{x\in X} \left(\tfrac{-|\lambda(x)-\mu(x)|-|A(x)^- - B(x)^-|+\lambda(x)-A(x)^- + \mu(x)-B(x)^-}{2}\right)\\
&= -\tfrac{1}{2}\sum_{x\in X} \big(|\lambda(x)-\mu(x)| + |A(x)^- - B(x)^-|\big)\\
&\qquad + \tfrac{1}{2}\sum_{x\in X}\big(\lambda(x) - A(x)^-\big) + \tfrac{1}{2}\sum_{x\in X}\big(\mu(x) - B(x)^-\big)\\
&\geq -\tfrac{1}{2}\left(\sum_{x\in X}\big(\lambda(x) - A(x)^-\big) + \sum_{x\in X}\big(\mu(x) - B(x)^-\big)\right)\\
&+ \tfrac{1}{2}\sum_{x\in X}\big(\lambda(x) - A(x)^-\big) + \tfrac{1}{2}\sum_{x\in X}\big(\mu(x) - B(x)^-\big) = 0.
\end{aligned}$$

Using (3.12), we have

$$\begin{aligned}
\sum_{x\in X} r(E_{\mathscr{A}\Cup\mathscr{B}}(x)) &= \sum_{x\in X}\big((A\cup B)(x)^+ - (\lambda\wedge\mu)(x)\big)\\
&= \sum_{x\in X}\big(\max\{A(x)^-, B(x)^-\} - \min\{\lambda(x),\mu(x)\}\big)\\
&= \sum_{x\in X}\Big(\tfrac{|A(x)^+ - B(x)^+| + A(x)^+ + B(x)^+}{2} - \tfrac{-|\lambda(x)-\mu(x)|+\lambda(x)+\mu(x)}{2}\Big)\\
&= \tfrac{1}{2}\sum_{x\in X}\big(|\lambda(x)-\mu(x)| + |A(x)^+ - B(x)^+|\big)\\
&- \tfrac{1}{2}\left(\sum_{x\in X}\big(\lambda(x) - A(x)^+\big) + \sum_{x\in X}\big(\mu(x) - B(x)^+\big)\right) \geq 0.
\end{aligned}$$

Hence $\mathscr{A}\Cup\mathscr{B} = \langle A\cup B, \lambda\wedge\mu\rangle$ is almost stable in X. □

The following examples show that the P-intersection and the R-intersection of almost stable cubic sets may not be almost stable.

Example 3.30. (1) Let $\mathscr{A} = \langle A,\lambda\rangle$ and $\mathscr{B} = \langle B,\mu\rangle$ be cubic sets in $X = \{a,b,c\}$ defined by Tables 14 and 15, respectively.

TABLE 14. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.7, 1.0]$	0.4
b	$[0.5, 1.0]$	0.8
c	$[0.6, 1.0]$	0.7

TABLE 15. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.5, 1.0]$	0.8
b	$[0.6, 1.0]$	0.7
c	$[0.7, 1.0]$	0.4

Then $\mathscr{A} = \langle A,\lambda\rangle$ and $\mathscr{B} = \langle B,\mu\rangle$ are almost stable cubic sets in X because

$$\sum_{x\in X} l(E_{\mathscr{A}}(x)) = 0.1,\ \sum_{x\in X} r(E_{\mathscr{A}}(x)) = 1.1,\ \sum_{x\in X} l(E_{\mathscr{B}}(x)) = 0.1, \text{ and } \sum_{x\in X} r(E_{\mathscr{B}}(x)) = 1.1.$$

But the P-intersection $\mathscr{A} \sqcap \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ is not almost stable because

$$\sum_{x\in X} l(E_{\mathscr{A}\sqcap\mathscr{B}}(x)) = \sum_{x\in X} \big(\min\{\lambda(x), \mu(x)\} - \min\{A(x)^-, B(x)^-\}\big) = -0.1 \ngeq 0.$$

(2) Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 16 and 17, respectively.

Table 16. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.2, 0.7]$	0.8
b	$[0.3, 0.6]$	0.5
c	$[0.1, 0.5]$	0.5

Table 17. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.2, 0.7]$	0.6
b	$[0.3, 0.6]$	0.7
c	$[0.1, 0.5]$	0.5

Then $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ are almost stable cubic sets in X because

$$\sum_{x\in X} l(E_{\mathscr{A}}(x)) = 1.2,\ \sum_{x\in X} r(E_{\mathscr{A}}(x)) = 0,\ \sum_{x\in X} l(E_{\mathscr{B}}(x)) = 1.2, \text{ and } \sum_{x\in X} r(E_{\mathscr{B}}(x)) = 0.$$

But the R-intersection $\mathscr{A} \Cap \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ is not almost stable since

$$\sum_{x\in X} r(E_{\mathscr{A}\Cap\mathscr{B}}(x)) = \sum_{x\in X} \big(\min\{A(x)^+, B(x)^+\} - \max\{\lambda(x), \mu(x)\}\big) = -0.2 \ngeq 0.$$

We now provide conditions for the P-intersection and the R-intersection of almost stable cubic sets to be almost stable.

Theorem 3.31. *Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be almost stable cubic sets in X.*
(i) *Assume that the following condition is valid.*

$$(\forall x \in X) \left(\begin{array}{l} \sum\limits_{x\in X} (|A(x)^- - B(x)^-| - |\lambda(x) - \mu(x)|) \geq 0, \\ \sum\limits_{x\in X} (|\lambda(x) - \mu(x)| - |A(x)^+ - B(x)^+|) \geq 0 \end{array} \right). \tag{3.13}$$

Then the P-intersection $\mathscr{A} \sqcap \mathscr{B} = \langle A \cap B, \lambda \wedge \mu\rangle$ of $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ is almost stable in X.

(ii) *If* $\mathscr{A} = \langle A, \lambda \rangle$ *and* $\mathscr{B} = \langle B, \mu \rangle$ *satisfy the following condition*

$$(\forall x \in X) \left(\sum_{x \in X} \left(|\lambda(x) - \mu(x)| + |A(x)^+ - B(x)^+| \right) = 0 \right), \tag{3.14}$$

then the R-intersection $\mathscr{A} \Cap \mathscr{B} = \langle A \cap B, \lambda \vee \mu \rangle$ *of* $\mathscr{A} = \langle A, \lambda \rangle$ *and* $\mathscr{B} = \langle B, \mu \rangle$ *is almost stable in* X.

Proof. Since $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are almost stable in X, there exist stable degrees $SD_{\mathscr{A}}$ and $SD_{\mathscr{B}}$, respectively, such that

$$\begin{gathered}
\sum_{x \in X} l(E_{\mathscr{A}}(x)) = \sum_{x \in X} (\lambda(x) - A(x)^-) \geq 0, \sum_{x \in X} r(E_{\mathscr{A}}(x)) = \sum_{x \in X} (A(x)^+ - \lambda(x)) \geq 0, \\
\sum_{x \in X} l(E_{\mathscr{B}}(x)) = \sum_{x \in X} (\mu(x) - B(x)^-) \geq 0, \text{and} \sum_{x \in X} r(E_{\mathscr{B}}(x)) = \sum_{x \in X} (B(x)^+ - \mu(x)) \geq 0.
\end{gathered}$$

(i) We have to show that $\sum_{x \in X} l(E_{\mathscr{A} \sqcap \mathscr{B}}(x)) \geq 0$ and $\sum_{x \in X} r(E_{\mathscr{A} \sqcap \mathscr{B}}(x)) \geq 0$ in the stable degree $SD_{\mathscr{A} \sqcap \mathscr{B}}$ of $\mathscr{A} \sqcap \mathscr{B}$. Using (3.13), we have

$$\begin{aligned}
&\sum_{x \in X} l(E_{\mathscr{A} \sqcap \mathscr{B}}(x)) = \sum_{x \in X} \left((\lambda \wedge \mu)(x) - (A \cap B)(x)^- \right) \\
&= \sum_{x \in X} \left(\min\{\lambda(x), \mu(x)\} - \min\{A(x)^-, B(x)^-\} \right) \\
&= \sum_{x \in X} \left(\tfrac{-|\lambda(x) - \mu(x)| + \lambda(x) + \mu(x)}{2} + \tfrac{|A(x)^- - B(x)^-| - A(x)^- - B(x)^-}{2} \right) \\
&= \sum_{x \in X} \left(\tfrac{-|\lambda(x) - \mu(x)| + |A(x)^- - B(x)^-| + \lambda(x) - A(x)^- + \mu(x) - B(x)^-}{2} \right) \\
&= \tfrac{1}{2} \sum_{x \in X} \left(-|\lambda(x) - \mu(x)| + |A(x)^- - B(x)^-| + \lambda(x) - A(x)^- + \mu(x) - B(x)^- \right) \\
&= \tfrac{1}{2} \sum_{x \in X} \left(|A(x)^- - B(x)^-| - |\lambda(x) - \mu(x)| \right) \\
&\qquad + \tfrac{1}{2} \sum_{x \in X} \left((\lambda(x) - (A(x)^- \right) + \tfrac{1}{2} \sum_{x \in X} \left((\mu(x)) - B(x)^-) \right) \geq 0.
\end{aligned}$$

Similarly, we have $\sum_{x \in X} r(E_{\mathscr{A} \sqcap \mathscr{B}}(x)) \geq 0$. Therefore $\mathscr{A} \sqcap \mathscr{B} = \langle A \cap B, \lambda \wedge \mu \rangle$ is almost stable in X.

(ii) We have

$$
\begin{aligned}
\sum_{x\in X} l(E_{\mathscr{A}\Cap\mathscr{B}}(x)) &= \sum_{x\in X}\big((\lambda\vee\mu)(x)-(A\cap B)(x)^-\big)\\
&= \sum_{x\in X}\big(\max\{\lambda(x),\mu(x)\}-\min\{A(x)^-,B(x)^-\}\big)\\
&= \sum_{x\in X}\left(\tfrac{|\lambda(x)-\mu(x)|+\lambda(x)+\mu(x)}{2}+\tfrac{|A(x)^--B(x)^-|-A(x)^--B(x)^-}{2}\right)\\
&= \sum_{x\in X}\left(\tfrac{|\lambda(x)-\mu(x)|+|A(x)^--B(x)^-|+\lambda(x)-A(x)^-+\mu(x)-B(x)^-}{2}\right)\\
&= \tfrac{1}{2}\sum_{x\in X}\big(|\lambda(x)-\mu(x)|+|A(x)^--B(x)^-|\big)\\
&\qquad + \tfrac{1}{2}\sum_{x\in X}\big(\lambda(x)-A(x)^-\big)+\tfrac{1}{2}\sum_{x\in X}\big(\mu(x)-B(x)^-\big)\\
&\geq \tfrac{1}{2}\left(\sum_{x\in X}\big(\lambda(x)-A(x)^-\big)+\sum_{x\in X}\big(\mu(x)-B(x)^-\big)\right)\geq 0.
\end{aligned}
$$

Using (3.14), we have

$$
\begin{aligned}
\sum_{x\in X} r(E_{\mathscr{A}\Cap\mathscr{B}}(x)) &= \sum_{x\in X}\big((A\cap B)(x)^+-(\lambda\vee\mu)(x)\big)\\
&= \sum_{x\in X}\big(\min\{A(x)^+,B(x)^+\}-\max\{\lambda(x),\mu(x)\}\big)\\
&= \sum_{x\in X}\left(\tfrac{-|A(x)^+-B(x)^+|+A(x)^++B(x)^+}{2}-\tfrac{|\lambda(x)-\mu(x)|+\lambda(x)+\mu(x)}{2}\right)\\
&= \tfrac{1}{2}\sum_{x\in X}\big(-|\lambda(x)-\mu(x)|-|A(x)^+-B(x)^+|\big)\\
&\qquad + \tfrac{1}{2}\left(\sum_{x\in X}\big(A(x)^+-\lambda(x)\big)+\sum_{x\in X}\big(B(x)^+-\mu(x)\big)\right)\\
&= \tfrac{1}{2}\left(\sum_{x\in X}\big(A(x)^+-\lambda(x)\big)+\sum_{x\in X}\big(B(x)^+-\mu(x)\big)\right)\geq 0.
\end{aligned}
$$

Hence $\mathscr{A}\Cap\mathscr{B}=\langle A\cap B,\lambda\vee\mu\rangle$ is almost stable in X. □

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SOME IDENTITIES OF CHEBYSHEV POLYNOMIALS ARISING FROM NON-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate some properties of Chebyshev polynomials arising from non-linear differential equations. From our investigation, we derive some new and interesting identities on Chebyshev polynomials.

1. Introduction

As is well known, the Chebyshev polynomials of the first kind, $T_n(x)$, $(n \geq 0)$, are defined by the generating function

$$\frac{1-t^2}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}, \quad (\text{see [1, 3, 5, 8, 17, 21]}). \tag{1.1}$$

The higher-order Chebyshev polynomials are given by the generating function

$$\left(\frac{1-t^2}{1-2xt+t^2}\right)^{\alpha} = \sum_{n=0}^{\infty} T_n^{(\alpha)}(x) t^n, \tag{1.2}$$

and Chebyshev polynomials of the second kind are denoted by U_n and given by generating function

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x) t^n, \quad (\text{see [1, 7, 12, 17]}). \tag{1.3}$$

The higher-order Chebyshev polynomials of the second kind are also defined by

$$\left(\frac{1}{1-2xt+t^2}\right)^{\alpha} = \sum_{n=0}^{\infty} U_n^{(\alpha)}(x) t^n. \tag{1.4}$$

The Chebyshev polynomials of the third kind are defined by the generating function

$$\frac{1-t}{1-2xt+t^2} = \sum_{n=0}^{\infty} V_n(x) t^n, \quad (\text{see [1, 7, 8, 17]}). \tag{1.5}$$

and the higher-order Chebyshev polynomials of the third kind are also given by the generating function

$$\left(\frac{1-t}{1-2xt+t^2}\right)^{\alpha} = \sum_{n=0}^{\infty} V_n^{(\alpha)}(x) t^n. \tag{1.6}$$

2010 *Mathematics Subject Classification.* 05A19, 33C45, 34A34.
Key words and phrases. Chebyshev polynomials of the first kind, Chebyshev polynomials of the second kind, Chebyshev polynomials of the third kind, Chebyshev polynomials of the fourth kind, non-linear differential equation.

Finally, we introduce the Chebyshev polynomials of the fourth kind defined by the generating function

$$\frac{1+t}{1-2xt+t^2}=\sum_{n=0}^{\infty}W_n\left(x\right)t^n. \tag{1.7}$$

The higher-order Chebyshev polynomials of the fourth kind are defined by

$$\left(\frac{1+t}{1-2xt+t^2}\right)^{\alpha}=\sum_{n=0}^{\infty}W_n^{(\alpha)}\left(x\right)t^n. \tag{1.8}$$

It is well known that the Legendre polynomials are defined by the generating function

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}p_n\left(x\right)t^n,\quad \left(\text{see [2, 20]}\right). \tag{1.9}$$

Chebyshev polynomials are important in approximation theory because the roots of the Chebyshev polynomials of the first kind, which are also called Chebyshev nodes, are used as nodes in polynomial nodes (see [19]).

The Chebyshev polynomials of the first kind and of the second kind are solutions of the following Chebyshev differential equations

$$\left(1-x^2\right)y''-xy'+n^2y=0, \tag{1.10}$$

and

$$\left(1-x^2\right)y''-3xy'+n\left(n+2\right)y=0. \tag{1.11}$$

These equations are special cases of the Strum-Liouville differential equation (see [1–3]).

The Chebyshev polynomials of the first kind can be defined by the contour integral

$$T_n\left(z\right)=\frac{1}{4\pi i}\oint\frac{\left(1-t^2\right)}{1-2tz+t^2}t^{-n-1}dt, \tag{1.12}$$

where the contour encloses the origin and is traversed in a counterclockwise direction (see [1, 19, 21]). The formula for $T_n\left(x\right)$ is given by

$$T_n\left(x\right)=\sum_{m=0}^{\left[\frac{n}{2}\right]}\binom{n}{2m}x^{n-2m}\left(x^2-1\right)^m. \tag{1.13}$$

From (1.3), we note that

$$2\left(x-t\right)\left(1-2xt+t^2\right)^{-2}=\sum_{n=0}^{\infty}nU_n\left(x\right)t^{n-1}. \tag{1.14}$$

Thus, by (1.14), we get

$$\left(2xt-2t^2\right)\left(1-2xt+t^2\right)^{-2}=\sum_{n=0}^{\infty}nU_n\left(x\right)t^n. \tag{1.15}$$

From (1.3) and (1.15), we can derive the following equation:

$$\frac{\left(2xt-2t^2\right)+\left(1-2xt+t^2\right)}{\left(1-2xt+t^2\right)^2}=\frac{1-t^2}{\left(1-2xt+t^2\right)^2} \tag{1.16}$$

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$$= \sum_{n=0}^{\infty} (n+1) U_n(x) t^n.$$

Note that

$$\begin{aligned} &\frac{1-t^2}{(1-2xt+t^2)^2} \\ &= \left(\frac{1-t^2}{1-2xt+t^2}\right)\left(\frac{1}{1-2xt+t^2}\right) \\ &= \left(\sum_{l=0}^{\infty} T_l(x) t^l\right)\left(\sum_{m=0}^{\infty} U_m(x) t^m\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} T_l(x) U_{n-l}(x)\right) t^n. \end{aligned} \tag{1.17}$$

From (1.16) and (1.17), we have

$$U_n(x) = \frac{1}{n+1}\sum_{l=0}^{n} T_l(x) U_{n-l}(x).$$

The Chebyshev polynomials have been studied by many authors in the several areas (see [1–21]).

In [11], Kim-Kim studied non-linear differential equations arising from Changhee polynomials and numbers related to Chebyshev poynomials.

In this paper, we study non-linear differential equations arising from Chebyshev polynomials and give some new and explicit formulas for those polynomials.

2. Differential equations arising from Chebyshev polynomials and their applications

Let

$$F = F(t,x) = \frac{1}{1-2tx+t^2}. \tag{2.1}$$

Then, by (1.1), we get

$$F^{(1)} = \frac{d}{dt}F(t,x) = 2(x-t)F^2. \tag{2.2}$$

From (2.2), we note that

$$2F^2 = (x-t)^{-1} F^{(1)}. \tag{2.3}$$

By using (2.3) and (2.2), we obtain the following equations:

$$2^2 \cdot 2F^3 = (x-t)^{-3} F^{(1)} + (x-t)^{-2} F^{(2)}, \tag{2.4}$$

$$2^3 \cdot 2 \cdot 3F^4 = 3(x-t)^{-5} F^{(1)} + 3(x-t)^{-4} F^{(2)} + (x-t)^{-3} F^{(3)} \tag{2.5}$$

and

$$\begin{aligned} 2^4 \cdot 2 \cdot 3 \cdot 4F^5 &= 3 \cdot 5(x-t)^{-6} F^{(1)} + 3 \cdot 5(x-t)^{-6} F^{(2)} \\ &\quad + (3 \cdot 2)(x-t)^{-5} F^{(3)} + (x-t)^{-4} F^{(4)}, \end{aligned} \tag{2.6}$$

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where

$$F^N = \underbrace{F \times \cdots \times F}_{N-\text{times}} \quad \text{and} \quad F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x).$$

Continuing this process, we set

$$2^N N! F^{N+1} = \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N} F^{(i)}, \tag{2.7}$$

where $N \in \mathbb{N}$.

From (2.7), we note that

$$\begin{aligned} &2^N N! F^N (N+1) F^{(1)} \\ &= \sum_{i=1}^{N} a_i(N)(2N-i)(x-t)^{i-2N-1} F^{(i)} + \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N} F^{(i+1)}. \end{aligned} \tag{2.8}$$

By (2.2) and (2.8), we get

$$\begin{aligned} &2^N N! (N+1) F^N \left(2(x-t)F^2\right) \\ &= \sum_{i=1}^{N} a_i(N)(2N-i)(x-t)^{i-2N-1} F^{(i)} \\ &+ \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N} F^{(i+1)}. \end{aligned} \tag{2.9}$$

Thus, from (2.9), we have

$$\begin{aligned} &2^{N+1}(N+1)! F^{N+2} \\ &= \sum_{i=1}^{N} a_i(N)(2N-i)(x-t)^{i-2(N+1)} F^{(i)} \\ &+ \sum_{i=2}^{N+1} a_{i-1}(N)(x-t)^{i-2(N+1)} F^{(i)}. \end{aligned} \tag{2.10}$$

On the other hand, by replacing N by $N+1$, in (2.7), we get

$$2^{N+1}(N+1)! F^{N+2} = \sum_{i=1}^{N+1} a_i(N+1)(x-t)^{i-2(N+1)} F^{(i)}. \tag{2.11}$$

Comparing the coefficients on both sides of (2.10) and (2.11), we have

$$a_1(N+1) = (2N-1) a_1(N), \tag{2.12}$$

$$a_{N+1}(N+1) = a_N(N), \tag{2.13}$$

and

$$a_i(N+1) = a_{i-1}(N) + (2N-i) a_i(N), \quad (2 \le i \le N). \tag{2.14}$$

Moreover, by (2.4) and (2.7), we get

$$2F^2 = (x-t)^{-1} F^{(1)} = a_1(1)(x-t)^{-1} F^{(1)}. \tag{2.15}$$

By comparing the coefficients on both sides of (2.15), we get

$$a_1(1) = 1. \tag{2.16}$$

Now, by (2.12) and (2.16), we have

$$\begin{aligned} a_1(N+1) &= (2N-1)\,a_1(N) \\ &= (2N-1)(2N-3)\,a_1(N-1) \\ &= (2N-1)(2N-3)(2N-5)\,a_1(N-2) \\ &\vdots \\ &= (2N-1)(2N-3)(2N-5)\cdots 1\cdot a_1(1) \\ &= (2N-1)!!, \end{aligned} \tag{2.17}$$

where $(2N-1)!!$ is Arfken's double factorial.

From (2.13), we easily note that

$$a_{N+1}(N+1) = a_N(N) = \cdots = a_1(1) = 1. \tag{2.18}$$

For $2 \le i \le N$, from (2.14), we can derive the following equation:

(2.19)

$$\begin{aligned} a_i(N+1) &= a_{i-1}(N) + (2N-i)\,a_i(N) \\ &= a_{i-1}(N) + (2N-i)\,a_{i-1}(N-1) + (2N-i)(2N-2-i)\,a_i(N-1) \\ &\vdots \\ &= \sum_{k=0}^{N-i}\left(\prod_{l=0}^{k-1}(2(N-l)-i)\right)a_{i-1}(N-k) + \prod_{l=0}^{N-i}(2(N-l)-i)\,a_i(i) \\ &= \sum_{k=0}^{N-i} 2^k\left(N-\frac{i}{2}\right)_k a_{i-1}(N-k) + 2^{N-i+1}\left(N-\frac{i}{2}\right)_{N-i+1} \\ &= \sum_{k=0}^{N-i+1} 2^k\left(N-\frac{i}{2}\right)_k a_{i-1}(N-k), \end{aligned}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \ge 1)$ and $(x)_0 = 1$.

As the above is also valid for $i = N+1$, by (2.19), we get

$$a_i(N+1) = \sum_{k=0}^{N+1-i} 2^k\left(N-\frac{i}{2}\right)_k a_{i-1}(N-k), \tag{2.20}$$

where $2 \le i \le N+1$.

Now, we give an explicit expression for $a_i(N+1)$.

From (2.17) and (2.20), we can derive the following equations:

$$\begin{aligned} a_2(N+1) &= \sum_{k_1=0}^{N-1} 2^{k_1}\left(N-\frac{2}{2}\right)_{k_1} a_1(N-k_1) \\ &= \sum_{k_1=0}^{N-1} 2^{k_1}\left(N-\frac{2}{2}\right)_{k_1} (2(N-k_1-1)-1)!!, \end{aligned} \tag{2.21}$$

(2.22)

$$\begin{aligned} a_3(N+1) &= \sum_{k_2=0}^{N-2} 2^{k_2}\left(N-\frac{3}{2}\right)_{k_2} a_2(N-k_2) \\ &= \sum_{k_2=0}^{N-2}\sum_{k_1=0}^{N-2-k_2} 2^{k_1+k_2}\left(N-\frac{3}{2}\right)_{k_2}\left(N-k_2-\frac{4}{2}\right)_{k_1}(2(N-2-k_1-k_2)-1)!!, \end{aligned}$$

and

(2.23)

$$\begin{aligned} a_4(N+1) &= \sum_{k_3=0}^{N-3} 2^{k_3}\left(N-\frac{4}{2}\right)_{k_3} a_3(N-k_3) \\ &= \sum_{k_3=0}^{N-3}\sum_{k_2=0}^{N-3-k_3}\sum_{k_1=0}^{N-3-k_3-k_2} 2^{k_1+k_2+k_3}\left(N-\frac{4}{2}\right)_{k_3}\left(N-k_3-\frac{5}{2}\right)_{k_2}\left(N-k_3-k_2-\frac{6}{2}\right)_{k_1} \\ &\quad\times(2(N-3-k_1-k_2-k_3)-1)!!. \end{aligned}$$

Thus, we see that, for $2 \le i \le N+1$,

(2.24)

$$\begin{aligned} a_i(N+1) = &\sum_{k_{i-1}=0}^{N-i+1}\sum_{k_{i-2}=0}^{N-i+1-k_{i-1}}\cdots\sum_{k_1=0}^{N-i+1-k_{i-1}-\cdots-k_2} 2^{\sum_{j=1}^{i-1}k_j} \\ &\times\prod_{j=2}^{i}\left(N-\sum_{l=j}^{i-1}k_l-\frac{2i-j}{2}\right)_{k_{j-1}}\left(2\left(N-i+1-\sum_{j=1}^{i-1}k_j\right)-1\right)!!. \end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 1. *The nonlinear differential equations*

$$2^N N! F^{N+1} = \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N} F^{(i)}, \quad (N\in\mathbb{N})$$

has a solution $F = F(t,x) = \frac{1}{1-2tx+t^2}$, *where*

$a_1(N) = (2N-3)!!$,

$$\begin{aligned} a_i(N) = &\sum_{k_{i-1}=0}^{N-i}\sum_{k_{i-2}=0}^{N-i-k_{i-1}}\cdots\sum_{k_1=0}^{N-i-k_{i-1}-\cdots-k_2} 2^{\sum_{j=1}^{i-1}k_j} \\ &\times\prod_{j=2}^{i}\left(N-\sum_{l=j}^{i-1}k_l-\frac{2i+2-j}{2}\right)_{k_{j-1}}\left(2\left(N-i-\sum_{j=1}^{i-1}k_j\right)-1\right)!! \end{aligned}$$

$(2 \le i \le N)$.

From (1.3) and (1.9), we note that

$$\begin{aligned} &\sum_{n=0}^{\infty} U_n(x)t^n \\ &= \frac{1}{1-2xt+t^2} \end{aligned} \tag{2.25}$$

$$= \left(\frac{1}{\sqrt{1-2xt+t^2}}\right)^2$$
$$= \left(\sum_{l=0}^{\infty} p_l(x) t^l\right)\left(\sum_{m=0}^{\infty} p_m(x) t^m\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} p_l(x) p_{n-l}(x)\right) t^n.$$

Thus, from (2.25), we have

$$U_n(x) = \sum_{l=0}^{n} p_l(x) p_{n-l}(x).$$

From (1.4), we obtain

$$2^N N! F^{N+1} = 2^N N! \sum_{n=0}^{\infty} U_n^{(N+1)}(x) t^n. \tag{2.26}$$

On the other hand, by Theorem 1, we get

(2.27)

$$2^N N! F^{N+1} = \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N} F^{(i)}$$
$$= \sum_{i=1}^{N} a_i(N)\left(\sum_{m=0}^{\infty}\binom{2N+m-i-1}{m} x^{i-2N-m} t^m\right)\left(\sum_{l=0}^{\infty} U_{i+l}(x)(l+i)_i t^l\right)$$
$$= \sum_{i=1}^{N} a_i(N)\sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\binom{2N+n-l-i-1}{n-l} x^{i-2N-n+l} U_{l+i}(x)(l+i)_i\right\} t^n$$
$$= \sum_{n=0}^{\infty}\left\{\sum_{i=1}^{N} a_i(N)\sum_{l=0}^{n}\binom{2N+n-l-i-1}{n-l} x^{i+l-2N-n} U_{i+l}(x)(l+i)_i\right\} t^n.$$

Comparing the coefficients on the both sides of (2.26) and (2.27), we obtain the following theorem.

Theorem 2. *For $N \in \mathbb{N}$, and $n \in \mathbb{N} \cup \{0\}$, the following identity holds.*

$$U_n^{(N+1)}(x) = \frac{1}{2^N N!}\sum_{i=1}^{N} a_i(N)\sum_{l=0}^{n}\binom{2N+n-l-i-1}{n-l} U_{l+i}(x) x^{i+l-2N-n}(l+i)_i.$$

The higher-order Legendre polynomials are given by the generating function

$$\left(\frac{1}{\sqrt{1-2xt+t^2}}\right)^{\alpha} = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x) t^n. \tag{2.28}$$

Thus, by 1.4 and (2.27), we get

$$\sum_{n=0}^{\infty} U_n^{(\alpha)}(x) t^n \tag{2.29}$$
$$= \left(\frac{1}{1-2xt+t^2}\right)^{\alpha}$$

$$
\begin{aligned}
&= \left(\frac{1}{\sqrt{1-2xt+t^2}}\right)^{2\alpha} \\
&= \left(\sum_{l=0}^{\infty} p_l^{(\alpha)}(x)\, t^l\right)\left(\sum_{m=0}^{\infty} p_m^{(\alpha)}(x)\, t^m\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} p_l^{(\alpha)}(x)\, p_{n-l}^{(\alpha)}(x)\right) t^n.
\end{aligned}
$$

From (2.29), we note that

$$
U_n^{(\alpha)}(x) = \sum_{l=0}^{n} p_l^{(\alpha)}(x)\, p_{n-l}^{(\alpha)}(x)\,. \tag{2.30}
$$

Therefore, we obtian the following corollaries.

Corollary 3. *For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n} p_l^{(N+1)} p_{n-l}^{(N+1)}(x) \\
&= \frac{1}{2^N N!}\sum_{i=1}^{N} a_i(N)\sum_{l=0}^{n}\binom{2N+n-l-i-1}{n-l} U_{l+i}(x)\,(l+i)_i\, x^{i+l-2N-n}.
\end{aligned}
$$

Corollary 4. *For $N \in \mathbb{N}$ and $n \in \mathbb{N}$, we have*

$$
\begin{aligned}
&U_n^{(N+1)}(x) \\
&= \frac{1}{2^N N!}\sum_{i=1}^{N} a_i(N)\sum_{l=0}^{n}\sum_{j=0}^{l+i}\binom{2N+n-l-i-1}{n-l} x^{i+l-2N-n}\,(l+i)_i\,(x)\, p_{l+i-j}(x)\,.
\end{aligned}
$$

By (1.6), we get

$$
\begin{aligned}
&2^N N! F^{N+1} \\
&= 2^N N!\,(1-t)^{-N-1}\left(\frac{1-t}{1-2xt+t^2}\right)^{N+1} \\
&= 2^N N!\left(\sum_{m=0}^{\infty}\binom{N+m}{m} t^m\right)\left(\sum_{l=0}^{\infty} V_l^{(N+1)}(x)\, t^l\right) \\
&= 2^N N!\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{N+n-l}{n-l} V_l^{(N+1)}(x)\right) t^n.
\end{aligned} \tag{2.31}
$$

On the other hand, by Theorem 1, we have

$$
\begin{aligned}
2^N N! F^{N+1} &= \sum_{i=1}^{N} a_i(N)\,(x-t)^{i-2N} F^{(i)} \\
&= \sum_{i=1}^{N} a_i(N)\,(x-t)^{i-2N}\left(\frac{d}{dt}\right)^i\left(\frac{1}{1-t}\cdot\frac{1-t}{1-xt+t^2}\right).
\end{aligned} \tag{2.32}
$$

From Leibniz formula, we note that

$$
\left(\frac{d}{dt}\right)^i\left(\frac{1-t}{1-2xt+t^2}\cdot\frac{1}{1-t}\right) \tag{2.33}
$$

$$=\sum_{l=0}^{i}\binom{i}{l}\left(\left(\frac{d}{dt}\right)^{i-l}\frac{1}{1-t}\right)\left(\left(\frac{d}{dt}\right)^{l}\frac{1-t}{1-2xt+t^2}\right)$$
$$=\sum_{l=0}^{i}\binom{i}{l}(i-l)!\,(1-t)^{-i+l-1}\left(\frac{d}{dt}\right)^{l}\left(\frac{1-t}{1-2xt+t^2}\right)$$
$$=\sum_{l=0}^{i}\binom{i}{l}(i-l)!\sum_{s=0}^{\infty}\binom{i-l+s}{s}t^s\sum_{p=0}^{\infty}V_{p+l}(x)\,(p+l)_l\,t^p$$
$$=\sum_{l=0}^{i}\frac{i!}{l!}\sum_{s=0}^{\infty}\binom{i-l+s}{s}t^s\sum_{p=0}^{\infty}V_{p+l}(x)\,(p+l)_l\,t^p.$$

By (2.32) and (2.33), we get

$$2^N N!F^{N+1} \tag{2.34}$$
$$=\sum_{n=0}^{\infty}\left\{\sum_{i=1}^{N}\sum_{l=0}^{i}a_i(N)\frac{i!}{l!}\sum_{m+s+p=n}\binom{2N+m-i-1}{m}\binom{i-l+s}{s}\right.$$
$$\left.\times(p+l)_l\,x^{i-2N-m}V_{p+l}(x)\right\}t^n.$$

Therefore, by (2.31) and (2.34), we obtain the following theorem.

Theorem 5. *For $N\in\mathbb{N}$ and $n\in\mathbb{N}\cup\{0\}$, we have the following identity:*

$$\sum_{l=0}^{n}\binom{N+n-l}{n-l}V_l^{(N+1)}(x)$$
$$=\frac{1}{2^N N!}\sum_{i=1}^{N}\sum_{l=0}^{i}a_i(N)\frac{i!}{l!}\sum_{m+s+p=n}\binom{2N+m-i-1}{m}\binom{i-l+s}{s}(p+l)_l$$
$$\times x^{i-2N-m}V_{p+l}(x).$$

From (1.8), we note that

$$2^N N!F^{N+1} \tag{2.35}$$
$$=2^N N!\,(1+t)^{-N-1}\left(\frac{1+t}{1-2xt+t^2}\right)^{N+1}$$
$$=2^N N!\left(\sum_{m=0}^{\infty}\binom{N+m}{m}(-1)^m t^m\right)\left(\sum_{l=0}^{\infty}W_l^{(N+1)}(x)\,t^l\right)$$
$$=2^N N!\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}(-1)^{n-l}\binom{N+n-l}{n-l}W_l^{(N+1)}(x)\right)t^n.$$

On the other hand, by Theorem 1, we get

$$2^N N!F^{N+1}=\sum_{i=1}^{N}a_i(N)\,(x-t)^{i-2N}\left(\frac{d}{dt}\right)^{i}\left\{\frac{1}{1+t}\cdot\frac{1+t}{1-2xt+t^2}\right\}. \tag{2.36}$$

Now, we observe that

$$\left(\frac{d}{dt}\right)^{i}\left\{\left(\frac{1}{1+t}\right)\left(\frac{1+t}{1-2xt+t^2}\right)\right\} \tag{2.37}$$

$$= \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} (i-l)! \left(\frac{1}{1+t} \right)^{i-l+1} \left(\frac{d}{dt} \right)^{l} \left(\frac{1+t}{1-2xt+t^2} \right)$$
$$= \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} (i-l)! \sum_{s=0}^{\infty} \binom{i-l+s}{s} (-1)^s t^s \sum_{p=0}^{\infty} W_{p+l}(x) (p+l)_l t^p.$$

From (2.36) and (2.37), we have

$$\begin{aligned}(2.38)\qquad & 2^N N! F^{N+1} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{i} (-1)^{i-l} \frac{i!}{l!} \sum_{m+s+p=n} (-1)^s \binom{2N+m-i-1}{m} \right. \\ &\quad \left. \times \binom{i-l+s}{s} (p+l)_l x^{i-2N-m} W_{p+l}(x) \right\} t^n.\end{aligned}$$

Therefore, by (2.35) and (2.38), we obtain the following theorem.

Theorem 6. *For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, the following identity is valid:*

$$\begin{aligned}&\sum_{l=0}^{n} (-1)^{n-l} \binom{N+n-l}{n-l} W_l^{(N+1)}(x) \\ &= \frac{1}{2^N N!} \sum_{i=1}^{N} \sum_{l=0}^{i} (-1)^{i-l} a_i(N) \frac{i!}{l!} \sum_{m+s+p=n} (-1)^s \binom{2N+m-i-1}{m} \\ &\quad \times \binom{i-l+s}{s} (p+l)_l x^{i-2N-m} W_{p+l}(x).\end{aligned}$$

From (1.1), we have

(2.39)

$$\begin{aligned}&2^N N! F^{N+1} \\ &= 2^N N! \left(\frac{1}{1-t^2} \cdot \frac{1-t^2}{1-2xt+t^2} \right)^{N+1} \\ &= 2^N N! \left(\frac{1}{1-t} \right)^{N+1} \left(\frac{1}{1+t} \right)^{N+1} \left(\frac{1-t^2}{1-2xt+t^2} \right)^{N+1} \\ &= 2^N N! \left(\sum_{l=0}^{\infty} \binom{N+l}{l} t^l \right) \left(\sum_{m=0}^{\infty} \binom{m+N}{m} (-1)^m t^m \right) \left(\sum_{p=0}^{\infty} T_p^{(N+1)}(x) t^p \right) \\ &= 2^N N! \sum_{n=0}^{\infty} \left(\sum_{l+m+p=n} \binom{N+l}{l} \binom{m+N}{m} (-1)^m T_p^{(N+1)}(x) \right) t^n.\end{aligned}$$

On the other hand, by Theorem 1, we get

$$\begin{aligned}(2.40)\qquad & 2^N N! F^{N+1} \\ &= \sum_{i=1}^{N} a_i(N) (x-t)^{i-2N} F^{(i)}\end{aligned}$$

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$$=\frac{1}{2}\sum_{i=1}^{N}a_i(N)(x-t)^{i-2N}\left(\frac{d}{dt}\right)^i\left\{\left(\frac{1}{1-t}+\frac{1}{1+t}\right)\frac{1-t^2}{1-2xt+t^2}\right\}.$$

From Leibniz formula, we note that the following equations:

$$\left(\frac{d}{dt}\right)^i\left\{\left(\frac{1}{1-t}\right)\cdot\left(\frac{1-t^2}{1-2xt+t^2}\right)\right\} \tag{2.41}$$
$$=\sum_{l=0}^{i}\binom{i}{l}(i-l)!\sum_{s=0}^{\infty}\binom{i+s-l}{s}t^s\sum_{p=0}^{\infty}T_{p+l}(x)(p+l)_l t^p,$$

and

$$\left(\frac{d}{dt}\right)^i\left\{\left(\frac{1}{1+t}\right)\left(\frac{1-t^2}{1-2xt+t^2}\right)\right\} \tag{2.42}$$
$$=\sum_{l=0}^{i}\binom{i}{l}(i-l)!(-1)^{i-l}\sum_{s=0}^{\infty}\binom{i-l+s}{s}(-1)^s t^s\sum_{p=0}^{\infty}T_{p+l}(x)(p+l)_l t^p.$$

By (2.40), (2.41), and (2.42), we obtain

(2.43)

$$\begin{aligned}
&2^N N! F^{N+1}\\
&=\frac{1}{2}\sum_{i=1}^{N}a_i(N)(x-t)^{i-2N}\sum_{l=0}^{i}\binom{i}{l}(i-l)!\sum_{s=0}^{\infty}\binom{i+s-l}{s}t^s\sum_{k=0}^{\infty}T_{p+l}(x)(p+l)_l t^p\\
&+\frac{1}{2}\sum_{i=1}^{N}a_i(N)(x-t)^{i-2N}\sum_{l=0}^{i}\binom{i}{l}(i-l)!(-1)^{i-l}\sum_{s=0}^{\infty}\binom{i-l+s}{s}(-1)^s t^s\\
&\times\sum_{p=0}^{\infty}T_{p+l}(x)(p+l)_l t^p\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{i=1}^{N}\sum_{l=0}^{i}a_i(N)\frac{i!}{l!}\sum_{m+s+p=n}\binom{2N+m-i-1}{m}\binom{i+s-l}{s}(p+l)_l\\
&\times x^{i-2N-m}T_{p+l}(x)t^n+\frac{1}{2}\sum_{n=0}^{\infty}\sum_{i=1}^{N}\sum_{l=0}^{i}a_i(N)\frac{i!}{l!}(-1)^{i-l}\\
&\times\sum_{m+s+p=n}(-1)^s\binom{2N+m-i-1}{m}\binom{i+s-l}{s}(p+l)_l x^{i-2N-m}T_{p+l}(x)t^n.
\end{aligned}$$

Therefore, by (2.39) and (2.43), we obtain the following theorem.

Theorem 7. *For $n\in\mathbb{N}\cup\{0\}$ and $N\in\mathbb{N}$, we have the following identity*

$$\begin{aligned}
&2^{N+1}N!\sum_{s+m+p=n}\binom{N+s}{s}\binom{m+N}{m}(-1)^m T_p^{(N+1)}(x)\\
&=\sum_{i=1}^{N}\sum_{l=0}^{i}a_i(N)\frac{i!}{l!}\sum_{m+s+p=n}\binom{2N+m-i-1}{m}\binom{i+s-l}{s}(p+l)_l
\end{aligned}$$

$$\times x^{i-2N-m} T_{p+l}(x) + \sum_{i=1}^{N} \sum_{l=0}^{i} a_i(N) \frac{i!}{l!} (-1)^{i-l} \sum_{m+s+p=n} (-1)^s \binom{2N+m-i-1}{m}$$
$$\times \binom{i+s-l}{s} (p+l)_l \, x^{i-2N-m} T_{p+l}(x).$$

Acknowledgements. This paper is supported by grant NO 14-11-00022 of Russian Scientific Fund.

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Blowup singularity for a degenerate and singular parabolic equation with nonlocal boundary *

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Abstract

In this paper, we are interested in the blowup behavior of the solution to a degenerate and singular parabolic equation

$$u_t = (x^\alpha u_x)_x + \int_0^l u^p dx - ku^q, \quad (x,t) \in (0,l) \times (0,+\infty)$$

with nonlocal boundary condition

$$u(0,t) = \int_0^l f(x)\, u(x,t) dx, \quad u(l,t) = \int_0^l g(x)\, u(x,t) dx, \quad t \in (0,+\infty),$$

where $p,\ q \in [1,\infty)$, $\alpha \in [0,1)$ and $k \in (0,\infty)$. In view of comparison principle, we investigate the conditions on the global existence and blowup of the solutions. Moreover, under some suitable hypotheses, we discuss the global blowup and the uniform blowup profile of the blowup solution.

Keywords: Degenerate and singular parabolic equation; Nonlocal boundary; Global existence; Blowup singularity

Mathematics Subject Classification(2000) : 35K50, 35K55, 35K65

1 Introduction

The main purpose of this paper is to deal with the blowup singularity of the following degenerate and singular parabolic equation with nonlocal source and nonlocal boundary condition

$$\begin{cases} u_t = (x^\alpha u_x)_x + \int_0^l u^p dx - ku^q, & (x,t) \in (0,l) \times (0,+\infty), \\ u(0,t) = \int_0^l f(x)\, u(x,t) dx, & t \in (0,+\infty), \\ u(l,t) = \int_0^l g(x)\, u(x,t) dx, & t \in (0,+\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in [0,l], \end{cases} \tag{1.1}$$

*This work is supported by National Natural Science Foundation of China (11426099, 11526076, 11571102), Scientific Research Fund of Hunan Provincial Education Department (14B067, 15A062)

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where $0 \le \alpha < 1$, p, $q \ge 1$, $k > 0$, the weight functions $f(x)$ and $g(x)$ in the boundary condition are nonnegative continuous on $[0, l]$ and not identically zero, and the initial value $u_0(x) \in C^{2+\delta}(0, l) \cap C[0, l]$ with $0 < \delta < 1$, and satisfies the compatibility conditions. It is obvious that the equation in problem (1.1) is singular and degenerate because the coefficients of u_x and u_{xx} may tend to ∞ and 0 as $x \to 0$.

This type equation in problem (1.1) can be viewed as a model which describes the conduction of heat related to the geometric shape of the body (see [1] and the references therein for more details of the physical background). On the other hand, lots of physical phenomena were formulated into nonlocal mathematical models, for example, Day [4, 5] derived a heat equation with nonlocal boundary in the study of the heat conduction with thermoelastcity. From then on, a lot of mathematicians devoted to studying the blowup behavior of the solutions of various parabolic problems with nonlocal boundary conditions (see [6, 7, 8, 9, 10, 11, 13, 15, 16, 21]).

The blowup phenomenon related to problem (1.1) attracted extensive attention of mathematicians in the past several decades (see [2, 3, 12, 18, 20, 22, 23]), but most of them considered the problems with null Dirichlet boundary conditions. Inspired by the works mentioned above, we consider problem (1.1), and our main attention is focused on evaluating the effects of the weighted nonlocal boundary, the nonlocal source and absorption term on the asymptotic blowup behavior of the solution $u(x, t)$ of problem (1.1). Compared with [3] and [18], we need more skills to handle the difficulties, which are produced by the degeneration and singularity of problem (1.1), and the appearance of the nonlinear nonlocal boundary condition.

Before stating our main results, for the sake of convenience, we denote

$$\mathcal{N} = \max\left\{\int_0^l f(x)dx, \int_0^l g(x)dx\right\},$$

and let λ_1 be the first eigenvalue and $\zeta_1(x)$ be the corresponding eigenfunction of the following eigenvalue problem

$$-\left(x^{\alpha}\zeta_x\right)_x = \lambda_1\zeta, \quad 0 < x < l; \quad \zeta(0) = \zeta(l) = 0. \tag{1.2}$$

Indeed, from [3, 14], we know that the principle eigenvalue λ_1 of the eigenvalue problem (1.2) is the first zero of

$$J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda}}{2-\alpha}l^{\frac{2-\alpha}{2}}\right) = 0,$$

and $\zeta_1(x)$ can be expressed in an explicit form as follows

$$\zeta_1(x) = ax^{\frac{1-\alpha}{2}}J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda_1}}{2-\alpha}x^{\frac{2-\alpha}{2}}\right), \tag{1.3}$$

where $J_{\frac{1-\alpha}{2-\alpha}}$ is Bessel function of the first kind of order $\frac{1-\alpha}{2-\alpha}$, and a is an appropriate positive parameter such that $\|\zeta_1(x)\|_{L^1([0,l])} = 1$. Furthermore, we know easily that $\zeta_1(x)$ is a positive smooth function in $(0, l)$, and in light of

$$\frac{d}{d\tau}J_{\vartheta}(\tau) = \frac{\vartheta}{2}J_{\vartheta}(\tau) - J_{\vartheta+1}(\tau),$$

we can deduce that, for $x \in (0, l)$,

$$\frac{d}{dx}\zeta_1(x) = \frac{a(1-\alpha)}{2}\left(1 + \frac{\sqrt{\lambda_1}}{2-\alpha}x^{\frac{2-\alpha}{2}}\right)x^{-\frac{1+\alpha}{2}}J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda_1}}{2-\alpha}x^{\frac{2-\alpha}{2}}\right) - a\sqrt{\lambda_1}x^{\frac{1-2\alpha}{2}}J_{\frac{3-2\alpha}{2-\alpha}}\left(\frac{\sqrt{\lambda_1}}{2-\alpha}x^{\frac{2-\alpha}{2}}\right).$$

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And hence, by making use of

$$J_{\vartheta}(\tau) \to \frac{1}{\Gamma(\vartheta+1)}\left(\frac{\tau}{2}\right)^{\vartheta} \text{ as } \tau \to 0,$$

where $\Gamma(\cdot)$ is the Gamma function, we find that

$$\lim_{x\to 0^+} \zeta_1(x) = 0$$

and

$$\lim_{x\to 0^+} \frac{d}{dx}\zeta_1(x) = \frac{a(1-\alpha)}{2\Gamma\left(\frac{3-2\alpha}{2-\alpha}\right)}\left(\frac{2\sqrt{\lambda_1}}{2-\alpha}\right)^{\frac{1-\alpha}{2-\alpha}},$$

which imply that

$$\sup_{x\in[0,l]} \zeta_1(x) < \infty \text{ and } \sup_{x\in[0,l]} \frac{d}{dx}\zeta_1(x) < \infty. \tag{1.4}$$

The main results of this paper are stated as follows.

Theorem 1.1. *Assume that $q > p \geq 1$, then all the solutions of problem (1.1) exist globally.*

Theorem 1.2. *Assume that $p > q \geq 1$, then problem (1.1) admits blowup solutions as well as global solutions. More precisely,*

(i) if $\mathcal{N} \leq 1$, then the solution exists globally provided that $u_0(x) \leq \left(\frac{k}{l}\right)^{\frac{1}{p-q}}$;

(ii) if $\mathcal{N} > 1$, then the solution of problem (1.1) blows up in finite time provided that $u_0(x) > \eta_1$, where $\eta_1 > 1$ is an appropriate constant;

(iii) there is a suitable positive small constant η_2 such that the solution $u(x,t)$ of problem (1.1) blows up in finite time for any $f(x)$ and $g(x)$ provided that

$$u_0(x) > \eta_2^{-\xi}\left(\frac{l}{2-\alpha}x^{1-\alpha} - \frac{1}{2-\alpha}x^{2-\alpha}\right),$$

where $\xi > \frac{1}{p-1}$.

Theorem 1.3. *Assume that $p = q > 1$. The solution $u(x,t)$ of problem (1.1) exists globally provided that $\mathcal{N} < 1$ and $u_0(x) \leq \epsilon_1 \mathcal{N}$, where ϵ_1 is given by (3.13). For any nonnegative weight functions $f(x)$ and $g(x)$, the solution $u(x,t)$ of problem (1.1) blows up in finite time provided that the initial value $u_0(x)$ is sufficiently large.*

Remark 1.1. *If $p = q = 1$, one can show that problem (1.1) has no blowup solution.*

The remaining part is devote to discussing the global blowup and the uniform blowup profile of the blowup solution, to this end, we assume that $p > q \geq 1$ (or $p = q > 1$), $\mathcal{N} \leq 1$ and $u_0(x)$ is large enough in some suitable sense. Moreover, we assume that $u_0(x)$ satisfies extra

$$(x^{\alpha}u_{0x})_x + \int_0^l u_0^p dx - ku_0^q \geq 0 \text{ for } x \in (0,l), \tag{1.5}$$

$$(x^{\alpha}u_{0x})_x \leq 0 \text{ in } (0,l), \tag{1.6}$$

and

$$\lim_{x\to 0^+}\left[(x^{\alpha}u_{0x})_x + \int_0^l u_0^p dx - ku_0^q\right] = \lim_{x\to l^-}\left[(x^{\alpha}u_{0x})_x + \int_0^l u_0^p dx - ku_0^q\right] = 0. \tag{1.7}$$

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Theorem 1.4. *Assume that $p > q \geq 1$ and $\mathcal{N} \leq 1$. Suppose that hypotheses* (1.5)*,* (1.6) *and* (1.7) *hold. Then*

$$u(x,t) \sim [l(p-1)(T-t)]^{-\frac{1}{p-1}} \qquad a.e.\ in\ (0,l)\ as\ t \to T,$$

where T is the blowup time.

Corollary 1.1. *Under the assumptions of Theorem 1.4, we see that the blowup set of the blowup solution $u(x,t)$ of problem (1.1) is the whole interval $(0,l)$.*

Theorem 1.5. *Assume that $p = q > 1$, $\mathcal{N} \leq 1$ and $0 < k < l$. Suppose that hypotheses* (1.5)*,* (1.6) *and* (1.7) *hold. Then*

$$u(x,t) \sim [l(p-1)(T-t)]^{-\frac{1}{p-1}} \qquad a.e.\ in\ (0,l)\ as\ t \to T,$$

where T is the blowup time.

Corollary 1.2. *Under the assumptions of Theorem 1.5, we know that the blowup set of the blowup solution $u(x,t)$ of problem (1.1) is the whole interval $(0,l)$.*

The rest of this paper is organized as follows. In Section 2, we shall state the comparison principle and local existence theorem for problem (1.1). In section 3, we shall concern with the conditions on the global existence of solution and prove Theorems 1.1, 1.2 and 1.3. In section 4, we shall estimate the uniform blowup profile and give the proofs of Theorems 1.4 and 1.5.

2 Comparison principle and local existence

In this section, we will establish a suitable comparison principle for problem (1.1) and state the existence and uniqueness result on the local solution. For the sake of simplify, we denote $I_T = (0,l) \times (0,T)$ and $\overline{I}_T = [0,l] \times [0,T)$. First, we give the definitions of the super-solution and sub-solution to problem (1.1).

Definition 2.1. *A nonnegative function $\overline{u}(x,t)$ is called a super-solution of problem (1.1) if $\overline{u}(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ satisfies*

$$\begin{cases} \overline{u}_t \geq (x^\alpha \overline{u}_x)_x + \int_0^l \overline{u}^p dx - k\overline{u}^q, & (x,t) \in I_T, \\ \overline{u}(0,t) \geq \int_0^l f(x)\,\overline{u}(x,t)dx, & t \in (0,T), \\ \overline{u}(l,t) \geq \int_0^l g(x)\,\overline{u}(x,t)dx, & t \in (0,T), \\ \overline{u}(x,0) \geq \overline{u}_0(x), & x \in [0,l]. \end{cases} \tag{2.1}$$

Similarly, $\underline{u}(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ is called a sub-solution of problem (1.1) if it satisfies all the reversed inequalities in (2.1). We say that $u(x,t)$ is a solution of problem (1.1) if it is both a sub-solution and a super-solution of problem (1.1).

Now, by using the similar arguments as those in [6] (or [10]), we give directly the following maximum principle.

Lemma 2.1. *Let $\omega(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ satisfy*

$$\begin{cases} \omega_t - (x^\alpha \omega_x)_x \geq \theta_1(x,t)\,\omega + \int_0^l \theta_1(x,t)\,\omega(x,t)\,dx, & (x,t) \in I_T, \\ \omega(0,t) \geq \int_0^l \theta_3(x)\,\omega(x,t)dx, & t \in (0,T), \\ \omega(l,t) \geq \int_0^l \theta_4(x)\,\omega(x,t)dx, & t \in (0,T), \end{cases} \tag{2.2}$$

where $\theta_i(x,t)$, $i = 1,2,3,4$, are bounded functions, $\theta_2(x,t)$ is nonnegative for $(x,t) \in I_T$, $\theta_3(x)$ and $\theta_4(x)$ are nonnegative, nontrivial in $(0,l)$. Then $\omega(x,0) > 0$ in $[0,l]$ implies that $\omega(x,t) > 0$ for $(x,t) \in I_T$. Moreover, if one of the following conditions holds, (i) $\theta_3(x) = \theta_4(x) \equiv 0$ for $x \in (0,l)$; (ii) $\theta_3(x)$, $\theta_4(x) \geq 0$ for $x \in (0,l)$ and $\max\left\{\int_0^l \theta_3(x)\,dx, \int_0^l \theta_4(x)\,dx\right\} \leq 1$, then $\omega(x,0) \geq 0$ in $[0,l]$ leads to $\omega(x,t) \geq 0$ for $(x,t) \in I_T$.

Based on the idea of [10], we can establish the comparison principle for problem (1.1) as follows, which is the main tool of establishing the conditions on the global existence and blowup of the solution.

Proposition 2.1 (Comparison principle)**.** *Let $\overline{u}(x,t)$ and $\underline{u}(x,t)$ be a nonnegative super-solution and sub-solution of problem (1.1), respectively. Then $\overline{u}(x,t) \geq \underline{u}(x,t)$ holds in $\overline{I}_T$ if $\overline{u}(x,0) \geq \underline{u}(x,0)$ for $x \in [0,l]$.*

Next, we state the result on the existence and uniqueness of the local solution of problem (1.1) at the end of this section.

Theorem 2.1 (Local existence and uniqueness)**.** *Assume that (1.5) holds, then there exists a small positive real number T such that problem (1.1) admits a unique nonnegative solution $u(x,t) \in C(\overline{I}_T) \cap C^{2,1}(I_T)$.*

Remark 2.1. *We can get the proof of Theorem 2.1 by using regularization method and Schauder's fixed point theorem. For more details, we refer the readers to [3, 23].*

3 Global existence of solution

The main goal of this section is to discuss the global existence and blowup property of the solution $u(x,t)$ to the problem (1.1). To this end, by Proposition 2.1, we only need to construct some suitable global super-solutions (or blowup sub-solutions).

Proof of Theorem 1.1. Let T be any positive number and $\overline{u}_1(x,t)$ be defined as

$$\overline{u}_1(x,t) = \frac{\chi_2}{\chi_1 \zeta_1(x) + 1} e^{\chi_3 t}$$

where χ_1 is large enough such that

$$\int_0^l \frac{1}{1 + \chi_1 \zeta_1(x)} dx \leq \max\left\{\max_{x\in[0,l]} f(x), \max_{x\in[0,l]} g(x)\right\},$$

and

$$\chi_2 = \max\left\{\max_{x\in[0,l]} (u_0(x)+1)(\chi_1\zeta_1(x)+1), \max_{x\in[0,l]} \left[\frac{(\chi_1\zeta_1(x)+1)^q}{k} \int_0^l \frac{1}{(1+\chi_1\zeta_1(x))^p} dx\right]^{\frac{1}{q-p}}\right\},$$

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$$\chi_3 = \lambda_1 + \max_{x\in[0,l]} \frac{2l^{\alpha}\chi_1^2}{(\chi_1\zeta_1(x)+1)^2}\left|\frac{d\zeta_1(x)}{dx}\right|^2.$$

By the direct calculation, one has

$$\begin{aligned} P\overline{u}_1 :&= \overline{u}_{1t} - (x^{\alpha}\overline{u}_{1x})_x - \int_0^l \overline{u}_1^p dx + k\overline{u}_1^q \\ &= \overline{u}_1\left[\chi_3 - \left(\frac{\lambda_1\chi_1\zeta_1(x)}{1+\chi_1\zeta_1(x)} + \frac{2x^{\alpha}\chi_1^2}{(\chi_1\zeta_1(x)+1)^2}\left|\frac{d}{dx}\zeta_1(x)\right|^2\right)\right] \\ &\quad + k\left(\frac{\chi_2 e^{\chi_3 t}}{1+\chi_1\zeta_1(x)}\right)^q - \left(\chi_2 e^{\chi_3 t}\right)^p \int_0^l \frac{1}{(1+\chi_1\zeta_1(x))^p}dx \\ &\geq 0, \end{aligned} \tag{3.1}$$

and

$$\overline{u}_1(x,0) = \frac{\chi_2}{1+\chi_1\zeta_1(x)} \geq \frac{\max\limits_{x\in[0,l]}(u_0(x)+1)(1+\chi_1\zeta_1(x))}{1+\chi_1\zeta_1(x)} > u_0(x). \tag{3.2}$$

On the other hand, we can verify that

$$\overline{u}_1(0,t) = \chi_2 e^{\chi_3 t} \geq \chi_2 e^{\chi_3 t}\max_{x\in[0,l]} f(x)\int_0^l \frac{1}{1+\chi_1\zeta_1(x)}dx \geq \int_0^l \frac{f(x)\chi_2 e^{\chi_3 t}}{1+\chi_1\zeta_1(x)}dx = \int_0^l f(x)\overline{u}_1(x,t)\,dx, \tag{3.3}$$

and

$$\overline{u}_1(l,t) \geq \int_0^l g(x)\overline{u}_1(x,t)\,dx. \tag{3.4}$$

Combining now from (3.1) to (3.4), we know that $\overline{u}_1(x,t)$ is a global super-solution of (1.1) in I_T and the solution $u(x,t)$ of (1.1) exists globally by Proposition 2.1. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. **(i)** If $p > q$ and $\mathcal{N} > 1$, then it is easy to check that $\overline{u}_2(x) = \left(\frac{k}{l}\right)^{\frac{1}{p-q}}$ is a global super-solution of problem (1.1) provided that $u_0(x) \leq \left(\frac{k}{l}\right)^{\frac{1}{p-q}}$.

(ii) Consider the following ordinary differential equation

$$\begin{cases} \underline{v}_1{}'(t) = l\underline{v}_1^p - k\underline{v}_1^q, & t > 0, \\ \underline{v}_1(0) = \underline{v}_{10}. \end{cases} \tag{3.5}$$

From $p > q \geq 1$, it follows that $\underline{v}_1^q \leq \underline{v}_1^p + 1$, and hence, we have

$$l\underline{v}_1^p - k\underline{v}_1^q \geq (l-k)\underline{v}_1^p - k,$$

which tells us that the solution $\underline{v}_1(t)$ of (3.5) is a super-solution of the following problem

$$\begin{cases} \underline{v}_2{}'(t) = (l-k)\underline{v}_2^p - k, & t > 0, \\ \underline{v}_2(0) = \underline{v}_{10} \end{cases} \tag{3.6}$$

provided $l > k$. Noticing that $(l-k)\underline{v}_2^p$ is convex, then there exists $\eta_1 > 1$ such that $(l-k)\underline{v}_2^p \geq 2k$ holds for $\underline{v}_2 \geq \eta_1$. It follows easily that if $\underline{v}_2(0) = \underline{v}_{10} > \eta_1$, then $\underline{v}_2(t)$ is increasing on its interval of the existence and

$$\underline{v}_2{}'(t) \geq \frac{l-k}{2}\underline{v}_2^p. \tag{3.7}$$

From the above inequality it follows that

$$\underline{v}_2(t) \to \infty \text{ as } t \to \frac{2}{(l-k)(p-1)\underline{v}_{10}^{p-1}}, \tag{3.8}$$

which leads to that $\underline{v}_1(t)$ will become infinite in a finite time. Recalling that $\mathcal{N} > 1$, then $\underline{v}_1(t)$ is a blowup sub-solution of problem (1.1) when $u_0(x) \geq \underline{v}_{10} > \eta$, so the solution $u(x,t)$ of problem (1.1) blows up in finite time for sufficiently large initial value.

(iii) Let $v(x,t)$ be the solution of the following auxiliary problem

$$\begin{cases} v_t = (x^\alpha v_x)_x + \int_0^l v^p(x,t)dx - kv^q, & 0 < x < l, t > 0, \\ v(0,t) = v(l,t) = 0, & t > 0, \\ v(x,0) = u_0(x), & 0 < x < l, \end{cases} \tag{3.9}$$

then $v(x,t)$ is a sub-solution of problem (1.1). Set

$$\underline{v}_3(x,t) = (\eta_2 - t)^{-\xi}\left(\frac{l}{2-\alpha}x^{1-\alpha} - \frac{1}{2-\alpha}x^{2-\alpha}\right) := (\eta_2 - t)^{-\xi}\mu(x),$$

where η_2 and $\xi > 0$ will be chosen later. Calculating directly, we have

$$\begin{aligned} P\underline{v}_3 :&= \underline{v}_{3t} - (x^\alpha \underline{v}_{3x})_x - \int_0^l \underline{v}_3^p(x,t)dx + k\underline{v}_3^q \\ &= (\eta_2 - t)^{-\xi p}\left[\xi(\eta_2 - t)^{\xi p - \xi - 1}\mu(x) + (\eta_2 - t)^{\xi(p-1)} + k(\eta_2 - t)^{\xi(p-q)}\mu^q(x) - \int_0^l \mu^p(x)\,dx\right]. \end{aligned}$$

Since $p > q \geq 1$, we can take ξ large enough such that $\xi p - \xi - 1 > 0$, then we have $P\underline{v}_3 \leq 0$ with $\eta_2 - t$ small enough, which implies that $\underline{v}_3(x,t)$ is a blowup sub-solution to problem (3.9) provided that $v(x,0) = u_0(x) > \mu(x)\eta_2^{-\xi}$. And hence, Proposition 2.1 tells us that the solution $u(x,t)$ of problem (1.1) blows up in finite time for large initial value. The proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3. For any given constant

$$\epsilon_0 \in \left(0, \frac{(1-\mathcal{N})(2-\alpha)^{3-\alpha}}{l^{2-\alpha}(1-\alpha)^{1-\alpha}}\right), \tag{3.10}$$

let $\sigma(x)$ be the unique positive solution of the following ordinary differential equation

$$\begin{cases} -(x^\alpha \sigma_x)_x = \epsilon_0, & 0 < x < l, \\ \sigma(0) = \sigma(l) = \mathcal{N}. \end{cases} \tag{3.11}$$

In fact, we can solve the explicit expression of $\sigma(x)$ as follows

$$\sigma(x) = \frac{l\epsilon_0}{2-\alpha}x^{1-\alpha} - \frac{\epsilon_0}{2-\alpha}x^{2-\alpha} + \mathcal{N}, \quad x \in [0,l].$$

Moreover, according to $\mathcal{N} < 1$, we can verify that

$$0 < \min_{x\in[0,l]} \sigma(x) = \mathcal{N} < \max_{x\in[0,l]} \sigma(x) = \mathcal{N} + \frac{\epsilon_0 l^{2-\alpha}(1-\alpha)^{1-\alpha}}{(2-\alpha)^{3-\alpha}} < 1. \tag{3.12}$$

Define

$$\overline{u}_3(x,t) = \epsilon_1 \sigma(x),$$

where

$$\epsilon_1 = \begin{cases} \left(\frac{\epsilon_0}{k\mathcal{N}^p - l}\right)^{\frac{1}{p-1}}, & \text{if } k\mathcal{N}^p - l > 0, \\ \text{any fixed positive constant}, & \text{if } k\mathcal{N}^p - l \leq 0. \end{cases} \tag{3.13}$$

Calculating directly, one has

$$\begin{aligned} P\overline{u}_3 :&= \overline{u}_{3t} - (x^\alpha \overline{u}_{3x})_x - \int_0^l \overline{u}_3^p dx + k\overline{u}_3^p \\ &= \epsilon_0\epsilon_1 - \epsilon_1^p \int_0^l \sigma^p dx + k\epsilon_1^p \sigma^p \\ &\geq \epsilon_0\epsilon_1 - l\epsilon_1^p \left[\max_{x\in[0,l]} \sigma(x)\right]^p + k\epsilon_1^p \left[\min_{x\in[0,l]} \sigma(x)\right]^p \\ &> \epsilon_0\epsilon_1 - \epsilon_1^p (k\mathcal{N}^p - l) \\ &\geq 0. \end{aligned} \tag{3.14}$$

Meanwhile, we can prove that

$$\overline{u}_3(0,t) = \epsilon_1 \mathcal{N} \geq \int_0^l \epsilon_1 f(x)\, dx > \int_0^l \epsilon_1 \sigma(x) f(x)\, dx = \int_0^l \overline{u}_3(x,t) f(x)\, dx \tag{3.15}$$

and

$$\overline{u}_3(l,t) > \int_0^l \overline{u}_3(x,t) f(x)\, dx. \tag{3.16}$$

Then $\overline{u}_3(x,t)$ is a global super-solution of problem (1.1) if $u_0(x) \leq \epsilon_1 \mathcal{N}$, and hence, we obtain our global existence result by Proposition 2.1.

The proof of blowup conclusion in this case is similar to the arguments of (iii) in Theorem 1.2, we omit the details here. The proof of Theorem 1.3 is completed. □

4 Global blowup set and uniform blowup profile

This section is mainly about the global blowup and the uniform blowup profile of the blowup solution for problem (1.1). Throughout this section, we assume that $p > q \geq 1$ (or $p = q > 1$), $\mathcal{N} \leq 1$ and $u_0(x)$ is large enough in some suitable sense. From Theorems 1.2 and 1.3, it follows that the solution $u(x,t)$ of problem (1.1) blows up in finite. For convenience, we denote T the blowup time.

From the assumptions on the initial value $u_0(x)$ and (1.5), (1.6) and (1.7), we can find a sufficiently small positive constant ε_1 and a nonnegative function $w_{0\varepsilon}(x)$ such that

(1) $w_{0\varepsilon} \in C^{2+\delta}(\varepsilon, l-\varepsilon) \cap C[\varepsilon, l-\varepsilon]$ with $\delta \in (0,1)$ and $\varepsilon \in (0, \varepsilon_1]$.

(2) $w_{0\varepsilon}(\varepsilon) = \int_\varepsilon^{l-\varepsilon} f(x)\, w_{0\varepsilon}(x)\, dx$ and $w_{0\varepsilon}(l-\varepsilon) = \int_\varepsilon^{l-\varepsilon} g(x)\, w_{0\varepsilon}(x)\, dx$.

(3) $w_{0\varepsilon}(x) < u_0(x)$ for $x \in (\varepsilon, 2\varepsilon) \cup (l-2\varepsilon, l-\varepsilon)$, and $w_{0\varepsilon}(x) = u_0(x)$ for $x \in [2\varepsilon, l-2\varepsilon]$.

(4) $(x^\alpha w_{0\varepsilon x})_x \leq 0$ for $x \in (\varepsilon, l-\varepsilon)$.

(5) $(x^\alpha w_{0\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_{0\varepsilon}^p dx - kw_{0\varepsilon}^q \geq 0$ for $\varepsilon \in (0, \varepsilon_1]$ and $x \in (\varepsilon, l-\varepsilon)$.

(6) $w_{0\varepsilon}$ is non-increasing with respect to ε in $(0, \varepsilon_1]$. Moreover

$$\lim_{x\to\varepsilon^+}\left[(x^\alpha w_{0\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_{0\varepsilon}^p dx - kw_{0\varepsilon}^q\right] = \lim_{x\to(l-\varepsilon)^-}\left[(x^\alpha w_{0\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_{0\varepsilon}^p dx - kw_{0\varepsilon}^q\right] = 0.$$

It is obvious that

$$\lim_{\varepsilon\to 0^+} w_{0\varepsilon}(x) = u_0(x).$$

Now, we consider the following regularized problem

$$\begin{cases} w_{\varepsilon t} = (x^\alpha w_{\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p dx - kw_\varepsilon^q, & (x,t) \in (\varepsilon, l-\varepsilon) \times (0,+\infty), \\ w_\varepsilon(\varepsilon, t) = \int_\varepsilon^{l-\varepsilon} f(x) w_\varepsilon(x,t) dx, & t \in (0,+\infty), \\ w_\varepsilon(l-\varepsilon, t) = \int_\varepsilon^{l-\varepsilon} g(x) w_\varepsilon(x,t) dx, & t \in (0,+\infty), \\ w_\varepsilon(x,0) = w_{0\varepsilon}(x), & x \in [0,l]. \end{cases} \tag{4.1}$$

Then it is not difficult to show that there exists a unique solution $w_\varepsilon(x,t)$ for problem (4.1). In addition, from the arguments of Section 2 in [23], it follows that

$$\lim_{\varepsilon\to 0^+} w_\varepsilon(x,t) = u(x,t),$$

where $u(x,t)$ is the solution of problem (1.1).

Lemma 4.1. *Suppose that hypotheses* (1.5), (1.6) *and* (1.7) *hold, and assume that* $p \geq q > 1$ *and* $\mathcal{N} \leq 1$. *Then* $(x^\alpha u_x)_x \leq 0$ *holds for* $(x,t) \in I_T$.

Proof. Taking $\eta = (x^\alpha w_{\varepsilon x})_x$, then from (4.1), we have

$$\eta_t = \left\{x^\alpha\left[(x^\alpha w_{\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p dx - kw_\varepsilon^q\right]_x\right\}_x = (x^\alpha \eta_x)_x - kqw_\varepsilon^{q-1}\eta - kq(q-1)w_\varepsilon^{q-2}|w_{\varepsilon x}|^2 \tag{4.2}$$

holds for any $(x,t) \in (\varepsilon, l-\varepsilon) \times (0,T)$, which tells us that

$$\eta_t - (x^\alpha \eta_x)_x + kqw_\varepsilon^{q-1}\eta \leq 0. \tag{4.3}$$

On the other hand, for any $t \in (0,T)$, we have

$$\begin{aligned} \eta(\varepsilon,t) &= \int_\varepsilon^{l-\varepsilon} f(x) w_{\varepsilon t}(x,t)\, dx - \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\, dx + k\left(\int_\varepsilon^{l-\varepsilon} f(x) w_\varepsilon(x,t)\, dx\right)^q \\ &= \int_\varepsilon^{l-\varepsilon} f(x)\left((x^\alpha w_{\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\, dx - kw_\varepsilon^q\right) dx \\ &\quad - \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\, dx + k\left(\int_\varepsilon^{l-\varepsilon} f(x) w_\varepsilon(x,t)\, dx\right)^q \\ &= \int_\varepsilon^{l-\varepsilon} f(x)\eta(x,t)\, dx + \left(\int_\varepsilon^{l-\varepsilon} f(x)\, dx - 1\right)\int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\, dx \\ &\quad - k\left[\int_\varepsilon^{l-\varepsilon} f(x) w_\varepsilon^q(x,t)\, dx - \left(\int_\varepsilon^{l-\varepsilon} f(x) w_\varepsilon(x,t)\, dx\right)^q\right]. \end{aligned} \tag{4.4}$$

It follows from Jensen's inequality that

$$\begin{aligned}&\int_{\varepsilon}^{l-\varepsilon} f(x)\, w_{\varepsilon}^{q} dx-\left(\int_{\varepsilon}^{l-\varepsilon} f(x)\, w_{\varepsilon}(x,t)\, dx\right)^{q}\\ &\geq \int_{\varepsilon}^{l-\varepsilon} f(x)\, dx\left(\frac{\int_{\varepsilon}^{l-\varepsilon} f(x)\, w_{\varepsilon}(x,t)\, dx}{\int_{\varepsilon}^{l-\varepsilon} f(x)\, dx}\right)^{q}-\left(\int_{\varepsilon}^{l-\varepsilon} f(x)\, w_{\varepsilon}(x,t)\right)^{q}\\ &\geq 0.\end{aligned}$$

Exploiting the above inequality and the assumption $\mathcal{N}\leq 1$ to (4.4), we can claim that

$$\eta(\varepsilon,t)\leq \int_{\varepsilon}^{l-\varepsilon} f(x)\,\eta(x,t)\, dx,\quad t\in(0,T). \tag{4.5}$$

By the analogous arguments, one can also show that

$$\eta(l-\varepsilon,t)\leq \int_{\varepsilon}^{l-\varepsilon} g(x)\,\eta(x,t)\, dx \tag{4.6}$$

holds for all $t\in(0,T)$.

Moreover, noticing that $\eta(x,0)=(x^{\alpha}w_{0\varepsilon x})_{x}\leq 0$ holds for $x\in(\varepsilon,l-\varepsilon)$. Then, maximum principle tells us that $\eta(x,t)=(x^{\alpha}w_{\varepsilon x})_{x}\leq 0$ holds for all $(x,t)\in(\varepsilon,l-\varepsilon)\times(0,T)$. In addition, by the arbitrariness of ε, we know that $(x^{\alpha}u_{x})_{x}\leq 0$ holds in I_T. The proof of Lemma 4.1 is complete. □

In what follows, for the sake of simplicity, we denote

$$\psi(t)=\int_{0}^{l} u^{p}(x,t)\, dx \text{ and } \Psi(t)=\int_{0}^{t}\psi(\tau)\, d\tau.$$

Lemma 4.2. *Assume that* (1.5), (1.6) *and* (1.7) *hold,* $p>q\geq 1$ *and* $\mathcal{N}\leq 1$, *then there exists a positive constant* C *such that*

$$\sup_{x\in K_d}(\Psi(t)-u(x,t))\leq \frac{C}{d^{2}}\left(1+Z(t)+\int_{0}^{t}\Psi(\tau)\, d\tau\right)$$

in $[0,l]\times[\frac{T}{2},T)$, *where*

$$Z(t)=o(\Psi(t)) \text{ as } t\to T,$$

and

$$K_d=\{x\in(0,l): dist(x,0)\geq d, dist(x,l)\geq d\}\subset(0,l).$$

Proof. Put

$$\mathfrak{F}(t)=\int_{0}^{l}(\Psi(t)-u(x,t))\,\zeta_{1}(x)\, dx, \tag{4.7}$$

where $\zeta_1(x)$ is given by (1.3). Taking the derivative of $\mathfrak{F}(t)$ with respect to t, we arrive at

$$\begin{aligned}\mathfrak{F}'(t) &= \int_0^l (\psi(t) - u_t)\,\zeta_1(x)\,dx \\ &= \int_0^l \left(-(x^\alpha u_x)_x + ku^q\right)\zeta_1(x)\,dx \\ &= \lambda_1 \int_0^l u(x,t)\,\zeta_1(x)\,dx + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx \\ &\quad + l^\alpha \zeta_{1x}\mid_{x=l} \int_0^l g(x)\,u(x,t)\,dx \\ &\le \lambda_1 \int_0^l u(x,t)\,\zeta_1(x)\,dx + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx \\ &= -\lambda_1 \mathfrak{F}(t) + \lambda_1 \Psi(t) + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx.\end{aligned} \tag{4.8}$$

On the other hand, it follows from Lemma 4.1 that

$$u_t \le \psi(t) - ku^q,$$

which implies that

$$-\max_{x\in[0,l]} u_0(x) \le \Psi(t) - u(x,t). \tag{4.9}$$

Then (4.9) and (4.8) lead to

$$\mathfrak{F}'(t) \le \lambda_1 \max_{x\in[0,l]} u_0(x) + \lambda_1 \Psi(t) + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx.$$

Integrating above inequality over from 0 to t, one has

$$\mathfrak{F}(t) \le \max\left\{\lambda_1, k\max_{x\in[0,l]}\zeta_1(x), \mathfrak{F}(0) + \lambda_1 T \max_{x\in[0,l]} u_0(x)\right\}\left(1 + \int_0^t \Psi(\tau)\,d\tau + \int_0^t\int_0^l u^q(x,\tau)\,dxd\tau\right). \tag{4.10}$$

Further, since $p > q \ge 1$, Hölder's inequality implies that

$$\int_0^t\int_0^l u^q(y,\tau)\,dyd\tau \le (lT)^{\frac{p-q}{p}}\left(\int_0^t\int_0^l u^p(y,\tau)\,dyd\tau\right)^{\frac{q}{p}} := Z(t). \tag{4.11}$$

It is not difficult to verify that

$$Z(t) = o(\Psi(t)) \text{ as } t \to T. \tag{4.12}$$

Combining (4.13), (4.11) with (4.12), we see that

$$\mathfrak{F}(t) \le \max\left\{\lambda_1, k\max_{x\in[0,l]}\zeta_1(x), \mathfrak{F}(0) + \lambda_1 T \max_{x\in[0,l]} u_0(x)\right\}\left(1 + Z(t) + \int_0^t \Psi(\tau)\,d\tau\right). \tag{4.13}$$

Now, by Lemma 4.5 in [17], we can claim that

$$\sup_{x\in K_d}(\Psi(t) - u(x,t)) \le \frac{C}{d^2}\left(1 + \int_0^t \Psi(\tau)\,d\tau + o(\Psi(t))\right)$$

holds for $(x,t) \in [0,l] \times \left[\frac{T}{2}, T\right)$, where C is an appropriate positive constant. The proof of Lemma 4.2 is complete. □

In view of Lemma 4.2, and by a slight variant of the proof of Lemma 4.5 in [17], we have the following Lemma.

Lemma 4.3. *Assume that* (1.6) *and* (1.7) *hold,* $p > q \geq 1$ *and* $\mathcal{N} \leq 1$*, then*

$$\lim_{t\to T} \sup_{[0,l]} |u(\cdot,t)| = +\infty \tag{4.14}$$

is equivalent to

$$\lim_{t\to T} \Psi(t) = +\infty \tag{4.15}$$

Moreover, if (4.14) or (4.15) is fulfilled, then

$$\lim_{t\to T} \frac{u(x,t)}{\Psi(t)} = \lim_{t\to T} \frac{|u(\cdot,t)|_\infty}{\Psi(t)} = 1 \tag{4.16}$$

uniformly on any compact subset of $(0,l)$.

Next, we give the proofs of Theorems 1.4 and Theorem 1.5, respectively.

Proof of Theorem 1.4. It follows from (4.16) that

$$u^p(x,t) \sim \Psi^p(t), \quad t \to T.$$

By the Lebesgue's dominated convergence theorem, we have

$$\Psi'(t) = \psi(t) = \int_0^l u^p(x,t)\,dx \sim l\Psi^p(t), \quad t \to T.$$

Therefore, by integrating the above equality, we can claim that

$$\Psi(t) \sim (l(p-1)(T-t))^{-\frac{1}{p-1}}. \tag{4.17}$$

Combining (4.16) with (4.17), we find that

$$u(x,t) \sim (l(p-1)(T-t))^{-\frac{1}{p-1}}, \quad t \to T, \tag{4.18}$$

which means that

$$\lim_{t\to T} (T-t)^{\frac{1}{p-1}} u(x,t) = \lim_{t\to T} (T-t)^{\frac{1}{p-1}} |u(\cdot,t)|_\infty = (l(p-1))^{-\frac{1}{p-1}}.$$

The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.5. Denote

$$\varphi(t) = \int_0^l u^p(y,t)\,dy - k\left(\max_{x\in[0,l]} u(x,t)\right)^p \text{ and } \Phi(t) = \int_0^t g(\tau)\,d\tau.$$

Similar to Lemma 4.3, we can get

$$\lim_{t\to T} \frac{u(x,t)}{\Phi(t)} = \lim_{t\to T} \frac{|u(\cdot,t)|_\infty}{\Phi(t)} = 1, \tag{4.19}$$

uniformly on any compact subset of $(0,l)$.

Since, the remaining arguments are the same as those in the proof of Theorem 1.4, we omit it here. The proof of Theorem 1.5 is complete. □

Acknowledgements

The authors are sincerely grateful to professor Chunlai Mu of Chongqing University for his encouragements and discussions.

Competing interests

The authors declare that they have no competing interests.

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Approximation properties of Kantorovich-type q-Bernstein-Stancu-Schurer operators

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Abstract. In this paper, we introduce a Kantorovich-type Bernstein-Stancu-Schurer operators $K_{n,p,q}^{\alpha,\beta}$ based on the concept of q-integers. We investigate statistical approximation properties and establish a local approximation theorem, we also give a convergence theorem for the Lipschitz continuous functions. Finally, we give some graphics to illustrate the convergence properties of operators to some functions.

2000 Mathematics Subject Classification: 41A10, 41A25, 41A36.
Key words and phrases: q-integers, Bernstein-Stancu-Schurer operators, A-statistical convergence, rate of convergence, Lipschitz continuous functions.

1 Introduction

In 2013, Özarslan and Vedi [7] introduced the q-Bernstein-Schurer-Kantorovich operators as follows:

$$K_n^p(f;q;x)=\sum_{r=0}^{n+p}\left[\begin{array}{c} n+p \\ r \end{array}\right]_q x^r\prod_{s=o}^{n+p-r-1}(1-q^sx)\int_0^1 f\left(\frac{[r]_q}{[n+1]_q}+\frac{1+(q-1)[r]_q}{[n+1]_q}t\right)d_qt$$

for any real number $0<q<1$, fixed $p\in\mathbb{N}_0$ and $f\in C[0,p+1]$. They gave the Korovkin-type approximation theorem, obtained the rate of convergence of the operators and so on. In 2014, Ren and Zeng [8] introduced two kinds of Kantorovich-type q-Bernstein-Stancu operators based on q-Jackson integral and Riemann-type q-integral respectively and got some approximation properties. In 2015, Acu [1] introduced and studied q analogue of Stancu-Schurer-Kantorovich operators. They proved a convergence theorem, established the rate of convergence, obtained a Voronovskaya type result and so on, they constructed

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the operators as follows:

$$K_{n,p}^{\alpha,\beta}(f;x)=\sum_{k=0}^{n+p}\begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k(1-x)_q^{n+p-k}\int_0^1 f\left(\frac{[k]_q+q^kt+\alpha}{[n+1]_q+\beta}\right)d_qt.$$

In 2015, Agrawal, Finta and Kumar [2] introduced a new Kantorovich-type generalization of the q-Bernstein-Schurer operators, they gave the basic convergence theorem, obtained the local direct results, estimated the rate of convergence and so on. The operators are defined as

$$K_{n,p}(f;q;x)=[n+1]_q\sum_{k=0}^{n+p}b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}}f(t)d_q^Rt, \tag{1}$$

where $b_{n+p,k}(q;x)$ is defined by

$$b_{n+p,k}(q;x)=\begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k(1-x)_q^{n+p-k}. \tag{2}$$

Motivated by above investigations, it seems there have no papers mentioned about the Stancu-type of the operators defined in (1). In present paper, we will introduce the Kantorovich-type q-Bernstein-Stancu-Schurer operators $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ which will be defined in (4). We will investigate statistical approximation properties, establish a local approximation theorem and give a convergence theorem for the Lipschitz continuous functions. Furthermore, we will give some graphics to illustrate the convergence properties of operators to some functions.

Before introducing the operators, we mention certain definitions based on q-integers, detail can be found in [5, 6]. For any fixed real number $0<q\leq 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q=\begin{cases}\frac{1-q^k}{1-q}, & q\neq 1;\\ k, & q=1.\end{cases}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q!=\begin{cases}[k]_q[k-1]_q...[1]_q, & k=1,2,...;\\ 1, & k=0,\end{cases},\quad \begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!},\quad (n\geq k\geq 0).$$

For $x\in[0,1]$ and $n\in\mathbb{N}_0$, we recall that

$$(1-x)_q^n=\begin{cases}1, & n=0;\\ \prod_{j=0}^{n-1}\left(1-q^jx\right)=(1-x)(1-qx)...\left(1-q^{n-1}x\right), & n=1,2,...\end{cases}.$$

The Riemann-type q-integral is defined by

$$\int_a^b f(t)d_q^Rt=(1-q)(b-a)\sum_{j=0}^{\infty}f\left(a+(b-a)q^j\right)q^j, \tag{3}$$

where the real numbers a, b and q satisfy that $0 \leq a < b$ and $0 < q < 1$.

For $f \in C(I)$, $I = [0, 1+p]$, $p \in \mathbb{N}_0$, $0 \leq \alpha \leq \beta$, $q \in (0,1)$ and $n \in \mathbb{N}$, we introduce the Kantorovich-type q-Bernstein-Stancu-Schurer operators as follows:

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) = ([n+1]_q + \beta) \sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^{-k} \int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} f(t) d_q^R t, \tag{4}$$

where $b_{n+p,k}(q;x)$ is defined by (2).

2 Auxiliary Results

In order to obtain the approximation properties, We need the following lemmas:

Lemma 2.1. *Using the definition* (3)*, we easily get*

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} d_q^R t = \frac{q^k}{[n+1]_q + \beta}, \tag{5}$$

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} t d_q^R t = \frac{([k]_q + \alpha) q^k}{([n+1]_q + \beta)^2} + \frac{q^{2k}}{[2]_q([n+1]_q + \beta)^2}, \tag{6}$$

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} t^2 d_q^R t = \frac{q^k([k]_q + \alpha)^2}{([n+1]_q + \beta)^3} + \frac{2q^{2k}([k]_q + \alpha)}{[2]_q([n+1]_q + \beta)^3} + \frac{q^{3k}}{[3]_q([n+1]_q + \beta)^3}. \tag{7}$$

Lemma 2.2. *(See [2], Lemma 2.1) The following equalities hold*

$$\sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^k = 1 - (1-q)[n+p]_q x, \tag{8}$$

$$\sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^{2k} = 1 - \left(1 - q^2\right)[n+p]_q x + q(1-q)^2 [n+p]_q [n+p-1]_q x^2. \tag{9}$$

Lemma 2.3. *For the Kantorovich-type q-Bernstein-Stancu-Schurer operators* (4)*, we have*

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(1;x) = 1, \tag{10}$$

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(t;x) = \frac{2q[n+p]_q x + 1 + [2]_q \alpha}{[2]_q([n+1]_q + \beta)}, \tag{11}$$

$$\begin{aligned} \widetilde{K_{n,p,q}^{\alpha,\beta}}\left(t^2;x\right) &= \frac{\left(q^2[3]_q + 3q^4\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q+\beta)^2} x^2 + \frac{[2]_q[3]_q\alpha^2 + 2[3]_q\alpha + [2]_q}{[2]_q[3]_q([n+1]_q+\beta)^2} \\ &\quad + \frac{\left(4q[3]_q\alpha + 3q + 5q^2 + 4q^3\right)[n+p]_q}{[2]_q[3]_q([n+1]_q+\beta)^2} x. \end{aligned} \tag{12}$$

Proof. (10) is easily obtained from (4) and (5). Using (4), (6) and (8), we have

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(t;x)$$

$$
\begin{aligned}
&= \sum_{k=0}^{n+p} b_{n+p,q}(k;x)\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}+\frac{q^k}{[2]_q([n+1]_q+\beta)}\right)\\
&= \frac{[n+p]_q}{[n+1]_q+\beta}\sum_{k=0}^{n+p} b_{n+p,q}(k;x)\frac{[k]_q}{[n+p]_q}+\frac{\alpha}{[n+1]_q+\beta}+\frac{1-(1-q)[n+p]_qx}{[2]_q([n+1]_q+\beta)}\\
&= \frac{[n+p]_q}{[n+1]_q+\beta}\sum_{k=0}^{n+p-1}\left[\begin{array}{c} n+p-1\\ k\end{array}\right]_q x^{k+1}(1-x)_q^{n+p-k-1}+\frac{1-(1-q)[n+p]_qx}{[2]_q([n+1]_q+\beta)}\\
&\quad +\frac{\alpha}{[n+1]_q+\beta}\\
&= \frac{[n+p]_q}{[n+1]_q+\beta}x-\frac{(1-q)[n+p]_q}{[2]([n+1]_q+\beta)}x+\frac{1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}.
\end{aligned}
$$

Thus, (11) is proved. Finally, from (4) and (7), we have

$$
\begin{aligned}
&\widetilde{K_{n,p,q}^{\alpha,\beta}}\left(t^2;x\right)\\
&= \sum_{k=0}^{n+p} b_{n+p,q}(k;x)\left(\frac{[k]_q^2+2\alpha[k]_q+\alpha^2}{([n+1]_q+\beta)^2}+\frac{2q^k([k]_q+\alpha)}{[2]_q([n+1]_q+\beta)^2}+\frac{q^{2k}}{[3]_q([n+1]_q+\beta)^2}\right),
\end{aligned}
$$

since $[k]_q^2=[k]_q[k-1]_q+q^{k-1}[k]_q$, and from lemma 2.2, we have

$$
\begin{aligned}
&\widetilde{K_{n,p,q}^{\alpha,\beta}}\left(t^2;x\right)\\
&= \sum_{k=0}^{n+p} b_{n+p,k}(q;x)\frac{[k]_q[k-1]_q}{([n+1]_q+\beta)^2}+\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\frac{2\alpha[k]_q}{([n+1]_q+\beta)^2}\\
&\quad +\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\frac{q^{k-1}[k]_q}{([n+1]_q+\beta)^2}+\frac{\alpha^2}{([n+1]_q+\beta)^2}+\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\frac{2q^k[k]_q}{[2]_q([n+1]_q+\beta)^2}\\
&\quad +\frac{2\alpha}{[2]_q([n+1]_q+\beta)^2}\sum_{k=0}^{n+p} b_{n+p,k}(q;x)q^k+\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\frac{q^{2k}}{[3]([n+1]_q+\beta)^2}\\
&= \frac{[n+p]_q[n+p-1]_qx^2}{([n+1]_q+\beta)^2}+\frac{2\alpha[n+p]_qx}{([n+1]_q+\beta)^2}+\frac{[n+p]_qx}{([n+1]_q+\beta)^2}-\frac{(1-q)[n+p]_q[n+p-1]_qx^2}{([n+1]_q+\beta)^2}\\
&\quad +\frac{\alpha^2}{([n+1]_q+\beta)^2}+\frac{2q[n+p]_qx}{[2]_q([n+1]_q+\beta)^2}-\frac{2q(1-q)[n+p]_q[n+p-1]_qx^2}{[2]_q([n+1]_q+\beta)^2}\\
&\quad +\frac{2\alpha\left(1-(1-q)[n+p]_qx\right)}{[2]_q([n+1]_q+\beta)^2}+\frac{1-\left(1-q^2\right)[n+p]_qx+q(1-q)^2[n+p]_q[n+p-1]_qx^2}{[3]_q([n+1]_q+\beta)^2}\\
&= \frac{[n+p]_q[n+p-1]_q}{([n+1]_q+\beta)^2}x^2+\frac{(2[2]_q\alpha+[2]_q+2q)[n+p]_q}{[2]_q([n+1]_q+\beta)^2}x+\frac{[2]_q[3]_q\alpha^2+2[3]_q\alpha+[2]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}\\
&\quad -\frac{(1-q)\left(1-q+4q[3]_q\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}x^2-\frac{(1-q)\left(2\alpha[3]_q+[2]_q^2\right)[n+p]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}x\\
&= \frac{\left(q^2[3]_q+3q^4\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}x^2+\frac{\left(4q[3]_q\alpha+3q+5q^2+4q^3\right)[n+p]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}x
\end{aligned}
$$

$$+\frac{[2]_q[3]_q\alpha^2+2[3]_q\alpha+[2]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}.$$

Thus, (12) is proved. □

Remark 2.4. *From lemma 2.3, it is observed that for $\alpha=\beta=0$, we get the moments for the operators defined in* (1)*, which are the corresponding results of lemma 2.1 in [2].*

Lemma 2.5. *Using lemma 2.3 and easily computations, we have*

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(t-x;x)=\left[\frac{2q[n+p]_q}{[2]_q([n+1]_q+\beta)}-1\right]x+\frac{1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}\doteq A_{n,p,q}^{\alpha,\beta}(x), \tag{13}$$

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}\left((t-x)^2;x\right)\leq\left[\frac{\left(q^2[3]_q+3q^4\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}+1-\frac{4q[n+p]_q}{[2]_q([n+1]_q+\beta)}\right]x^2$$
$$+\frac{[2]_q[3]_q\alpha^2+2[3]_q\alpha+[2]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}+\frac{\left(4q[3]_q\alpha+3q+5q^2+4q^3\right)[n+p]_q}{[2]_q[3]_q([n+1]_q+\beta)^2}x\doteq B_{n,p,q}^{\alpha,\beta}(x). \tag{14}$$

3 Statistical approximation properties

In this section, we present the statistical approximation properties of the operator $\widetilde{K_{n,p,q}^{\alpha,\beta}}$ by using the Korovkin-type statistical approximation theorem proved in [4].

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K is defined by $\delta(K):=\lim_n\frac{1}{n}\sum_{k=1}^{n}\chi_K(k)$ provided the limit exists, where χ_K is the characteristic function of K. A sequence $x:=\{x_n\}$ is called statistically convergent to a number L if, for every $\varepsilon>0$, $\delta\{n\in\mathbb{N}:|x_n-L|\geq\varepsilon\}=0$. Let $A:=(a_{jn}), j,n=1,2,...$ be an infinite summability matrix. For a given sequence $x:=\{x_n\}$, the $A-$transform of x, denoted by $Ax:=((Ax)_j)$, is given by $(Ax)_j=\sum_{k=1}^{\infty}a_{jn}x_n$ provided the series converges for each j. We say that A is regular if $\lim_n(Ax)_j=L$ whenever $\lim x=L$. Assume that A is a non-negative regular summability matrix. A sequence $x=\{x_n\}$ is called A-statistically convergent to L provided that for every $\varepsilon>0$, $\lim_j\sum_{n:|x_n-L|\geq\varepsilon}a_{jn}=0$. We denote this limit by $st_A-\lim_n x_n=L$. For $A=C_1$, the Cesàro matrix of order one, A-statistical convergence reduces to statistical convergence. It is easy to see that every convergent sequence is statistically convergent but not conversely.

We consider a sequence $q:=\{q_n\}$ for $0<q_n<1$ satisfying

$$st_A-\lim_n q_n=1, \tag{15}$$

If $e_i=t^i,\ t\in\mathbb{R}^+,\ i=0,1,2,...$ stands for the ith monomial, then we have

Theorem 3.1. *Let $A=(a_{nk})$ be a non-negative regular summability matrix and $q:=\{q_n\}$ be a sequence satisfying* (15)*, then for all $f\in C(I)$, $x\in[0,1]$, we have*

$$st_A-\lim_n\left\|\widetilde{K_{n,p,q}^{\alpha,\beta}}f-f\right\|_{C(I)}=0. \tag{16}$$

Proof. Obviously

$$st_A - \lim_n \left\|\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_0) - e_0\right\|_{C(I)} = 0. \tag{17}$$

By (13), we have

$$\left|\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_1;x) - e_1(x)\right| \leq \left|\frac{2q_n[n+p]_{q_n}}{[2]_{q_n}([n+1]_{q_n}+\beta)} - 1\right| + \frac{1+[2]_{q_n}\alpha}{[2]_{q_n}([n+1]_{q_n}+\beta)}.$$

Now for a given $\varepsilon > 0$, let us define the following sets:

$$U := \left\{k : \left\|\widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_1) - e_1\right\|_{C(I)} \geq \varepsilon\right\},\ U_1 := \left\{k : \left|\frac{2q_k[n+p]_{q_k}}{[2]_{q_k}([n+1]_{q_k}+\beta)} - 1\right| \geq \frac{\varepsilon}{2}\right\},$$
$$U_2 := \left\{k : \frac{1+[2]_{q_k}\alpha}{[2]_{q_k}([n+1]_{q_k}+\beta)} \geq \frac{\varepsilon}{2}\right\}.$$

Then one can see that $U \subseteq U_1 \cup U_2$, so we have

$$\begin{aligned}\delta\left\{k \leq n : \left\|\widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_1) - e_1\right\|_{C(I)}\right\} &\leq \delta\left\{k \leq n : \left|\frac{2q_k[n+p]_{q_k}}{[2]_{q_k}([n+1]_{q_k}+\beta)} - 1\right| \geq \frac{\varepsilon}{2}\right\} \\ &\quad +\delta\left\{k \leq n : \frac{1+[2]_{q_k}\alpha}{[2]_{q_k}([n+1]_{q_k}+\beta)} \geq \frac{\varepsilon}{2}\right\},\end{aligned}$$

since $st_A - \lim_n q_n = 1$, we have

$$st_A - \lim_n \left|\frac{[n+p]_{q_n}}{[n+1]_{q_n}+\beta} - 1\right| = 0, \quad st_A - \lim_n \frac{1+[2]_{q_n}\alpha}{[2]_{q_n}([n+1]_{q_n}+\beta)} = 0,$$

which implies that the right-hand side of the above inequality is zero, thus we have

$$st_A - \lim_n \left\|\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_1) - e_1\right\|_{C(I)} = 0. \tag{18}$$

Finally, by (10) and (12), we get

$$\begin{aligned}&\left|\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_2;x) - e_2(x)\right| \\ \leq\ &\left|\frac{\left(q_n^2[3]_{q_n}+3q_n^4\right)[n+p]_{q_n}[n+p-1]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n}+\beta)^2} - 1\right| + \frac{\left(4q_n[3]_{q_n}\alpha+3q_n+5q_n^2+4q_n^3\right)[n+p]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n}+\beta)^2} \\ &+\frac{[2]_{q_n}[3]_{q_n}\alpha^2+2[3]_{q_n}\alpha+[2]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n}+\beta)^2} \doteq \alpha_n + \beta_n + \gamma_n.\end{aligned}$$

Since $st_A - \lim_n q_n = 1$, one can see that

$$st_A - \lim_n \alpha_n = st_A - \lim_n \beta_n = st_A - \lim_n \gamma_n = 0. \tag{19}$$

For $\varepsilon > 0$, we define the following four sets

$$V := \left\{k : \left\|\widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_2) - e_2\right\|_{C(I)} \geq \varepsilon\right\},\quad V_1 := \left\{k : \alpha_k \geq \frac{\varepsilon}{3}\right\},\quad V_2 := \left\{k : \beta_k \geq \frac{\varepsilon}{3}\right\},$$

$$V_3 := \left\{k : \gamma_k \geq \frac{\varepsilon}{3}\right\}.$$

Hence, from (19) we obtain the right-hand side of the above inequality is zero, so we have

$$\delta\left\{k \leq n : \left|\left|\widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_2) - e_2\right|\right|_{C(I)} \geq \varepsilon\right\} = 0,$$

thus

$$st_A - \lim_n \left|\left|\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_2) - e_2\right|\right|_{C(I)} = 0. \tag{20}$$

Combining (17), (18) and (20), theorem 3.1 follows from the Korovkin-type statistical approximation theorem established in [4], the proof is completed. □

4 Local approximation properties

Let $f \in C(I)$, endowed with the norm $||f|| = \sup_{x\in I}|f(x)|$. The Peetre's $K-$functional is defined by

$$K_2(f;\delta) = \inf_{g\in C^2}\left\{||f-g|| + \delta||g''||\right\},$$

where $\delta > 0$ and $C^2 = \{g \in C(I) : g', g'' \in C(I)\}$. By [3, p.177, Theorem 2.4], there exits an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{21}$$

where

$$\omega_2(f;\delta) = \sup_{0<h\leq\delta}\ \sup_{x,x+h,x+2h\in I}|f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C(I)$. We denote the usual modulus of continuity of $f \in C(I)$ by

$$\omega(f;\delta) = \sup_{0<h\leq\delta}\ \sup_{x,x+h\in I}|f(x+h) - f(x)|.$$

Now we give a direct local approximation theorem for the operators $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f,x)$.

Theorem 4.1. *For $q \in (0,1)$, $x \in [0,1]$ and $f \in C(I)$, we have*

$$\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f,x) - f(x)\right| \leq C\omega_2\left(f; \sqrt{\left(A_{n,p,q}^{\alpha,\beta}(x)\right)^2 + B_{n,p,q}^{\alpha,\beta}(x)/2}\right) + \omega\left(f; \left|A_{n,p,q}^{\alpha,\beta}(x)\right|\right), \tag{22}$$

where C is a positive constant, $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14).

Proof. We define the auxiliary operators

$$\overline{K_{n,p,q}^{\alpha,\beta}}(f;x) = \widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) - f\left(\frac{2q[n+p]_qx + 1 + [2]_q\alpha}{[2]_q([n+1]_q+\beta)}\right) + f(x), \tag{23}$$

$x \in [0,1]$. The operators $\overline{K_{n,p,q}^{\alpha,\beta}}(f;x)$ are linear and preserve the linear functions:

$$\overline{K_{n,p,q}^{\alpha,\beta}}(t-x;x)=0 \tag{24}$$

(see Lemma 2.3).
Let $g \in C^2$. By Taylor's expansion

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t (t-u)g''(u)du,$$

and (24), we get

$$\overline{K_{n,p,q}^{\alpha,\beta}}(g;x)=g(x)+\overline{K_{n,p,q}^{\alpha,\beta}}\left(\int_x^t (t-u)g''(u)du;x\right).$$

Hence, by (23), (13) and (14), we have

$$\begin{aligned}
\left|\overline{K_{n,p,q}^{\alpha,\beta}}(g;x)-g(x)\right| &\leq \left|\widetilde{K_{n,p,q}^{\alpha,\beta}}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\
&\quad+\left|\int_x^{\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}}\left(\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}-u\right)g''(u)du\right| \\
&\leq \widetilde{K_{n,p,q}^{\alpha,\beta}}\left(\left|\int_x^t (t-u)|g''(u)|du\right|;x\right) \\
&\quad+\int_x^{\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}}\left|\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}-u\right||g''(u)|du \\
&\leq \left\{\widetilde{K_{n,p,q}^{\alpha,\beta}}\left((t-x)^2;x\right)+\left[\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}-x\right]^2\right\}||g''|| \\
&\leq \left[\left(A_{n,p,q}^{\alpha,\beta}(x)\right)^2+B_{n,p,q}^{\alpha,\beta}(x)\right]||g''||,
\end{aligned}$$

where $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14). On the other hand, by (23), (4) and lemma 2.3, we have

$$\left|\overline{K_{n,p,q}^{\alpha,\beta}}(f;x)\right| \leq \left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)\right|+2||f|| \leq ||f||\widetilde{K_{n,p,q}^{\alpha,\beta}}(1;x)+2||f|| \leq 3||f||. \tag{25}$$

Now (23) and (25) imply

$$\begin{aligned}
\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)-f(x)\right| &\leq \left|\overline{K_{n,p,q}^{\alpha,\beta}}(f-g;x)-(f-g)(x)\right|+\left|\overline{K_{n,p,q}^{\alpha,\beta}}(g;x)-g(x)\right| \\
&\quad+\left|f\left(\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}\right)-f(x)\right| \\
&\leq 4||f-g||+\left[\left(A_{n,p,q}^{\alpha,\beta}(x)\right)^2+B_{n,p,q}^{\alpha,\beta}(x)\right]||g''||+\omega\left(f;\left|A_{n,p,q}^{\alpha,\beta}(x)\right|\right).
\end{aligned}$$

Hence taking infimum on the right hand side over all $g \in C^2$, we get

$$\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)-f(x)\right| \leq 4K_2\left(f;\left[\left(A_{n,p,q}^{\alpha,\beta}(x)\right)^2+B_{n,p,q}^{\alpha,\beta}(x)\right]/4\right)+\omega\left(f;\left|A_{n,p,q}^{\alpha,\beta}(x)\right|\right).$$

By (21), for every $q \in (0,1)$, we have

$$\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)-f(x)\right| \leq C\omega_2\left(f;\sqrt{\left(A_{n,p,q}^{\alpha,\beta}(x)\right)^2+B_{n,p,q}^{\alpha,\beta}(x)/2}\right)+\omega\left(f;\left|A_{n,p,q}^{\alpha,\beta}(x)\right|\right),$$

where $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14). This completes the proof of Theorem 4.1. □

Remark 4.2. *For any fixed $x \in [0,1]$, $0 \leq \alpha \leq \beta$, $p \in \mathbb{N}_0$ and $n \in \mathbb{N}$, let $q := \{q_n\}$ be a sequence satisfying $0 < q_n < 1$ and $\lim_n q_n = 1$, we have*

$$\lim_n A_{n,p,q}^{\alpha,\beta}(x)=0 \quad \textit{and} \quad \lim_n B_{n,p,q}^{\alpha,\beta}(x)=0,$$

where $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14). These give us a rate of pointwise convergence of the operators $\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(f;x)$ to $f(x)$.

Next we study the rate of convergence of the operators $K_{n,q}(f;x)$ with the help of functions of Lipschitz class $Lip_M(\xi)$, where $M>0$ and $0<\xi\leq 1$. A function f belongs to $Lip_M(\xi)$ if

$$|f(y)-f(x)| \leq M|y-x|^{\xi} \quad (y,x \in \mathbb{R}). \tag{26}$$

We have the following theorem.

Theorem 4.3. *Let $q := \{q_n\}$ be a sequence satisfying $0<q_n<1$, $\lim\limits_n q_n = 1$ and $f \in Lip_M(\xi)$, $0<\xi\leq 1$. Then we have*

$$\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)-f(x)\right| \leq M\left(B_{n,p,q}^{\alpha,\beta}(x)\right)^{\frac{\xi}{2}}, \tag{27}$$

where $B_{n,p,q}^{\alpha,\beta}(x)$ is defined in (14).

Proof. Since $\widetilde{K_{n,p,q}^{\alpha,\beta}}$ is a linear positive operator and $f \in Lip_M(\xi)$ $(0<\xi\leq 1)$, we have

$$\begin{aligned}
&|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)-f(x)| \\
\leq\ & \widetilde{K_{n,p,q}^{\alpha,\beta}}(|f(t)-f(x)|;x) \\
=\ & ([n+1]_q+\beta)\sum_{k=0}^{n+p} b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} |f(t)-f(x)|d_q^R t \\
\leq\ & M([n+1]_q+\beta)\sum_{k=0}^{n+p} b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} |t-x|^{\xi}d_q^R t
\end{aligned}$$

$$\begin{aligned}
&\leq M([n+1]_q+\beta)\sum_{k=0}^{n+p} b_{n+p,k}(q;x)q^{-k}\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}\left[(t-x)^{\xi}\right]^{\frac{2}{\xi}} d_q^R t\right)^{\frac{\xi}{2}}\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} d_q^R t\right)^{\frac{2-\xi}{2}} \\
&= M([n+1]_q+\beta)\sum_{k=0}^{n+p} b_{n+p,k}(q;x)q^{-k}\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2 d_q^R t\right)^{\frac{\xi}{2}}\left(\frac{q^k}{[n+1]_q+\beta}\right)^{\frac{2-\xi}{2}} \\
&= M\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2 d_q^R t\right)^{\frac{\xi}{2}}\left(\frac{[n+1]_q+\beta}{q^k}\right)^{\frac{\xi}{2}} \\
&= M\sum_{k=0}^{n+p}\left[b_{n+p,k}(q;x)\right]^{\frac{2-\xi}{2}}\left(([n+1]_q+\beta)b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2 d_q^R t\right)^{\frac{\xi}{2}}.
\end{aligned}$$

Applying Hölder's inequality for sums, we obtain

$$\begin{aligned}
&\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)-f(x)\right| \\
&\leq M\left(\sum_{k=0}^{n+p} b_{n+p,k}(q;x)\right)^{\frac{2-\xi}{2}}\left(\sum_{k=0}^{n+p}([n+1]_q+\beta)b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2 d_q^R t\right)^{\frac{\xi}{2}} \\
&= M\left(\widetilde{K_{n,p,q}^{\alpha,\beta}}\left((t-x)^2;x\right)\right)^{\frac{\xi}{2}} = M\left(B_{n,p,q}^{\alpha,\beta}(x)\right)^{\frac{\xi}{2}}.
\end{aligned}$$

Thus, theorem 4.3 is proved. □

5 Graphical analysis

In this section, we will illustrate two examples to state the convergence of operators $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ to $f(x)$ by means of Graphs.

Example 1: From figure 1, we can observe that as q increases, $n = 50$ be fixed, Kantorovich-type q-Bernstein-Stancu-Schurer operators given by (4) converge to the function $f(x) = \sin(2\pi x)$.

In comparison to figure 1, let $q = 0.99$ be fixed, as n increases, operators given by (4) converge to the function as shown in figure 2.

Example2: Similarly for different values of parameters q and n, let $p = 1$, $\alpha = 2$ and $\beta = 3$, convergence of operators to the function $f(x) = 1 - \cos(4e^x)$ is shown in figure 3 and 4, respectively.

Acknowledgement

This work is supported by the National Natural Science Foundation of China (Grant No. 61572020, 11601266), the China Postdoctoral Science Foundation funded project (Grant No. 2015M582036), the Natural Science Foundation of Fujian Province of China (Grant No. 2016J05017),

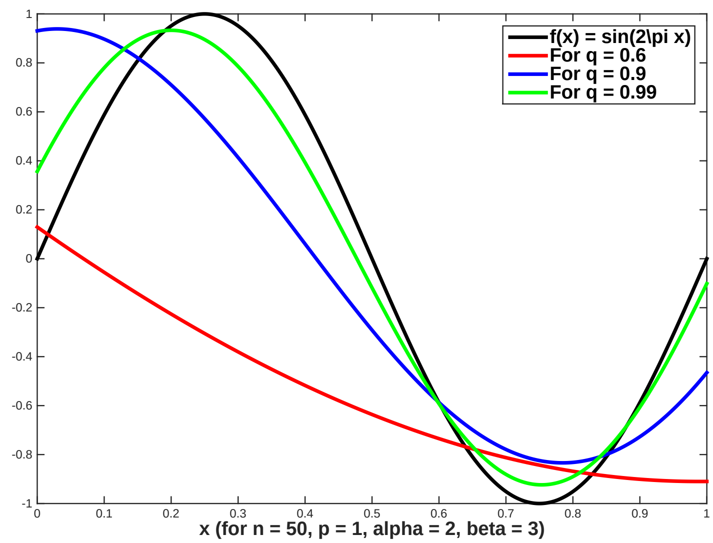

Figure 1: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $n = 50$, $p = 1$, $\alpha = 2$, $\beta = 3$ and different values of q.

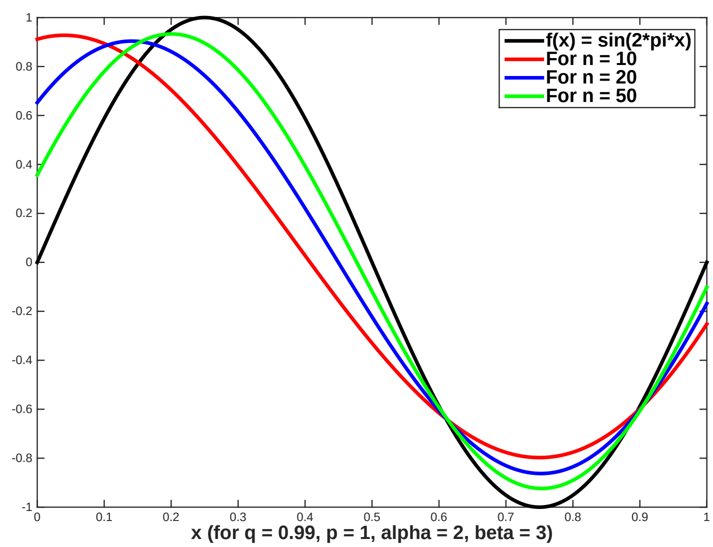

Figure 2: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $q = 0.99$, $p = 1$, $\alpha = 2$, $\beta = 3$ and different values of n.

the Startup Project of Doctor Scientific Research and Young Doctor Pre-Research Fund Project of Quanzhou Normal University (Grant No. 2015QBKJ01), Fujian Provincial Key Laboratory of Data Intensive Computing and Key Laboratory of Intelligent Computing and Information Processing, Fujian Province University.

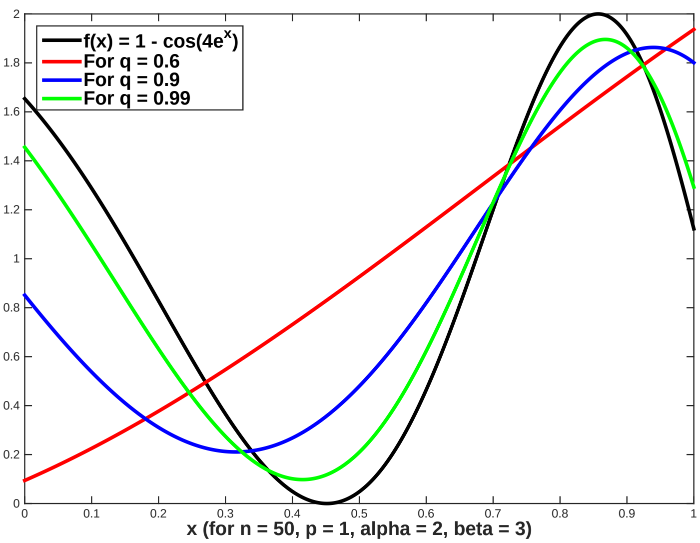

Figure 3: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $n=50$, $p=1$, $\alpha=2$, $\beta=3$ and different values of q.

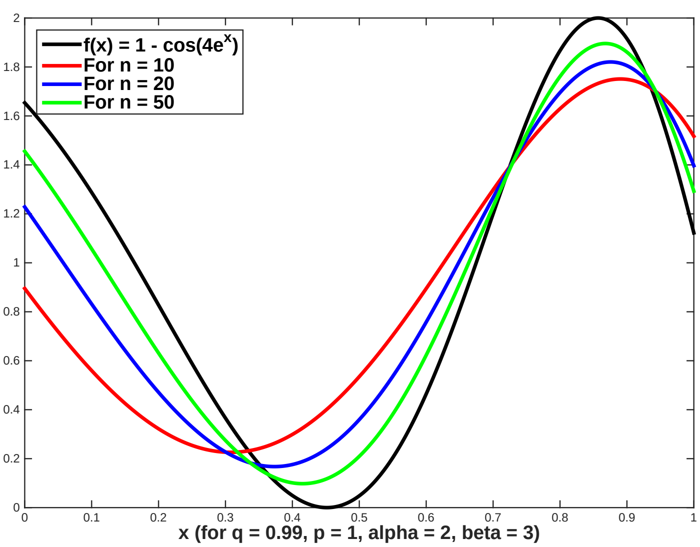

Figure 4: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $q=0.99$, $p=1$, $\alpha=2$, $\beta=3$ and different values of n.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.5, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

On the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$

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Abstract: In this paper, we study the exact values of the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ for X being $l_\infty - l_1$ and $l_q - l_1$ spaces. Moreover, we shown that some new conditions for uniformly normal structure of a Banach space X.
Keywords: generalized von Neumann-Jordan constant; $l_\infty - l_1$ and $l_q - l_1$ space; uniformly normal structure
2000 Mathematics subject classification : 46B20.

1. Introduction

In order to study the geometric structure of a Banach space, many geometric constant have been investigated. In particular, the von Neuman-Jordan constant $C_{NJ}(X)$ is widely treated. In[1], as a generalization of the von Neuman-Jordan constant, a new geometric constant called the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ was introduced. It is proved that the $C_{NJ}^{(p)}(X)$ is strongly connected with geometric structure, such as uniformly non-square, uniformly normal structure. Hence it's necessary to compute the $C_{NJ}^{(p)}(X)$ for some concrete spaces.

Throughout this paper, let $X = (X, \|\cdot\|)$ be a real Banach spaces. We will use B_X, S_X and $ex(B_X)$ to denote unit ball, unit sphere of X and the set of extreme points of B_X, respectively.

Recall that the von Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space X was introduced by Clarkson[3], as the smallest constant C for which,

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C,$$

holds for all $x, y \in X$.

An equivalent definition of the constant is

$$C_{NJ}(X) = \sup\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X\}.$$

The properties of $C_{NJ}(X)$ have been investigated in many papers(see for instances [2],[4],[8],[9],[10]). Recently, a generalized form of this constant was introduced as following

Definition 1.[1] The generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ is defined by

$$C_{NJ}^{(p)}(X) := \sup\{\frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}(\|x\|^p + \|y\|^p)} : x, y \in X, (x,y) \neq (0,0)\},$$

where $1 \leq p < \infty$.

It's equivalent to

$$C_{NJ}^{(p)}(X) = \sup\{\frac{\|x+ty\|^p + \|x-ty\|^p}{2^{p-1}(1+t^p)} : x, y \in S_X, 0 \leq t \leq 1\},$$

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[1]supported by the National Natural Science Foundation of P. R. China(11271112; 11201127), Innovation Scientists and Technicians Troop Construction Projects of Henan Province(114200510011).

where $1 \le p < \infty$.

Now let us collect some properties of this constant (see [1]):
(i) $1 \le C_{NJ}^{(p)}(X) \le 2$;
(ii) X is uniformly non-square if and only if $C_{NJ}^{(p)}(X) < 2$;
(iii) Let $r \in (1,2]$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Then for $X = L_r[0,1]$,
(1) if $1 < p \le r$ then $C_{NJ}^{(p)}(X) = 2^{2-p}$ and if $r < p \le r'$ then $C_{NJ}^{(p)}(X) = 2^{\frac{p}{r}-p+1}$,
(2) if $r' < p < \infty$ then $C_{NJ}^{(p)}(X) = 1$.

In this paper, we study the exact values of the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ for X being $l_\infty - l_1$ and $l_q - l_1$ space. Moreover, we shown that some new conditions for uniformly normal structure of a Banach space X.

2. Main Results

Firstly, we consider $l_\infty - l_1$ space. As $C_{NJ}^{(1)}(X) = 2$ for any Banach space X, we only consider the case $p > 1$.
Theorem 2.1. ($l_\infty - l_1$ spaces). Let $p > 1$ and $X = l_\infty - l_1$ which is $\mathbb{R}^2$ endowed with the norm

$$\|x\| = \begin{cases} \|x\|_\infty, \ if \ x_1x_2 \ge 0, \\ \|x\|_1, \ if \ x_1x_2 \le 0. \end{cases}$$

Then

$$C_{NJ}^{(p)}(l_\infty - l_1) = \frac{(1+t_0)^p + 1}{2^{p-1}(1+t_0^p)} = \frac{1}{2^{p-1}(1-t_0^{p-1})}, \tag{2.1}$$

where $t_0 \in (0,1)$ is the unique solution of the equation

$$(1+t)^{p-1} - t^{p-1} - t^{p-1}(1+t)^{p-1} = 0. \tag{2.2}$$

Proof. Firstly we shall show that $\|x+ty\|^p + \|x-ty\|^p \le 1 + (1+t)^p$ for any $x, y \in S_X$ and every $t \in [0,1]$.
By Minkowski inequality, for any $\alpha, \beta \in [0,1]$ and any $x_1, x_2, y_1, y_2 \in B_X$ with $x = \alpha X_1 + (1-\alpha)x_2, y = \beta y_2 + (1-\beta)y_2$, we have

$$\begin{aligned}
&\|x+ty\|^p + \|x-ty\|^p \\
&= \|\alpha(x_1+ty) + (1-\alpha)(x_2+ty)\|^p + \|\alpha(x_1-ty) + (1-\alpha)(x_2-ty\|^p \\
&\le \alpha\|x_1+ty\|^p + (1-\alpha)\|x_2+ty\|^p + \alpha\|x_1-ty\|^p + (1-\alpha)\|x_2-ty\|^p \\
&= \alpha[\|\beta(x_1+ty_1) + (1-\beta)(x_1+ty_2)\|^p + \|\beta(x_1-ty_1) + (1-\beta)(x_1-ty_2)\|^p] \\
&+(1-\alpha)[\|\beta(x_2+ty_1) + (1-\beta)(x_2+ty_2)\|^p + \|\beta(x_2-ty_1) + (1-\beta)(x_2-ty_2)\|^p] \\
&\le \alpha\beta[\|x_1+ty_1\|^p + \|x_1-ty_1\|^p] + \alpha(1-\beta)[\|x_1+ty_2\|^p + \|x_1-ty_2\|^p] \\
&+(1-\alpha)\beta[\|x_2+ty_1\|^p + \|x_2-ty_1\|^p] + (1-\alpha)(1-\beta)[\|x_2+ty_2\|^p + \|x_2-ty_2\|^p]
\end{aligned}$$

Hence, we only need to prove $\|x+ty\|^p + \|x-ty\|^p \le 1 + (1+t)^p$ for any $x, y \in ex(B_X)$ and every $t \in [0,1]$.
Since $ex(B_X) = \{(1,0),(0,1),(1,1),(-1,0),(-1,-1),(0,-1)\}$ and we can change x into $-x$ or y into $-y$. So we may assume that $x, y = (0,1),(1,0)$ or $(1,1)$. Obviously, for these x, y we easily have $\|x+ty\|^p + \|x-ty\|^p \le 1 + (1+t)^p$ for every $t \in [0,1]$. Therefore,

$$C_{NJ}^{(p)}(l_\infty - l_1) \le \sup_{t\in[0,1]}\{\frac{(1+t)^p+1}{2^{p-1}(1+t^p)}\}.$$

Let $f(t) = \frac{(1+t)^p+1}{1+t^p}$, then

$$f^{'}(t) = \frac{p(1+t)^{p-1}}{(1+t^p)^2}[1-t^{p-1}-(\frac{t}{1+t})^{p-1}].$$

Defining $h(t) = 1-t^{p-1}-(\frac{t}{1+t})^{p-1}$, we have $h(t)$ is decreasing from 1 to $-\frac{1}{2^{p-1}}$ on $[0,1]$. Whence there exists an unique $t_0 \in (0,1)$ such that $h(t_0) = 0$. Therefore,

$$C_{NJ}^{(p)}(l_\infty - l_1) \le \frac{(1+t_0)^p+1}{2^{p-1}(1+t_0^p)}.$$

On the other hand, by taking $x_0 = (1,0), y_0 = (t_0,t_0)$, we have

$$C_{NJ}^{(p)}(l_\infty - l_1) \ge \frac{(1+t_0)^p+1}{2^{p-1}(1+t_0^p)}.$$

Hence,

$$C_{NJ}^{(p)}(l_\infty - l_1) = \frac{(1+t_0)^p+1}{2^{p-1}(1+t_0^p)},$$

where $t_0 \in (0,1)$ is the unique solution of $1-t^{p-1} = (\frac{t}{1+t})^{p-1}$.
From (2.2), we also have

$$(1+t_0)^p+1 = (1+t_0)\frac{t_0^{p-1}}{1-t_0^{p-1}}+1 = \frac{1+t_0^p}{1-t_0^{p-1}}.$$

Therefore (2.1) is obtained.
Corollary 2.2. For $X = l_\infty - l_1$, we have

$$C_{NJ}^{(\frac{3}{2})}(X) = \frac{1}{\sqrt{2}-\sqrt{2\sqrt{2}+1-\sqrt{5+4\sqrt{2}}}} \approx 1.5077. \tag{2.3}$$

and

$$C_{NJ}^{(3)}(X) = \frac{3+2\sqrt{2}+\sqrt{5+4\sqrt{2}}}{8} \approx 1.1366. \tag{2.4}$$

Proof. (1) For $p = \frac{3}{2}$, (2.2) is equivalent to $t^4+1-2t^3-2t-5t^2 = 0$.
that is

$$t^2+\frac{1}{t^2}-2(t+\frac{1}{t}) = 5.$$

Hence, we can get $t = \frac{2\sqrt{2}+1-\sqrt{5+4\sqrt{2}}}{2}$ and (2.3) is valid by(2.1).
(2) For $p = 3$, (2.2) is equivalent to $t^2 = (1+t)^2(1-t^2)$. Letting $t = u-1$, we have

$$u^4+1-2u^3-2u+u^2 = 0.$$

that is

$$u^2+\frac{1}{u^2}-2(u+\frac{1}{u}) = -1.$$

Hence, $u = \frac{\sqrt{2}+1+\sqrt{2\sqrt{2}-1}}{2}$ and $t = \frac{\sqrt{2}-1+\sqrt{2\sqrt{2}-1}}{2}$. Therefore

$$C_{NJ}^{(3)}(X) = \frac{1}{4(1-t^2)} = \frac{1}{2-2(\sqrt{2}-1)\sqrt{2\sqrt{2}-1}} = \frac{3+2\sqrt{2}+\sqrt{5+4\sqrt{2}}}{8} \approx 1.1366.$$

Theorem 2.3. ($l_q - l_1$ spaces). If $p \geq q > 1$. Let $X = \mathbb{R}^2$ endowed with the norm

$$\|x\| = \begin{cases} \|x\|_q, if\ x_1x_2 \geq 0 \\ \|x\|_1, if\ x_1x_2 \leq 0 \end{cases},$$

then

$$C_{NJ}^{(p)}(l_q - l_1) = 1 + 2^{\frac{p}{q}-p}.$$

In order to prove this theorem, firstly we give the following lemma.

Lemma2.4. Let $a, b, c, d \geq 0$ and $p \geq q > 1$ such that $a^q + b^q = 1$ and $c^q + d^q = 1$. If $0 \leq t \leq 1$, $a \geq ct$ and $b \leq dt$, then

$$[(a+ct)^q + (b+dt)^q]^{\frac{p}{q}} + (a - ct + dt - b)^p \leq (1+t)^p + (1+t^q)^{\frac{p}{q}}.$$

Proof. Clearly, $0 \leq a - ct + dt - b \leq 1 + t$. So we will consider the following two cases.

Case I. if $0 \leq a - ct + dt - b \leq (1+t^q)^{\frac{1}{q}}$, then

$$\begin{aligned}
&[(a+ct)^q + (b+dt)^q]^{\frac{p}{q}} + (a - ct + dt - b)^p \\
&\leq [(a^q + b^q)^{\frac{1}{q}} + t(c^q + d^q)^{\frac{1}{q}}]^p + (1+t^q)^{\frac{p}{q}} \\
&= (1+t)^p + (1+t^q)^{\frac{p}{q}}.
\end{aligned}$$

Case II. if $(1+t^q)^{\frac{1}{q}} \leq a - ct + dt - b \leq 1 + t$, then

$$\begin{aligned}
&[(a+ct)^q + (b+dt)^q]^{\frac{1}{q}} + (a - ct + dt - b) \\
&\leq (a^q + d^q t^q)^{\frac{1}{q}} + (c^q t^q + b^q)^{\frac{1}{q}} + a - ct + dt - b \\
&\leq (1+t^q)^{\frac{1}{q}} + ct + b + a - ct + dt - b \\
&\leq (1+t^q)^{\frac{1}{q}} + 1 + t.
\end{aligned}$$

So,

$$[(a+ct)^q + (b+dt)^q]^{\frac{1}{q}} \leq (1+t^q)^{\frac{1}{q}} + 1 + t - (a - ct + dt - b).$$

Thus,

$$\begin{aligned}
&[(a+ct)^q + (b+dt)^q]^{\frac{p}{q}} + (a - ct + dt - b)^p \\
&\leq [(1+t^q)^{\frac{1}{q}} + 1 + t - (a - ct + dt - b)]^p + (a - ct + dt - b)^p \\
&\leq \max_{u \in [(1+t^q)^{\frac{1}{q}}, 1+t]} [(1+t^q)^{\frac{1}{q}} + 1 + t - u]^p + u^p \\
&= (1+t)^p + (1+t^q)^{\frac{p}{q}}.
\end{aligned}$$

Proof of Theorem 2.3

Note that $ex(B_X) = \{(x_1, x_2) : x_1^q + x_2^q = 1, x_1x_2 \geq 0\}$.

Now we prove that

$$\|x + ty\|^p + \|x - ty\|^p \leq (1+t)^p + (1+t^q)^{\frac{p}{q}},$$

holds for any $x, y \in ex(B_X)$ and any $t \in [0, 1]$.

Case I. If $(a-ct)(b-dt)\geq 0$. By Minkowski inequality, we have

$$\begin{aligned}
&\|x+ty\|^p+\|x-ty\|^p\\
&=\|x+ty\|_q^p+\|x-ty\|_q^p\\
&=[(a+ct)^q+(b+dt)^q]^{\frac{p}{q}}+[|a-ct|^q+|b-dt|^q]^{\frac{p}{q}}\\
&\leq[(a^q+b^q)^{\frac{1}{q}}+(c^qt^q+d^qt^q)^{\frac{1}{q}}]^p+1\\
&\leq(1+t)^p+1\\
&\leq(1+t)^p+(1+t^q)^{\frac{p}{q}}.
\end{aligned}$$

Case II. If $(a-ct)(b-dt)\leq 0$. By Lemma2.4, we have that

$$\begin{aligned}
&\|x+ty\|^p+\|x-ty\|^p\\
&=\|x+ty\|_q^p+\|x-ty\|_1^p\\
&=[(a+ct)^q+(b+dt)^q]^{\frac{p}{q}}+(a-ct+dt-b)^p\\
&\leq(1+t)^p+(1+t^q)^{\frac{p}{q}}.
\end{aligned}$$

Therefore, $\|x+ty\|^p+\|x-ty\|^p\leq(1+t)^p+(1+t^q)^{\frac{p}{q}}$ is also valid for any $x,y\in S_X$. Hence ,

$$C_{NJ}^{(p)}(l_q-l_1)\leq\frac{(1+t)^p+(1+t^q)^{\frac{p}{q}}}{2^{p-1}(1+t^p)}.$$

On the other hand, for every $t\in[0,1]$, taking $x_0=(1,0),y_0=(0,1)$, we have

$$\begin{aligned}
&C_{NJ}^{(p)}(l_q-l_1)\\
&\geq\tfrac{\|x_0+ty_0\|^p+\|x_0-ty_0\|^p}{2^{p-1}(1+t^p)}\\
&=\tfrac{(1+t)^p+(1+t^q)^{\frac{p}{q}}}{2^{p-1}(1+t^p)}.
\end{aligned}$$

Hence,

$$C_{NJ}^{(p)}(l_q-l_1)=\max_{t\in[0,1]}\frac{(1+t)^p+(1+t^q)^{\frac{p}{q}}}{2^{p-1}(1+t^p)}.$$

We let $f(t)=\frac{(1+t)^p+(1+t^q)^{\frac{p}{q}}}{1+t^p}$, so

$$f'(t)=\frac{p\{(1+t^q)^{\frac{p}{q}-1}(t^{q-1}-t^{p-1})+(1+t)^{p-1}(1-t^{p-1})\}}{(1+t^p)^2}\geq 0.$$

That imply $f(t)$ is not decreasing. Hence,

$$\begin{aligned}
&C_{NJ}^{(p)}(l_q-l_1)\\
&=2^{1-p}\max\nolimits_{t\in[0,1]}f(t)\\
&=2^{1-p}f(1)=1+2^{\frac{p}{q}-p}.
\end{aligned}$$

Lemma2.6. Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$C_{NJ}^{(p)}(X)=2^{1-\frac{p}{q}}C_{NJ}^{(q)}(X^*)^{\frac{p}{q}}$$

and

$$C_{NJ}^{(p)}(X)=C_{NJ}^{(p)}(X^{**}),$$

where X^* is the dual of X.

Proof. Let $l_p(X)=\{(x_1,x_2):\|(x_1,x_2)\|=(\|x_1\|^p+\|x_2\|^p)^{\frac{1}{p}}\}$ and define the operator $A:l_p(X)\to l_p(X)$ by $(x_1,x_2)\mapsto(x_1+x_2,x_1-x_2)$. Then we easily have $C_{NJ}^{(p)}(X)=\frac{\|A\|^p}{2^{p-1}}$. Similarly, $C_{NJ}^{(q)}(X^*)=$

$\frac{\|A^*\|^q}{2^{q-1}}$. So $C_{NJ}^{(p)}(X) = 2^{1-\frac{p}{q}} C_{NJ}^{(q)}(X^*)^{\frac{p}{q}}$ by $\|A\| = \|A^*\|$, and hence $C_{NJ}^{(q)}(X^*) = 2^{1-\frac{q}{p}} C_{NJ}^{(p)}(X^{**})^{\frac{q}{p}}$. Therefore, we have $C_{NJ}^{(p)}(X) = C_{NJ}^{(p)}(X^{**})$.

The relationship between the constant $C_{NJ}^{(p)}(X)$ and the uniformly normal structure of X as follows:

Theorem 2.7. The Banach space X has uniformly normal structure if any one of the following conditions is valid

(i)$C_{NJ}^{(p)}(X) < \frac{\left(1+\sqrt{1+2^{\frac{2p-3}{p-1}}}\right)^{p-1}}{2^{2p-3}}$ for some $p \in \left(1, \frac{3-log_2 3}{2-log_2 3}\right)$;

(ii)$C_{NJ}^{(q)}(X^*) < \frac{1+(1+2^{3-q})^{\frac{1}{2}}}{2}$ for some $q > 1$,

where $p^{-1} + q^{-1} = 1$.

Proof. According to $C_{NJ}^{(p)}(X) < 2$, we have X is uniformly non-square, so we only need to prove X has weak normal structure.

Assume that X has no weak normal structure. Then it is well known (see[5])that for any $\varepsilon > 0$ there exists $z_1, z_2, z_3 \in S_X$ and $g_1, g_2, g_3 \in S_{X^*}$ satisfying the following statements:

(i) for all $i \neq j$, we have $|\|z_i - z_j\| - 1| < \varepsilon, |g_i(z_j)| < \varepsilon$,

(ii) $g_i(z_j) = 1$ for $i = 1, 2, 3$,

(iii) $\|z_3 - (z_2 + z_1)\| \geq \|z_2 + z_1\| - \varepsilon$.

Let us fix $\varepsilon > 0$ as small as needed. Then, we can find $z_1, z_2, z_3 \in S_X$ and $g_1, g_2, g_3 \in S_{X^*}$ satisfying the above properties.

(1)Taking $\alpha = \frac{\left(1+\sqrt{1+2^{\frac{2p-3}{p-1}}}\right)^{p-1}}{2^{2p-3}}$. We will consider the following two cases:

Case I. If $\|z_2 + z_1\| \leq \alpha$. Then,

$$\begin{aligned}
&\frac{\|g_1+g_2\|^q+\|g_2-g_1\|^q}{2^{q-1}(\|g_2\|^q+\|g_1\|^q)} \\
\geq\ &\frac{[(g_1+g_2)(\frac{z_2+z_1}{\alpha})]^q+[(g_2-g_1)(\frac{z_2-z_1}{\|z_2-z_1\|})]^q}{2^q} \\
\geq\ &\frac{(\frac{2-2\varepsilon}{\alpha})^q+(\frac{2-2\varepsilon}{1+\varepsilon})^q}{2^q} \\
=\ &(\frac{1-\varepsilon}{\alpha})^q + (\frac{1-\varepsilon}{1+\varepsilon})^q.
\end{aligned}$$

Case II. If $\|z_2 + z_1\| > \alpha$. Then, the contains two sub-cases:

(i) If $\|z_3 - z_2 + z_1\| \leq \alpha$. Then,

$$\begin{aligned}
&\frac{\|g_1+g_3\|^q+\|g_3-g_1\|^q}{2^{q-1}(\|g_3\|^q+\|g_1\|^q)} \\
\geq\ &\frac{[(g_1+g_3)(\frac{z_3-z_2+z_1}{\alpha})]^q+[(g_3-g_1)(\frac{z_3-z_1}{\|z_3-z_1\|})]^q}{2^q} \\
\geq\ &\frac{(\frac{2-4\varepsilon}{\alpha})^q+(\frac{2-2\varepsilon}{1+\varepsilon})^q}{2^q} \\
=\ &(\frac{1-2\varepsilon}{\alpha})^q + (\frac{1-\varepsilon}{1+\varepsilon})^q.
\end{aligned}$$

(ii) If $\|z_3 - z_2 + z_1\| > \alpha$. Then,

$$\begin{aligned}
&\frac{\|z_3-z_2+z_1\|^p+\|z_3-z_2-z_1\|^p}{2^{p-1}(\|z_3-z_2\|^p+\|z_1\|^p)} \\
\geq\ &\frac{\alpha^p+(\|z_2+z_1\|-\varepsilon)^p}{2^{p-1}[(1+\varepsilon)^p+1]} \\
\geq\ &\frac{\alpha^p+(\alpha-\varepsilon)^p}{2^{p-1}[(1+\varepsilon)^p+1]}.
\end{aligned}$$

Letting $\varepsilon \to 0$, and by lemma2.6 we have

$$C_{NJ}^{(p)}(X) \geq \min\{2^{1-\frac{p}{q}}(\frac{1}{\alpha^q}+1)^{\frac{p}{q}}, \frac{\alpha^p}{2^{p-1}}\} = \frac{\left(1+\sqrt{1+2^{\frac{2p-3}{p-1}}}\right)^{p-1}}{2^{2p-3}},$$

which contradicts to the hypothesis(i).

(2)Taking $\alpha = \frac{1+(1+2^{3-q})^{\frac{1}{2}}}{2}$. By the proof of (1), we have

$$C_{NJ}^{(q)}(X^*) \geq \min\{\frac{1}{\alpha^q}+1, 2^{1-\frac{q}{p}}(\frac{\alpha^p}{2^{p-1}})^{\frac{q}{p}}\} = \min\{\frac{1}{\alpha^q}+1, \frac{\alpha^q}{2^{q-1}}\} = \frac{1+(1+2^{3-q})^{\frac{1}{2}}}{2},$$

which contradicts to the hypothesis(ii).

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Discrete dynamical systems in soft topological spaces *

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Abstract In this paper the iteration of soft continuous functions is investigated and their discrete dynamical systems in soft topological spaces are defined. Some basic concepts related to discrete dynamical systems (such as soft ω-limit set, soft invariant set, soft periodic point, soft nonwandering point, and soft recurrent point) are introduced into soft topological spaces. Soft topological mixing and soft topological transitivity are also studied. At last, soft topological entropy is defined and several properties of it are discussed.

Keywords Soft point, Soft ω-limit set, Soft nonwandering point, Soft topological mixing, Soft topological transitivity, Soft topological entropy

1 Introduction and preliminaries

The real world is too complex for our immediate and direct understanding, so we create *models* which are simplifications of the real word. In 1999, Molodtsov [1] introduced the concept of soft set which gives a new approach to modeling uncertainties. And he also discussed the application of soft set theory in many fields, such as: operations analysis, game theory, the smoothness of function, and so on[2]. Maji et al.[3] and Ali et al.[4] defined some operators of soft sets. Beyond these theoretical works of soft set, research works on its applications in various fields are progressing rapidly, and great progress has been achieved, including soft set theory in abstract algebras[5−10], decision making, data analysis, information system, and so on[11−14]. The application of soft set theory in algebraic structures was introduced by Aktaş and Çağman[5], they defined the notion of soft groups and progressed some basic properties. Jun[6,7] investigated soft BCK/BCI-algebras and its application in ideal theory. Dudek et al.[8] discussed soft ideals in BCC-algebras. Zhang[9] studied intuitionistic fuzzy soft rings. Feng et al.[10] worked on soft semirings, soft ideals and idealistic soft semirings. Maji et al.[11] first applied soft sets to solve the decision making problem that is based on the concept of knowledge reduction in the theory of rough sets[12]. Based on the analysis of the rough set model on a tolerance relation and the fuzzy rough set, two types of fuzzy rough sets models on tolerance relations are constructed and researched by Xu et al.[13]. Chen et al.[14] presented a

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new definition of soft set parametrization reduction so as to improve the soft set based decision making in [11]. Yang[15] combined the multi-fuzzy set and soft set, from which they obtained a new soft set model named multi-fuzzy soft set, and applied it to decision making. Soft set theory is also be used in topology. Shabir and Naz's work[16] on soft topological spaces defined over an initial universe with a fixed set of parameters. The notions of soft open set, soft closed set, soft closure, soft interior point, soft neighborhood of a point, and soft separation axioms (such as soft T_i-space for $i = 1, 2, 3, 4$, soft normal space, and soft regular space) were also introduced and their basic properties were investigated. Min[17] pointed out some mistakes of [16] and investigated some properties of the soft separation axioms defined in [15]. Zorlutuna etc.[18] introduced some new concepts in soft topological spaces (such as soft point, interior point, interior, neighborhood, continuity, and compactness).

Motivated by Chen etc.[19] and Liu[20], this paper will investigate iteration of soft continuous functions and their discrete dynamical systems in soft topological spaces. Some basic concepts on dynamical systems (such as soft ω-limit set, soft invariant set, soft periodic point, soft nonwandering point, and soft recurrent point) are introduced in soft topological spaces, soft topological mixing, soft topological transitivity, soft topological entropy and its several properties are studied. As a result, some conclusions of discrete dynamical systems in ordinary topological spaces are generalized. Now we give some definitions and results to be used in this paper.

Definition 1[1] A soft set on a set X is a triple (M, E, X), where $M : E \longrightarrow 2^X$ (the set of all subsets of X) is a mapping. The set of all soft sets on X is denoted by $\mathbb{S}(X, E)$.

Roughly speaking, a soft set on a set X is just a family $\{M_e\}_{e\in E}$ of subsets of X; it can be looked to be a subset of X if E is a singleton.

Let $(M, E, X), (N, E, X) \in \mathbb{S}(X, E)$. If $M(e) \subseteq N(e)$ $(\forall e \in E)$, then (M, E, X) is called a soft subset of (N, E, X), denoted by $(M, E, X)\widetilde{\subseteq}(N, E, X)$. If $(M, E, X)\widetilde{\subseteq}(N, E, X)$ and $(M, E, X)\widetilde{\supseteq}(N, E, X)$, then (M, E, X) and (N, E, X) are said to be soft equal, denoted by $(M, E, X) = (N, E, X)$.

Remark 1[16] (1) Let X be a set, and $A \in 2^X$. Define $\widetilde{A} : E \longrightarrow 2^X$ as $\widetilde{A}(e) = A$ $(\forall e \in E)$, then $(\widetilde{A}, E, X) \in \mathbb{S}(X, E)$; we use $\widetilde{A}$ to denote this soft set (particularly, we use $\widetilde{x}$ to denote the soft set $\widetilde{\{x\}}$).

(2) Let X be a set, and $(M, E, X) \in \mathbb{S}(X, E)$. Then $(M', E, X) \in \mathbb{S}(X, E)$, where $M' : E \longrightarrow 2^X$ is defined as

$$M'(e) = X - M(e)\ (\forall e \in E).$$

Sometimes we use $(M, E, X)'$ to replace (M', E, X).

(3) Let X be a set, $\{(H_j, E, X)\}_{j\in J} \subseteq \mathbb{S}(X, E)$. Then $(M, E, X), (N, E, X) \in \mathbb{S}(X, E)$, called the union (denoted as $\widetilde{\bigcup}_{j\in J}(H_j, E, X)$) and intersection (denoted as $\widetilde{\bigcap}_{j\in J}(H_j, E, X)$)

of the family $\{(H_j, E, X)\}_{j\in J}$ respectively, where

$$M(e) = \bigcup_{j\in J} H_j(e) \ (\forall e \in E)$$

and

$$N(e) = \bigcap_{j\in J} H_j(e) \ (\forall e \in E).$$

(4) Let X be a set, $(H, E, X) \in \mathbb{S}(X, E)$, and $x \in X$. Write $x \in (H, E, X)$ if $x \in H(e)$ $(\forall e \in E)$, and $x \notin (H, E, X)$ if $x \notin H(e)$ for some $e \in E$.

(5) Let X be a set. The difference of the two soft sets (M, E, X) and (N, E, X) is a soft set (H, E, X) over X (usually, denoted by $(M, E, X) - (N, E, X)$) which is defined by $H(e) = M(e) - N(e)$ $(\forall e \in E)$.

(6) Let X be a set, and $(M, E, X), (N, E, X) \in \mathbb{S}(X, E)$. Then

(i) $((M, E, X)\widetilde{\cup}(N, E, X))' = (M, E, X)'\widetilde{\cap}(N, E, X)'$;

(ii) $((M, E, X)\widetilde{\cap}(N, E, X))' = (M, E, X)'\widetilde{\cup}(N, E, X)'$.

Definition 2[18] (1) A soft set $(M, E, X) \in \mathbb{S}(X, E)$ is called elementary (or a soft point in $\widetilde{X}$, denoted by e_M) if $M(e) \neq \emptyset$ for some $e \in E$ and $M(e') = \emptyset$ for all $e' \in E - \{e\}$.

(2) Let e_M be a soft point in $\widetilde{X}$, and (N, E, X) is a soft set. If $M(e) \subseteq N(e)$, then e_M is said to be in (N, E, X), denoted by $e_M\widetilde{\in}(N, E, X)$.

Definition 3[17] Let X and Y be two sets, E and F be two nonempty parameter sets, and $f : E \longrightarrow F$ and $g : X \longrightarrow Y$ are mappings. For each $(M, E, X) \in \mathbb{S}(X, E)$, define

$$(f, g)(M, E, X) = (g^{\rightarrow}(M), f(E), Y),$$

where

$$g^{\rightarrow}(M)(\alpha) = \bigcup_{f(e)=\alpha} g(M(e)) \ (\forall \alpha \in F).$$

Then we obtain a mapping

$$(f, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, F).$$

For each $(N, F, Y) \in \mathbb{S}(Y, F)$, define

$$(f, g)^{-1}(N, F, Y) = (g^{-1} \circ N \circ f, f^{-1}(F), X),$$

where

$$(g^{-1} \circ N \circ f)(e) = g^{-1}(N(f(e))) \ \ (\forall e \in f^{-1}(F)).$$

Then we obtain another mapping

$$(f, g)^{-1} : \mathbb{S}(Y, F) \longrightarrow \mathbb{S}(X, E).$$

Definition 4[16] (1) Let X be a set, and $\mathscr{T} \subseteq \mathbb{S}(X, E)$ satisfies

(i) $\emptyset$ and $X \in \mathscr{T}$;

(ii) $\mathscr{T}$ is closed under arbitrary unions;

(ii) $\mathscr{T}$ is closed under finite intersections.

Then $\mathscr{T}$ is called a soft topology on X, and $(X, \mathscr{T}, E)$ is called a soft topological space. The members of $\mathscr{T}$ are called soft open sets, members of $\mathscr{T}' = \{(M', E, X) \mid (M, E, X) \in \mathscr{T}\}$ are called soft closed sets.

(2) Let $(X, \mathscr{T}, E)$ be a soft topological space, and Y be a non-empty subset of X. Then

$$\mathscr{T}_Y = \{(M_Y, E, X) \mid (M, E, X) \in \mathscr{T}\}$$

is a soft topology on Y, it is called the soft relative topology on Y, and $(Y, \mathscr{T}_Y, E)$ is called a soft subspace of $(X, \mathscr{T}, E)$, where

$$(M_Y, E, X) = \widetilde{Y} \widetilde{\cap} (M, E, X) \ (\forall (M, E, X) \in \mathscr{T}).$$

Example 1 (1) Let $X = \{x_1, x_2, x_3\}$ be a 3-element set, $E = \{e_1, e_2\}$ be a 2-element set, and

$$\mathscr{T} = \{(M_i, E, X) \mid i = 1, 2, \cdots, 6\} \cup \{\widetilde{\emptyset}, \widetilde{X}\},$$

where (M_i, E, X) $(i = 1, 2, \cdots, 6)$ are defined as follows:

$$M_1(e) = \begin{cases} \{x_2\}, & \text{if } e = e_1; \\ \{x_1\}, & \text{if } e = e_2. \end{cases}$$

$$M_2(e) = \begin{cases} \{x_1\}, & \text{if } e = e_1; \\ \{x_3\}, & \text{if } e = e_2. \end{cases}$$

$$M_3(e) = \begin{cases} \{x_3\}, & \text{if } e = e_1; \\ \{x_2\}, & \text{if } e = e_2. \end{cases}$$

$$M_4(e) = \begin{cases} \{x_2, x_3\}, & \text{if } e = e_1; \\ \{x_1, x_2\}, & \text{if } e = e_2. \end{cases}$$

$$M_5(e) = \begin{cases} \{x_1, x_2\}, & \text{if } e = e_1; \\ \{x_1, x_3\}, & \text{if } e = e_2. \end{cases}$$

$$M_6(e) = \begin{cases} \{x_1, x_3\}, & \text{if } e = e_1; \\ \{x_2, x_3\}, & \text{if } e = e_2. \end{cases}$$

Then $\mathscr{T}$ is a soft topology on X and hence $(X, \mathscr{T}, E)$ is a soft topological space.

(2) Let $X = R$ (the set of all real numbers), $E = \{e_1, e_2\}$ be a 2-element set,

$$\mathcal{J} = \{A \subseteq X \mid X - A \text{ is a finite subset of } X\} \cup \{\emptyset, X\}$$

(i.e. the finite complement topology on X), and

$$\mathscr{T} = \{(M, E, X) \mid M(e_1), M(e_2) \in \mathcal{J}\}.$$

Then $\mathscr{T}$ is a soft topology on X and hence $(X, \mathscr{T}, E)$ is a soft topological space.

(3) Let $X = R$, $E = \{e_1, e_2\}$ be a 2-element set, $\mathcal{J}$ be the ordinary topology on X (i.e. $\mathcal{J}$ is the topology on X generated by the basis $\mathcal{B} = \{(a,b) \mid a, b \in R, a < b\}$), and

$$\mathscr{T} = \{(M, E, X) \mid M(e_1), M(e_2) \in \mathcal{J}\}.$$

Then $\mathscr{T}$ is a soft topology on X and hence $(X, \mathscr{T}, E)$ is a soft topological space.

(4) Let $X = [0,1]$, $E = \{e_1, e_2\}$ be a 2-element set, $\mathcal{J}$ be the ordinary topology on X (i.e. $\mathcal{J}$ is the topology on $[0,1]$ generated by the basis

$$\mathcal{B} = \{(a,b) \mid a \in [0,1), b \in (0,1], a < b\}),$$

and

$$\mathscr{T} = \{(M, E, X) \mid M(e_1), M(e_2) \in \mathcal{J}\}.$$

Then $\mathscr{T}$ is a soft topology on X and hence $(X, \mathscr{T}, E)$ is a soft topological space.

Remark 2 (1)[16] Let $(X, \mathscr{T}, E)$ be a soft topological space, e_M is a soft point in $\widetilde{X}$, $(N, E, X) \in \mathbb{S}(X, E)$. If there exists a $(A, E, X) \in \mathscr{T}$ such that

$$e_M \widetilde{\in} (A, E, X) \widetilde{\subseteq} (N, E, X),$$

then (N, E, X) is called a neighborhood of e_M.

(2) It can be easily seen that $\widetilde{\emptyset}, \widetilde{X} \in \mathscr{T}'$, and $\mathscr{T}'$ is closed under the operations of arbitrary intersections and finite unions. It can be also seen that $(N, E, X) \in \mathscr{T}'$ if and only if

$$((A, E, X) - e_M) \widetilde{\cap} (N, E, X) \neq \widetilde{\emptyset}$$

for any$e_M \in \widetilde{X}$ and any neighborhood (A, E, X) of e_M.

(3)[16] Let$(X, \mathscr{T}, E)$ be a soft topological space, and $(M, E, X) \in \mathbb{S}(X, E)$. Then

$$\overline{(M, E, X)} = \widetilde{\bigcap} \{(N, E, X) \mid (M, E, X) \widetilde{\subseteq} (N, E, X),$$
$$(N, E, X) \in \mathscr{T}'_X\}$$

is called the closure of (M, E, X). Clearly, $(M, E, X) \in \mathbb{S}(X, E)$ is a soft closed set of $(X, \mathscr{T}, E)$ if and only if $\overline{(M, E, X)} = (M, E, X)$.

(4)[16] Let $(X, \mathscr{T}, E)$ be a soft topological space over X, then $\mathscr{T}^e = \{M(e) \mid (M, E, X) \in \mathscr{T}\}$ is a topology on X $(e \in E)$.

(5) If E is a single point set, then a soft topological space $(X, \mathscr{T}, E)$ can be seen as a common topological space.

Definition 5 Let $(X, \mathscr{T}_X, E)$ and $(Y, \mathscr{T}_Y, E)$ be soft topological spaces. A soft function

$$(f, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, E)$$

is said to be a soft continuous function from $(X, \mathscr{T}_X, E)$ to $(Y, \mathscr{T}_Y, E)$ if

$$(f, g)^{-1}(N, E, Y) \in \mathscr{T}_X \quad (\forall (N, E, Y) \in \mathscr{T}_Y).$$

Remark 3 Let $(X, \mathscr{T}_X, E)$ and $(Y, \mathscr{T}_Y, E)$ be soft topological spaces, and

$$(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, E)$$

be a soft continuous function from $(X, \mathscr{T}_X, E)$ to $(Y, \mathscr{T}_Y, E)$. Then $g : X \longrightarrow Y$ is a continuous function from $(X, \mathscr{T}_X^e)$ to $(Y, \mathscr{T}_Y^e)$ $(\forall e \in E)$.

Definition 6[18] (1) Let $(X, \mathscr{T}, E)$ be a soft topological space, $(P, E, X) \in \mathbb{S}(X, E)$, and $\mathscr{A} \subseteq \mathscr{T}$. If

$$\widetilde{\bigcup} \mathscr{A} = (P, E, X),$$

then $\mathscr{A}$ is called an soft open cover of (P, E, X).

(2) Let $(X, \mathscr{T}, E)$ be a soft topological space, and $(P, E, X) \in \mathbb{S}(X, E)$. (P, E, X) is said to be soft compact if every open soft cover of it has a finite subcover. If $\widetilde{X}$ is compact, then $(X, \mathscr{T}, E)$ is called a soft compact topological space.

Theorem 1[18] Let $(X, \mathscr{T}, E)$ be a soft compact topological space, then each soft closed subset (P, E, X) is a soft compact subset of $\widetilde{X}$.

Theorem 2 Let $(X, \mathscr{T}_X, E)$ and $(Y, \mathscr{T}_Y, E)$ be soft topological spaces, and

$$(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, E)$$

is a soft function. Then the following conditions are equivalent:

(1) (id_E, g) is a soft continuous function from $(X, \mathscr{T}_X, E)$ to $(Y, \mathscr{T}_Y, E)$.

(2) $(id_E, g)^{-1}(N, E, Y) \in \mathscr{T}_X' \quad (\forall\ (N, E, Y) \in \mathscr{T}_Y')$.

(3) $(id_E, g)\overline{(M, E, X)} \widetilde{\subseteq} \overline{(id_E, g)(M, E, X)} \quad (\forall (M, E, X) \in \mathbb{S}(X, E))$.

(4) $(id_E, g)^{-1}\overline{(P, E, Y)} \widetilde{\supseteq} \overline{(id_E, g)^{-1}(P, E, Y)} \quad (\forall (P, E, Y) \in \mathbb{S}(Y, E))$.

Proof Straightforward. □

2 Discrete dynamical systems in soft topological spaces

Let X be a topological space, and $g : X \longrightarrow X$ a continuous mapping, then the family $\{g^n\}_{n \in N}$ defines a (discrete) semi-dynamical system in topological space X, where N stands for the set of all nonnegative integers. In addition, if g is a homeomorphism (i.e. g is a one-to-one correspondence and both g and its inverse mapping g^{-1} are continuous), then we can define g^{-n} by $g^{-n} = (g^{-1})^n \ (\forall n \in N)$, then $\{g^n\}_{n \in Z}$ defines a discrete dynamical system in topological space X, where Z stands for the set of all integers.

Let $(X, \mathscr{T}, E)$ be a soft topological space and

$$(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(X, E)$$

be a soft continuous function from $(X, \mathscr{T}, E)$ to $(X, \mathscr{T}, E)$. It can be seen from definition 3 that

$$(g^n)^{\rightarrow} = (g^{\rightarrow})^n,$$

so we can define the n-th iterate of (id_E, g) $(n \in N)$ as follows:

$$\begin{aligned}(id_E, g)^n &= (id_E, g) \circ (id_E, g)^{n-1} = (id_E \circ id_E, g \circ g^{n-1})\\ &= (id_E, g^n),\end{aligned}$$

$(id_E, g)^0 = (id_E, g^0) = (id_E, id_X)$,

where id_E (resp. id_X) denotes the identity mapping of E (resp., X) onto itself. Then the family $\{(id_E, g)^n\}_{n\in N}$ defines a (discrete) semi-dynamical system in soft topological space $(X, \mathscr{T}, E)$, where N stands for the set of all nonnegative integers. If g is a one-to-one correspondence and both (id_E, g) and its inverse mapping $(id_E, g)^{-1}$ are continuous, it can be seen from definition 3 that

$$(g^{\leftarrow})^n = (g^n)^{\leftarrow} \ (\forall n \in N - \{0\})$$

and

$$((g^n)^{\leftarrow})^m = (g^{\leftarrow})^{nm} \quad (\forall n \in N - \{0\}, \forall m \in N).$$

Let

$$(id_E, g)^{-n} = (id_E, g^{-n}) = (id_E, (g^n)^{-1}) \quad (\forall n \in N),$$

then $\{(id_E, g)^n\}_{n\in Z}$ defines a discrete dynamical system in soft topological space, and it is denoted by $(X, (id_E, g))$. If $(X, \mathscr{T}, E)$ is a soft compact topological space, then $(X, (id_E, g))$ is called a soft compact discrete topological dynamical system. It is easy to show that $(id_E, g)^n(e_M)$
$(\forall n \in Z)$ is a soft point when e_M is a soft point.

Example 2 Let us consider the soft topological space in Example 1(1). Define $g : X \longrightarrow X$ as follows:

$$g(x_1) = x_2, \quad g(x_2) = x_3, \quad g(x_3) = x_1.$$

We will verify that both (id_E, g) and its inverse mapping $(id_E, g)^{-1}$ are continuous. In fact,

$$(id_E, g)^{-1}(M_1, E, X) = (g^{-1} \circ M_1 \circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1} \circ M_1 \circ id_E(e) &= g^{-1}((M_1)(e))\\ &= \begin{cases} g^{-1}(\{x_2\}), & \text{if } e = e_1;\\ g^{-1}(\{x_1\}), & \text{if } e = e_2.\end{cases}\\ &= \begin{cases} \{x_1\}, & \text{if } e = e_1;\\ \{x_3\}, & \text{if } e = e_2.\end{cases}\\ &= M_2(e)\end{aligned}$$

Thus $(id_E, g)^{-1}(M_1, E, X) = (M_2, E, X) \in \mathscr{T}$.

$$(id_E, g)^{-1}(M_2, E, X) = (g^{-1} \circ M_2 \circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1}\circ M_2\circ id_E(e) &= g^{-1}((M_2)(e)) \\ &= \begin{cases} g^{-1}(\{x_1\}), & \text{if } e=e_1; \\ g^{-1}(\{x_3\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_3\}, & \text{if } e=e_1; \\ \{x_2\}, & \text{if } e=e_2. \end{cases} \\ &= M_3(e) \end{aligned}$$

Thus $(id_E, g)^{-1}(M_2, E, X) = (M_3, E, X) \in \mathscr{T}$.

$$(id_E, g)^{-1}(M_3, E, X) = (g^{-1}\circ M_3\circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1}\circ M_3\circ id_E(e) &= g^{-1}((M_3)(e)) \\ &= \begin{cases} g^{-1}(\{x_3\}), & \text{if } e=e_1; \\ g^{-1}(\{x_2\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_2\}, & \text{if } e=e_1; \\ \{x_1\}, & \text{if } e=e_2. \end{cases} \\ &= M_1(e) \end{aligned}$$

Thus $(id_E, g)^{-1}(M_3, E, X) = (M_1, E, X) \in \mathscr{T}$.

$$(id_E, g)^{-1}(M_4, E, X) = (g^{-1}\circ M_4\circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1}\circ M_4\circ id_E(e) &= g^{-1}((M_4)(e)) \\ &= \begin{cases} g^{-1}(\{x_2, x_3\}), & \text{if } e=e_1; \\ g^{-1}(\{x_1, x_2\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_1, x_2\}, & \text{if } e=e_1; \\ \{x_3, x_1\}, & \text{if } e=e_2. \end{cases} \\ &= M_5(e) \end{aligned}$$

Thus $(id_E, g)^{-1}(M_4, E, X) = (M_5, E, X) \in \mathscr{T}$.

$$(id_E, g)^{-1}(M_5, E, X) = (g^{-1}\circ M_5\circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1}\circ M_5\circ id_E(e) &= g^{-1}((M_5)(e)) \\ &= \begin{cases} g^{-1}(\{x_1, x_2\}), & \text{if } e=e_1; \\ g^{-1}(\{x_3, x_1\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_3, x_1\}, & \text{if } e=e_1; \\ \{x_2, x_3\}, & \text{if } e=e_2. \end{cases} \\ &= M_6(e) \end{aligned}$$

Thus $(id_E, g)^{-1}(M_5, E, X) = (M_6, E, X) \in \mathscr{T}$.

$$(id_E, g)^{-1}(M_6, E, X) = (g^{-1}\circ M_6\circ id_E, E, X),$$

where

$$\begin{aligned} g^{-1}\circ M_6\circ id_E(e) &= g^{-1}((M_6)(e)) \\ &= \begin{cases} g^{-1}(\{x_1, x_3\}), & \text{if } e=e_1; \\ g^{-1}(\{x_2, x_3\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_3, x_2\}, & \text{if } e=e_1; \\ \{x_1, x_2\}, & \text{if } e=e_2. \end{cases} \\ &= M_4(e) \end{aligned}$$

Thus $(id_E, g)^{-1}(M_6, E, X) = (M_4, E, X) \in \mathscr{T}$. It is easy to see that

$$(id_E, g)^{-1}(\widetilde{\emptyset}) = \widetilde{\emptyset} \in \mathscr{T}$$

and

$$(id_E, g)^{-1}(\widetilde{X}) = \widetilde{X} \in \mathscr{T}.$$

Therefore, (id_E, g) is continuous.

From the above, it is easy to see that

$$(id_E, g)^{-1} = (id_E, g^{-1}),$$

since for any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g^{-1})(M, E, X) = ((g^{-1})^{\rightarrow}(M), E, X),$$

where

$$(g^{-1})^{\rightarrow}(M)(e) = g^{-1}(M)(e) = g^{-1} \circ M \circ id_E(e).$$

Thus for any $(M, E, X) \in \mathscr{T}$,

$$\begin{aligned} & ((id_E, g)^{-1})^{-1}(M, E, X) \\ = \ & (id_E, g^{-1})^{-1}(M, E, X) \\ = \ & ((g^{-1})^{-1} \circ M \circ id_E, E, X) \\ = \ & (g \circ M \circ id_E, E, X) \end{aligned}$$

Hence

$$((id_E, g)^{-1})^{-1}(M_1, E, X) = (g \circ M_1 \circ id_E, E, X),$$

where

$$\begin{aligned} g \circ M_1 \circ id_E(e) &= g((M_1)(e)) \\ &= \begin{cases} g(\{x_2\}), & \text{if } e = e_1; \\ g(\{x_1\}), & \text{if } e = e_2. \end{cases} \\ &= \begin{cases} \{x_3\}, & \text{if } e = e_1; \\ \{x_2\}, & \text{if } e = e_2. \end{cases} \\ &= M_3(e) \end{aligned}$$

Thus $((id_E, g)^{-1})^{-1}(M_1, E, X) = (M_3, E, X) \in \mathscr{T}$.

$$((id_E, g)^{-1})^{-1}(M_2, E, X) = (g \circ M_2 \circ id_E, E, X),$$

where

$$\begin{aligned} g \circ M_2 \circ id_E(e) &= g((M_2)(e)) \\ &= \begin{cases} g(\{x_1\}), & \text{if } e = e_1; \\ g(\{x_3\}), & \text{if } e = e_2. \end{cases} \\ &= \begin{cases} \{x_2\}, & \text{if } e = e_1; \\ \{x_1\}, & \text{if } e = e_2. \end{cases} \\ &= M_1(e) \end{aligned}$$

Thus $((id_E, g)^{-1})^{-1}(M_2, E, X) = (M_1, E, X) \in \mathscr{T}$.

$$((id_E, g)^{-1})^{-1}(M_3, E, X) = (g \circ M_3 \circ id_E, E, X),$$

where

$$\begin{aligned} g\circ M_3\circ id_E(e) &= g((M_3)(e)) \\ &= \begin{cases} g(\{x_3\}), & \text{if } e=e_1; \\ g(\{x_2\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_1\}, & \text{if } e=e_1; \\ \{x_3\}, & \text{if } e=e_2. \end{cases} \\ &= M_2(e) \end{aligned}$$

Thus $((id_E,g)^{-1})^{-1}(M_3,E,X)=(M_2,E,X)\in\mathscr{T}$.

$$((id_E,g)^{-1})^{-1}(M_4,E,X)=(g\circ M_4\circ id_E,E,X),$$

where

$$\begin{aligned} g\circ M_4\circ id_E(e) &= g((M_4)(e)) \\ &= \begin{cases} g(\{x_2,x_3\}), & \text{if } e=e_1; \\ g(\{x_1,x_2\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_1,x_3\}, & \text{if } e=e_1; \\ \{x_2,x_3\}, & \text{if } e=e_2. \end{cases} \\ &= M_6(e) \end{aligned}$$

Thus $((id_E,g)^{-1})^{-1}(M_4,E,X)=(M_6,E,X)\in\mathscr{T}$.

$$((id_E,g)^{-1})^{-1}(M_5,E,X)=(g\circ M_5\circ id_E,E,X),$$

where

$$\begin{aligned} g\circ M_5\circ id_E(e) &= g((M_5)(e)) \\ &= \begin{cases} g(\{x_1,x_2\}), & \text{if } e=e_1; \\ g(\{x_3,x_1\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_2,x_3\}, & \text{if } e=e_1; \\ \{x_1,x_2\}, & \text{if } e=e_2. \end{cases} \\ &= M_4(e) \end{aligned}$$

Thus $((id_E,g)^{-1})^{-1}(M_5,E,X)=(M_4,E,X)\in\mathscr{T}$.

$$((id_E,g)^{-1})^{-1}(M_6,E,X)=(g\circ M_6\circ id_E,E,X),$$

where

$$\begin{aligned} g\circ M_6\circ id_E(e) &= g((M_6)(e)) \\ &= \begin{cases} g(\{x_1,x_3\}), & \text{if } e=e_1; \\ g(\{x_2,x_3\}), & \text{if } e=e_2. \end{cases} \\ &= \begin{cases} \{x_2,x_1\}, & \text{if } e=e_1; \\ \{x_3,x_1\}, & \text{if } e=e_2. \end{cases} \\ &= M_5(e) \end{aligned}$$

Thus $((id_E,g)^{-1})^{-1}(M_6,E,X)=(M_5,E,X)\in\mathscr{T}$. It is easy to see that

$$((id_E,g)^{-1})^{-1}(\widetilde{\emptyset})=\widetilde{\emptyset}\in\mathscr{T}$$

and

$$((id_E,g)^{-1})^{-1}(\widetilde{X})=\widetilde{X}\in\mathscr{T}.$$

Therefore, $(id_E,g)^{-1}$ is continuous. Hence, $(X,(id_E,g))$ is a soft topological dynamical system.

Example 3 Let us consider the soft topological space in Example 1(2). Let $g : X \longrightarrow X$ be an arbitrary one-to-one respondence on X. Then for any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g)^{-1}(M, E, X) = (g^{-1} \circ M \circ id_E, E, X),$$

where

$$g^{-1} \circ M \circ id_E(e) = g^{-1}(M(e)) \quad (\forall e \in E),$$

the complement $X - g^{-1} \circ M \circ id_E(e)$ is still a finite subset of X since g is an one-to-one respondence, thus $(id_E, g)^{-1}(M, E, X) \in \mathscr{T}$. Therefore, (id_E, g) is continuous.

On the other hand, for any $(M, E, X) \in \mathscr{T}$,

$$\begin{aligned} & ((id_E, g)^{-1})^{-1}(M, E, X) \\ = \; & (id_E, g^{-1})^{-1}(M, E, X) \\ = \; & ((g^{-1})^{-1} \circ M \circ id_E, E, X) \\ = \; & (g \circ M \circ id_E, E, X) \end{aligned}$$

where

$$g \circ M \circ id_E(e) = g(M(e)) \quad (\forall e \in E),$$

the complement $X - g \circ M \circ id_E(e)$ is still a finite subset of X since g is an one-to-one respondence, thus

$$(id_E, g)(M, E, X) \in \mathscr{T}.$$

Therefore, $(id_E, g)^{-1}$ is continuous. Hence, $(X, (id_E, g))$ is a soft topological dynamical system.

Example 4 Let us consider the soft topological space in Example 1(3). Define $g : X \longrightarrow X$ as follows:

$$g(x) = x + 1 \quad (\forall x \in X).$$

Then for every $(a, b) \in \mathcal{B}$, $g(a, b) = (a + 1, b + 1)$, and $g^{-1}(a, b) = (a - 1, b - 1)$, thus $g(\mathcal{B}) = g^{-1}(\mathcal{B}) = \mathcal{B}$. Denote the topology on X generated by $g(\mathcal{B})$ and $g^{-1}(\mathcal{B})$ by $g(\mathcal{J})$ and $g^{-1}(\mathcal{J})$. Then $g(\mathcal{J}) = g^{-1}(\mathcal{J}) = \mathcal{J}$.

For any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g)^{-1}(M, E, X) = (g^{-1} \circ M \circ id_E, E, X),$$

where

$$g^{-1} \circ M \circ id_E(e) = g^{-1}(M(e)) \quad (\forall e \in E),$$

since $M(e) \in \mathcal{J}$, we have $g^{-1}(M(e)) \in g^{-1}(\mathcal{J}) = \mathcal{J}$, thus $(id_E, g)^{-1}(M, E, X) \in \mathscr{T}$. Therefore, (id_E, g) is continuous.

On the other hand, for any $(M, E, X) \in \mathscr{T}$,

$$\begin{aligned} & ((id_E, g)^{-1})^{-1}(M, E, X) \\ = \; & (id_E, g^{-1})^{-1}(M, E, X) \\ = \; & ((g^{-1})^{-1} \circ M \circ id_E, E, X) \\ = \; & (g \circ M \circ id_E, E, X) \end{aligned}$$

where

$$g \circ M \circ id_E(e) = g(M(e)) \quad (\forall e \in E),$$

since $M(e) \in \mathcal{J}$, we have $g(M(e)) \in g(\mathcal{J}) = \mathcal{J}$, thus $(id_E, g)(M, E, X) \in \mathscr{T}$. Therefore, $(id_E, g)^{-1}$ is continuous. Hence, $(X, (id_E, g))$ is a soft topological dynamical system.

Example 5 Let us consider the soft topological space in Example 1(4). Define $g : X \longrightarrow X$ as follows:

$$g(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}]; \\ 2 - 2x, & x \in (\frac{1}{2}, 1]. \end{cases}$$

For every $(a, b) \in \mathcal{B}$,

$$g^{-1}(a, b) = \begin{cases} (\frac{a}{2}, \frac{b}{2}), & b \leq \frac{1}{2}; \\ (\frac{2-a}{2}, \frac{2-b}{2}), & a \geq \frac{1}{2}; \\ (\frac{a}{2}, \frac{2-b}{2}), & a < \frac{1}{2} < b. \end{cases}$$

Thus $g^{-1}(\mathcal{B}) \subseteq \mathcal{B}$. Let $g^{-1}(\mathcal{J})$ be the topology on X generated by $g^{-1}(\mathcal{B})$, then $g^{-1}(\mathcal{J}) \subseteq \mathcal{J}$.

For any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g)^{-1}(M, E, X) = (g^{-1} \circ M \circ id_E, E, X),$$

where

$$g^{-1} \circ M \circ id_E(e) = g^{-1}(M(e)) \quad (\forall e \in E),$$

since $M(e) \in \mathcal{J}$, we have $g^{-1}(M(e)) \in g^{-1}(\mathcal{J}) \subseteq \mathcal{J}$, thus $(id_E, g)^{-1}(M, E, X) \in \mathscr{T}$. Therefore, (id_E, g) is continuous. Hence, $(X, (id_E, g))$ is a semi-soft topological dynamical system.

Definition 7 Let $(X, (id_E, g))$ be a soft discrete topological dynamical system and $e_M \in \widetilde{X}$ is a soft point. Define several soft sets as follows:

$$Orb_{(id_E, g)}(e_M) = \{(id_E, g)^n(e_M) \mid n \in Z\},$$

$$Orb^{+}_{(id_E, g)}(e_M) = \{(id_E, g)^n(e_M) \mid n \in N - \{0\}\}.$$

$$Orb^{-}_{(id_E, g)}(e_M) = \{(id_E, g)^{-n}(e_M) \mid n \in N - \{0\}\}.$$

Then we call $Orb_{(id_E, g)}(e_M)$ (resp., $Orb^{+}_{(id_E, g)}(e_M)$, $Orb^{-}_{(id_E, g)}(e_M)$) the soft orbit (resp., soft positive semi-orbit, soft negative semi-orbit) of the soft dynamical system of (id_E, g).

Let $e_M \in \widetilde{X}$, if $(id_E, g)^n(e_M) = e_M$ for some $n \in N - \{0\}$, then e_M is called a soft periodic point of (id_E, g), the smallest one of such integers is referred to as the soft period of e_M. In particular, if $(id_E, g)(e_M) = e_M$, then e_M is called a soft fixed point of (id_E, g). Let $Per(id_E, g)$ (resp. $Fix(id_E, g)$) be the set of all soft periodic points (resp. all soft fixed points) of (id_E, g). Then $Fix(id_E, g) \subseteq Per(id_E, g)$.

Definition 8 Let $e_M \in \widetilde{X}$ be a soft point, then the soft set

$$\omega(e_M) = \widetilde{\bigcap}_{n \in N - \{0\}} \overline{\widetilde{\bigcup}\{(id_E, g)^k(e_M) \mid k \geq n\}},$$

is called a soft ω-limit set.

Obviously $\omega(e_M)$ is a soft closed set of $(X, \mathscr{T}, E)$. If the soft topological space $(X, \mathscr{T}, E)$ is soft compact, then $\omega(e_M) \neq \widetilde{\emptyset}$ by Theorem 7.4 in [20].

Definition 9 Let $(X, (id_E, g))$ be a soft discrete topological dynamical system, and $e_M \in \widetilde{X}$ a soft point.

(1) If for each soft open neighborhood (N, E, X) of e_M, there exists an $n \in N - \{0\}$ such that $(id_E, g)^n(e_M)$
$\widetilde{\in}(N, E, X)$, then e_M is called a soft recurrent points of (id_E, g). The set of all soft recurrent points of (id_E, g) is denoted by $Rec(id_E, g)$. Clearly, $Per(id_E, g) \subseteq Rec(id_E, g)$.

(2) If for each soft open neighborhood (N, E, X) of e_M, there exists an $n \in N - \{0\}$ such that

$$(id_E, g)^{-n}(N, E, X)\widetilde{\cap}(N, E, X) \neq \widetilde{\emptyset}.$$

Then e_M is called a soft nonwandering point of (id_E, g). The set of all soft nonwandering points of (id_E, g) is denoted by $\Omega(id_E, g)$, i.e.,

$\Omega(id_E, g) = \{e_M \in \widetilde{X} \mid e_M$ be a soft nonwandering point of $(id_E, g)\}$.
Each soft point of $\widetilde{X} - \Omega(id_E, g)$ is called a soft wandering point.

Definition 10 Let (id_E, g) be a soft continuous function from $(X, \mathscr{T}, E)$ to $(X, \mathscr{T}, E)$.

(1) (id_E, g) is called soft topological mixing if, for any pair (M, E, X) and $(N, E, X) \in \mathscr{T}$ of nonempty soft open sets of $(X, \mathscr{T}, E)$, there exists an $n \in N - \{0\}$ such that
$(id_E, g)^n(M, E, X)\widetilde{\cap}(N, E, X) \neq \widetilde{\emptyset}$.

(2) (id_E, g) is called soft topological transitivity if there exists a soft point $e_M \in \widetilde{X}$ such that $Orb_{(id_E,g)}(e_M)$ is dense in $\widetilde{X}$ (i.e. $\overline{Orb_{(id_E,g)}(e_M)} = \widetilde{X}$).

(3) A soft set (N, E, X) is said to be soft invariant of (id_E, g) if $(id_E, g)(N, E, X)\widetilde{\subseteq}(N, E, X)$ (i.e. $g(N(e)) \subseteq N(e)$ for each $e \in E$).

Theorem 3 Let $(X, \mathscr{T}, E)$ be a soft topological space, and $(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(\mathbb{X}, \mathbb{E})$ be a soft continuous function from $(X, \mathscr{T}, E)$ to $(X, \mathscr{T}, E)$. Then

(1) $\Omega(id_E, g)$ is a soft closed set of $\widetilde{X}$, and $Rec(id_E, g)$
$\widetilde{\subseteq}\Omega(id_E, g)$.

(2) $Orb_{(id_E,g)}(e_M)$, $\omega(e_M)$, $Per(id_E, g)$, $Fix(id_E, g)$ and $\Omega(id_E, g)$ are invariant of (id_E, g).

(3) $\Omega((id_E, g)^m)$ is an invariant and closed soft set, and

$$\Omega((id_E, g)^m)\widetilde{\subseteq}\Omega(id_E, g) \quad (m \in N - \{0\}).$$

(4) Each soft point $e_M \in \widetilde{X}$ is a soft nonwandering point if one of the following conditions is satisfied:

(i) (id_E, g) is soft topological mixing, g is a one-to-one correspondence, and both (id_E, g) and its inverse mapping $(id_E, g)^{-1}$ are continuous

(ii) $Per(id_E, g) = \widetilde{X}$.

Proof (1) Suppose that a soft point e_M is not a soft wandering point of (id_E, g), then there exists some soft open neighborhood (N, E, X) and some $n \in N - \{0\}$ such that

$$(id_E, g)^{-n}(N, E, X)\widetilde{\cap}(N, E, X) = \widetilde{\emptyset}.$$

So all the soft points in (N, E, X) are not soft wandering points of (id_E, g), it follows that $\Omega(id_E, g)$ be a soft closed set of $\widetilde{X}$.

Now let soft point $e_M \in Rec(id_E, g)$, then for each soft open neighborhood (N, E, X) of e_M, there exists some $n \in N - \{0\}$ such that $(id_E, g)^n(e_M)\widetilde{\subseteq}(N, E, X)$, so for any $e \in E$, $g^n(M(e)) \subseteq N(e)$, thus $M(e) \subseteq g^{-n}(N(e))$, it implies that

$$e_M\widetilde{\in}(id_E, g^{-n})(N, E, X) = (id_E, g)^{-n}(N, E, X),$$

then

$$e_M\widetilde{\in}(id_E, g)^{-n}(N, E, X)\widetilde{\cap}(N, E, X),$$

hence

$$Rec(id_E, g)\widetilde{\subseteq}\Omega(id_E, g).$$

(2) We only show that $\omega(e_M)$ and $\Omega(id_E, g)$ are invariant sets of (id_E, g). Firstly, we have

$$\begin{array}{rl} & (id_E, g)(\omega(e_M)) \\ = & (id_E, g)(\widetilde{\bigcap}_{n\in N-\{0\}}\overline{\widetilde{\bigcup}\{(id_E, g)^k(e_M) \mid k \geq n\}}) \\ \widetilde{\subseteq} & \widetilde{\bigcap}_{n\in N-\{0\}}(id_E, g)\overline{\widetilde{\bigcup}\{(id_E, g)^k(e_M) \mid k \geq n\}}) \\ \widetilde{\subseteq} & \widetilde{\bigcap}_{n\in N-\{0\}}\overline{\widetilde{\bigcup}\{(id_E, g)^{k+1}(e_M) \mid k \geq n\}}) \\ \widetilde{\subseteq} & \widetilde{\bigcap}_{n\in N-\{0\}}\overline{\widetilde{\bigcup}\{(id_E, g)^k(e_M) \mid k \geq n\}} = \omega(e_M) \end{array}$$

Now let soft point $e_M\widetilde{\in}\Omega(id_E, g)$ and (N, E, X) a soft open neighborhood of soft point $(id_E, g)(e_M)$, we can obtain that $(id_E, g)^{-1}(N, E, X)$ is a soft open neighborhood of soft point e_M since (id_E, g) is a soft continuous function, then there exists some $n \in N - \{0\}$ such that

$$\begin{array}{rl} & (id_E, g)^{-1}((id_E, g)^{-n}(N, E, X))\widetilde{\cap}(N, E, X)) \\ = & (id_E, g)^{-n}((id_E, g)^{-1}(N, E, X))\widetilde{\cap}(id_E, g)^{-1}(N, E, X) \\ \neq & \widetilde{\emptyset} \end{array}$$

So

$$(id_E, g)^{-n}(N, E, X))\widetilde{\cap}(N, E, X) \neq \widetilde{\emptyset}.$$

Therefore

$$(id_E, g)(e_M)\widetilde{\in}\Omega(id_E, g),$$

Hence

$$(id_E, g)(\Omega(id_E, g))\widetilde{\subseteq}\Omega(id_E, g).$$

(3) Straightforward.

(4) Let (i) hold, $e_M\widetilde{\in}\widetilde{X}$ be a soft point and $(N,E,X)\in\mathscr{T}$ be a soft open neighborhood of e_M. Because (id_E,g) is soft topological mixing, there exists some $n\in N-\{0\}$ such that

$$(id_E,g)^n(M,E,X)\widetilde{\cap}(M,E,X)\neq\widetilde{\emptyset}.$$

Then

$$(id_E,g)^{-n}(M,E,X)\widetilde{\cap}(M,E,X)\neq\widetilde{\emptyset}$$

since g is a one-to-one correspondence and both (id_E,g) and its inverse mapping $(id_E,g)^{-1}$ are continuous. Thus $e_M\in\Omega(id_E,g)$.

Let (ii) hold. Then

$$\widetilde{X}=\overline{Per(id_E,g)}\widetilde{\subseteq}\overline{Rec(id_E,g)}\widetilde{\subseteq}\overline{\Omega(id_E,g)}=\Omega(id_E,g)\widetilde{\subseteq}\widetilde{X}.$$

Therefore $\Omega(id_E,g)=\widetilde{X}$. $\square$

Remark 4 If g is a one-to-one correspondence, both

$$(id_E,g):\mathbb{S}(X,E)\longrightarrow\mathbb{S}(X,E)$$

and its inverse mapping

$$(id_E,g)^{-1}:\mathbb{S}(X,E)\longrightarrow\mathbb{S}(X,E)$$

are continuous, and $(M,E,X)\in\mathbb{S}(X,E)$. Then

$$(id_E,g)^n(M,E,X)\widetilde{\cap}(M,E,X)\neq\widetilde{\emptyset}$$

if and only if

$$(id_E,g)^{-n}(M,E,X)\widetilde{\cap}(M,E,X)\neq\widetilde{\emptyset}\ (\forall n\in N-\{0\}).$$

So $\Omega(id_E,g)=\Omega(id_E,g)^{-1}$.

Definition 11 Let $(X,\mathscr{T}_X,E)$ and $(Y,\mathscr{T}_Y,E)$ be soft topological spaces,

$$(id_E,g):\mathbb{S}(X,E)\longrightarrow\mathbb{S}(X,E)$$

be a soft continuous function from $(X,\mathscr{T}_X,E)$ to $(X,\mathscr{T}_X,E)$,

$$(id_E,f):\mathbb{S}(Y,E)\longrightarrow\mathbb{S}(Y,E))$$

be a soft continuous function from $(Y,\mathscr{T}_Y,E)$ to $(Y,\mathscr{T}_Y,E)$). If there exists a soft continuous function $(id_E,h):\mathbb{S}(X,E)\longrightarrow\mathbb{S}(Y,E)$ from $(X,\mathscr{T}_X,E)$ to $(Y,\mathscr{T}_Y,E)$ such that

$$(id_E,h)\circ(id_E,f)=(id_E,g)\circ(id_E,h)$$

(i.e. $(id_E,h\circ f)=(id_E,g\circ h)$), then (id_E,h) is said to be soft topology semi-conjugate from (id_E,g) to (id_E,f). If g is a one-to-one correspondence and both (id_E,g) and its inverse

mapping $(id_E, g)^{-1}$ are continuous, then (id_E, h) is said to soft topological conjugate from (id_E, g) to (id_E, f). Here, we denote $(id_E, g) \cong (id_E, f)$.

$$
\begin{tikzcd}
\mathbb{S}(X,E) \arrow[r, "{(id_E, g)}"] \arrow[d, "{(id_E, h)}"'] & \mathbb{S}(X,E) \arrow[d, "{(id_E, h)}"] \\
\mathbb{S}(Y,E) \arrow[r, "{(id_E, f)}"'] & \mathbb{S}(Y,E)
\end{tikzcd}
$$

fig.1

Remark 5 (1) $\cong$ is an equivalence relation.

(2) If (id_E, h) is a soft topological conjugate mapping from (id_E, g) to (id_E, f), then for each soft point $e_M \in \widetilde{X}$ and $n \in N - \{0\}$, we have

$$(id_E, h)((id_E, f)^n(e_M)) = (id_E, g^n)((id_E, h)(e_M)),$$

it follows that

$$(id_E, h)(Orb_{(id_E,g)}(e_M)) = Orb_{(id_E,g)}((id_E, h)(e_M)),$$

and it is easy to show that

$$(id_E, h)(\omega(e_M)) = \omega((id_E, h)(e_M));$$

$$(id_E, h)(Per(id_E, g)) = Per(id_E, f);$$

$$(id_E, h)(Fix(id_E, g)) = Fix(id_E, f);$$

$$(id_E, h)(Rec(id_E, g)) = Rec(id_E, f);$$

$$(id_E, h)(\Omega(id_E, g)) = \Omega(id_E, f).$$

3 Soft topological entropy

In this section, the definition of soft topological entropy will be given and some fundamental properties of the soft topological entropy will be studied.

Definition 12 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, and α be a soft open cover of $\widetilde{X}$. Denote the smallest cardinality of all subcovers (for $\widetilde{X}$) of α by $N_{\widetilde{X}}(\alpha)$, i.e.,

$$N_{\widetilde{X}}(\alpha) = \min\left\{|\beta| \mid \beta \subseteq \alpha \text{ and } \widetilde{X} = \widetilde{\bigcup}\beta\right\}.$$

Since $\widetilde{X}$ is compact soft set, $N_{\widetilde{X}}(\alpha)$ is a positive integer. Let $H_{\widetilde{X}}(\alpha) = \log N_{\widetilde{X}}(\alpha)$.

Let α and β be two soft open covers of $\widetilde{X}$. Define their join by

$$\alpha\widehat{\cup}\beta = \{(P, E, X)\widetilde{\cap}(Q, E, X) \mid (P, E, X) \in \alpha, (Q, E, X) \in \beta\}.$$

Clearly, the join $\alpha \widehat{\bigcup} \beta$ is a soft open cover of $\widetilde{X}$. It is well known that β is called a refinement of α (denoted by $\alpha \prec \beta$) if for each $(Q, E, X) \in \beta$, there exists a $(P, E, X) \in \alpha$ such that $(Q, E, X) \widetilde{\subseteq} (P, E, X)$.

Theorem 4 Let$(X, (id_E, g))$ be a soft compact discrete topological dynamical system, α and β be two soft open covers of $\widetilde{X}$. Then the following hold.

(1) $H_{\widetilde{X}}(\alpha) \geq 0$.

(2) if $\beta \prec \alpha$,then $H_{\widetilde{X}}(\alpha) \leq H_{\widetilde{X}}(\beta)$.

(3) $H_{\widetilde{X}}(\alpha \widehat{\bigcup} \beta) \leq H_{\widetilde{X}}(\alpha) + H_{\widetilde{X}}(\beta)$.

(4) $H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) = H_{\widetilde{X}}(\alpha)$.

Proof we only prove (4). Let $N_{\widetilde{X}}(\alpha) = n$, then any subcover of α containing less than n elements of α would not cover $\widetilde{X}$. Let

$$\{(P_1, E, X), (P_2, E, X), \cdots, (P_n, E, X)\}$$

be a subcover (for $\widetilde{X}$) of α with a cardinality n, since (id_E, g) is continuous,

$$\{(id_E, g)^{-1}(P_1, E, X), (id_E, g)^{-1}(P_2, E, X),$$
$$\cdots, (id_E, g)^{-1}(P_n, E, X)\}$$

is a subcover (for $(id_E, g)^{-1}(\widetilde{X})$) of $(id_E, g)^{-1}(\alpha)$. By $(id_E, g)(\widetilde{X}) = \widetilde{X}$ we can know $\widetilde{X} = (id_E, g)^{-1}(\widetilde{X})$, so

$$\{(id_E, g)^{-1}(P_1, E, X), (id_E, g)^{-1}(P_2, E, X),$$
$$\cdots, (id_E, g)^{-1}(P_n, E, X)\}$$

is a finite open subcover (for $\widetilde{X}$) of $(id_E, g)^{-1}(\alpha)$. Therefore,

$$N_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \leq n = N_{\widetilde{X}}(\alpha)$$

which implies $H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \leq H_{\widetilde{X}}(\alpha)$.

Now, suppose that $N_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) = m$. Let

$$\{(id_E, g)^{-1}(Q_1, E, X), (id_E, g)^{-1}(Q_2, E, X),$$
$$\cdots, (id_E, g)^{-1}(Q_m, E, X)\}$$

be a finite open subcover (for $\widetilde{X}$) of $(id_E, g)^{-1}(\alpha)$. Therefore,

$$\widetilde{X} = \widetilde{\bigcup}_{i=1}^{m} \{(id_E, g)^{-1}(Q_i, E, X)\}.$$

Since $(id_E, g)(\widetilde{X}) = \widetilde{X}$, then

$$\begin{aligned} \widetilde{X} &= (id_E, g)(\widetilde{X}) = \widetilde{\bigcup}_{i=1}^{m} \{(id_E, g)^{-1}(Q_i, E, X)\} \\ &= \widetilde{\bigcup}_{i=1}^{m} \{(id_E, g)(id_E, g)^{-1}((Q_i, E, X))\} \\ &= \widetilde{\bigcup}_{i=1}^{n} \{(Q_i, E, X)\}. \end{aligned}$$

So

$$\{(Q_i, E, X) \mid i = 1, 2, \cdots, m\}$$

is a finite open subcover (for $\widetilde{X}$) of α, Hence, $m \geq N_{\widetilde{X}}(\alpha)$, i.e.,

$$N_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \geq N_{\widetilde{X}}(\alpha)$$

which implies

$$H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \geq H_{\widetilde{X}}(\alpha).$$

By the above, we can get that

$$H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) = H_{\widetilde{X}}(\alpha). \quad \square$$

Theorem 5 Let$(X, (id_E, g))$ be a soft compact discrete topological dynamical system, α be a soft open cover of $\widetilde{X}$. Then the limit

$$\lim_{n\to\infty} \frac{1}{n} H_{\widetilde{X}}(\widehat{\bigcup}_{k=1}^{n-1}\{(id_E, g)^{-k}(\alpha)\})$$

exists.

Proof. Let

$$a_n = H_{\widetilde{X}}(\widehat{\bigcup}_{k=1}^{n-1}\{(id_E, g)^{-k}(\alpha)\}).$$

We only need to show that

$$a_{n+p} \leq a_n + a_p \ (\forall n, p \in N - \{0\}).$$

From theorem 2.7(3) and (4), we have

$$\begin{aligned}
a_{n+p} &= H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=0}^{n+p-1}\left\{(id_E, g)^{-k}(\alpha)\right\}\right) \\
&= H_{\widetilde{X}}(\left(\widehat{\bigcup}_{k=0}^{n-1}\left\{(id_E, g)^{-k}(\alpha)\right\}\right) \\
&\qquad \widehat{\bigcup}\left(\widehat{\bigcup}_{k=n}^{n+p-1}\left\{(id_E, g)^{-k}(\alpha)\right\}\right)) \\
&= H_{\widetilde{X}}(\left(\widehat{\bigcup}_{k=0}^{n-1}\left\{(id_E, g)^{-k}(\alpha)\right\}\right) \\
&\qquad \widehat{\bigcup}\left((id_E, g)^{-n}\widehat{\bigcup}_{k=0}^{p-1}\left\{(id_E, g)^{-k}(\alpha)\right\}\right)) \\
&\leq H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=0}^{n-1}\left\{(id_E, g)^{-k}(\alpha)\right\}\right) \\
&\qquad + H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=0}^{p-1}\left\{(id_E, g)^{-k}(\alpha)\right\}\right).
\end{aligned}$$

Thus $a_{n+p} \leq a_n + a_p$. $\square$

Definition 13 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, let α be a soft open cover of $\widetilde{X}$. Then

$$Ent((id_E, g), \alpha, \widetilde{X}) = \lim_{n\to\infty} \frac{1}{n} H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=1}^{n-1}\{(id_E, g)^{-k}(\alpha)\}\right)$$

is called the soft topological entropy of (id_E, g) on $\widetilde{X}$ relative to α, and

$$Ent(id_E, g) = \sup_{\alpha}\{Ent((id_E, g), \alpha, \widetilde{X}) \mid \alpha \text{ is a soft open cover of } \widetilde{X}\}$$

is called the soft topological entropy of (id_E, g).

By Theorem 1, each soft closed subset of $\widetilde{X}$ is a soft compact subset of $\widetilde{X}$, then the following theorem holds.

Theorem 6 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, α be a soft open cover of $\widetilde{X}$, (A_1, E, X) and (A_2, E, X) be two closed soft sets, and $(A_1, E, X)\widetilde{\subseteq}(A_2, E, X)$, Then

(1)

$$Ent((id_E, g), \alpha, (A_1, E, X)) \leq Ent((id_E, g), \alpha, (A_2, E, X)).$$

(2)

$$Ent((id_E, g), (A_1, E, X)) \leq Ent((id_E, g), (A_2, E, X)).$$

Proof. (1) Let

$$N_{(A_2,E,X)}(\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}) = s.$$

Then there exists a soft open subcover

$$\{(P_1, E, X), (P_2, E, X), \cdots, (P_s, E, X)\}$$

of

$$\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}$$

for (A_2, E, X). Since $(A_1, E, X) \widetilde{\subseteq}(A_2, E, X)$, we have

$$\{(P_1, E, X), (P_2, E, X), \cdots, (P_s, E, X)\}$$

is also a subcover of

$$\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}$$

for (A_1, E, X), and hence

$$N_{(A_1,E,X)}(\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}) \leq s$$

$$= N_{(A_2,E,X)}(\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}).$$

So

$$H_{(A_1,E,X)}(\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\})$$

$$\leq H_{(A_2,E,X)}(\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}).$$

Then

$$Ent((id_E, g), \alpha, (A_1, E, X))$$
$$\leq Ent((id_E, g), \alpha, (A_2, E, X)).$$

(2)

$$\begin{aligned} & Ent((id_E, g), (A_1, E, X)) \\ = \ & \sup_{\alpha}\{Ent((id_E, g), \alpha, (A_1, E, X)) \mid \alpha \text{ is a soft open} \\ & \qquad\qquad \text{cover of } \widetilde{X}\} \\ \leq \ & \sup_{\alpha}\{Ent((id_E, g), \alpha, (A_2, E, X)) \mid \alpha \text{ is a soft open} \\ & \qquad\qquad \text{cover of } \widetilde{X}\} \\ = \ & Ent((id_E, g), (A_2, E, X)). \ \square \end{aligned}$$

Theorem 7 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, and α be a soft open cover of $\widetilde{X}$. Then $Ent(id_E, id_X) = 0$.

Proof Straightforward.

Theorem 8 $Ent(id_E, g^m) \geq m \cdot Ent(id_E, g)$ ($\forall m \in N - \{0\}$).

Proof As

$$((g^n)^{\leftarrow})^m = (g^{\leftarrow})^{nm} \quad (\forall n \in N - \{0\}, \forall m \in N),$$

we have

$$\begin{aligned} & \widehat{\bigcup}_{t=0}^{n-1}\{(id_E, g^m)^{-s}\widehat{\bigcup}_{t=0}^{m-1}\{(id_E, g)^{-t}(\alpha)\}\} \\ = \ & \widehat{\bigcup}_{s=0}^{mn-1}\{(id_E, g)^{-s}(\alpha)\} \end{aligned}$$

Hence

$$H_{\widetilde{X}}(\widehat{\bigcup}_{t=0}^{n-1}\{(id_E, g^m)^{-s}\widehat{\bigcup}_{t=0}^{m-1}\{(id_E, g)^{-t}(\alpha))\}\}$$
$$= H_{\widetilde{X}}(\widehat{\bigcup}_{s=0}^{mn-1}\{(id_E, g)^{-s}(\alpha)\}).$$

Denote

$$\beta = \widehat{\bigcup}_{s=0}^{m-1}\{(id_E, g)^{-s}(\alpha)\}.$$

Then

$$\begin{aligned} & Ent(id_E, g^m) = Ent(id_E, g)^m \geq Ent((id_E, g)^m, \beta, \widetilde{X}) \\ = \ & \lim_{n\to\infty} \tfrac{1}{n} H_{\widetilde{X}}\left(\widehat{\bigcup}_{t=0}^{n-1}\{(id_E, g^m)^{-s}\widehat{\bigcup}_{t=0}^{m-1}\{(id_E, g)^{-t}(\alpha))\}\}\right) \\ = \ & \lim_{n\to\infty} m \cdot \tfrac{1}{mn} H_{\widetilde{X}}(\widehat{\bigcup}_{s=0}^{mn-1}\{(id_E, g)^{-s}(\alpha)\}) \\ = \ & m \cdot Ent((id_E, g), \alpha, \widetilde{X}). \end{aligned}$$

Hence,

$$Ent(id_E, g^m) \geq m \cdot \sup_{\alpha} Ent((id_E, g), \alpha, \widetilde{X})$$
$$= m \cdot Ent(id_E, g). \ \square$$

4 Conclusion

In this paper, the discrete dynamical systems in soft topological spaces are defined, and simple examples are also given. Some basic concepts (such as soft ω-limit set, soft invariant set, soft periodic point, soft nonwandering point, and soft recurrent point) of the discrete dynamical system are introduced into soft topological spaces. Soft topological mixing and soft topological transitivity are also studied. At last, soft topological entropy is defined and several properties of it are discussed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgement

This work was supported by the International Science and Technology Cooperation Foundation of China (Grant No. 2012DFA11270), the National Natural Science Foundation of China (Grant No. 11371292, 11071151), and Shaanxi Provincial Natural Science Foundation (Grant No. 2014JM1018).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.5, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

FUNCTIONAL INEQUALITIES IN VECTOR BANACH SPACE

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Abstract. In this paper, we prove that the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(ax+by+cz)+f(bx+ay+bz)+f(cx+cy+az)\| \leq \|(a+b+c)f(x+y+z)\|$$

in vector Banach space, where $a \neq b \neq c \in \mathbb{R}$ are fixed points with $3 > |a+b+c|$.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [24] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [24] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems for several functional equations or inequations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]–[8],[10], [12]–[16], [22]–[25],[26]-[31],[34]).

We recall some basic facts concerning generalized norm.

Definition 1.1 (see [15]). Let E be a real vector space. A generalized norm for E is a mapping $\|\cdot\|_G : E \to \mathbb{R}_+^k$ denoted by

$$\|x\|_G = (\alpha_1(x), \alpha_2(x), \alpha_3(x), \cdots, \alpha_k(x))$$

such that

(a) $\|x\|_G \geq 0$, that is, $\alpha_i(x) \geq 0$ for all $i = 1, 2, \cdots, k$;

(b) $\|x\|_G = 0$ if and only if $x = 0$, that is, $\alpha_i(x) = 0$ for all i, if and only if $x = 0$;

(c) $\|\lambda x\|_G = |\lambda| \|x\|_G$,that is,$\alpha_i(\lambda x) = |\lambda|\alpha_i(x)$;

(d) $\|x+y\|_G \leq \|x\|_G + \|y\|_G$, which means, $\alpha_i(x+y) \leq \alpha_i(x) + \alpha_i(y)$;

2010 *Mathematics Subject Classification.* Primary 39B62, 39B52, 46B25.

Key words and phrases. additive functional inequaties; Hyers-Ulam stability; vector Banach space

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Example 1.2. *In* $\mathbb{R}^2$, $\|x\|_G = (|x_1|, |x_2|)$.

Definition 1.3. Let $(X, \|\cdot\|_G)$ be a general normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} \alpha_i(x_n - x) = 0$ for all $i = 1, 2 \cdots, k$. In that case, x is called the limit of the sequence x_n and we denote it by $G\text{-}lim_{n\to\infty} x_n = x$.

Definition 1.4. A sequence x_n in X is called *Cauchy* if for each $\epsilon > 0$ and each $a > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $\|x_{n+p} - x_n\|_G \leq \epsilon$, that is, $\alpha_i(x_{n+p} - x_n) \leq \epsilon$.

It is known that every convergent sequence in the general normed space is Cauchy. If each Cauchy sequence is convergent, then the the general normed space is said to be complete and the general normed space is called a *vector Banach space*.

2. HYers-Ulam Stability In vector Banach Space

From now on , Let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a vector Banach space.

This paper,we prove that the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(ax + by + cz) + f(bx + cy + bz) + f(cx + ay + az)\|_G \leq \|(a + b + c)f(x + y + z)\|_G$$

in the vector Banach space, where $a \neq b \neq c \in \mathbb{R}$ are fixed points with $3 > |a + b + c|$.

Lemma 2.1. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping. If it satisfies*

$$\begin{aligned} &\|f(ax + by + cz) + f(bx + cy + bz) + f(cx + ay + az)\|_G \\ &\leq \|(a + b + c)f(x + y + z)\|_G \end{aligned} \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$ *and* a, b, c *are fixed real numbers with* $3 > |a + b + c|$. *Then* f *is additive.*

Proof. Letting $x = y = z = 0$ in (2.1) for all $x, y, z \in \mathcal{X}$, we get

$$\|3f(0)\|_G \leq \|(a + b + c)f(0)\|_G \tag{2.2}$$

for $a, b, c \in \mathbb{R}$.

For any $i = 1, 2, \cdots, k$,

$$\alpha_i(3f(0)) \leq \alpha_i((a + b + c)f(0))$$

we get

$$3\alpha_i(f(0)) \leq |a + b + c|\alpha_i(f(0)),$$

Thus $f(0) = 0$.

Letting $x = 0$ and Replacing z by $-y$ in (2.1), we get

$$\|f((b - c)y) + f((c - b)y)\|_G \leq \|(a + b + c)f(0)\|_G = |a + b + c|\alpha_i(f(0)) = 0$$

and so $f(-x) = -f(x)$ for all $x \in \mathcal{X}$.

Replacing x by $-y-z$ in (2.1), we have

$$\|f((b-a)y + (c-a)z) + f((a-b)y) + f((a-c)z)\|_G \le 0$$

for all $y, z \in \mathcal{X}$. Then we can obtain

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathcal{X}$. □

Theorem 2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0,\infty)$ such that*

$$\begin{aligned} &\|f(ax+by+cz) + f(bx+cy+bz) + f(cx+ay+az)\|_G \\ &\le \|(a+b+c)f(x+y+z)\|_G + (\underbrace{\varphi(x,y,z), \varphi(x,y,z), \cdots, \varphi(x,y,z)}_{k}) \end{aligned} \tag{2.3}$$

and

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left((-2)^j x, (-2)^j x, (-2)^j x\right) < \infty \tag{2.4}$$

for all $x, y, z \in \mathcal{X}$ and $a \neq b \neq c \in \mathbb{R}$ are fixed points with $3 > |a+b+c|$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned} &\|f(x) - A(x)\|_G \\ &\le \left(\underbrace{\widetilde{\varphi}\left(\frac{b+c-2a}{(a-b)(a-c)}x, \frac{1}{a-b}x, \frac{1}{a-c}x\right), \cdots, \widetilde{\varphi}\left(\frac{b+c-2a}{(a-b)(a-c)}x, \frac{1}{a-b}x, \frac{1}{a-c}x\right)}_{k}\right) \end{aligned} \tag{2.5}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x = -y-z$ in (2.3), we get

$$\begin{aligned} &\|f((b-a)y + (c-a)z) + f((a-b)y) + f((a-c)z)\|_G \\ &\le \underbrace{(\varphi(-y-z,y,z), \cdots, \varphi(-y-z,y,z))}_{k} \end{aligned} \tag{2.6}$$

for all $y, z \in \mathcal{X}$.

Letting $y = \frac{x}{b-a}, z = \frac{y}{c-a}$ in (2.6), we get

$$\begin{aligned} &\|f(x+y) + f(-x) + f(-y)\|_G \\ &\le \underbrace{\left(\varphi\left(\frac{x}{a-b} + \frac{y}{a-c}, \frac{x}{b-a}, \frac{y}{c-a}\right), \cdots, \varphi\left(\frac{x}{a-b} + \frac{y}{a-c}, \frac{x}{b-a}, \frac{y}{c-a}\right)\right)}_{k} \end{aligned} \tag{2.7}$$

for all $x, z \in \mathcal{X}$.

Letting $x = y$ in (2.7) we get

$$\begin{aligned}
&\|2f(-x) + f(2x)\|_G \\
&\leq \left(\varphi\left(\frac{2a-b-c}{(a-b)(a-c)}x, \frac{1}{b-a}x, \frac{1}{c-a}x\right), \cdots,\right. \\
&\left.\varphi\left(\frac{2a-b-c}{(a-b)(a-c)}x, \frac{1}{b-a}x, \frac{1}{c-a}x\right)\right)
\end{aligned}$$

for all $x \in \mathcal{X}$. Thus

$$\begin{aligned}
&\left\|f(x) - \frac{f(-2x)}{-2}\right\|_G \\
&\leq \frac{1}{2}\left(\varphi\left(\frac{b+c-2a}{(a-b)(a-c)}x, \frac{1}{a-b}x, \frac{1}{a-c}x\right), \cdots,\right. \\
&\left.\varphi\left(\frac{b+c-2a}{(a-b)(a-c)}x, \frac{1}{a-b}x, \frac{1}{a-c}x\right)\right)
\end{aligned}$$

for all $x \in \mathcal{X}$.

Hence one may have the following formula for positive integers m, l with $m > l$,

$$\begin{aligned}
&\left\|\frac{1}{(-2)^l} f\left((-2)^l x\right) - \frac{1}{(-2)^m} f\left((-2)^m x\right)\right\|_G \\
&\leq \sum_{i=l}^{m-1} \frac{1}{2^i}\left(\varphi\left(\frac{(-2)^i(b+c-2a)}{(a-b)(a-c)}x, \frac{(-2)^i}{a-b}x, \frac{(-2)^i}{a-c}x\right), \cdots,\right. \\
&\left.\varphi\left(\frac{(-2)^i(b+c-2a)}{(a-b)(a-c)}x, \frac{(-2)^i}{a-b}x, \frac{(-2)^i}{a-c}x\right)\right)
\end{aligned}$$

for all $x \in \mathcal{X}$. That is,

$$\begin{aligned}
&\alpha_i\left(\frac{1}{(-2)^l} f\left((-2)^l x\right) - \frac{1}{(-2)^m} f\left((-2)^m x\right)\right) \\
&\leq \sum_{i=l}^{m-1} \frac{1}{2^i}\varphi\left(\frac{(-2)^i(b+c-2a)}{(a-b)(a-c)}x, \frac{(-2)^i}{a-b}x, \frac{(-2)^i}{a-c}x\right)
\end{aligned} \tag{2.8}$$

for all $x \in mathcalX$. It follows from (2.8) that the sequence $\left\{\frac{f((-2)^k x)}{(-2)^k}\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is a generalized norm space, the sequence $\left\{\frac{f((-2)^k x)}{(-2)^k}\right\}$ converges. So one may define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := G - \lim_{k\to\infty}\left\{\frac{f((-2)^k x)}{(-2)^k}\right\}, \quad \forall x \in \mathcal{X}.$$

Taking $m = 0$ and letting l tend to ∞ in (2.8), we have the inequality (2.5).

It follows from (2.3) that

$$\begin{aligned}
&\|A(ax+by+cz)+A(bx+ay+bz)+A(cx+cy+az)\|_G\\
&=\lim_{k\to\infty}\left|\frac{1}{(-2)^k}\right|\big\|f((-2)^k(ax+by+cz))+f((-2)^k(bx+ay+bz))\\
&+f((-2)^k(cx+cy+az))\big\|_G\\
&\leq \lim_{k\to\infty}\left|\frac{1}{(-2)^k}\right|\big\|(a+b+c)f((-2)^k(x+y+z))\big\|_G \qquad (2.9)\\
&+\lim_{k\to\infty}\left|\frac{1}{(-2)^k}\right|\left(\underbrace{\varphi((-2)^kx,(-2)^ky,(-2)^kz),\cdots,\varphi((-2)^kx,(-2)^ky,(-2)^kz)}_{k}\right)\\
&\leq \|(a+b+c)A(x+y+z)\|_G
\end{aligned}$$

for all $x,y,z\in\mathcal{X}$. One see that A satisfies the inequality (2.1) and so it is additive by Lemma (2.1).

Now, we show that the uniqueness of A. Let $T:X\to Y$ be another additive mapping satisfying (2.5). Then one has

$$\begin{aligned}
&\|A(x)-T(x)\|_G=\left\|\frac{1}{(-2)^k}A\left((-2)^kx\right)-\frac{1}{(-2)^k}T\left((-2)^kx\right)\right\|_G\\
&\leq\frac{1}{2^k}\left(\left\|A\left((-2)^kx\right)-f\left((-2)^kx\right)\right\|_G\right.\\
&\left.+\left\|T\left((-2)^kx\right)-f\left((-2)^kx\right)\right\|_G\right)\\
&\leq 2\frac{1}{2^k}\left(\underbrace{\widetilde{\varphi}\left(\frac{(b+c-2a)(-2)^k}{(a-b)(a-c)}x,\frac{(-2)^k}{a-b}x,\frac{(-2)^k}{a-c}x\right),\cdots}_{k}\right)
\end{aligned}$$

which tends to zero as $k\to\infty$ for all $x\in X$. So we can conclude that $A(x)=T(x)$ for all $x\in X$. □

Theorem 2.3. *Let $f:\mathcal{X}\to\mathcal{Y}$ be a mapping with $f(0)=0$. If there is a function $\varphi:X^3\to[0,\infty)$ satisfying (2.3) such that*

$$\widetilde{\varphi}(x,y,z):=\sum_{j=1}^{\infty}2^j\varphi\left(\frac{x}{(-2)^j},\frac{y}{(-2)^j},\frac{z}{(-2)^j}\right)<\infty \qquad (2.10)$$

for all $x,y,z\in\mathcal{X}$, then there exists a unique additive mapping $A:\mathcal{X}\to\mathcal{Y}$ such that

$$\|f(x)-A(x)\|_G\leq\widetilde{\varphi}(x,x,-2x) \qquad (2.11)$$

for all $x\in\mathcal{X}$.

Proof. The proof is similar with Theorem (2.2), we can get

$$\left\| f(x) - (-2) f\left(\frac{x}{-2}\right) \right\|_G \leq \left(\underbrace{\varphi\left(\frac{(2a-b-c)x}{2(a-b)(a-c)}, \frac{x}{2(b-a)}, \frac{x}{2(c-a)}\right) \cdots \varphi\left(\frac{(2a-b-c)x}{2(a-b)(a-c)}, \frac{x}{2(b-a)}, \frac{x}{2(c-a)}\right)}_{k} \right)$$

for all $x \in \mathcal{X}$.

Next, we can prove that the sequence $\{(-2)^n f\left(\frac{x}{(-2)^n}\right)$ is a Cauchy sequence for all $x \in \mathcal{X}$,and define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} (-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x \in \mathcal{X}$ that is similar to the corresponding part of the proof of Theorem (2.2). □

Acknowledgments

G. Lu was supported by Doctoral Science Foundation of Shengyang University of Technology(No.521101302)and the Project Sponsored by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry. Y.jin was supported by National Natural Science Foundation of China(11361066)The study of high-precision algorithm for high dimensional delay partial differential equations 2014-2017.

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Coupled fixed point theorems for generalized $(\psi, \phi)-$ weak contraction in partially ordered G-metric spaces

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In this manuscript, we give coupled fixed point results for generalized $(\psi, \phi)-$ weak contraction, satisfying rational type expression in the context of partially ordered G-metric spaces. The derived results generalize the result of K. Chakrabarti (K. Chakrabarti, Coupled fixed point theorems with rational type contractive condition in a partially ordered G-metric space, Journal of Mathematics, Volume 2014, Article ID 785357, 7 pages). To demonstrate our result and also to demonstrate the authenticity of our result from the previous one, we give suitable example.

Key Words: Coupled fixed point, Mixed monotone property, Partially ordered G-metric space, $(\psi, \phi)-$weak contraction.
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1 Introduction and preliminaries

Fixed point theory provide one of the most important and useful technique for the existence of fixed point, coincidence point, common fixed point and coupled fixed pint for self map under different condition. It is used for the existence and uniqueness of the solution of mathematical model which may be in the form of differential equations, matrix equations, integral equations, functional equations, linear inequalities or mixed see ([5], [17], [19], [30]). In this area the first well known result proved by Banach [8] known as Banach contraction principle. Many authors generalized this principle in various spaces by using different contractive conditions ([6], [13], [15], etc.).

In recent years, metric fixed point theory has been developed rapidly in partially ordered metric space. Ran and Reurings [30] extended the Banach contraction principle in partially ordered sets and also discuss some applications to linear and nonlinear matrix equations. Nieto and Rodriguez-Lopez [23]

extended the result of Ran and Reurings and used their established result to obtain a unique solution for first order ODEs. Jaggi [15] construct rational type contraction in complete metric space. Harjani et al [13] extend the result of Jaggi to partial ordered complete metric space. For more details (see[13], [33]).

Alber and Gurre [6] gave the concept of weak contraction as a generalization of contraction and established the existence of fixed points for a self map in a Hilbert space. Rhoades [31] extended this concept to metric spaces and defined ϕ-weak contraction. Dutta and Choudhury [12] generalized ϕ-weak contraction to the concept of (ψ, ϕ) weak contraction and studies some fixed point results. Zhang and Song [34] extend weak contraction for the study of two self map. Furthermore Djorić [11] generalized the result of Zhang and Song and studied common fixed point for generalized (ψ, ϕ) weak contraction. For some other similar results see [22], [25], [29], [32].

The concept of mixed monotone mappings introduced by Bhaskar and Lakshmikantham [9] and derived some coupled fixed point results. Furthermore, they applied their results on a first order differential equation with periodic boundary conditions [14]. Lakshmikanthem and Ćirić [17] generalized the concept of mixed monotone mapping and established a coupled fixed point theorem for nonlinear contractions in partially ordered metric spaces. Recently Chakrabarti [10] investigated coupled fixed point theorems for map satisfying nonlinear rational type contraction and mixed monotone property in partially ordered G-metric space.

In this work, using the concept of generalized rational type $(\psi, \phi)-$weak contraction condition, coupled fixed point results in the framework of complete partially ordered generalized metric spaces are investigated. Through out the paper $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ will denote the set of all non-negative real numbers, the set of positive integer and the set of non-negative integer respectively.

Definition 1. *[20] Let $(X, \preceq)$ be a partially ordered set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following conditions:*

1. $G(u, v, w) = 0$ *if* $u = v = w$;
2. $0 < G(u, u, v)$ *for all* $u, v \in X$ *with* $u \neq v$;
3. $G(u, u, v) \leq G(u, v, w)$ *for all* $u, v, w \in X$ *with* $v \neq w$;
4. $G(u, v, w) = G(u, w, v) = G(v, w, u) = \ldots$ *(symmetry in all three variables);*
5. $G(u, v, w) \leq G(u, p, p) + G(p, v, w)$ *for all* $u, v, w, p \in X$ *(rectangle inequality).*

Then it is called a G-metric on X and the triple $(X, G, \preceq)$ is called partially ordered G-metric space.

Definition 2. *[20] The pair (X, G) is said to be symmetric G-metric space if $G(u, v, v) = G(u, u, v)$ for all $u, v \in X$.*

Example 1. *(1) Let $X = \mathbb{R}^+$ and $G : X \times X \times X \to \mathbb{R}^+$ be the function defined is follows $G(u, v, w) = \max\{|u - v|, |v - w|, |w - u|\}$, for all $u, v, w \in X$. Then G is symmetric G-metric on X.*

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(2) Let $X = \{a, b\}$. Define $G(a, a, a) = G(b, b, b) = 0, G(a, a, b) = 1, G(a, b, b) = 2,$ and extend G to X^3 by using the symmetry in the variables. Then it is clear that (X, G) is an asymmetric $G-$metric space.

(3) Also see examples of asymmetric $G-$metric spaces in ([2], Example 2.6; [3], Example 2.2; [18], Example 2.2; [22], Example 3.4.).

Definition 3. *[20] Let (X, G) be a G-metric space and let α_n be a sequence in X. A point $\alpha \in X$ is said to be the limit of the sequence α_n if*

$$\lim_{n,m\to\infty} G(\alpha_n, \alpha_m, \alpha) = 0$$

and the sequence α_n is said to be G-convergent in X.

Definition 4. *[20] A sequence α_n is called a G-Cauchy sequence if for every $\varepsilon > 0$, there is a positive integer $\mathbb{N}$ such that $G(\alpha_n, \alpha_m, \alpha_l) < \varepsilon$ for all $n, m, l > \mathbb{N}$.*

Definition 5. *[20] A metric space (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in X.*

Definition 6. *[9] Let $(X, \preceq)$ be a partially ordered set, $T : X \times X \to X$. Then T is said to have mixed-monotone property if $T(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y. That is., for all $x, y \in X$*

Definition 7. *[17] Let $(X, \preceq)$ be a partially ordered set, $T : X \times X \to X$ and $g : X \to X$. We say T is the g-mixed monotone property if T is monotone g-nondecreasing in its first argument and monotone g-non-increasing in its second argument. That is., for all $x, y \in X$*

$$x_1, x_2 \in X, \ gx_1 \preceq gx_2 \Rightarrow T(x_1, y) \preceq T(x_2, y),$$
$$y_1, y_2 \in X, \ gy_1 \preceq gy_2 \Rightarrow T(x, y_1) \succeq T(x, y_2).$$

Definition 8. *[9] Let $T : X \times X \to X$ be a map such that $T(x, y) = x$ and $T(y, x) = y$ then the pair $(x, y) \in X \times X$ is called a coupled fixed point of T. It is clear that $(x, y)$is a coupled fixed point if and only if (y, x) is such.*

Definition 9. *[17] Let $T : X \times X \to X$ and $g : X \to X$ be two map such that $T(x, y) = gx$ and $T(y, x) = gy$ then the pair $(x, y) \in X \times X$ is called a coupled coincidence point of T and g.*

Definition 10. *[17] Two maps $T : X \times X \to X$ and $g : X \to X$ are said to be commutative if $g(T(x, y)) = T(gx, gy)$.*

Chakrababati [10] proved the following results.

Theorem 1. *[10]Let $(X, \preceq)$ be a partially ordered set and let (X, G) be a $G-$complete G-metric space. Suppose $T : X \times X \longrightarrow X$ be a continuous mapping on X having the mixed monotone property. Suppose for all $(x, y), (u, v), (w, z) \in X \times X$ with $(x, y) \preceq (u, v) \preceq (w, z)$ holds*

$$\begin{aligned} &G(T(x, y), T(u, v), T(w, z)) \\ &\le \alpha \frac{G(x, T(x, y), T(x, y))\, G(u, T(u, v), T(u, v))\, G(w, T(w, z), T(w, z))}{G^2(x, u, w)} \\ &\quad + \beta G(x, u, w), \end{aligned}$$

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where $8\alpha + \beta < 1$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$, then T has a coupled fixed point $(x_*, y_*) \in X$.

Theorem 2. *[10] Let $(X, \preceq)$ be a partially ordered set and let (X, G) be a $G-$complete G-metric space. Suppose $T : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be a continues mappings on X such that T has the mixed g-monotone property. Suppose that $T(X \times X) \subseteq g(X)$, g commute with T and for $(x, y), (u, v), (w, z) \in X \times X$ with $(x, y) \preceq (u, v) \preceq (w, z)$ and $gx \preceq gu \preceq gw$ or $gy \succeq gu \succeq gz$ holds*

$$\begin{aligned} & G(T(x,y), T(u,v), T(w,z)) \\ \leq\ & \alpha \frac{G\left(gx, T(x,y), T(x,y)\right) G\left(gu, T(u,v), T(u,v)\right) G\left(gw, T(w,z), T(w,z)\right)}{G^2(gx, gu, gw)} \\ & + \beta G(gx, gu, gw), \end{aligned}$$

where $8\alpha + \beta < 1$. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$ then T and g have a coupled coincidence point $(x_*, y_*) \in X \times X$, that is., (x_*, y_*) satisfies $gx_* = T(x_*, y_*)$, $gy_* = T(y_*, x_*)$.

2 Main Results

In our main results we used the following two classes.

$\psi \in \Psi$ if and only if $\psi : [0, \infty) \to [0, \infty)$, ψ is continuous and non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$.

$\phi \in \Phi$ if and only if $\phi : [0, \infty) \to [0, \infty)$, ψ is a lower semi continuous and non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

Also, for more details of G-metric spaces see ([1]-[4], [7], [16], [18], [21], [26]-[28]).

Remark 1. *It is worth to noticing that both results in [10] without the conditions $G^2(x, u, w) \neq 0$ that is., $G(gx, gu, gw) \neq 0$ are not correct.*

Now, we announce the first our result.

Theorem 3. *Let $(X, \preceq)$ be a partially ordered set and let (X, G) be a $G-$complete symmetric G-metric space. Suppose $T : X \times X \longrightarrow X$ be a continuous mapping having the mixed monotone property and satisfying*

$$\psi(G(T(x,y), T(u,v), T(w,z)) \leq \psi(M(x,u,w,y,v,z)) - \phi(M(x,u,w,y,v,z)), \tag{2.1}$$

for all $x, y, z, u, v, w \in X$ with $G(x, u, w) \neq 0$ and $(x, y) \preceq (u, v) \preceq (w, z)$ or $(x, y) \succeq (u, v) \succeq (w, z)$, where

$$\begin{aligned} & M(x,u,w,y,v,z) \\ =& \max\left\{ \frac{[G(x, T(x,y), T(x,y) G(u, T(u,v), T(u,v) G(w, T(w,z), T(w,z)]}{G^2(x,u,w)}, \right. \\ & \left. G(x,u,w) \right\}, \end{aligned}$$

$\psi \in \Psi$ and $\phi \in \Phi$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$. Then T has a coupled fixed point $(x_*, y_*) \in X$.

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Proof. Suppose that there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$. Further, define $x_{n+1} = T(x_n, y_n)$ and $y_{n+1} = T(y_n, x_n)$. Using the mixed monotone property and the mathematical induction we obtain that $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$ for all $n \in \mathbb{N}$ (very known method).

Consider now

$$\psi\left(G\left(x_{n+1}, x_n, x_n\right)\right) = \psi\left(G\left(T\left(x_n, y_n\right), T\left(x_{n-1}, y_{n-1}\right), T\left(x_{n-1}, y_{n-1}\right)\right)\right).$$

Using (2.1) we have that

$$\begin{aligned}\psi\left(G\left(x_{n+1}, x_n, x_n\right)\right) \leq &\psi\left(M\left(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1}\right)\right) \\ &- \phi\left(M\left(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1}\right)\right)\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}&M\left(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1}\right) \\ =&\max\left\{\frac{G\left(x_n, T\left(x_n, y_n\right), T\left(x_n, y_n\right)\right) G^2\left(x_{n-1}, T\left(x_{n-1}, y_{n-1}\right) T\left(x_{n-1}, y_{n-1}\right)\right)}{G^2\left(x_n, x_{n-1}, x_{n-1}\right)},\right. \\ &\left. G\left(x_n, x_{n-1}, x_{n-1}\right)\right\}.\end{aligned}$$

Let $G_n = G(x_n, x_{n-1}, x_{n-1})$ then,

$$M(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1}) = \max\{G_{n+1}, G_n\}.$$

Further we show that G_n is non-incresing. Suppose their exist n_0 such that $G_{n_0+1} > G_{n_0}$ then from (2.2)

$$\psi(G_{n_0+1}) \leq \psi(G_{n_0+1}) - \phi(G_{n_0+1}).$$

Which implies that $\phi(G_{n_0+1}) \leq 0$. A contradiction. Hence $G_n \geq G_{n+1}$ for all $n \geq 1$. Since $\{G_n\}$ is a non-increasing sequence of positive real numbers there exists $r \geq 0$ such that

$$\lim_{n\to\infty} G_n = r. \tag{2.3}$$

We shall show that $r = 0$. Suppose $r > 0$ then applying limit in (2.2) and using (2.3), we have

$$\psi(r) \leq \psi(r) - \phi(r) < \psi(r).$$

We obtain a contradiction. Therefore $r = 0$ that is.,

$$\lim_{n\to\infty} G_n = 0. \tag{2.4}$$

Now, we show that $\{x_n\}$ is a G-Cauchy sequence. Suppose that, $\{x_n\}$ is not G-Cauchy. Then, there exist $\epsilon > 0$ and subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that,

$$G(x_{m(k)}, x_{m(k)}, x_{n(k)}) \geq \epsilon, \quad \forall\, k \in \mathbb{N}. \tag{2.5}$$

Furthermore, corresponding to $m(k)$ one can choose $n(k)$ such that, it is the smallest integer with $n(k) > m(k)$ satisfying (2.5) then,

$$G(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) < \epsilon, \quad \forall\, k \in \mathbb{N} \tag{2.6}$$

Now

$$\begin{aligned}\epsilon &\leq G(x_{m(k)}, x_{m(k)}, x_{n(k)}) = G(x_{n(k)}, x_{m(k)}, x_{m(k)})\\ &\leq G(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)}),\end{aligned}$$

Taking limit $k \to \infty$ and using (2.4) we get

$$\lim_{k\to\infty} G(x_{m(k)}, x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.7}$$

Now

$$\begin{aligned}G(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) &= G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1})\\ &\leq G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1})\\ &\leq G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{m(k)}, x_{m(k)})\\ &\quad + G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}), \qquad (2.8)\end{aligned}$$

and

$$\begin{aligned}&G(x_{n(k)}, x_{m(k)}, x_{m(k)})\\ &\leq G(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{m(k)}, x_{m(k)})\\ &\leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1})\\ &\quad + G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}), \qquad (2.9)\end{aligned}$$

Using limit $k \to \infty$ in (2.8) and (2.9) and using (2.4) and (2.7) we get

$$\lim_{k\to\infty} G(x_{m(k)-1}, x_{m(k)-1} x_{n(k)-1}) = \epsilon. \tag{2.10}$$

Consider

$$\begin{aligned}&\psi\Big(G(x_{m(k)}, x_{m(k)}, x_{n(k)})\Big)\\ &\leq \psi\Big(M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1})\Big)\\ &\quad - \phi\Big(M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1})\Big), \qquad (2.11)\end{aligned}$$

where

$$\begin{aligned}&M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1})\\ &= \max\left\{\frac{[[G(x_{m(k)-1}, x_{m(k)}, x_{m(k)})]^2 G(x_{n(k)-1}, x_{n(k)}, x_{n(k)})]}{G(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})^2}, \right. \qquad (2.12)\\ &\qquad \left. G(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})\right\}. \qquad (2.13)\end{aligned}$$

Applying limit $k \to \infty$ in (2.13), using (2.7), (2.10) and (2.4) we get

$$\lim_{k\to\infty} M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1}) = \epsilon. \tag{2.14}$$

Taking limit of (2.11) using (2.7), (2.14) and lower semi continuity of ϕ we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon) < \psi(\epsilon),$$

which is contradiction. So $\epsilon = 0$. Therefore x_n is a G-Cauchy sequence. Similarly by the same argument we can show that y_n is a G-Cauchy sequence. By completeness of X, there is $x_*,\ y_* \in X$ such that $x_n \to x_*$ and $y_n \to y_*$ as $n \to \infty$.
Now we have to show that (x_*, y_*) is a coupled fixed point of T. Since T is continuous on X and G is also continuous in each of its variable, so

$$G(T(x_*, y_*), x_*, x_*) = G(\lim_{n\to\infty} T(x_n, y_n), x_*, x_*) = G(x_*, x_*, x_*) = 0.$$

Hence, we proved that $T(x_*, y_*) = x_*$ Similarly by the same argument we obtain that $T(y_*, x_*) = y_*$. So (x_*, y_*) is a coupled fixed point of T. □

Theorem 4. *Suppose that the conditions of Theorem 3 are valid. In addition suppose that for each $(x, y), (u, v) \in X \times X$ exists $(w, z) \in X \times X$ which is comparable to (x, y) and (u, v). Then coupled fixed point of T is unique.*

Proof. Suppose that $(x_*, y_*), (x^{'}, y^{'}) \in X \times X$ are two coupled fixed points.

Case 1

If (x_*, y_*), $(x\prime, y\prime)$ are comparable then from (2.1)

$$\psi(G(T(x_*, y_*), T(x^{'}, y^{'}), T(x^{'}, y^{'}) \leq \psi(M(x_*, x^{'}, x^{'}, y_*, y^{'}, y^{'}) - \phi(M(x_*, x^{'}, x^{'}, y_*, y^{'}, y^{'}), \qquad (2.15)$$

where

$$\begin{aligned} &M(x_*, x^{'}, x^{'}, y_*, y^{'}, y^{'}) \\ &= \max\left\{\frac{G(x_*, T(x_*, y_*), T(x_*, y_*))[G(x^{'}, T(x^{'}, y^{'}), T(x^{'}, y^{'})]^2]}{G(x_*, x^{'}, x^{'})}, G(x_*, x^{'}, x^{'})\right\} \\ &= \max\left\{\frac{G(x_*, x_*, x_*)[G(x^{'}, x^{'}, x^{'}]^2}{G(x_*, x^{'}, x^{'})}, G(x_*, x^{'}, x^{'})\right\}. \end{aligned}$$

Which implies that

$$M(x_*, x^{'}, x^{'}, y_*, y^{'}, y^{'}) = G(x_*, x^{'}, x^{'}).$$

From (2.15) we have

$$\psi(G(x_*, x^{'}, x^{'}) = \psi(G(T(x_*, y_*), T(x^{'}, y^{'}), T(x^{'}, y^{'}) < \phi(G(x_*, x^{'}, x^{'}),$$

which is contradiction. Hence we must have $x_* = x^{'}$. Similarly we can easily show that $y_* = y^{'}$ so couple fixed point is unique.

Case 2

If (x_*, y_*), $(x^{'}, y^{'})$ are not comparable by Theorem 3 there is a $(u, v) \in X \times X$ comparable to (x_*, y_*) and $(x^{'}, y^{'})$ if there is $m_0 \in \mathbb{N}$ such that $T^{m_0}(u, v) = (x_*, y_*)$, then
$T^{m_0+1}(u, v) = T(x_*, y_*) = x_*$, in last we get $T^m(u, v) = x_*$ for $m \geq m_0$ this mean $T^m(u, v) \to x_*$ for $m \to \infty$
if there is no such m_0 then for any $m \geq 1$

$$\begin{aligned} \psi(G(T^m(u, v), x_*, x_*) &= \psi(G(T^m(u, v), T^m(x_*, y_*), T^m(x_*, y_*)) \\ &\leq \psi(M(u, x_*, x_*, v, y_*, y_*) - \phi(M(u, x_*, x_*, v, y_*, y_*), \qquad (2.16) \end{aligned}$$

where

$$
\begin{aligned}
&M(u, x_*, x_*, v, y_*, y_*)\\
&= \max\left\{ \frac{G(T^{m-1}(u,v), T^m(u,v), T^m(u,v))[G(T^{m-1}(x_*,y_*), T^m(x_*,y_*), T^m(x_*,y_*))]^2}{G(T^{m-1}(u,v), T^{m-1}(x_*,y_*), T^{m-1}(x_*,y_*))}, \right.\\
&\qquad \left. G(T^{m-1}(u,v), T^{m-1}(x_*,y_*), T^{m-1}(x_*,y_*)) \right\}\\
&= \max\left\{ \frac{G(T^{m-1}(u,v), x_*, x_*)[G(x_*, x_*, x_*)]^2}{G(T^{m-1}(u,v), x_*, x_*)}, G(T^{m-1}(u,v), x_*, x_*) \right\}.
\end{aligned}
$$

Which implies that

$$M(u, x_*, x_*, v, y_*, y_*) = G(T^{m-1}(u,v), x_*, x_*).$$

Putting M in (2.16), we have

$$
\begin{aligned}
\psi(G(T^m(u,v), x_*, x_*) \leq \psi(G(T^{m-1}(u,v), x_*, x_*))\\
\phi(G(T^{m-1}(u,v), x_*, x_*)).
\end{aligned} \tag{2.17}
$$

This implies that

$$\psi(G(T^m(u,v), x_*, x_*) < \psi(G(T^{m-1}(u,v), x_*, x_*),$$

since ψ is non-decreasing therefore,

$$G(T^m(u,v), x_*, x_*) < G(T^{m-1}(u,v), x_*, x_*)$$

that is, $\{G(T^m(u,v), x_*, x_*)\}$ is a decreasing sequence of positive real numbers. Therefore, there is an α_1 such that $\{G(T^m(u,v), x_*, x_*)\} \to \alpha_1$. We shall show that $\alpha_1 = 0$. Suppose, to the contrary, that $\alpha_1 > 0$. Taking the limit in equation (2.17) we get contradiction. So α_1 =0. Implies $G(T^m(u,v), x_*, x_*)$=0, that is., $T^m(u,v) = x_*$. Similarly we can show that $T^m(u,v) = y_*$, $(T^m(u,v) = x^{'}$ and $(T^m(u,v) = y^{'}$. Hence the coupled fixed point is unique. □

The next result is the generalization of Theorem 3. Because the proof is similar, then it is omitted.

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set and let (X, G) be a $G-$complete symmetric G-metric space. Suppose that $T : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are a continues mappings such that T has the $g-$mixed monotone property. Suppose that $T(X \times X) \subseteq g(X)$, g commute with T and satisfying*

$$\psi(G(T(x,y), T(u,v), T(w,z)) \leq \psi(M(x,u,w,y,v,z) - \phi(M(x,u,w,y,v,z), \tag{2.18}$$

for all $x, y, z, u, v, w \in X$ with $G(gx, gu, gw) \neq 0$ and $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ or $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$, where

$$
\begin{aligned}
&M(x,u,w,y,v,z)\\
&= \max\left\{ \frac{G\left(gx, T(x,y), T(x,y)\right) G\left(gu, T(u,v), T(u,v)\right) G\left(gw, T(w,z), T(w,z)\right)}{G^2(gx, gu, gw)}, \right.\\
&\qquad \left. G(gx, gu, gw) \right\},
\end{aligned}
$$

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$\psi \in \Psi$ and $\phi \in \Phi$. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$ then T and g have a coupled coincidence point $(x_*, y_*) \in X \times X$, that is., (x_*, y_*) satisfies $gx_* = T(x_*, y_*)$, $gy_* = T(y_*, x_*)$.

Corollary 1. *Let (X, G) be a partially ordered set and let (X, G) be a $G-$complete symmetric G-metric space. Suppose that $T : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are a continues mappings such that T has the $g-$mixed monotone property. Suppose that $T(X \times X) \subseteq g(X)$, g commute with T and for $0 < k < 1$ satisfying*

$$G(T(x,y), T(u,v), T(w,z)) \leq k(M(x,u,w,y,v,z),$$

for all $x, y, z, u, v, w \in X$ with $G(gx, gu, gw) \neq 0$ and $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ or $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$, where

$$\begin{aligned} &M(x,u,w,y,v,z) \\ = \ &\max\left\{\frac{G\left(gx, T(x,y), T(x,y)\right) G\left(gu, T(u,v), T(u,v)\right) G\left(gw, T(w,z), T(w,z)\right)}{G^2(gx,gu,gw)},\right. \\ &\left. G(gx,gu,gw)\right\}. \end{aligned}$$

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$ then T and g have a coupled coincidence point $(x_*, y_*) \in X \times X$, that is., (x_*, y_*) satisfies $gx_* = T(x_*, y_*)$, $gy_* = T(y_*, x_*)$.

Proof. The proof follows by taking $\psi(t) = t$, $\phi(t) = (1-k)t$ where $0 < k < 1$ in Theorem 5. ☐

Remark 2. *For $0 < \alpha < \frac{1}{8}$, $0 < \beta < \frac{1}{16}$ and for all $x, y, z, u, v, w \in X$ with $G(gx, gu, gw) \neq 0$ and $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ or $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$ we have*

$$\begin{aligned} &G(T(x,y), T(u,v), T(w,z)) \\ \leq &\alpha\frac{[G(gx, T(x,y), T(x,y)G(gu, T(u,v), T(u,v)G(gw, T(w,z), T(w,z)]}{G(gx,gu,gw)^2} \\ &+ \beta G(gx,gu,gw), \\ \leq &(\alpha+\beta)\max\left\{\frac{G\left(gx, T(x,y), T(x,y)\right) G\left(gu, T(u,v), T(u,v)\right) G\left(gw, T(w,z), T(w,z)\right)}{G^2(gx,gu,gw)},\right. \\ &\left. G(gx,gu,gw)\right\}. \end{aligned}$$

where $k = \alpha + \beta < 1$. Clearly, the relation $0 < 8\alpha + \beta < 1$ implies that Corollary 1 is the generalization of Theorem 2. Therefore Theorem 5 is the generalization of Theorem 2.

Now we give example which satisfying Theorem 5 but does not Theorem 2.

Example 2. *Let $X = [0,1]$ and consider the natural ordered relation in X, defined $G : X \times X \times X \to \mathbb{R}^+$ by*

$$G(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x,y,z\}. & \end{cases}$$

Then (X,G) is $G-$complete symmetric G-metric space. Let $T : X\times X \to X$, $g : X \to X$, $\phi : [0,\infty) \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ define by,

$$T(x,y) = \begin{cases} \frac{x^3-y^3}{4}, & \text{if } x \geq y, \\ 0, & \text{if } x < y, \end{cases}$$

$$g(x) = x^2, \quad \phi(t) = \frac{t}{2}, \quad \psi(t) = \frac{t}{4}.$$

We discuss the following cases.

Case 1. $(x,y) = (0,0), (u,v) = (0,0), (w,z) = (1,0)$ *it is clear that* $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$ *or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$ *and*

$$\psi(G(T(0,0),T(0,0),T(1,0)) \leq \psi(M(0,0,1,0,0,0) - \phi(M(0,0,1,0,0,0),$$

where $G(T(0,0),T(0,1),T(0,1)) = 1$ *and* $M(0,1,1,1,1,1) = 1$.

Case 2. $(x,y) = (0,1), (u,v) = (1,1), (w,z) = (1,1)$ *it is clear that* $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$*or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$ *and*

$$\psi(G(T(0,1),T(1,1),T(1,1)) \leq \psi(M(0,1,1,1,1,1) - \phi(M(0,1,1,1,1,1),$$

where $G(T(0,1),T(1,1),T(1,1)) = 0$ *and* $M(0,1,1,1,1,1) = 1$.

Case 3. $(x,y) = (0,0), (u,v) = (1,0), (w,z) = (1,0)$ *it is clear that* $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$ *or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$ *and*

$$\psi(G(T(0,0),T(1,0),T(1,0)) \leq \psi(M(0,1,1,0,0,0) - \phi(M(0,1,1,0,0,0),$$

where $G(T(0,0),T(1,0),T(1,0)) = \frac{1}{4}$ *and* $M(0,1,1,0,0,0) = 1$.

Case 4. $(x,y) = (0,1), (u,v) = (1,1), (w,z) = (1,1)$ *again it is clear that* $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$ *or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$*and*

$$\psi(G(T(0,1),T(1,1),T(1,1)) \leq \psi(M(0,1,1,1,1,1) - \phi(M(0,1,1,1,1,1),$$

where $G(T(0,1),T(1,1),T(1,1)) = 0$ *and* $M(0,1,1,1,1,1) = 1$.

Case 5. $(x,y) = (u,v) = (0,1), (w,z) = (1,1)$ *also it is clear that* $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$ *or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$ *and*

$$\psi(G(T(0,1),T(0,1),T(1,1)) \leq \psi(M(0,1,1,1,1,1) - \phi(M(0,1,1,1,1,1),$$

where $G(T(0,1),T(0,1),T(1,1)) = 0$ *and* $M(0,0,1,1,1,1) = 1$.

Clearly for $(gx,gy) \preceq (gu,gv) \preceq (gw,gz)$ *or* $(gx,gy) \succeq (gu,gv) \succeq (gw,gz)$*all the conditions of Theorem 5 hold. So* $(0,0)$ *is the unique common coupled fixed point of* T *and* g*. On the other side if we taking in the Case 3* $\alpha = \beta = \frac{1}{6}$ *then Theorem 2 fail to satisfy.*

Acknowledgments. The first author were supported in part by the Serbian Ministry of Science and Technological Developments (Project: Methods of Numerical and Nonlinear Analysis with Applications, grant number #174002)

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.5, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

TRIANGULAR NORMS BASED ON INTUITIONISTIC FUZZY BCK-SUBMODULES

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Abstract: *We introduce the concept of intuitionistic fuzzy BCK-submodules of a BCK-module with respect to a t-norm and a s-norm and present some basic properties.*

Keywords : Intuitionistic fuzzy BCK-submodules, Triangular Norms, (Imaginable) Intuitionistic (T, S)-fuzzy BCK-submodules.

1. Introduction

The theory of fuzzy sets proposed by Zadeh [11] in 1965, and later on several researchers worked in this field. As a natural advancement of these research works we get one of the interesting generalizations of the theory of fuzzy sets that is the theory of intuitionstic fuzzy sets propounded by Atanassov [1, 2]. In 1966 Imai and Iseki [5] proposed the concept of BCK-algebra. Xi [10] applied the concept of fuzzy set to BCK-algebras. Also Bakhshi [3] in 2011 introduced the concept of fuzzy BCK-submodule of BCK-module and gave some related results. Recently, Badhurays and Bashammakh [4] considered the intuitionistic fuzzification of the concept of BCK-submodules in a BCK-module and investigated some properties of such BCK-modules. In this paper, we are going to introduce the notion of intuitionistic (T,S)-fuzzy BCK-submodules by using triangular norms, say T and S, and investigate several properties. We obtain some results on level sets of an intuitionistic (T,S)-fuzzy BCK-submodule by using the concept of level sets and triangular norms.
For the notations and terminology not given in this paper, the reader is referred to Atanassov [1, 2] (1986, 1994), Jun [8] (2001), Janiŝ [6] (2010), and Zadeh [11] (1965).

2. Preliminaries

First we present the fundamental definitions.

Definition 2.1. (Imai and Iseki [5]) a BCK-algebra is a set X with a binary operation $*$ and a constant 0 satisfying the following axioms :
(BCK1) $((x * y) \star (x * z)) \star (z * y) = 0$

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(BCK2) $(x * (x * y)) * y = 0$,
(BCK3) $x * x = 0$,
(BCK4) $0 * x = 0$,
(BCK5) $x * y = 0$ and $y * x = 0$ imply that $x = y$,
for all $x, y, z \in X$.
A partial ordering "$\leq$" is defined on X by $x \leq y$ iff $x * y = 0$.

Definition 2.2. (Zadeh [11]) By a fuzzy set μ in a nonempty set X we mean a function $\mu\colon X \longmapsto [0,1]$, and the complement of μ denoted by $\bar{\mu}$ is the fuzzy set in X given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.

Definition 2.3. (Atanassov [1]) An intuitionistic fuzzy set *(IFS)* in a universe X is an object of the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\},$$

where the functions $\mu\colon X \longmapsto [0,1]$ and $\lambda\colon X \longmapsto [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x)) + \lambda_A(x) \leq 1$ for all $x \in X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A(x), \lambda_A(x))$ for the *IFS*

$$A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\}$$

Definition 2.4. (Atanassov [1]) Let X be a non-empty set and $A = (\mu_A(x), \lambda_A(x))$, $B = (\mu_B(x), \lambda_B(x))$ be *IFS*'s of X. Then

(1) $A \subset B$ iff $\mu_A(x) < \mu_B(x)$ and $\lambda_A(x) > \lambda_B(x)$ for all $x \in X$.
(2) $A = B$ iff $\mu_A(x) = \mu_B(x)$ and $\lambda_A(x) = \lambda_B(x)$ for all $x \in X$
(3) $A^C = (\lambda_A, \mu_A)$.
(4) $A \cap B = \{x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\} : x \in X\}$.
(5) $A \cup B = \{x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\} : x \in X\}$.
(6) $\Box A = \{(x, \mu_A(x), \bar{\mu_A}(x)) | x \in X\}$.
(7) $\Diamond A = \{(x, \bar{\lambda}_A(x), \lambda_A(x)) | x \in X\}$.

Definition 2.5. (Atanassov [1]) Let $A = (\mu_A(x), \lambda_A(x))$ be an intuitionistic fuzzy set in M and let $\alpha \in [0,1]$. Then the sets

$$U(\mu_A, \alpha) = \{x \in M : \mu_A(x) \geq \alpha\},$$
$$L(\lambda_A, \alpha) = \{x \in M : \lambda_A(x) \leq \alpha\}$$

are called a μ-level α-cut and a λ-level α-cut of A, respectively.

Theorem 2.1. (Bakhshi [3]) *Let X be a bounded implicative BCK-algebra. Then $(X, +, 0)$ is an X-module where "$+$" is defined as $x + y = (x \star y) \vee (y \star x)$ and $xy = x \wedge y$.*

Theorem 2.2. (Bakhshi [3]) *A subset A of a BCK-module M is a BCK-submodule of M iff $a - b, xa \in A$, for every $a, b \in A$ and $x \in X$.*

Definition 2.6. (Bakhshi [3]) A fuzzy subset A of M is said to be a fuzzy *BCK*-submodule if for all $m, m_1, m_2 \in M$ and $x \in X$, the following axioms hold :

(1) $A(m_1 + m_2) \geq \min\{A(m_1), A(m_2)\}$

(2) $A(m) = A(-m)$
(3) $A(xm) \geq A(m)$

Definition 2.7. (Badhurays and Bashammakh [4]) An intuitionistic fuzzy subset $A = (\mu_A(x), \lambda_A(x))$ of M is said to be an intuitionistic fuzzy BCK-submodule of M if for all $m, m_1, m_2 \in M$ and $x \in X$, the following axioms hold :

(1) $\mu_A(m_1 + m_2) \geq \min\{\mu_A(m_1), \mu_A(m_2)\}$,
$\lambda_A(m_1 + m_2) \leq \max\{\lambda_A(m_1), \lambda_A(m_2)\}$.
(2) $\mu_A(m) = \mu_A(-m), \lambda_A(m) = \lambda_A(-m)$,
(3) $\mu_A(xm) \geq \mu_A(m), \lambda_A(xm) \leq \lambda_A(m)$.

Definition 2.8 (Klir and Yuan [9]) a triangular norm (or t-norm) T is a mapping $T\colon [0,1] \times [0,1] \longmapsto [0,1]$, which satisfies the following axioms for every $x, y, z, \in [0,1]$:

(T1) $T(x,1) = x$ (boundary condition);
(T2) $y \leq z$ implies $T(x,y) \leq T(x,z)$ (monotonicity);
(T3) $T(x,y) = T(y,x)$ (commutativity);
(T4) $T(x, T(y,z)) = T(T(x,y),z)$ (associativity).

Definition 2.9. (Klir and Yuan [9]) a triangular conorm (or t-conorm) S is a mapping $S\colon [0,1] \times [0,1] \longmapsto [0,1]$, which satisfies the following axioms for every $x, y, z, \in [0,1]$:

(S1) $S(x,0) = x$ (boundary condition);
(S2) $y \leq z$ implies $S(x,y) \leq S(x,z)$ (monotonicity);
(S3) $S(x,y) = S(y,x)$ (commutativity);
(S4) $S(x, S(y,z)) = S(S(x,y),z)$ (associativity).

Both t-norm and s-norm are called triangular norms. For all $\alpha, \beta \in [0,1]$, It is clear that

$$T(\alpha,\beta) \leq \min\{\alpha,\beta\} \leq \max\{\alpha,\beta\} \leq S(\alpha,\beta).$$

Definition 2.10. (Jun and Hong [7]) For a t-norm T and a s-norm S, we use the symbols Δ_T and Δ_S as the sets :

$$\Delta_T = \{a \in [0,1] | T(a,a) = a\},$$
$$\Delta_S = \{a \in [0,1] | S(a,a) = a\},$$

respectively.

Definition 2.11. (Jun and Hong [7]) We say that the intuitionistic fuzzy set $A = (\mu_A(x), \lambda_A(x))$ in M satisfies the imaginable property if

$$Im(\mu_A) \subseteq \Delta_T \text{ and } Im(\lambda_A) \subseteq \Delta_S.$$

Definition 2.12. (Klir and Yuan [9]) The norms T and S are called dual if and only if

D1) $\bar{T}(x,y) = S(\bar{x}, \bar{y})$,
D2) $\bar{S}(x,y) = T(\bar{x}, \bar{y})$ for all $x, y \in [0,1]$

A few t-norms which are frequently encountered are T_l, T_m, and T_w defined by $T_l(a,b) = \max\{a+b-1, 0\}$ (Lukasiewicz), $T_m(a,b) = \min\{a,b\}$ (minimum) and

$$T_w(a,b) := \{ \begin{array}{ll} \min\{a,b\} & \text{if } a=1 \text{ or } b=1, \\ 0 & \text{otherwise (weak).} \end{array} ,$$

A few s-norms which are frequently encountered are S_l,S_m, and S_w defined by $S_l(a,b) = \min\{a+b, 1\}$ (Lukasiewicz), $S_m(a,b) = \max\{a,b\}$ (maximum) andæ

$$S_w(a,b) := \{ \begin{array}{ll} \max\{a,b\} & \text{if } a=0 \text{ or } b=0, \\ 1 & \text{otherwise (strong).} \end{array} ,$$

3. Intuitionistic (T,S)-fuzzy BCK-submodules

Throughout this paper, M is a BCK-module and T is a t-norm and S is a s-norm unless otherwise specified. we can extend the concept of the intuitionistic fuzzy BCK-submodules of M to the concept of intuitionistic (T,S)-fuzzy BCK-submodules in the following way:

Definition 3.1. Let T be a t-norm and S be a s-norm on $[0,1]$. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in M is called an intuitionistic fuzzy BCK-submodule of M with respect to t-norm and s-norm (briefly, intuitionistic (T,S)-fuzzy BCK-submodule of M) if it satisfies the following conditions for all $m, m_1, m_2 \in M$:

(1) $\mu_A(m_1+m_2) \geq T\{\mu_A(m_1), \mu_A(m_2)\}$,
$\lambda_A(m_1+m_2) \leq S\{\lambda_A(m_1), \lambda_A(m_2)\}$.
(2) $\mu_A(m) = \mu_A(-m), \lambda_A(m) = \lambda_A(-m)$,
(3) $\mu_A(xm) \geq \mu_A(m), \lambda_A(xm) \leq \lambda_A(m)$.

Example 3.2. Let $X = \{0,1,2,3\}$ and consider the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then $(X,*)$ is a BCK-module over itself. Define a fuzzy set $\mu_A : M \longmapsto [0,1]$ by $\mu(0) = 0.5$, $\mu(m) = 0.3$, $m \in M$ and $\lambda_A : M \longmapsto [0,1]$ by $\lambda_A(0) = 0.3$, $\lambda_A(m) = 0.5$, $m \in M$. Let $T_l : [0,1] \times [0,1] \longmapsto [0,1]$ be a function defined by $T_l(a,b) = \max(a+b-1, 0)$ for all $a,b \in [0,1]$ and let $S_l : [0,1] \times [0,1] \longmapsto [0,1]$ be a function defined by $S_l(a,b) = \min(a+b, 1)$ for all $a,b \in [0,1]$. Then T_l is a t-norm and S_l is a s-norm. By routine calculations, we know that $A = (\mu_A(x), \lambda_A(x))$ is an intuitionistic (T_l, S_l)-fuzzy BCK-submodule of M.

Theorem 3.3. *An intuitionistic fuzzy subset A of M is an intuitionistic (T,S)-fuzzy BCK-submodule of M if and only if*

(1) $\mu_A(m_1-m_2) \geq T\{\mu_A(m_1), \mu_A(m_2)\}$,
$\lambda_A(m_1-m_2) \leq S\{\lambda_A(m_1), \lambda_A(m_2)\}$.
(2) $\mu_A(xm) \geq \mu_A(m), \lambda_A(xm) \leq \lambda_A(m)$.

proof. Let A be an intuitionistic (T, S)-fuzzy BCK-submodule of M, then

$$\begin{aligned}\mu_A(m_1 - m_2) &= \mu_A(m_1 + (-m_2))\\ &\geq T(\mu_A(m_1), \mu_A(-m_2))\\ &= T(\mu_A(m_1), \mu_A(m_2)),\end{aligned}$$

Similarly, $\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2))$. Condition 2 is hold by definition. Conversely suppose A satisfies 1 and 2. Then we have by 2

$$\mu_A(-m) = \mu_A((-1).m) \geq \mu_A(m),$$

and

$$\mu_A(m) = \mu_A((-1).(-1).m) \geq \mu_A(-m).$$

Thus ${}_A(m) = {}_A(-m)$. Similarly, $\lambda_A(m) = \lambda_A(-m)$.
Also we have

$$\begin{aligned}\mu_A(m_1 + m_2) &= \mu_A(m_1 - (-m_2))\\ &\geq T(\mu_A(m_1), \mu_A(-m_2))\\ &\geq T(\mu_A(m_1), \mu_A(m_2))\end{aligned}$$

Similarly,

$$\lambda_A(m_1 + m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)).$$

Thus A is an intuitionistic (T, S)-fuzzy BCK-submodule of M.

Proposition 3.4. *Let T and S be dual norms. If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M, then so is $\Box A = (\mu_A, \overline{\mu}_A)$.*

Proof. For all $m_1, m_2 \in M$, we have

$$T(\mu_A(m_1), \mu_A(m_2)) \leq \mu_A(m_1 + m_2)$$

and so

$$T(1 - \overline{\mu}_A(m_1), 1 - \overline{\mu}_A(m_2)) \leq 1 - \overline{\mu}_A(m_1 + m_2)$$

hence

$$1 - T(1 - \overline{\mu}_A(m_1), 1 - \overline{\mu}_A(m)) \geq 1 - (1 - \overline{\mu}_A(m_1 + m_2)$$

which implies

$$\overline{T}(1 - \overline{\mu}_A(m_1), 1 - \overline{\mu}_A(m_2)) \geq \overline{\mu}_A(m_1 + m_2)$$

since T and S are dual, we get

$$S(\overline{\mu}_A(m_1), \overline{\mu}_A(m_2)) \geq \overline{\mu}_A(m_1 + m_2)\ ,$$

Moreover $\mu_A(m) = \mu_A(-m)$ imply that

$$1 - \mu_A(m) = 1 - \mu_A(-m),$$

Thus $\overline{\mu}_A(m) = \overline{\mu}_A(-m)$. Now, let $m \in M$ and $x \in X$, since μ_A is T-fuzzy BCK-submodule of M, we have $\mu_A(x.m) \geq \mu_A(m)$. Hence $1 - \mu_A(x.m) \leq 1 - \mu_A(m)$ which implies $\overline{\mu}_A(xm) \leq \overline{\mu}_A(m)$. Therefore $\Box A = (\mu_A, \overline{\mu}_A)$ is an intuitionistic (T, S) - fuzzy BCK-submodule of M.

Proposition 3.5. *Let T and S be dual norms. If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK- submodule of M, then so is $\Diamond A = (\overline{\lambda}_A, \lambda_A)$.*

Proof. For all $m_1, m_2 \in M$, we have

$$S(\lambda_A(m_1), \lambda_A(m_2)) \geq \lambda_A(m_1 + m_2)$$

and so

$$S(1 - \overline{\lambda}_A(m_1), 1 - \overline{\lambda}_A(m_2)) \geq 1 - \overline{\lambda}_A(m_1 + m_2)$$

hence

$$1 - S(1 - \overline{\lambda}_A(m_1), 1 - \overline{\lambda}_A(m_2)) \leq 1 - (1 - \overline{\lambda}_A(m_1 + m_2))$$

which implies

$$1 - S(\overline{\overline{\lambda}}_A(m_1), \overline{\overline{\lambda}}_A(m_2)) \leq \overline{\lambda}_A(m_1 + m_2)$$

since T and S are dual

$$1 - \overline{T}(\overline{\lambda}_A(m_1), \overline{\lambda}_A(m_2)) \leq \overline{\lambda}_A(m_1 + m_2)$$

that is

$$T(\overline{\lambda}_A(m_1), \overline{\lambda}_A(m_2)) \leq \overline{\lambda}_A(m_1 + m_2).$$

Moreover

$$\overline{\lambda}_A(m) = \overline{\lambda}_A(-m)$$

imply that $1 - \lambda_A(m) = 1 - \lambda_A(-m)$, Thus $\lambda_A(m) = \lambda_A(-m)$. Now, let $m \in M$ and $x \in X$, since λ_A is T-fuzzy BCK-submodule of M we have $\lambda_A(x.m) \leq \lambda_A(m)$. Hence $1 - \lambda_A(x.m) \geq 1 - \lambda_A(m)$ which implies $\overline{\lambda}_A(xm) \geq \overline{\lambda}_A(m)$. Therefore $\Diamond A = (\overline{\lambda}_A, \lambda_A)$ is an intuitionistic (T, S) - fuzzy BCK-submodule of M.

Combining the above two Propositions it is not difficult to verify that the following theorem is valid.

Theorem 3.6. *Let T and S be dual norms. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M if and only if $\Box A$ and $\Diamond A$ are intuitionistic (T, S)-fuzzy BCK-submodule of M.*

Corollary 3.7. *Let T and S be dual norms. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M if and only if μ_A and $\overline{\lambda_A}$ are T-fuzzy BCK-submodule of M.*

From corollary 3.7 we immediately obtain the following result.

Theorem 3.8. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_m, S_m)- fuzzy BCK- submodule of M if and only if the fuzzy sets μ_A and $\overline{\lambda}_A$ are fuzzy BCK-submodule of M.*

Theorem 3.9. *An intuitionistic fuzzy set $A = (\mu_A\ ,\ \lambda_A)$ is an intuitionistic (T_m, S_m)-fuzzy BCK- submodule of M if and only if $\Box A = (\mu_A, \bar{\mu}_A)$ and $\Diamond A = (\overline{\lambda}_A, \lambda_A)$ are intuitionistic (T_m, S_m)-fuzzy BCK- submodule of M.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M. By Theorem 3.8, we get $\mu_A = \overline{\overline{\mu}}_A$and $\overline{\lambda}_A$ are fuzzy BCK-submodule of M.

Therefore $\Box A = (\mu_A, \overline{\mu}_A)$ and $\Diamond A = (\overline{\lambda}_A, \lambda_A)$ are intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M. Conversely, assume that $A = (\mu_A, \lambda_A)$ and $\Box A = (\mu_A, \overline{\mu}_A)$ and $\Diamond A = (\overline{\lambda}_A, \lambda_A)$ are intuitionistic (T_m, S_m)-fuzzy BCKsubmodule of M. Then the fuzzy sets μ_A and $\overline{\lambda}_A$ are fuzzy BCK-submodule of M. Therefore $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_m, S_m)-fuzzy BCK- submodule of M.

Definition 3.10. An intutionistic (T, S)-fuzzy BCK-submodule of M is called an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M if it satisfies the imaginable property.

Proposition 3.11. *Every imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M is an intuitionistic fuzzy BCK-submodule of M.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M. Then

$$\mu_A(m_1 + m_2) \geq T(\mu_A(m_1), \mu_A(m_2))$$

and

$$\lambda_A(m_1 + m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2))$$

for all $m_1, m_2 \in M$.
Since $A = (\mu_A, \lambda_A)$ is imaginable, we have

$$\begin{aligned}
&\min\{\mu_A(m_1), \mu_A(m_2)\} \\
&= T(\min\{\mu_A(m_1), \mu_A(m_2)\}, \min\{\mu_A(m), \mu_A(m_2)\}) \\
&\leq T(\mu_A(m_1), \mu_A(m_2)) \\
&\leq \min\{\mu_A(m_1), \mu_A(m_2)\},
\end{aligned}$$

and

$$\begin{aligned}
&\max\{\lambda_A(m_1), \lambda_A(m_2)\} \\
&= S(\max\{\lambda_A(m_1), \lambda_A(m_2)\}, \max\{\lambda_A(m), \lambda_A(m_2)\}) \\
&\geq S(\lambda_A(m_1), \lambda_A(m_2)) \\
&\geq \max \lambda_A(m_1), \lambda_A(m_2).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mu_A(m_1 - m_2) \\
&\geq T(\mu_A(m_1), \mu_A(m_2)) \\
&= \min\{\mu_A(m_1), \mu_A(m_2)\},
\end{aligned}$$

and

$$\begin{aligned}
&\lambda_A(m_1 - m_2) \\
&\leq S(\lambda_A(m_1), \lambda_A(m_2)) \\
&= \max\{\lambda_A(m_1), \lambda_A(m_2)\}.
\end{aligned}$$

Now let $x \in X$ and $m \in M$. Since $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M, we have $\mu_A(xm) \geq \mu_A(m)$, $\lambda_A(xm) \leq \lambda_A(m)$. Therefore $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M.

Note that every intuitionistic fuzzy BCK-submodule is an intuitionistic (T,S)-fuzzy BCK-submodule but the converse is not true as seen in the following Example.

Example 3.12. We consider the BCK-module M which is given in Example 3.2. Define an intuitionistic fuzzy set $A=(\mu_A,\lambda_A)$ in M

$$\mu_A(m)=\begin{cases} 0.2 & \text{if} \quad m=1 \\ 0.3 & \text{if} \quad m=2,3 \\ 0.5 & \text{if} \quad m=0 \end{cases} \;;\quad \lambda_A(m)=\begin{cases} 0.5 & \text{if} \quad m=1 \\ 0.3 & \text{if} \quad m=2,3 \\ 0.1 & \text{if} \quad m=0 \end{cases}$$

Then $A=(\mu_A,\lambda_A)$ is an intuitionistic (T_w,S_w)-fuzzy BCK-submodule of M, but it is not an intuitionistic fuzzy BCK-submodule of M since

$$\mu_A(2+3)=\mu_A(1)=0.2<0.3=\min(\mu_A(2),\mu_A(3)).$$

Proposition 3.13. *If an intuitionistic fuzzy set $A=(\mu_A,\lambda_A)$ in M is an imaginable intuitionistic (T,S)-fuzzy BCK-submodule of M, then for all $m\in M$, $\mu_A(0)\geq\mu_A(m)$ and $\lambda_A(0)\leq\lambda_A(m)$.*

Proof. From Definition 3.1 (3) it follows that

$$\mu_A(0)=\mu_A(0.m)\geq\mu_A(m)$$

and

$$\lambda_A(0)=\lambda_A(0.m)\leq\lambda_A(m)$$

for all $m\in M$.

Theorem 3.14. *If $A=(\mu_A,\lambda_A)$ is an imaginable intuitionistic (T,S)-fuzzy BCK-submodule of M, then the set $H=\{m\in M|\mu(m)=\mu(0)\}$ and $K=\{m\in M|\lambda_A(m)=\lambda_A(0)\}$ are BCK-submodule of M.*

Proof. Assume that $A=(\mu_A,\lambda_A)$ is an imaginable intuitionistic (T,S)-fuzzy BCK-submodule of M, and let $m_1,m_2\in M$. Since $A=(\mu_A,\lambda_A)$ is an imaginable intuitionistic (T,S)-fuzzy BCK-submodule of M, we have

$$\begin{aligned}\mu_A(m_1-m_2)&\geq T(\mu_A(m),\mu_A(m))\\&=T(\mu_A(0),\mu_A(0))\\&=\mu_A(0)\end{aligned}$$

for all $m_1,m_2\in M$, Using Lemma Proposition 3.11., we get $\mu_A(m_1-m_2)=\mu_A(0)$. Hence $m_1-m_2\in H$. Now let $x\in X$and $m\in M$. Since $A=(\mu_A,\lambda_A)$ is an intuitionistic (T,S)-fuzzy BCK-submodule of M, we have $\mu_A(x.m)\geq\mu_A(m)=\mu_A(0)$. Using Lemma Proposition 3.11., we get $\mu_A(x.m)=\mu_A(0)$ and so $x.m\in H$. Therefore H is a BCK-submodule of M. By similar method, we get K is a BCK-submodule of M.

Definition 3.15. Let $A=(\mu_A,\lambda_A)$ be an intuitionistic fuzzy set in BCK-submodule M and let $\alpha,\beta\in[0,1]$ with $\alpha+\beta\leq 1$. Then the set

$$A_{(\alpha,\beta)}:=\{m\in M|\mu_A(m)\geq\alpha,\lambda_A(m)\leq\beta\}$$

is called an (α,β)-level set of $A=(\mu_A,\lambda_A)$.

Theorem 3.16. *Let $A=(\mu_A,\lambda_A)$ be an intuitionistic fuzzy set in M such that*

$A_{(\alpha,\beta)}$ is a BCK-submodule of M , for all $(\alpha, \beta) \in [0,1]$ with $\alpha + \beta \leq 1$. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M .

Proof. Let $m_1, m_2, m \in M$ and $x \in X$ be such that $A(m_1) = (\alpha_1, \beta_1)$, $A(m_2) = (\alpha_2, \beta_2)$ where $\alpha_i + \beta_i \leq 1$ for $i = 1, 2$. Then $m_1, m_2 \in A_{(\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))}$, and so $m_1 - m_2 \in A_{\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))}$.
Hence

$$\mu_A(m_1 - m_2) \geq \min(\alpha_1, \alpha_2) \geq T(\alpha_1, \alpha_2),$$

and

$$\lambda_A(m_1 - m_2)) \leq \max(\beta_1, \beta_2) \leq S(\beta_1, \beta_2).$$

Also, if we put $s' = {}_A(m), t' = {}_A(m)$ where $s' + t' \leq 1$. Then $m \in A_{(s',t')}$. Since $A_{(s',t')}$ is a *BCK*- submodule of M, we have $xm \in A_{(s',t')}$. It follows that

$$\mu_A(xm) \geq s' = \mu_A(m)$$

and

$$\lambda_A(xm) \leq t' = \lambda_A(m)$$

Therefore $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy *BCK*-submodule of M.

The following Example shows that the converse of Theorem 3.16 is not true.

Example 3.17. We consider the intuitionistic (T_w, S_w)-fuzzy *BCK*-submodule A of M which is given in Example 3.2. Then $A_{(0.3,0.5)} = \{2, 3, 0\}$ is not *BCK*-submodule of M since $2 + 3 = 1 \notin A_{(0.3,0.5)}$

Theorem 3.18. *If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M, then $A_{(1,0)}$ is either empty or a BCK-submodule of M .*

Proof. Let $m_1, m_2 \in A_{(1,0)}$. Then $\mu_A(m_1) \geq 1$, $\mu_A(m_2) \geq 1$, $\lambda_A(m_1) \leq 0$ and $\lambda_A(m_2) \leq 0$. It follows from Definitions 2.10 and Theorem 3.3 that

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2)) \geq T(1, 1) = 1$$

and

$$\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)) \leq S(0, 0) = 0,$$

so $m_1 - m_2 \in A_{(1,0)}$. Let $m \in A_{(1,0)}$ and $x \in X$. Then

$$\mu_A(xm) \geq \mu_A(m) \geq 1$$

and

$$\lambda_A(xm) \leq \lambda_A(m) \leq 0,$$

so $xm \in A_{(1,0)}$.

As a generalization of Theorem 3.18, we get the following Theorem.

Theorem 3.19. *If $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M, then $A_{(\alpha,\beta)}$ is either empty or a BCK-submodule of M for all $\alpha \in \Delta_T$ and $\beta \in \Delta_S$. with $\alpha + \beta \leq 1$.*

Proof. Let $m_1, m_2 \in A_{(\alpha,\beta)}$ where $\alpha \in \Delta_T$, $\beta \in \Delta_S$ and $\alpha + \beta \leq 1$. Then

$$\begin{aligned} &\mu_A(m_1 - m_2) \\ &\geq T(\mu_A(m_1), \mu_A(m_2)) \\ &\geq T(\alpha, \alpha) = \alpha \end{aligned}$$

and

$$\begin{aligned} &\lambda_A(m_1 - m_2) \\ &\leq S(\lambda_A(m_1), \lambda_A(m_2)) \\ &\leq S(\beta, \beta) = \beta, \end{aligned}$$

and so $m_1 - m_2 \in A_{(\alpha,\beta)}$. Let $m \in A_{(\alpha,\beta)}$ and $x \in X$. Then

$$\mu_A(xm) \geq \mu_A(m) \geq \alpha$$

and

$$\lambda_A(xm) \leq \lambda_A(m) \leq \beta,$$

so $xm \in A_{(\alpha,\beta)}$.Hence $A_{(\alpha,\beta)}$ is a BCK-submodule of M .

Proposition 3.20. (Bakhshi [3]) *A fuzzy set in M is a fuzzy BCK-submodule of M if and only if the non-empty $U(\mu, \alpha)$, $\alpha \in [0, 1]$ is a BCK-submodule of M.*

By the above Proposition , we get the following result.

Corollary 3.21. *If $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic fuzzy set in M. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M if and only if the non-empty sets $U(\mu, \alpha)$ and $L(\lambda, \alpha)$ are BCK-submodules of M, for every $(\alpha, \beta) \in [0, 1]$.*

From corollary 3.21 we immediately obtain the following Theorem.

Theorem 3.22. *Let T be the minimum t-norm and let S the maximum s-norm dual of T . Then an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ of M is is an intuitionistic (T, S)-fuzzy BCK-submodule of M if and only if*

$$A_{(\alpha,\beta)} := \{m \in M | \mu_A(m) \geq \alpha, \lambda_A(m) \leq \beta\}$$

is a BCK-submodule of M, where $(\alpha, \beta) \in [0, 1]$.

Proposition 3.23. *Let S be a non-empty subset of a BCK-module M. Then an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ defined by*

$$\mu_A(m) = \begin{cases} 1 & \text{if} \quad m \in S, \\ \alpha & \text{otherwise.} \end{cases}, \lambda_A(m) = \begin{cases} 0 & \text{if} \quad m \in S, \\ \beta & \text{otherwise.} \end{cases}$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ *and $\alpha + \beta \leq 1$ is an intuitionistic (T, S)-fuzzy BCK -submodule of M if and only if S is a BCK-submodule of M.*

Proof. Let S be a BCK-submodule of M. Let $m_1, m \in M$. If $m_1, m_2 \in S$,

then $m_1 - m_2 \in S$, and so

$$\begin{aligned}\mu_A(m_1 - m_2) = 1 \geq 1 \\ &= T(1, 1) \\ &= T(\mu_A(m_1), \mu_A(m_2))\end{aligned}$$

and

$$\begin{aligned}\lambda_A(m_1 - m_2) &= 0 \\ &= S(0, 0) \\ &= S(\lambda_A(m_1), \lambda_A(m_2))\end{aligned}$$

For $m_1 \in S$, $m_2 \notin S$, we have

$$\begin{aligned}\mu_A(m_1 - m_2) = \alpha \geq \alpha \\ &= T(1, \alpha) \\ &= T(\mu_A(m_1), \mu_A(m_2))\end{aligned}$$

and

$$\begin{aligned}\lambda_A(m_1 - m_2) = \beta \leq \beta \\ &= S(0, \beta) \\ &= S(\lambda_A(m_1), \lambda_A(m_2))\end{aligned}$$

Similarly, for the case $m_1 \notin S$, $m_2 \in S$, we have

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2))$$

and

$$\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)).$$

For $m_1 \notin S$, $m_2 \notin S$,

$$\begin{aligned}\mu_A(m_1 - m_2) &\geq \alpha \\ &= T(1, \alpha) \\ &\geq T(\alpha, \alpha) \\ &= T(\mu_A(m_1), \mu_A(m_2)),\end{aligned}$$

and

$$\begin{aligned}\lambda_A(m_1 - m_2) &\leq \beta \\ &= S(0, \beta) \\ &\leq S(\beta, \beta) \\ &= S(\lambda_A(m_1), \lambda_A(m_2)).\end{aligned}$$

Thus for all cases,

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2))$$

and

$$\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)).$$

Next, let $m \in M$ and $x \in X$, Then, if $m \in S$ then $xm \in S$ and so,

$$\mu_A(xm) = 1 \geq 1 = \mu_A(m)$$

and

$$\lambda_A(xm) = 0 \leq 0 = \lambda_A(m).$$

If $m \notin S$, then

$$\mu_A(xm) \geq \alpha = {}_A(m)$$

and

$$\lambda_A(xm) \leq \beta = \lambda_A(m).$$

Therefore $\mu_A(xm) \geq \mu_A(m)$ and $\lambda_A(xm) \leq \lambda_A(m)$. Thus $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M .

Conversely, we assume $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M. Let $m_1, m_2 \in S$, $x \in X$. Then,

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2)) = T(1, 1) = 1,$$

hence $\mu_A(m_1 - m_2) = 1$. Thus $m_1 - m_2 \in S$. Also, $\mu_A(xm) \geq \mu_A(m) = 1$ implies $\mu_A(xm) = 1$ implies $xm \in S$. Hence, S is a BCK-submodule of M .

Corollary 3.24. *Let S be a non-empy subset of a BCK-module M and let χ_s be the characteristic function of S. Then $A = (\chi_s, \chi_s^c)$ is an intutionistic (T, S)-fuzzy BCK-submodule of M if and only if S is a BCK-submodule of M.*

Definition 3.25. (Janiŝ [6]) Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of X and let T be a t-norm. Then $A_{T,\alpha}$ is a subset of X defined by

$$A_{T,\alpha} = \{x \in X | T(\mu_A(x), 1 - \lambda_A(x)) \geq \alpha\},$$

for every $\alpha \in [0, 1]$

Theorem 3.26. *Let T and S be dual norms. If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M. Then*

$$A_{T,1} = \{m \in M | T(\mu_A(m), 1 - \lambda_A(m)) = 1\}$$

is a BCK-submodule of M.

Proof. Let $m_1, m_2 \in A_{T,1}$. Then,

$$\begin{aligned}
&T(\mu_A(m_1 - m_2), 1 - {}_A(m_1 - m_2)) \\
&\geq T(T(\mu_A(m_1), {}_A(m_2)), 1 - S({}_A(m_1), {}_A(m_2))) \\
&= T(T(\mu_A(m_2), (\mu_A(m_1)), T(1 - \lambda_A(m_1), 1 - \lambda_A(m_2))) \\
&= T(\mu_A(m_2), T(\mu_A(m_1), T(1 - \lambda_A(m_1), 1 - \lambda_A(m_2)))) \\
&= T(\mu_A(m_2), T(T(\mu_A(m_1), 1 - \lambda_A(m_1)), 1 - \lambda_A(m_2))) \\
&= T(\mu_A(m_2), T(1 - \lambda_A(m_2), T(\mu_A(m_1), 1 - \lambda_A(m_1)))) \\
&= T(T(\mu_A(m_2), 1 - \lambda_A(m_2)), T(\mu_A(m_1), 1 - \lambda_A(m_1))) \\
&= T(1, 1) = 1
\end{aligned}$$

Thus, we have $T(\mu_A(m_1 - m_2), 1 - \lambda_A(m_1 - m_2)) = 1$ Therefore $m_1 - m_2 \in A_{T,1}$. Also, let $x \in X$ and $m \in A_{T,1}$. Then $T(\mu_A(m), 1 - \lambda_A(m)) = 1$. Further, $T(\mu_A(xm), 1-\lambda_A(xm)) \geq T(\mu_A(m), 1-\lambda_A(m)) = 1$. Therefore $xm \in A_{T,1}$. Hence, $A_{T,1}$ is a is a BCK-submodule of M.

For any tringular norm T, the level set $A_{T,\alpha}$ of an intuitionistic (T,S)-fuzzy BCK-submodule of M is not necessarily to be a BCK-submodule of M. However, if T is the minimum tringular norm, then all level sets $A_{T,\alpha}$ of an intuitionistic (T,S)-fuzzy BCK-submodule of M are BCK-submodules of M.

Theorem 3.27. *Let $A=(\mu_A,\lambda_A)$ be an intuitionistic (T_m,S_m)-fuzzy BCK-submodule of M such that T_m,S_m are dual. Then for every $\alpha\in[0,1]$,*

$$A_{T_m,\alpha}=\{m\in M|T(\mu_A(m),1-\lambda_A(m))\geq\alpha\}$$

is a BCK-submodule of M.

Proof. Let $A=(\mu_A(x),\lambda_A(x))$ is an intuitionistic (T_m,S_m)-fuzzy BCK-submodule of M. Let $m_1,m_2\in A_{T,\cdot}$. Then,

$$\begin{aligned}
&T_m(\mu_A(m_1-m_2),1-{}_A(m_1-m_2))\\
&\geq T_m(T_m(\mu_A(m_1),\mu_A(m_2)),1-S_m(\lambda_A(m_1),\lambda_A(m_2)))\\
&=T_m(T_m(\mu_A(m_2),(\mu_A(m_1)),T_m(1-\lambda_A(m_1),1-\lambda_A(m_2)))\\
&=T_m(\mu_A(m_2),T_m(\mu_A(m_1),T_m(1-\lambda_A(m_1),1-\lambda_A(m_2))))\\
&=T_m(\mu_A(m_2),T_m(T_m(\mu_A(m_1),1-\lambda_A(m_1)),1-\lambda_A(m_2)))\\
&=T_m(\mu_A(m_2),T_m(1-\lambda_A(m_2),T_m(\mu_A(m_1),1-\lambda_A(m_1))))\\
&=T_m(T_m(\mu_A(m_2),1-\lambda_A(m_2)),T_m(\mu_A(m_1),1-\lambda_A(m_1)))\\
&\geq T_m(\alpha,\alpha)=\alpha
\end{aligned}$$

Thus, we have

$$T_m(\mu_A(m_1-m_2),1-{}_A(m_1-m_2))\geq\alpha$$

Therefore, $m_1-m_2\in A_{T_m,\alpha}$. Also, let $x\in X$ and $m\in A_{T_m,\alpha}$. Then

$$T_m(\mu_A(m),1-\lambda_A(m))\geq\alpha$$

Further,

$$T_m(\mu_A(xm),1-\lambda_A(xm))\geq T_m(\mu_A(m),1-\lambda_A(m))\geq\alpha$$

Therefore we have $T_m(\mu_A(xm),1-\lambda_A(xm))\geq\alpha$. Hence $xm\in A_{T_m,\alpha}$. Thus $A_{T_m,\alpha}$ is a is a BCK-submodule of M.

Definition 3.28. Let $A=(\mu_A,\lambda_A)$ be an intuitionistic fuzzy set of X, let T and S be dual norms. Then $A_{T,S,\alpha}$ is a subset of X defined by

$$A_{T,S,1}=\{x\in XT(\mu_A(x),S(\mu_A(x),\lambda_A(x)))\geq\alpha\}$$

for every $\alpha\in[0,1]$.

Theorem 3.29. *Let $A=(\mu_A,\lambda_A)$ be an intuitionistic (T,S)-fuzzy BCK-submodule of M, then*

$$A_{T,S,1}=\{m\in M|T(\mu_A(m),S(\mu_A(m),\lambda_A(m)))=1\}$$

is a BCK-submodule of M.

Proof. Let $A=(\mu_A,\lambda_A)$ be an intuitionistic (T,S)-fuzzy BCK-submodule of M. Let $m_1,m_2\in A_{T,S,1}$, then

$$T(\mu_A(m_1),S(\mu_A(m_1),\lambda_A(m_1)))=1$$

and

$$T(\mu_A(m_2), S(\mu_A(m_2), \lambda_A(m_2))) = 1.$$

Therefore $\mu_A(m_1) \geq 1$ and $\mu_A(m_2) \geq 1$ which mean that $\mu_A(m_1) = 1$ and $\mu_A(m_2) = 1$. From monotonicity of T, we have,

$$\begin{aligned}
&T(\mu_A(m_1 - m_2), S(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) \\
&\geq T(T(\mu_A(m_1 - m_2)), T(\mu_A(m_1 - m_2))) \\
&\geq T(T(\mu_A(m), \mu_A(m)), T(\mu_A(m), \mu_A(m))) \\
&= T(T(1,1), T(1,1)) \\
&= T(1,1) = 1
\end{aligned}$$

Therefore, $T(\mu_A(m_1 - m_2), S(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) = 1$ implies $m_1, m_2 \in A_{T,S,1}$. Also, let $x \in X$ and $m \in A_{T,S,1}$. Then, $T(\mu_A(m), S(\mu_A(m), \lambda_A(m))) = 1$. which impliese $\mu_A(m) = 1$. Now,

$$\begin{aligned}
&T(\mu_A(xm), S(\mu_A(xm), \lambda_A(xm))) \\
&\geq T(\mu_A(xm), \mu_A(xm)) \\
&\geq T(\mu_A(m), \mu_A(m)) \\
&= T(1,1) = 1
\end{aligned}$$

Thus, we have, $T(\mu_A(xm), S(\mu_A(xm), \lambda_A(xm))) = 1$. Therefore, $xm \in A_{T,S,1}$. Hence, $A_{T,S,1}$ is a BCK-submodule of M.

Theorem 3.30. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of Msuch that T_m, S_m are dual. Then for every $\alpha \in [0,1]$,*

$$A_{T,S,\alpha} = \{m \in M | T(\mu_A(m), S(\mu_A(m), \lambda_A(m))) \geq \alpha\}$$

is a BCK-submodule of M.

Proof. Let $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M. Let $m_1, m_2 \in A_{T,S,\alpha}$, then

$$T_m(\mu_A(m_1), S_m(\mu_A(m_1), \lambda_A(m_1))) \geq \alpha$$

and

$$T_m(\mu_A(m_2), S_m(\mu_A(m_2), \lambda_A(m_2))) \geq \alpha.$$

Therefore $\mu_A(m_1) \geq \alpha$ and $\mu_A(m_2 \geq \alpha$. Due monotonicity of T_m, we have,

$$\begin{aligned}
&T_m(\mu_A(m_1 - m_2), S_m(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) \\
&\geq T_m(\mu_A(m_1 - m_2)), (\mu_A(m_1 - m_2))) \\
&= \mu_A(m_1 - m_2) \\
&\geq T_m(\mu_A(m_1), \mu_A(m_2)) \\
&\geq T_m(\alpha, \alpha) \\
&= \alpha
\end{aligned}$$

Therefore, $T_m(\mu_A(m_1 - m_2), S_m(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) \geq \alpha$ and hence $m_1 - m_2 \in A_{T_m,S_m,\alpha}$. Also, let $m \in A_{T_m,S_m,\alpha}$ and $x \in X$. Then,

$$T_m(\mu_A(m), S_m(\mu_A(m), \lambda_A(m))) \geq \alpha.$$

which impliese $\mu_A(m) \geq \alpha$. From monotonicity of T_m, we have,

$$\begin{aligned}
&T_m(\mu_A(xm), S_m(\mu_A(xm), \lambda_A(xm)) \\
&\geq T_m(\mu_A(xm), \mu_A(xm)) \\
&= \mu_A(xm) \\
&\geq \mu_A(m) \\
&\geq \alpha
\end{aligned}$$

Thus $T_m(\mu_A(xm), S_m(\mu_A(xm), \lambda_A(xm)) \geq \alpha$. Therefore, $xm \in A_{T_m,S_m,\alpha}$. Hence, $A_{T_m,S_m,\alpha}$ is a BCK-submodule of M.

4. Conclusion

One of the generalizations of fuzzy BCK-submodules, namely, intuitionistic (T,S)-fuzzy BCK- submodules was defined and some properties of intuitionistic (T,S)-fuzzy BCK-submodules are investigated. Also, some related results on level sets of an intuitionistic (T,S)-fuzzy BCK-submodule are investigated. These investigations of generalized fuzzy on BCK-modules could be enable us to discuss further study in this field.

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On strongly almost generalized difference lacunary ideal convergent sequences of fuzzy numbers

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Abstract

The purpose of this paper is to introduce some new sequence spaces of fuzzy numbers defined by lacunary ideal convergence using generalized difference matrix and Orlicz functions. We also study some algebraic and topological properties of these classes of sequences. Moreover, some illustrative examples are given in support of our results.

Keywords and phrases: Ideal convergence; fuzzy number; difference sequence; Orlicz function; lacunary sequence.

AMS subject classification (2010): 40A05; 40C05; 40G15; 06B99.

1 Introduction and preliminaries

The concept of ideal convergence is the dual (equivelant) to the notion of filter convergence introduced by Cartan [4]. The filter convergence is a generalization of the classical notion of convergence of sequences of real or complex numbers and it has been an important tool in the study of functional analysis. Nowadays many authors studied this notion from various aspects and applied this notion to various problems arising in the convergence theory. Kostyrko et al. [13] and Nuray and Ruckle [23] independently studied in detalis about the notion of ideal convergence which is based upon the structure of the admissible ideal I of subsets $\mathbb{N}$ of natural numbers. Later on it was further investigated by many authors, e.g. Tripathy and Hazarika [26], Mursaleen and Mohiuddine [22] and references therein.

Let S be a non-empty set. Then a non empty class $I \subseteq P(S)$ is said to be an *ideal* on S if and only if (i) $\phi \in I$; (ii) I is additive; (iii) hereditary. An ideal $I \subseteq P(S)$ is said to be *non trivial* if $I \neq \phi$ and $S \notin I$. A non-empty family of sets $F \subseteq P(S)$ is said to be a *filter* on S if and only if (i) $\phi \notin F$ (ii) for each $A, B \in F$ we have $A \cap B \in F$; (iii) for each $A \in F$ and each $B \supset A$, we have $B \in F$. For each ideal I, there is a filter $F(I)$ corresponding to I i.e. $F(I) = \{K \subseteq S : K^c \in I\}$, where $K^c = S - K$. We say that a non-trivial ideal $I \subseteq P(S)$ is an *admissible ideal* on S if and only if it contains all singletons, i.e. if it contains $\{\{s\} : s \in S\}$. Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to the number ℓ (denoted by $I\text{-}\lim x_k = \ell$) if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$.

We used the standard notation $\theta = (k_r)$ to denote the *lacunary sequence*, where θ is a sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ $(r \neq 1)$ by q_r (see [8]).

The notion of lacunary ideal convergence for sequences of real numbers and fuzzy numbers, respectively, has been defined and studied in [27] and [9]. Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal. A real sequence

$x = (x_k)$ is said to be *lacunary I-convergent* to $L \in \mathbb{R}$, in symbol we shall write $I_\theta\text{-}\lim x = L$, if for every $\varepsilon > 0$, the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| \geq \varepsilon \right\} \in I.$$

Throughout the paper we use w to denotes the set of all real sequences $x = (x_k)$. The difference sequence spaces have been introduced by Kızmaz [12] by using the difference operator Δ as follows:

$$Z(\Delta) = \{(x_k) \in w : \Delta x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$ and $\Delta x_k = \Delta^1 x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$, where the standard notations ℓ_∞, c and c_0 are used to denote the set of bounded, convergent and null sequences, respectively. Later this idea was generalized by Et and Çolak [6] by considering Δ^n instead of Δ, where $(\Delta^n x_k) = \Delta^1(\Delta^{n-1} x_k)$ for $n \geq 2$ and all $k \in \mathbb{N}$. In case of $n = 0$ we obtain x_k. Tripathy et al. [28] presented another generalization of difference sequence spaces by introducing the operator Δ_m^n and is given by $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ so that $\Delta_m^n x_k$ has the following binomial representation:

$$\Delta_m^n x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k+m\nu},$$

for all $k \in \mathbb{N}$. If we take $n = 1$, then $Z(\Delta_m^n)$ is reduced to $Z(\Delta_m)$ which was introduced by Tripathy and Esi [25], in this case the operator $\Delta_m x$ is given by $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$ for all $k, m \in \mathbb{N}$. The choice of $m = 1$ in the definition of $Z(\Delta_m^n)$ gives us the difference sequence spaces introduced by Et and Colak [6]. Başar and Altay [1] introduced the generalized difference matrix $B(r, s) = (b_{nk}(r, s))$ by

$$b_{nk}(r,s) = \begin{cases} r, & \text{if } k = n; \\ s, & \text{if } k = n-1; \\ 0, & \text{if } 0 \leq k < n-1 \text{ or } k > n. \end{cases}$$

for all $k, n \in \mathbb{N}$ and all non-zero real numbers r, s. The generalized difference matrix B^n of order n has been recently defined by Başarir and Kayikçi [2] and its binomial representation is given by

$$B^n x_k = \sum_{\nu=0}^{n} \binom{n}{\nu} r^{n-\nu} s^\nu x_{k-\nu},$$

for all $n \in \mathbb{N}$ and $r, s \in \mathbb{R} - \{0\}$. Another generalization of above difference matrix was given by Başarir et al. [3] as $B_{(m)}^n$, where $B_{(m)}^n x = (B_{(m)}^n x_k) = (r B_{(m)}^{n-1} x_k + s B_{(m)}^{n-1} x_{k-m})$ and $B_{(m)}^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$B_{(m)}^n x_k = \sum_{\nu=0}^{n} \binom{n}{\nu} r^{n-\nu} s^\nu x_{k-m\nu}.$$

In [24], Orlicz introduced functions nowadays called Orlicz functions and constructed the sequence space (L^M). Krasnoselskii and Rutitsky further investigated the Orlicz space in [14]. Some recent related work we refer to Mohiuddine et al. [19,20]. A function $M : [0, \infty) \to [0, \infty)$ is said to be an *Orlicz function* if it is non-decreasing, continuous, convex with $M(0) = 0$, $M(x) > 0$ as $x > 0$ and $M(x) \to \infty$ as $x \to \infty$ (see [24]). It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all $\lambda \in (0, 1)$. An Orlicz function M is said to be satisfy Δ_2-*condition* for all values of u, if there exists a constant $K > 0$ such that $M(Lu) \leq KLM(u)$ for all values of $L > 1$ (see, Krasnoselskii and Rutitsky [14]).

Lindenstrauss and Tzafriri [16] introduced the sequence space ℓ_M by using the notion of Orlicz function by

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

and proved that this space is a Banach space with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Every space ℓ_M contains a subspace isomorphic to the classical sequence space ℓ_p for some $1 \leq p < \infty$. The space ℓ_p, $1 \leq p < \infty$ is itself an Orlicz sequence space with $M(t) = |t|^p$.

A sequence space E is said to be (i) *normal* (or *solid*) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, (ii) *symmetric* if $(x_{\pi(k)}) \in E$, whenever $(x_k) \in E$, where π is a permutation of $\mathbb{N}$.

Let E be a sequence space and $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$. A sequence space of the form $\lambda_K^E = \{(x_{k_n}) \in w : (k_n) \in E\}$ is called a *K-step space* of E. A canonical preimage of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in w$ and is defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E. We say that E is *monotone* if E contains the canonical pre-image of all its step spaces. Note that every normal space is monotone (see [11], pp. 53).

A sequence $x = (x_k) \in \ell_\infty$ (the space of bounded sequences) is said to be *almost convergent*, denoted by $\widehat{c}$, if all of its Banach limits coincide. Lorentz [17] introduced this sequence space as follows:

$$\widehat{c} = \left\{ x \in \ell_\infty : \lim_k t_{jk}(x) \text{ exists uniformly in } j \right\},$$

where

$$t_{jk}(x) = \frac{x_j + x_{j+1} + ... + x_{j+k}}{k+1}.$$

It is clear that

$$t_{jk}(x) = \begin{cases} \frac{1}{k} \sum\limits_{i=1}^{k} x_{j+i} & \text{for } k \geq 1; \\ x_j & \text{for } k = 0. \end{cases}$$

Zadeh [29] introduced the concept of fuzzy set theory and its applications can be found in many branches of mathematical and engineering sciences including management science, control engineering, computer science, artificial intelligence. Matloka [18] introduced the bounded and convergent sequences of fuzzy numbers and proved that every convergent sequence of fuzzy numbers is bounded. Later, various classes of sequences of fuzzy numbers have been defined and studied by Colak et al. [5], Et et al. [7], Mursaleen and Başarir [21], Hazarika [10] and references therein.

Now recalling some notions of fuzzy numbers which we will used to prove our main results. Throughout the paper we used w^F, ℓ_∞^F, c^F and c_0^F to denote the set of all, bounded, convergent and null sequence spaces of fuzzy numbers, respectively. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}$ i.e., a mapping $X : \mathbb{R} \to J(= [0, 1])$ associating each real number t with its grade of membership $X(t)$. A fuzzy number X is said to be (i) *upper-semi continuous* if for each $\varepsilon > 0, X^{-1}([0, a + \varepsilon))$ for all $a \in [0, 1]$ is

open in the usual topology of $\mathbb{R}$, (ii) *convex* if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$ for $s < t < r$ (iii) *normal* if there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$.

We used the notation X^α to denotes α-level set of a fuzzy number $X, 0 < \alpha \leq 1$ and is given by $X^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\}$. The set of all normal, convex and upper semi-continuous fuzzy number with compact support will be denoted by $\mathbb{R}(J)$ and the fuzzy number we mean that the number belongs to $\mathbb{R}(J)$. We used the symbol D to denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on $\mathbb{R}$. For any two sets $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. A metric d on D is given by $d(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. It is easy to see that (D, d) is a complete metric space. Also, the relation $\leq$ is a partial order on D.

The absolute value $|X|$ of $X \in \mathbb{R}(J)$ is given by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Suppose that $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}$ is a mapping such that $\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha)$. Then $(\mathbb{R}(J), \overline{d})$ is a complete metric space.

We define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$, for all $\alpha \in J$. By $\bar{0}$ and $\bar{1}$ we denotes the additive and multiplicative identities in $\mathbb{R}(J)$, respectively.

A sequence $u = (u_k)$ of fuzzy numbers is said to be (i) *bounded* if the set $\{u_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded, (ii) *convergent* to a fuzzy number u_0 if for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\bar{d}(u_k, u_0) < \varepsilon$, for all $k \geq n_0$, (iii) *I-convergent* (see [15]) if there exists a fuzzy number u_0 such that for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : \bar{d}(u_k, u_0) \geq \varepsilon\} \in I$. We write I-lim $u_k = u_0$, (iv) *I-bounded* if there exists $K > 0$ such that the set $\{k \in \mathbb{N} : \bar{d}(u_k, \bar{0}) \geq K\} \in I$.

2 Main results

Throughout the article we assume that I is an admissible ideal of $\mathbb{N}$. In this section, we introduce the following definitions. We introduce some new strongly almost ideal convergent sequence spaces using the generalized difference matrix $B^n_{(m)}$ and Orlicz function M. Let us consider a sequence $p = (p_k)$ of positive real numbers and let m, n be any nonnegative integers. For some $\rho > 0$, we define the following sequence spaces.

$$[\widehat{w}_0^{IF}(M, \theta, B^n_{(m)}, p)] = \Bigg\{(u_k) \in w^F : \Bigg\{r \in \mathbb{N} : \frac{1}{h_r} \times \sum_{k \in J_r} \left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon \Bigg\} \in I, \text{uniformly in } j \in \mathbb{N} \Bigg\}$$

$$[\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)] = \Bigg\{(u_k) \in w^F : \Bigg\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon \Bigg\} \in I, \text{uniformly in } j \in \mathbb{N} \text{ and for some } u_0 \in \mathbb{R}(J) \Bigg\}$$

$$[\widehat{w}_\infty^{F}(M, \theta, B^n_{(m)}, p)] = \Bigg\{(u_k) \in w^F : \sup_r \frac{1}{h_r}$$

$$\times \sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{\rho}\right)\right]^{p_k} < \infty, \text{ uniformly in } j\in\mathbb{N}\Bigg\}$$

$$[\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)},p)] = \Bigg\{(u_k)\in w^F : \exists K>0 \; s.t. \Bigg\{r\in\mathbb{N} : \frac{1}{h_r}$$

$$\times \sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{\rho}\right)\right]^{p_k} \geq K\Bigg\}\in I, \text{ uniformly in } j\in\mathbb{N}\Bigg\}.$$

Particular cases:

(i) If $p=(p_k)=1$ for all $k\in\mathbb{N}$, we denote $[\widehat{w}^{IF}_0(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_0(M,\theta,B^n_{(m)})]$, $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}(M,\theta,B^n_{(m)})]$, $[\widehat{w}^{F}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{F}_\infty(M,\theta,B^n_{(m)})]$ and $[\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)})]$.

(ii) If $M(x)=x$, we denote $[\widehat{w}^{IF}_0(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_0(\theta,B^n_{(m)},p)]$, $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}(\theta,B^n_{(m)},p)]$, $[\widehat{w}^{F}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{F}_\infty(\theta,B^n_{(m)},p)]$ and $[\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_\infty(\theta,B^n_{(m)},p)]$.

(iii) If $\theta=(2^r)$, we denote $[\widehat{w}^{IF}_0(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_0(M,B^n_{(m)},p)]$, $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}(M,B^n_{(m)},p)]$, $[\widehat{w}^{F}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{F}_\infty(M,B^n_{(m)},p)]$ and $[\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)},p)] = [\widehat{w}^{IF}_\infty(M,B^n_{(m)},p)]$.

Throughout the manuscript, we will used the following well-known inequality. Suppose that $p=(p_k)$ is a sequence of positive real numbers with $0<p_k\leq \sup_k p_k = H$, $D=\max\{1,2^{H-1}\}$. Then

$$|a_k+b_k|^{p_k} \leq D(|a_k|^{p_k}+|b_k|^{p_k}) \text{ for all } k\in\mathbb{N} \text{ and } a_k,b_k\in\mathbb{C}.$$

Also $|a|^{p_k}\leq \max\{1,|a|^H\}$ for all $a\in\mathbb{C}$.

Now we are ready to give our main results as follows.

Theorem 2.1. *Let $p=(p_k)$ be a bounded sequence of positive real numbers. The spaces $[\widehat{w}^{IF}_0(M,\theta,B^n_{(m)},p)]$, $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$, $[\widehat{w}^{F}_\infty(M,\theta,B^n_{(m)},p)]$, and $[\widehat{w}^{IF}_\infty(M,\theta,B^n_{(m)},p)]$ are closed with respect to addition and scalar multiplication.*

Proof. We prove the result only for the space $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$. The others can be treated similarly. Let $u=(u_k)$ and $v=(v_k)$ be two elements of $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$ and α_1,α_2 be scalars. Let $\varepsilon>0$ be given. Then there exist positive numbers ρ_1,ρ_2 such that

$$P=\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho_1}\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\}\in I \qquad (\text{uniformly in } j\in\mathbb{N})$$

and

$$Q=\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}v_k),v_0)}{\rho_2}\right)\right]^{p_k} \geq \frac{\varepsilon}{2}\right\}\in I \qquad (\text{uniformly in } j\in\mathbb{N}).$$

Let $\rho_3=\max(2|\alpha_1|\rho_1, 2|\alpha_2|\rho_2)$. Since M is non-decreasing and convex function, we have

$$\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}(\alpha_1u_k+\alpha_2v_k)),\alpha_1u_0+\alpha_2v_0)}{\rho_3}\right)\right]^{p_k}$$

$$\leq \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\alpha_1\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho_3}\right)\right]^{p_k}+\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\alpha_2\overline{d}(t_{jk}(B^n_{(m)}v_k),v_0)}{\rho_3}\right)\right]^{p_k}$$
$$\leq \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho_1}\right)\right]^{p_k}+\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}v_k),v_0)}{\rho_2}\right)\right]^{p_k},$$

uniformly in j. Therefore, we have

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}(\alpha_1u_k+\alpha_2v_k)),\alpha_1u_0+\alpha_2v_0)}{\rho_3}\right)\right]^{p_k}\geq\varepsilon\right\}\subseteq P\cup Q\in I.$$

uniformly in j. This yields $(\alpha_1u+\alpha_2v)\in[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$. This completes the proof. $\square$

Theorem 2.2. *Let M_1 and M_2 be two Orlicz functions. Then*

(i) $[Z(M_2,\theta,B^n_{(m)},p)]\subseteq[Z(M_1M_2,\theta,B^n_{(m)},p)]$.

(ii) $[Z(M_1,\theta,B^n_{(m)},p)]\cap Z(M_2,\theta,B^n_{(m)},p)]\subseteq[Z(M_1+M_2,\theta,B^n_{(m)},p)]$,

where $Z=\widehat{w}^{IF}_0,\widehat{w}^{IF},\widehat{w}^{IF}_\infty,\widehat{w}^{F}_\infty$.

Proof. (i) Let $u=(u_k)\in[\widehat{w}^{IF}(M_2,\theta,B^n_{(m)},p)]$ and let $\varepsilon>0$ be given. For some $\rho>0$, we have

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in J_r}\left[M_2\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\in I, \tag{2.1}$$

uniformly in $j\in\mathbb{N}$. Choose λ with $0<\lambda<1$ such that $M_1(t)<\varepsilon$ for $0\leq t\leq\lambda$. We define

$$v_k=\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho}$$

and consider

$$\lim_{k\in\mathbb{N};0\leq v_k\leq\lambda}[M_1(v_k)]^{p_k}=\lim_{k\in\mathbb{N};v_k\leq\lambda}[M_1(v_k)]^{p_k}+\lim_{k\in\mathbb{N};v_k>\lambda}[M_1(v_k)]^{p_k}.$$

Therefore, one obtains

$$\lim_{k\in\mathbb{N};v_k\leq\lambda}[M_1(v_k)]^{p_k}\leq[M_1(2)]^H\lim_{k\in\mathbb{N};v_k\leq\lambda}[v_k]^{p_k},\qquad (H=\sup_k p_k). \tag{2.2}$$

For the second summation (i.e. $v_k>\lambda$), we go through the following procedure. We have

$$v_k<\frac{v_k}{\lambda}<1+\frac{v_k}{\lambda}.$$

It follows from the fact that M_1 is convex and non-decreasing,

$$M_1(v_k)<M_1\left(1+\frac{v_k}{\lambda}\right)\leq\frac{1}{2}M_1(2)+\frac{1}{2}M_1\left(\frac{2v_k}{\lambda}\right).$$

Since M_1 satisfies Δ_2-condition, we can write

$$M_1(v_k)<\frac{1}{2}K\frac{v_k}{\lambda}M_1(2)+\frac{1}{2}K\frac{v_k}{\lambda}M_1(2)=K\frac{v_k}{\lambda}M_1(2).$$

This yields the following estimates:

$$\lim_{k\in\mathbb{N};v_k>\lambda}[M_1(v_k)]^{p_k}\leq\max\left\{1,(K\lambda^{-1}M_1(2))^H\right\}\lim_{k\in\mathbb{N};v_k>\lambda}[v_k]^{p_k}. \tag{2.3}$$

It follows from (2.1), (2.2) and (2.3) that

$$(u_k) \in [\widehat{w}^{IF}(M_1.M_2, \theta, B^n_{(m)}, p)].$$

Hence, $[\widehat{w}^{IF}(M_2, \theta, B^n_{(m)}, p)] \subseteq [\widehat{w}^{IF}(M_1.M_2, \theta, B^n_{(m)}, p)]$.

(ii) Let $(u_k) \in [\widehat{w}^{IF}(M_1, \theta, B^n_{(m)}, p)] \cap [\widehat{w}^{IF}(M_2, \theta, B^n_{(m)}, p)]$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_1 \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I \qquad \text{(uniformly in } j \in \mathbb{N})$$

and

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_2 \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I \qquad \text{(uniformly in } j \in \mathbb{N}).$$

The rest of the proof follows from the following relation:

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[(M_1 + M_2) \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \\ &\qquad \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_1 \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \\ &\qquad\quad \bigcup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_2 \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\}. \end{aligned}$$

□

Note that if we take $M_1(x) = M(x)$ and $M_2(x) = x$ for all $x \in [0, \infty)$ in the above theorem, then we obtain the following corollary:

Corollary 2.3. *One has* $[Z(\theta, B^n_{(m)}, p)] \subseteq [Z(M, \theta, B^n_{(m)}, p)]$, *where* $Z = \widehat{w}^{IF}_0$, $\widehat{w}^{IF}$, $\widehat{w}^{IF}_\infty$, $\widehat{w}^{F}_\infty$.

As in classical theory, the following is easy to prove.

Theorem 2.4. *(a) If* $M_1(x) \leq M_2(x)$ *for all* $x \in [0, \infty)$, *then* $[Z(M_1, \theta, B^n_{(m)}, p)] \subseteq [Z(M_2, \theta, B^n_{(m)}, p)]$ *for* $Z = \widehat{w}^{IF}_0, \widehat{w}^{IF}$ *and* $\widehat{w}^{F}_\infty$.

(b) If $n_1 < n_2$ *then* $[Z(\theta, B^{n_1}_{(m)}, p)] \subseteq [Z(\theta, B^{n_2}_{(m)}, p)]$ *for* $Z = \widehat{w}^{IF}_0, \widehat{w}^{IF}$ *and* $\widehat{w}^{F}_\infty$.

Theorem 2.5. *Let* M *be an Orlicz function. Then*

$$[\widehat{w}^{IF}_0(M, \theta, B^n_{(m)}, p)] \subset [\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)] \subset [\widehat{w}^{F}_\infty(M, \theta, B^n_{(m)}, p)]$$

and the inclusions are proper.

Proof. Suppose that $(u_k) \in [\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)]$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I.$$

Clearly,

$$M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{\rho}\right) \leq \frac{1}{2}M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho}\right) + \frac{1}{2}M\left(\frac{\overline{d}(u_0,\overline{0})}{\rho}\right).$$

Taking supremum over k on both sides of above inequalities implies that $(u_k) \in [\widehat{w}^F_\infty(M,\theta,B^n_{(m)},p)]$. Thus, we have $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)] \subset [\widehat{w}^F_\infty(M,\theta,B^n_{(m)},p)]$.

The inclusion $[\widehat{w}^{IF}_0(M,\theta,B^n_{(m)},p)] \subset [\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$ is obvious.

We now show that the inclusion is strict in the above theorem by constructing the following illustrative example.

Example 2.1. Suppose that $\theta = (2^r)$ and $M(x) = x$ for all $x \in [0,\infty)$. Suppose also that $r = 1$, $s = -1$, $n = 1$, $m = 2$. Let us define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} \frac{6}{k}t + 1 & \text{if} \quad -\frac{k}{6} \leq t \leq 0; \\ -\frac{6}{k}t + 1 & \text{if} \quad 0 < t \leq \frac{k}{6}; \\ 0 & , \quad \text{otherwise,} \end{cases}$$

where $k = 2^i$ $(i = 1,2,3,...)$, otherwise $u_k(t) = \overline{0}$. For $\alpha \in (0,1]$, the α-level sets of u_k and $B^1_{(m)}u_k$ are

$$[u_k]^\alpha = \begin{cases} [\frac{k}{6}(\alpha-1), \frac{k}{6}(1-\alpha)] & \text{if} \quad k = 2^i, i = 1,2,3,... \\ [0,0] & , \quad \text{otherwise} \end{cases}$$

and

$$[B^1_{(2)}u_k]^\alpha = \begin{cases} [\frac{1}{3}(\alpha-1), \frac{1}{3}(1-\alpha)] & \text{for} \quad k = 2^i \\ [0,0] & , \quad \text{otherwise .} \end{cases}$$

It is easy to prove that $-\overline{\frac{1}{3}} < [T_j]^\alpha < \overline{\frac{1}{3}}$ for $\alpha \in (0,1]$, where $[T_j]^\alpha = [t_{j,k}(B^1_{(2)}u_k)]^\alpha = [\frac{1}{j+1}\sum_{i=1}^{j} B^1_{(2)}u_k]^\alpha$. Because

$$[t_{j,k}(B^1_{(2)}u_k)]^\alpha = \begin{cases} \frac{1}{1+\frac{1}{j}}[\frac{1}{3}(\alpha-1), \frac{1}{3}(1-\alpha)] & \text{for} \quad k = 2^i;\ j \geq 1 \\ [0,0] & , \quad \text{otherwise} \end{cases}$$

and

$$[t_{j,k}(B^1_{(2)}u_k)]^\alpha = \begin{cases} [\frac{1}{3}(\alpha-1), \frac{1}{3}(1-\alpha)] & \text{if} \quad j = 0 \\ [0,0] & , \quad \text{otherwise .} \end{cases}$$

Thus (T_j) is I-bounded but not I-convergent. □

Theorem 2.6. *The inclusions* $[Z(M,\theta,B^{n-1}_{(m)},p)] \subseteq [Z(M,\theta,B^n_{(m)},p)]$ *are strict for* $n \geq 1$. *In general* $[Z(M,\theta,B^i_{(m)},p)] \subseteq [Z(M,\theta,B^n_{(m)},p)]$ $(i = 1,2,...,n-1)$ *and the inclusion is strict, where* $Z = \widehat{w}^{IF}_0, \widehat{w}^{IF}, \widehat{w}^{IF}_\infty, \widehat{w}^F_\infty$.

Proof. Suppose that $u = (u_k) \in [\widehat{w}^{IF}_0(M,\theta,B^{n-1}_{(m)},p)]$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k),\overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

Since M is non-decreasing and convex it follows that

$$\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{2\rho}\right)\right]^{p_k}$$

$$\begin{aligned}
&\leq \left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{2\rho}\right)\right]^{p_k} \\
&\leq D\left[\frac{1}{2}M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), \overline{0})}{\rho}\right)\right]^{p_k} + D\left[\frac{1}{2}M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{\rho}\right)\right]^{p_k} \\
&\leq DK\left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), \overline{0})}{\rho}\right)\right]^{p_k} + DK\left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{\rho}\right)\right]^{p_k},
\end{aligned}$$

where $K = \max\{1, \left(\frac{1}{2}\right)^H\}$. Therefore we have

$$\begin{aligned}
&\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^{n}_{(m)}u_k), \overline{0})}{2\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \\
&\qquad \subseteq \left\{r \in \mathbb{N} : DK\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \\
&\qquad\quad \bigcup\left\{r \in \mathbb{N} : DK\frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\},
\end{aligned}$$

i.e.,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^{n}_{(m)}u_k), \overline{0})}{2\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

Hence, $(u_k) \in [\widehat{w}_0^{IF}(M, \theta, B^n_{(m)}, p)]$.

We now show that the inclusion is strict in the above theorem (Theorem 2.6) by constructing the following illustrative example.

Example 2.2. Let $\theta = (2^r)$ and $M(x) = x$ for all $x \in [0, \infty)$ Suppose also that $r = 1$, $s = -1$, $n = 2$, $m = 2$ and $p_k = 1$ for all $k \in \mathbb{N}$. We now define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} -\frac{t}{k^2-1} + 1 & , \text{ if } k^2 - 1 \leq t \leq 0; \\ -\frac{t}{k^2+1} + 1 & , \text{ if } 0 < t \leq k^2 + 1; \\ 0 & , \text{ otherwise.} \end{cases}$$

For $\alpha \in (0, 1]$, the α-level sets of u_k, $B^1_{(2)}u_k$ and $B^2_{(2)}u_k$ are as follow:

$$[u_k]^\alpha = [(1-\alpha)(k^2-1), (1-\alpha)(k^2+1)],$$

and

$$[B^1_{(2)}u_k]^\alpha = [(1-\alpha)(4k-6), (1-\alpha)(4k-2)],$$
$$[B^2_{(2)}u_k]^\alpha = [4(1-\alpha), 12(1-\alpha)].$$

It is easy to verified that the sequence $[B^1_{(2)}u_k]^\alpha$ is not I-convergent but $[B^2_{(2)}u_k]^\alpha$ is I-convergent. $\square$

Theorem 2.7. *Let $0 < p_k \leq q_k < \infty$ for each k. Then $[Z(M, \theta, B^n_{(m)}, p)] \subseteq [Z(M, \theta, B^n_{(m)}, q)]$ for $Z = \widehat{w}_0^{IF}$ and $\widehat{w}^{IF}$.*

Proof. Let $(u_k) \in [\widehat{w}_0^{IF}(M,\theta,B^n_{(m)},p)$. Then there exists a number $\rho > 0$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k), \overline{0})}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I \qquad \text{(uniformly in } j \in \mathbb{N}).$$

For sufficiently large k, since $p_k \leq q_k$ for each k, therefore we obtain

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k), \overline{0})}{\rho} \right) \right]^{q_k} \geq \varepsilon \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k), \overline{0})}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I,$$

uniformly in $j \in \mathbb{N}$, i.e. $(u_k) \in [\widehat{w}_0^{IF}(M,\theta,B^n_{(m)},q)]$.

Similarly, we can show that $]\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)] \subseteq [\widehat{w}^{IF}(M,\theta,B^n_{(m)},q)]$. $\square$

Corollary 2.8. *(a) Let* $0 < \inf_k p_k \leq p_k \leq 1$. *Then* $[Z(M,\theta,B^n_{(m)},p)] \subseteq [Z(M,\theta,B^n_{(m)})]$ *for* $Z = \widehat{w}_0^{IF}$ *and* $\widehat{w}^{IF}$.

(b) Let $1 \leq p_k \leq \sup_k p_k < \infty$. *Then* $[Z(M,\theta,B^n_{(m)})] \subseteq [Z(M,\theta,B^n_{(m)},p)]$ *for* $Z = \widehat{w}_0^{IF}$ *and* $\widehat{w}^{IF}$.

Theorem 2.9. *If* I *is an admissible ideal and* $I \neq I_f$, *then the sequence spaces* $[\widehat{w}_0^{IF}(M,\theta,B^n_{(m)},p)]$ *and* $\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$ *are neither normal nor monotone, where* I_f *denotes the class of all finite subsets of* $\mathbb{N}$.

Proof. To prove our result, we construct the following example.

Example 2.3. Suppose that $M(x) = x$ for all $x \in [0,\infty)$ and $r = 1$, $s = -1$, $n = 1$, $m = 1$. Consider that $I = I_\delta$, where $I_\delta = \{A \subset \mathbb{N} : \text{asymptotic density of } A \text{ (in symbol, } \delta(A)) = 0\}$ and note that I_δ is an ideal of $\mathbb{N}$, and $p_k = 1$ for all $k \in \mathbb{N}$. We now define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} 1 + t - k & , \text{ if } t \in [k-1, k]; \\ 1 - t + k & , \text{ if } t \in [k, k+1]; \\ 0 & , \text{ otherwise.} \end{cases}$$

Let us define

$$\alpha_k = \begin{cases} 1 & , \text{ if } k \text{ is odd;} \\ 0 & , \text{ if } k \text{ is even.} \end{cases}$$

Thus $(\alpha_k u_k) \notin [\widehat{w}_0^{IF}(M,\theta,B^n_{(m)},p)]$ and $\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$. Therefore, we conclude that the spaces $[\widehat{w}_0^{IF}(M,\theta,B^n_{(m)},p)]$ and $\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$ are not normal and hence these spaces are not monotone. $\square$

Theorem 2.10. *If* I *is an admissible ideal and* $I \neq I_f$, *then the sequence space* $[Z(M,\theta,B^n_{(m)},p)]$ *is not symmetric, where* $Z = \widehat{w}_0^{IF}, \widehat{w}^{IF}$.

Proof. We shall prove the result only for the space $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$ with the help of the following example. For other space, the proof is similar so we omitted.

Example 2.4. Suppose that $M(x) = x$ for all $x \in [0, \infty)$ and $r = 1$, $s = -1$, $n = 1$, $m = 1$. Let $I = I_\delta$ and $p_k = 1$ for all $k \in \mathbb{N}$. We now define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} t - 4k + 1 & , \text{ if } t \in [4k-1, 4k]; \\ -t + 4k + 1 & , \text{ if } t \in [4k, 4k+1]; \\ 0 & , \text{ otherwise.} \end{cases}$$

Thus, we have $(u_k) \in [\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)]$. But the rearrangement (v_k) of (u_k) defined as

$$v_k = \{u_1, u_4, u_2, u_9, u_3, u_{16}, u_5, u_{25}, u_6, ...\}.$$

This implies that $(v_k) \notin [\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)]$. Hence $[\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)]$ is not symmetric. $\square$

3 Acknowledgement

This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (292-130-1436-G). The authors, therefore, acknowledge with thanks DSR for technical and financial support.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.5, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

The Catalan Numbers: a Generalization, an Exponential Representation, and some Properties

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Abstract

In the paper, the authors establish an exponential representation for a function involving the gamma function and originating from investigation of the Catalan numbers in combinatorics, find necessary and sufficient conditions for the function to be logarithmically completely monotonic, introduce a generalization of the Catalan numbers, derive an exponential representation for the generalization, and present some properties of the generalization.

2010 Mathematics Subject Classification: Primary 11R33; Secondary 11B75, 11B83, 11S23, 26A48, 33B15, 44A20.

Key words and phrases: exponential representation; necessary and sufficient condition; logarithmically completely monotonic function; gamma function; Catalan number; generalization; property; Catalan–Qi function.

1 Introduction

It is known [4, 21, 22] that, in combinatorics, the Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" whose solution is the Catalan number C_{n-2}. Explicit formulas of C_n for $n \ge 0$ include

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{2^n(2n-1)!!}{(n+1)!} = \frac{1}{n}\binom{2n}{n-1} = {}_2F_1(1-n,-n;2;1) = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \quad (1)$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t$ for $\Re(z) > 0$ is the classical Euler gamma function and

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{n=0}^\infty \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} \quad (2)$$

1

is the generalized hypergeometric series defined for $a_i \in \mathbb{C}$ and $b_i \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, for positive integers $p, q \in \mathbb{N}$, and in terms of the rising factorials $(x)_n = \prod_{k=0}^{n-1}(x+k)$. The asymptotic form for the Catalan function C_x is

$$C_x \sim \frac{4^x}{\sqrt{\pi}}\left(\frac{1}{x^{3/2}} - \frac{9}{8}\frac{1}{x^{5/2}} + \frac{145}{128}\frac{1}{x^{7/2}} + \cdots\right),$$

see [3, 4, 21, 22, 24]. Recently, among other things, the formula

$$C_n = (-1)^n \frac{2^n}{n!} \sum_{k=0}^{n} \frac{1}{2^k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{m=0}^{n-1} (\ell - 2m) = \frac{2^n}{n!} \sum_{k=0}^{n} \frac{k!}{2^k} \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!!$$

was found in [18, Theorem 3]. For more information on the Catalan numbers C_n, please refer to two monographs [2, 3] and references cited therein.

In the paper [20], motivated by the explicit expression (1), the authors established an integral representation of the Catalan function C_x for $x \ge 0$.

Theorem 1.1 ([20, Theorem 1])**.** *For $x \ge 0$, we have*

$$C_x = \frac{e^{3/2} 4^x (x+1/2)^x}{\sqrt{\pi}\,(x+2)^{x+3/2}} \exp\left[\int_0^\infty \frac{1}{t}\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)\left(e^{-t/2} - e^{-2t}\right) e^{-xt} \,\mathrm{d}\, t\right]. \tag{3}$$

Recall from [8, Chapter XIII], [19, Chapter 1], and [25, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $0 \le (-1)^k f^{(k)}(x) < \infty$ on I for all $k \ge 0$. Recall from [11] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if $0 \le (-1)^k [\ln f(x)]^{(k)} < \infty$ hold on I for all $k \in \mathbb{N}$. For more information on logarithmically completely monotonic functions, please refer to [14, 19].

The formula (3) can be rearranged as

$$\ln\left[\frac{\sqrt{\pi}\,(x+2)^{x+3/2}}{e^{3/2} 4^x (x+1/2)^x} C_x\right] = \int_0^\infty \frac{1}{t}\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)\left(e^{-t/2} - e^{-2t}\right) e^{-xt} \,\mathrm{d}\, t. \tag{4}$$

Since the function $\frac{1}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)$ is positive on $(0, \infty)$, the right-hand side of (4) is a completely monotonic function on $(0, \infty)$. This means that the function

$$\frac{(x+2)^{x+3/2}}{4^x (x+1/2)^x} C_x \tag{5}$$

is logarithmically completely monotonic on $(0, \infty)$. Because any logarithmically completely monotonic function must be completely monotonic, see [14, Eq. (1.4)] and references therein, the function (5) is also completely monotonic on $(0, \infty)$.

By virtue of (1), the function (5) can be rewritten as

$$\frac{(x+2)^{x+3/2}\Gamma(x+1/2)}{(x+1/2)^x \Gamma(x+2)}, \quad x > 0. \tag{6}$$

Hence, the logarithmically complete monotonicity of (5) implies the logarithmically complete monotonicity of (6). The function (6) is the special case $F_{1/2,2}(x)$ of the general function

$$F_{a,b}(x) = \frac{\Gamma(x+a)}{(x+a)^x} \frac{(x+b)^{x+b-a}}{\Gamma(x+b)}, \quad a, b \in \mathbb{R}, \quad a \ne b \quad x > -\min\{a, b\}. \tag{7}$$

We notice that the function $F_{a,b}(x)$ does not appear in the expository and survey articles [9, 14] and plenty of references therein. Therefore, it is significant to naturally pose an open problem below.

Open Problem 1.1 ([20, Open Problem 1]). *What are the necessary and sufficient conditions on $a, b \in \mathbb{R}$ such that the function $F_{a,b}(x)$ defined by (7) is (logarithmically) completely monotonic in $x \in (-\min\{a, b\}, \infty)$?*

This problem was answered in [6, Theorem 2] as follows.

Theorem 1.2 ([6, Theorem 2]). *The sufficient conditions on a, b such that the function $[F_{a,b}(x)]^{\pm 1}$ defined by (7) is logarithmically completely monotonic in $x \in (-\min\{a, b\}, \infty)$ are $(a, b) \in D_{\pm}(a, b)$, where*

$$D_{\pm}(a, b) = \{(a, b) : a \gtrless b, a \ge 1\} \cup \left\{(a, b) : a \lessgtr b, a \le \frac{1}{2}\right\}.$$

The necessary conditions on a, b for the function $[F_{a,b}(x)]^{\pm 1}$ to be logarithmically completely monotonic in $x \in (-\min\{a, b\}, \infty)$ are $a(a - b) \gtreqless \frac{a-b}{2}$.

The aims of this paper are to establish an exponential representation for the function $F_{a,b}(x)$, to find necessary and sufficient conditions on a, b for $[F_{a,b}(x)]^{\pm 1}$ to be logarithmically completely monotonic on $[0, \infty)$, to introduce a generalization of the Catalan numbers C_n, and to derive an exponential representation for the generalization of C_n.

The first main result in this paper can be stated as the following theorem.

Theorem 1.3. *For $a, b > 0$, the function $F_{a,b}(x)$ defined by (7) has the exponential representation*

$$F_{a,b}(x) = \exp\left[b - a + \int_0^\infty \frac{1}{t}\left(a + \frac{1}{t} - \frac{1}{1 - e^{-t}}\right)\left(e^{-bt} - e^{-at}\right)e^{-xt} \,\mathrm{d}\, t\right] \tag{8}$$

on $[0, \infty)$ and the function $[F_{a,b}(x)]^{\pm 1}$ is logarithmically completely monotonic on $[0, \infty)$ if and only if $(a, b) \in D_{\pm}(a, b)$.

Comparing (3) with (8) hints and stimulates us to consider the three-variable function

$$C(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z \frac{\Gamma(z + a)}{\Gamma(z + b)}, \quad \Re(a), \Re(b) > 0, \quad \Re(z) \ge 0. \tag{9}$$

Since $C\left(\frac{1}{2}, 2; n\right) = C_n$ for $n \ge 0$ is of the form (1), we can regard $C(a, b; x)$ as an analytical generalization of the Catalan numbers C_n. For uniqueness and convenience of referring to the quantity $C(a, b; x)$, we call $C(a, b; x)$ the Catalan–Qi function and, when taking $x = n \in \{0\} \cup \mathbb{N}$, call $C(a, b; n)$ the Catalan–Qi numbers.

By virtue of the integral representation (8) in Theorem 1.3, we immediately derive an integral representation for the Catalan–Qi function $C(a, b; x)$.

Theorem 1.4. *For $a, b > 0$ and $x \ge 0$, we have*

$$\begin{aligned} C(a, b; x) = {} & \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^x \frac{(x + a)^x}{(x + b)^{x+b-a}} \\ & \times \exp\left[b - a + \int_0^\infty \frac{1}{t}\left(\frac{1}{1 - e^{-t}} - \frac{1}{t} - a\right)\left(e^{-at} - e^{-bt}\right)e^{-xt} \,\mathrm{d}\, t\right]. \end{aligned} \tag{10}$$

3

Remark 1.1. Can one give a combinatorial interpretation of the Catalan–Qi function $C(a,b;x)$ defined by (9) and its integral representation (10)?

In [22] and related references therein, the following simple properties of the Catalan numbers C_n are listed:

$$C_{n+1}=\frac{2(2n+1)}{n+2}C_n, \quad C_n=\frac{1}{(n+1)!}\prod_{k=1}^{n}(4k-2), \quad \sum_{n=1}^{\infty}\frac{C_n}{4^n}=1, \tag{11}$$

$$\sum_{n=0}^{\infty}C_n\frac{x^{2n}}{(2n)!}=\frac{I_1(2x)}{x}, \quad e^{2x}\big[I_0(2x)-I_1(2x)\big]=\sum_{n=0}^{\infty}C_n\frac{x^n}{n!}, \tag{12}$$

where

$$I_\nu(z)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k+\nu}$$

for $\nu\in\mathbb{R}$ and $z\in\mathbb{C}$, see [1, p. 375, 9.6.10], is the modified Bessel function of the first kind. Corresponding to these properties, the following properties of the Catalan–Qi function $C(a,b;z)$ can be obtained.

Theorem 1.5. *For $n\ge 0$ and $\Re(z)\ge 0$, we have*

$$C(a,b;z+1)=\frac{b}{a}\frac{z+a}{z+b}C(a,b;z); \quad C(a,b;n)=\left(\frac{b}{a}\right)^n\prod_{k=0}^{n-1}\frac{a+k}{b+k};$$

$$\sum_{n=1}^{\infty}\left(\frac{a}{b}\right)^n C(a,b;n)=\frac{a}{b-a-1}, \quad b>a+1>1;$$

$$\sum_{n=0}^{\infty}C(a,b;n)\frac{x^{2n}}{(2n)!}={}_1F_2\left(a;\frac{1}{2},b;\frac{b}{4a}x^2\right); \quad \sum_{n=0}^{\infty}C(a,b;n)\frac{x^n}{n!}={}_1F_1\left(a;b;\frac{b}{a}x\right).$$

Remark 1.2. When $a=\frac{1}{2}$ and $b=2$, the formulas in Theorem 1.5 become those listed in (11) and (12).

Remark 1.3. The last two formulas in Theorem 1.5 show that the functions ${}_1F_2\big(a;\frac{1}{2},b;\frac{b}{4a}x^2\big)$ and ${}_1F_1\big(a;b;\frac{b}{a}x\big)$ can be regarded as the generating functions of the Catalan–Qi numbers $C(a,b;n)$.

2 Proofs of Theorems 1.3 to 1.5

We are now start out to prove Theorem 1.3 by two approaches and to prove Theorems 1.4 and 1.5.

First proof of Theorem 1.3. Taking the logarithm of $F_{a,b}(x)$ gives

$$\ln F_{a,b}(x)=\ln\Gamma(x+a)-x\ln(x+a)-\ln\Gamma(x+b)+(x+b-a)\ln(x+b)\triangleq f_a(x)-f_a(x+b-a).$$

Differentiating twice with respect to the variable x of $f_a(x)$ yields

$$f_a'(x)=\psi(x+a)-\ln(x+a)+\frac{a}{x+a}-1 \quad\text{and}\quad f_a''(x)=\psi'(x+a)-\frac{1}{x+a}-\frac{a}{(x+a)^2}.$$

By virtue of the formulas

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-zt}}{1-e^{-t}} \,\mathrm{d}\, t \quad \text{and} \quad \Gamma(z) = k^z \int_0^\infty t^{z-1} e^{-kt} \,\mathrm{d}\, t$$

for $\Re(z) > 0$, $\Re(k) > 0$, and $n \in \mathbb{N}$ in [1, p. 260, 6.4.1] and [1, p. 255, 6.1.1], we obtain

$$f_a''(x-a) = \psi'(x) - \frac{1}{x} - \frac{a}{x^2} = \int_0^\infty \biggl(\frac{1}{1-e^{-t}} - \frac{1}{t} - a\biggr) t e^{-xt} \,\mathrm{d}\, t.$$

Accordingly, we have

$$\begin{aligned} [\ln F_{a,b}(x)]'' = f_a''(x) - f_a''(x+b-a) &= \int_0^\infty \biggl(\frac{1}{1-e^{-t}} - \frac{1}{t} - a\biggr) t \bigl[e^{-(x+a)t} - e^{-(x+b)t}\bigr] \,\mathrm{d}\, t \\ &= \int_0^\infty \biggl(\frac{1}{1-e^{-t}} - \frac{1}{t} - a\biggr) t \bigl(e^{-at} - e^{-bt}\bigr) e^{-xt} \,\mathrm{d}\, t. \end{aligned} \tag{13}$$

The famous Bernstein-Widder theorem, [25, p. 161, Theorem 12b], states that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $(0,\infty)$ is that $f(x) = \int_0^\infty e^{-xt} \,\mathrm{d}\,\mu(t)$, where μ is a positive measure on $[0,\infty)$ such that the above integral converges on $(0,\infty)$. Hence, in order to find necessary and sufficient conditions on a, b such that the function $[\ln F_{a,b}(x)]''$ is completely monotonic on $(0,\infty)$, it is necessary and sufficient to discuss the positivity or negativity of the function

$$\biggl(\frac{1}{1-e^{-t}} - \frac{1}{t} - a\biggr) t \bigl(e^{-at} - e^{-bt}\bigr) \tag{14}$$

on $(0,\infty)$.

It is clear that the factor $e^{-at} - e^{-bt}$ is positive (or negative, respectively) if and only if $b > a$ (or $b < a$, respectively). Since the function $\frac{1}{1-e^{-t}} - \frac{1}{t} = \frac{1}{e^t-1} - \frac{1}{t} + 1$ is strictly increasing on $(0,\infty)$ and has the limits $\lim_{t\to0^+}\bigl(\frac{1}{1-e^{-t}} - \frac{1}{t}\bigr) = \frac{1}{2}$ and $\lim_{t\to\infty}\bigl(\frac{1}{1-e^{-t}} - \frac{1}{t}\bigr) = 1$, see [5, 15] and references therein, the factor $\frac{1}{1-e^{-t}} - \frac{1}{t} - a$ is positive (or negative, respectively) on $(0,\infty)$ if and only if $a \le \frac{1}{2}$ (or $a \ge 1$, respectively). Consequently, the function (14) is

1. positive if and only if either $b > a$ and $a \le \frac{1}{2}$ or $b < a$ and $a \ge 1$,
2. negative if and only if either $b < a$ and $a \le \frac{1}{2}$ or $b > a$ and $a \ge 1$.

As a result, the function $\pm[\ln F_{a,b}(x)]''$ is completely monotonic on $(0,\infty)$ if and only if $(a,b) \in D_\pm(a,b)$.

By a straightforward computation, we see that

$$\lim_{x\to\infty} [\ln F_{a,b}(x)]' = \lim_{x\to\infty} \biggl[\psi(x+a) - \psi(x+b) + \ln\frac{x+b}{x+a} + \frac{a(b-a)}{(x+a)(x+b)}\biggr] = 0 \tag{15}$$

for all $a, b \in \mathbb{R}$. This implies that, if and only if $(a,b) \in D_\pm(a,b)$, the first logarithmic derivative satisfies $[\ln F_{a,b}(x)]' \lesseqgtr 0$. By the definition of logarithmically completely monotonic functions, we conclude that, if and only if $(a,b) \in D_\pm(a,b)$, the function $[F_{a,b}(x)]^{\pm1}$ is logarithmically completely monotonic on $(0,\infty)$.

Integrating from u to ∞ with respect to x on the very ends of (13) and considering the limit (15) give

$$-[\ln F_{a,b}(u)]' = \int_0^\infty \biggl(\frac{1}{1-e^{-t}} - \frac{1}{t} - a\biggr)\bigl(e^{-at} - e^{-bt}\bigr)e^{-ut}\,\mathrm{d}\,t.$$

Further integrating with respect to u from x to ∞ on both sides of the above equality and employing the limit $\lim_{x\to\infty} F_{a,b}(x) = e^{b-a}$ reveal that

$$\ln F_{a,b}(x) = b - a + \int_0^\infty \frac{1}{t}\biggl(\frac{1}{1-e^{-t}} - \frac{1}{t} - a\biggr)\bigl(e^{-at} - e^{-bt}\bigr)e^{-xt}\,\mathrm{d}\,t.$$

The first proof of Theorem 1.3 is thus complete. □

Second proof of Theorem 1.3. As did in the proof of [20, Theorem 1], employing the formula

$$\ln\Gamma(z) = \ln\bigl(\sqrt{2\pi}\,z^{z-1/2}e^{-z}\bigr) + \int_0^\infty \frac{1}{t}\biggl(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\biggr)e^{-zt}\,\mathrm{d}\,t$$

in [23, (3.22)] and utilizing $\ln\frac{b}{a} = \int_0^\infty \frac{e^{-au}-e^{-bu}}{u}\,\mathrm{d}\,u$ in [1, p. 230, 5.1.32] yield

$$\begin{aligned}
\ln F_{a,b}(x) &= b - a + \biggl(a - \frac{1}{2}\biggr)\ln\frac{x+a}{x+b} + \int_0^\infty \biggl(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\biggr)\frac{e^{-xt}}{t}\bigl(e^{-at} - e^{-bt}\bigr)\,\mathrm{d}\,t \\
&= b - a + \biggl(a - \frac{1}{2}\biggr)\int_0^\infty \frac{e^{-xt}}{t}\bigl(e^{-bt} - e^{-at}\bigr)\,\mathrm{d}\,t + \int_0^\infty \biggl(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\biggr)\frac{e^{-xt}}{t}\bigl(e^{-at} - e^{-bt}\bigr)\,\mathrm{d}\,t \\
&= b - a + \int_0^\infty \frac{1}{t}\biggl(a - \frac{1}{2} - \frac{1}{2} + \frac{1}{t} - \frac{1}{e^t-1}\biggr)\bigl(e^{-bt} - e^{-at}\bigr)e^{-xt}\,\mathrm{d}\,t \\
&= b - a + \int_0^\infty \frac{1}{t}\biggl(a + \frac{1}{t} - \frac{1}{1-e^{-t}}\biggr)\bigl(e^{-bt} - e^{-at}\bigr)e^{-xt}\,\mathrm{d}\,t.
\end{aligned}$$

The rest of the second proof is the same as in the first proof after the equation (13). The second proof of Theorem 1.3 is complete. □

Proof of Theorem 1.4. This follows from straightforwardly combining (7) and (8) with (9). □

Proof of Theorem 1.5. It is easy to see that

$$C(a,b;z+1) = \frac{\Gamma(b)}{\Gamma(a)}\biggl(\frac{b}{a}\biggr)^{z+1}\frac{\Gamma(z+a+1)}{\Gamma(z+b+1)} = \frac{b}{a}\frac{z+a}{z+b}\frac{\Gamma(b)}{\Gamma(a)}\biggl(\frac{b}{a}\biggr)^{z}\frac{\Gamma(z+a)}{\Gamma(z+b)} = \frac{b}{a}\frac{z+a}{z+b}C(a,b;z).$$

Consequently, when taking $z = n-1$,

$$\begin{aligned}
C(a,b;n) &= \frac{b}{a}\frac{n+a-1}{n+b-1}C(a,b;n-1) = \biggl(\frac{b}{a}\biggr)^2\frac{n+a-1}{n+b-1}\frac{n+a-2}{n+b-2}C(a,b;n-2) \\
&= \cdots = \biggl(\frac{b}{a}\biggr)^n\frac{n+a-1}{n+b-1}\frac{n+a-2}{n+b-2}\cdots\frac{a+1}{b+1}\frac{a}{b}C(a,b;0) = \biggl(\frac{b}{a}\biggr)^n\prod_{k=0}^{n-1}\frac{a+k}{b+k}.
\end{aligned}$$

By (9), it follows that

$$\sum_{n=1}^{\infty}\left(\frac{a}{b}\right)^n C(a,b;n)=\frac{\Gamma(b)}{\Gamma(a)}\sum_{n=1}^{\infty}\frac{\Gamma(n+a)}{\Gamma(n+b)}=\frac{\Gamma(b)}{\Gamma(a)}\frac{\Gamma(a+1)\Gamma(b-a-1)}{\Gamma(b)\Gamma(b-a)}=\frac{a}{b-a-1}.$$

The last two formulas in Theorem 1.5 can be straightforwardly derived from the definition (2) of the generalized hypergeometric series. The proof of Theorem 1.5 is complete. □

Remark 2.1. This paper is a companion of the articles [6, 7, 12, 13, 16, 18, 20] and the preprints [10, 18] and is a revised version of the preprint [17].

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Semiring structures based on meet and plus ideals in lower BCK-semilattices

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Abstract. The notion of the meet set based on two subsets of a lower BCK-semilattice X is introduced, and related properties are investigated. Conditions for the meet set to be a (positive implicative, commutative, implicative) ideal are discussed. The meet ideal based on subsets, and the plus ideal of two subsets in a lower BCK-semilattice X are also introduced, and related properties are investigated. Using meet operation and addition, the semiring structure is induced.

1. Introduction

Ideal theory has an important role in the development BCK/BCI-algebras (see [1, 3, 4]). It was shown in [5] that if X is a BCK-algebra then $(X, \leq)$ is a poset, and moreover if X is a commutative BCK-algebra, i.e., $x * (x * y) = y * (y * x)$ holds in X, then $(X, \leq)$ is a lower semilattice. Pałasiński [7] discussed properties of certain ideals in BCK-algebras which are lower semilattices.

In this paper, we introduce the notion of the meet set based on two subsets of a lower BCK-semilattice X and we discuss conditions for the meet set to be a (positive implicative, commutative, implicative) ideal. We also introduced the meet ideal based on subsets, and the plus ideal of two subsets in a lower BCK-semilattice X. We investigate several related properties, and we induce the semiring structure by using meet operation and addition.

2. Prliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-*algebra* if it satisfies the following conditions

[0] **2010 Mathematics Subject Classification**: 06F35, 03G25.

[0] **Keywords**: Lower BCK-semilattice; meet set; meet ideal; plus ideal; meet operation; addition; semiring.

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(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is called a *lower BCK-semilattice* (see [6]) if X is a lower semilattice with respect to the BCK-order.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X (see [6]) if it satisfies

$$0 \in A, \tag{2.1}$$

$$(\forall x \in X)\, (\forall y \in A)\, (x * y \in A\ \Rightarrow\ x \in A)\,. \tag{2.2}$$

Note that every ideal A of a BCK/BCI-algebra X satisfies the following implication (see [6]).

$$(\forall x, y \in X)\, (x \leq y,\ y \in A\ \Rightarrow\ x \in A)\,. \tag{2.3}$$

For any subset A of X, the ideal generated by A is defined to be the intersection of all ideals of X containing A, and it is denoted by $\langle A \rangle$. If A is finite, then we say that $\langle A \rangle$ is *finitely generated ideal* of X (see [6]).

A subset A of a BCK-algebra X is called a *commutative ideal* of X (see [6]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A)\, ((x * y) * z \in A\ \Rightarrow\ x * (y * (y * x)) \in A)\,. \tag{2.4}$$

A subset A of a BCK-algebra X is called a *positive implicative ideal* of X (see [6]) if it satisfies (2.1) and

$$(\forall x, y, z \in X)\, ((x * y) * z \in A,\, y * z \in A\ \Rightarrow\ x * z \in A)\,. \tag{2.5}$$

A subset A of a BCK-algebra X is called an *implicative ideal* of X (see [6]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A)\, ((x * (y * y)) * z \in A\ \Rightarrow\ x \in A)\,. \tag{2.6}$$

A proper ideal P of a lower BCK-semilattice X is said to be *prime* if it satisfies

$$(\forall a, b \in X)\, (a \wedge b \in P\ \Rightarrow\ a \in P \text{ or } b \in P)\,. \tag{2.7}$$

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We refer the reader to the books [2, 6] for further information regarding BCK/BCI-algebras.

3. Meet and plus ideals

In what follows, let X be a lower BCK-semilattice unless otherwise specified. For any nonempty subsets A and B of X, we consider the set

$$K := \{a \wedge b \mid a \in A,\ b \in B\}$$

where $a \wedge b$ is the greatest lower bound of a and b. We say that K is the *meet set* based on A and B. Note that $A \cap B \subseteq K$, but the reverse inclusion is not true as seen in the following example.

Example 3.1. (1) Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

For $A = \{2, 3\}$ and $B = \{1, 4\}$, we have

$$K := \{a \wedge b \mid a \in A,\ b \in B\} = \{0, 1, 2\} \nsubseteq A \cap B.$$

(2) Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{1, 2, 3\}$ and $B = \{1, 3, 4\}$ of X, we have

$$K := \{a \wedge b \mid a \in A,\ b \in B\} = \{0, 1, 3\} \nsubseteq \{1, 3\} = A \cap B.$$

The following example shows that the set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ may not be an ideal of X for some subsets A and B of X.

Example 3.2. Let $X = \{0, 1, 2, 3, 4\}$ be a lower BCK-semilattice in Example 3.1(1). For $A = \{2, 3\}$ and $B = \{1, 4\}$, we have

$$\{a \wedge b \mid a \in A,\ b \in B\} = \{0, 1, 2\},$$

which is not an ideal of X.

We provide conditions for the meet set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ based on A and B to be an ideal.

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Theorem 3.3. *If A and B are ideals of X, then so is the meet set*

$$K := \{a \wedge b \mid a \in A,\ b \in B\}$$

based on A and B.

Proof. Obviously, $0 \in K$. Let $x \in K$ and $y * x \in K$ for $x, y \in X$. Then $x = a \wedge b$ and $y * x = a' \wedge b'$ where $a, a' \in A$ and $b, b' \in B$. Since $a \wedge b \leq a$ and A is an ideal, we have $x = a \wedge b \in A$. Similarly, we have

$$y * x = a' \wedge b' \leq a' \in A.$$

Since A is an ideal of X, it follows that $y \in A$. By the similar way, we get $y \in B$. Therefore,

$$y = y \wedge y \in \{a \wedge b \mid a \in A,\ b \in B\} = K$$

and K is an ideal of X. □

Lemma 3.4 ([6])**.** *For an ideal A of a BCK-algebra X, the following are equivalent.*

(i) *A is positive implicative.*

(ii) $(\forall x, y \in X)\,((x * y) * y \in A \;\Rightarrow\; x * y \in A)$.

Lemma 3.5 ([6])**.** *For an ideal A of a BCK-algebra X, the following are equivalent.*

(i) *A is commutative.*

(ii) $(\forall x, y \in X)\,(x * y \in A \;\Rightarrow\; x * (y * (y * x)) \in A)$.

Lemma 3.6 ([6])**.** *Let A be an ideal of a BCK-algebra X. Then A is implicative if and only if A is both positive implicative and commutative.*

Theorem 3.7. *If A and B are positive implicative (resp., commutative, implicative) ideals of X, then so is the meet set*

$$K := \{a \wedge b \mid a \in A,\ b \in B\}$$

based on A and B.

Proof. Assume that A and B are positive implicative ideals of X. Then A and B are ideals of X, and so the set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ is an ideal of X by Theorem 3.3. Let $(x * y) * y \in K$ for every $x, y \in X$. Then $(x * y) * y = a \wedge b$ for some $a \in A$ and $b \in B$. Since $a \wedge b \leq a$ and A is an ideal, we have $(x * y) * y \in A$. Similarly, $(x * y) * y \in B$. Since A and B are positive implicative ideals, it follows from Lemma 3.4 that $x * y \in A$ and $x * y \in B$. Therefore

$$x * y = (x * y) \wedge (x * y) \in \{a \wedge b \mid a \in A,\ b \in B\} = K,$$

and so K is a positive implicative ideal of X by Lemma 3.4.

Now suppose that A and B are commutative ideals of X. Then A and B are ideals of X, and so the set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ is an ideal of X by Theorem 3.3. Let $x * y \in K$ for every $x, y \in X$. Then $x * y = a \wedge b$ for some $a \in A$ and $b \in B$. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, it follows

that $x * y \in A \cap B$. Since A and B are commutative, we have $x * (y * (y * x)) \in A \cap B$ by Lemma 3.5. Hence

$$\begin{aligned} x * (y * (y * x)) &= (x * (y * (y * x))) \wedge (x * (y * (y * x))) \\ &\in \{a \wedge b \mid a \in A,\ b \in B\} = K, \end{aligned}$$

and therefore K is a commutative ideal if X.

Now, if A and B are implicative ideals of X, then they are both positive implicative and commutative by Lemma 3.6. Thus K is both a positive implicative ideal and a commutative ideal of X, and so it is an implicative ideal of X. □

Given two nonempty subsets A and B of X, we consider the ideal of X generated by the meet set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ based on A and B.

Definition 3.8. For any nonempty subsets A and B of X, we denote

$$A \wedge B := \langle \{a \wedge b \mid a \in A, b \in B\} \rangle$$

which is called the *meet ideal* of X generated by A and B. In this case, we say that the operation "$\wedge$" is a *meet operation*. If $A = \{a\}$, then $\{a\} \wedge B$ is denoted by $a \wedge B$. Also, if $B = \{b\}$, then $A \wedge \{b\}$ is denoted by $A \wedge b$.

Obviously, $A \wedge B = B \wedge A$ for any nonempty subsets A and B of X. If A and B are ideals of X, then

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\}.$$

Example 3.9. For two subsets $A = \{2, 3\}$ and $B = \{1, 4\}$ of X in Example 3.1, the meet ideal of X generated by A and B is $A \wedge B = \langle \{0, 1, 2\} \rangle = \{0, 1, 2, 3\}$.

For any nonempty subsets A, B and C of X, we have

$$A \subseteq B,\ A \subseteq C \ \Rightarrow\ A \subseteq B \wedge C. \tag{3.1}$$

The following example shows that there are subsets A, B and C of X such that $A \subseteq B$ and $A \subseteq C$, but $B \wedge C \nsubseteq A$.

Example 3.10. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{0, 1\}$, $B = \{0, 1, 2, 3\}$ and $C = \{0, 1, 2, 4\}$ of X, we have

$$B \wedge C = \langle \{b \wedge c \mid b \in B,\ c \in C\} \rangle = \{0, 1, 2\} \nsubseteq \{0, 1\} = A.$$

Proposition 3.11. *If A, B and C are ideals of X, then*

$$A \wedge \{0\} = \{0\}. \tag{3.2}$$

$$A \wedge B = A \cap B. \tag{3.3}$$

$$(A \wedge B) \wedge C = A \wedge (B \wedge C) = \{a \wedge b \wedge c \mid a \in A, b \in B, c \in C\}. \tag{3.4}$$

Proof. It is clear that $A \wedge \{0\} = \{0\}$. Using (3.1), we have $A \cap B \subseteq A \wedge B$. Let $x \in A \wedge B$. Then there exist $a \in A$ and $b \in B$ such that $x = a \wedge b$. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, we have $x \in A \cap B$ by (2.3). Hence $A \wedge B = A \cap B$. The result (3.4) is straightforward. □

Corollary 3.12. *If A, B and C are ideals of X, then the condition* (3.1) *is valid.*

By Proposition 3.11, we know that for ideals A_1, A_2, $\cdots$, A_n of X

$$\begin{aligned} \bigwedge_{i=1}^{n} A_i &:= A_1 \wedge A_2 \wedge \cdots \wedge A_n \\ &= \{a_1 \wedge a_2 \wedge \cdots \wedge a_n \mid a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n\} \\ &= \bigcap_{i=1}^{n} A_i. \end{aligned} \tag{3.5}$$

For any nonempty subsets A and B of X, denote by $A + B$ the ideal generated by $A \cup B$, and is called the *plus ideal* of A and B. The operation "+" is called the *addition*. Obviously, $A, B \subseteq A + B$, $A + \{0\} = A$ and $A + B = B + A$.

Example 3.13. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{1, 3\}$ and $B = \{2\}$ of X, we have

$$A + B = \langle A \cup B \rangle = \{0, 1, 2, 3\},$$

which is a plus ideal of X.

Proposition 3.14. *For any nonempty subsets A and B of X, we have $A \wedge B \subseteq A + B$.*

Proof. If $x \in A \wedge B$, then there exists $z_1, z_2, \cdots, z_n \in \{a \wedge b \mid a \in A,\ b \in B\}$ such that

$$(\cdots((x * z_1) * z_2) * \cdots) * z_n = 0. \tag{3.6}$$

For each $i \in \{1, 2, \cdots, n\}$, we have $z_i = a_i \wedge b_i$ where $a_i \in A$ and $b_i \in B$. Thus

$$a_i \wedge b_i \leq a_i \in A \subseteq A \cup B \subseteq A + B,$$

and so $z_i \in A + B$ for all $i \in \{1, 2, \cdots, n\}$. Since $0 \in A + B$, it follows from (3.6) and (2.2) that $x \in A + B$. Hence $A \wedge B \subseteq A + B$. □

Given two nonempty subsets A and B of X, we note that every ideal I of X is represented by the meet ideal based on some A and B, and every ideal J of X is represented by the plus ideal of A and B. But we know that they are different, that is, $I \neq J$ in general as seen in the following example.

Example 3.15. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For two subsets $A = \{1\}$ and $B = \{2, 3\}$ of X, the ideal $I = \{0, 1\}$ is represented by the meet ideal based on A and B as follows

$$I = \langle A \wedge B \rangle = \langle \{0, 1\} \rangle = \{0, 1\}.$$

Also the ideal $J = \{0, 1, 2, 3\}$ is represented by the plus ideal of A and B as follows:

$$J = A + B = \langle A \cup B \rangle = \langle \{1, 2, 3\} \rangle = \{0, 1, 2, 3\}.$$

We know that $I \neq J$.

The following example shows that the reverse inclusion in Proposition 3.14 is not true in general.

Example 3.16. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ which is given in Example 3.13. For subsets $A = \{1, 2\}$ and $B = \{1, 3\}$ of X, we have

$$A \wedge B = \langle \{0, 1\} \rangle = \{0, 1\}$$

and

$$A + B = \langle \{1, 2, 3\} \rangle = \{0, 1, 2, 3\}.$$

Thus $A + B \nsubseteq A \wedge B$.

For any nonempty subsets A, B and C of X, consider the following condition.

$$A \subseteq C,\ B \subseteq C\ \Rightarrow A + B \subseteq C. \tag{3.7}$$

The following example shows that the condition (3.7) is not valid in general.

Example 3.17. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{1, 3\}$, $B = \{2, 3\}$ and $C = \{1, 2, 3\}$ of X, we have

$$A + B = \langle A \cup B\rangle = \{0, 1, 2, 3\} \nsubseteq C.$$

We provide conditions for the implication (3.7) to be hold.

Proposition 3.18. *If A and B are nonempty subsets of X and C is an ideal of X, then the implication* (3.7) *is valid.*

Proof. Let A and B be subsets of X and C be an ideal of X such that $A \subseteq C$ and $B \subseteq C$. If $x \in A + B$, then

$$(\cdots((x * z_1) * z_2) * \cdots) * z_n = 0 \tag{3.8}$$

for some $z_1, z_2, \cdots, z_n \in A \cup B$. It follows that $z_i \in C$ for all $i = 1, 2, \cdots, n$ and $0 \in C$. Since C is an ideal of X, it follows from (3.8) and (2.2) that $x \in C$. Therefore $A + B \subseteq C$. □

Let A be an ideal of a BCI-algebra X and S be a subset of X with a nilpotent element. Then

$$x \in \langle A \cup S\rangle \text{ if and only if } (\cdots((x * s_1) * s_2) * \cdots) * s_n \in A$$

for some $s_1, s_2, \cdots, s_n \in S$ (see [2]). Since every element of a BCK-algebra is nilpotent, we can apply the result above to BCK-algebras as follows.

Lemma 3.19. *Let A an ideal of a BCK-algebra X. For any subset S of X, we have*

$$x \in \langle A \cup S\rangle \textit{ if and only if } (\cdots((x * s_1) * s_2) * \cdots) * s_n \in A$$

for some $s_1, s_2, \cdots, s_n \in S$.

Lemma 3.20 ([2])**.** *Let X be a commutative BCK-algebra and $x, y, z \in X$. Then*

$$(x \wedge y) * (x \wedge z) = (x \wedge y) * z.$$

Theorem 3.21. *For any ideals A, B and C of a commutative BCK-algebra X, we have*

$$A \wedge (B + C) = (A \wedge B) + (A \wedge C) \textit{ and } (B + C) \wedge A = (B \wedge A) + (C \wedge A).$$

Proof. Note that $A \wedge B \subseteq A$ and $A \wedge B \subseteq B \subseteq B + C$. It follows from (3.1) that

$$A \wedge B \subseteq A \wedge (B + C).$$

Similarly $A \wedge C \subseteq A \wedge (B + C)$, and thus

$$(A \wedge B) + (A \wedge C) \subseteq A \wedge (B + C)$$

by Proposition 3.18. Now let $x \in A \wedge (B + C)$. Then $x = a \wedge z$ for some $a \in A$ and $z \in B + C = \langle B \cup C \rangle$. It follows from Lemma 3.19 that there exist $c_1, c_2, \cdots, c_n \in C$ such that

$$(\cdots((z * c_1) * c_2) * \cdots) * c_n \in B. \tag{3.9}$$

Note that $a \wedge c_1, a \wedge c_2, \cdots, a \wedge c_n \in A \wedge C$. Using Lemma 3.20 and (a3) induces

$$\begin{aligned}((a \wedge z) * (a \wedge c_1)) * (a \wedge c_2) &= ((a \wedge z) * c_1) * (a \wedge c_2)\\ &= ((a \wedge z) * (a \wedge c_2)) * c_1\\ &= ((a \wedge z) * c_2) * c_1\\ &= ((a \wedge z) * c_1) * c_2\end{aligned}$$

which implies from Lemma 3.20 and (a3) again that

$$\begin{aligned}&(((a \wedge z) * (a \wedge c_1)) * (a \wedge c_2)) * (a \wedge c_3)\\ &= (((a \wedge z) * c_1) * c_2) * (a \wedge c_3)\\ &= (((a \wedge z) * (a \wedge c_3)) * c_1) * c_2\\ &= (((a \wedge z) * c_3) * c_1) * c_2\\ &= (((a \wedge z) * c_1) * c_2) * c_3.\end{aligned}$$

By the mathematical induction, we conclude that

$$\begin{aligned}&(\cdots(((a \wedge z) * (a \wedge c_1)) * (a \wedge c_2)) * \cdots) * (a \wedge c_n)\\ &= (\cdots(((a \wedge z) * c_1) * c_2) * \cdots) * c_n.\end{aligned} \tag{3.10}$$

The inequality $a \wedge z \leq z$ implies from (a2) that

$$(\cdots(((a \wedge z) * c_1) * c_2) * \cdots) * c_n \leq (\cdots((z * c_1) * c_2) * \cdots) * c_n. \tag{3.11}$$

Since $(\cdots((z * c_1)) * c_2) * \cdots) * c_n \in B$ and B is an ideal, it follows from (2.3) that

$$(\cdots(((a \wedge z) * c_1) * c_2) * \cdots) * c_n \in B. \tag{3.12}$$

Note that $(\cdots(((a \wedge z) * c_1) * c_2) * \cdots) * c_n \leq a \wedge z \leq a$ and $a \in A$, and so

$$(\cdots(((a \wedge z) * c_1) * c_2) * \cdots) * c_n \in A. \tag{3.13}$$

Combining (3.10), (3.12) and (3.13), we have

$$(\cdots(((a \wedge z) * (a \wedge c_1)) * (a \wedge c_2)) * \cdots) * (a \wedge c_n) \in A \wedge B. \tag{3.14}$$

Since $a \wedge c_1, a \wedge c_2, \cdots, a \wedge c_n \in A \wedge C$, it follows from Lemma 3.20 that

$$x = a \wedge z \in \langle (A \wedge B) \cup (A \wedge C) \rangle = (A \wedge B) + (A \wedge C). \tag{3.15}$$

Consequently $A \wedge (B + C) = (A \wedge B) + (A \wedge C)$. Similarly we have $(B + C) \wedge A = (B \wedge A) + (C \wedge A)$. $\square$

Through our discussion above, we make a semiring as follows.

Theorem 3.22. *Let $\mathcal{I}(X)$ be the set of all ideals of a commutative BCK-algebra X. Then $(\mathcal{I}(X), +, \wedge)$ is a semiring, that is, two operations $+$ and $\wedge$ are associative on $\mathcal{I}(X)$ such that*

(i) *addition $+$ is a commutative operation,*

(ii) *there exist $\{0\} \in \mathcal{I}(X)$ such that $A + \{0\} = A$ and $A \wedge \{0\} = \{0\} \wedge A = \{0\}$ for each $A \in \mathcal{I}(X)$, and*

(iii) *the meet operation $\wedge$ distributes over addition (+) both from the left and from the right.*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.5, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

The solutions of some types of q-shift difference differential equations *

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Abstract

In this paper, we investigate some properties of solutions of some types of q-shift difference differential equations. In addition, we also generalize the Rellich-Wittich-type theorem about differential equations to the case of q-shift difference differential equations. Moreover, we give some example to show the existence and growth of some q-shift difference differential equations.
Key words: q-shift; difference differential equation; zero order.
Mathematical Subject Classification (2010): 39A 50, 30D 35.

1 Introduction and Some Results

The main purpose of this paper is to investigate some properties of solutions of some q-shift difference differential equations by using Nevanlinna theory in the fields of complex analysis. Thus, we firstly assume that readers are familiar with the basic results and the notations of the Nevanlinna value distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $T(r, f)$, $\cdots$, (see Hayman [15], Yang [33] and Yi and Yang [34]). For a meromorphic function f, we use $S(r, f)$ to denote any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside a possible exceptional set of finite logarithmic measure, $\mathbb{S}(f)$ denotes the family of all meromorphic function $a(z)$ such that $T(r, a) = S(r, f) = o(T(r, f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. Besides, we use $S_1(r, f)$ to denote any quantity satisfying $S_1(r, f) = o(T(r, f))$ for all r on a set F of logarithmic density 1, the logarithmic density of a set F is defined by

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_{[1,r]\cap F} \frac{1}{t} dt.$$

For convenience, we claim that the set F of logarithmic density can be not necessarily the same at each occurrence.

*The author was supported by the NSF of China (11561033), the Natural Science Foundation of Jiangxi Province in China (20151BAB201008), and the Foundation of Education Department of Jiangxi Province (GJJ150902) of China.

About forty years ago, F. Rellich, H. Wittich and I. Laine investigated the existence or growth of solutions of some differential equations (see [17, 18, 20, 22]) and obtained the following results.

Theorem 1.1 *(see [17, Rellich]). Let the differential equation be the following form*

$$w'(z) = f(w), \tag{1}$$

If $f(w)$ is transcendental meromorphic function of w, then equation (1) has no non-constant entire solution.

Theorem 1.2 *(see [26, Wittich]). Let*

$$\Phi(z, w) = \sum a_{(i)}(z) w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n}$$

be differential polynomial, with coefficients $a_{(i)}(z)$ are polynomials of z. If the right-hand side of the differential equation

$$\Phi(z, w) = f(w), \tag{2}$$

$f(w)$ is the transcendental meromorphic function of w, then equation (2) has no non-constant entire solution.

Remark 1.1 *H. Wittich [26] studied the more general differential equation than equation (1).*

Later, Yanagihara and Shimomura extended the above type theorem to the case of difference equations (see [25, 31, 32]), and obtained the following two results

Theorem 1.3 *(see [25, Shimomura]). For any non-constant polynomial $P(w)$, the difference equation*

$$w(z + 1) = P(w(z))$$

has a non-trivial entire solution.

Theorem 1.4 *(see [31, Yanagihara]). For any non-constant rational function $R(w)$, the difference equation*

$$w(z + 1) = R(w(z))$$

has a non-trivial meromorphic solution in the complex plane.

After theirs work, by using Nevanlinna theory in complex difference equations (see [1, 3, 7, 8, 11, 12, 14]), many mathematicians have done a lot of researches in difference equations, difference product and q-difference in the complex plane $\mathbb{C}$, there were a number of articles (including [5, 13, 16, 19, 24, 36]) focused on the existence and growth of solutions of difference equations. In addition, K. Liu, H.Y. Xu and X. G. Qi investigated some properties of complex q-shift difference equations [23, 24, 28]. Inspired by these papers, the purpose of this paper is to study the above Rellich-Wittich-type theorem of q-shift difference differential equation.

Definition 1.1 *We call the equation as q-shift difference differential equation if a equation contains the q-shift term $f(z+c)$, q-difference term $f(qz)$ and differential term $f'(z)$ of one function $f(z)$ at the same time.*

We consider the q-shift difference differential equation of the form

$$\Omega(z,w) := \sum_J a_J(z) \prod_{j=1}^{n} \left(w^{(j)}(q_j z + c_j)^{i_j} \right) = P_s[f(w)], \tag{3}$$

where $a_J(z)$ are polynomials of z and $q_j, c_j \in \mathbb{C} \setminus \{0\}$, $P_m[f]$ is a polynomial of f of degree m,

$$P_m[f] = d_m(z) f^m + d_{m-1}(z) f^{m-1} + \cdots + d_0(z),$$

and $d_m(z), \ldots, d_0(z)$ are polynomials of z, and obtain the following results.

Theorem 1.5 *For equation (3), if $s \geq 1$ and f is a transcendental meromorphic function, then equation (3) has no non-constant transcendental entire solution with zero order.*

Theorem 1.6 *Under the assumptions of Theorem 1.5, the q-shift difference differential equation*

$$\sum_J a_J(z) \prod_{j=1}^{n} (w^{(j)}(q_j z + c_j))^{i_j} = \frac{P_s[f(w)]}{Q_t[f(w)]},$$

has no non-constant transcendental entire solution with zero order, where $s \geq 1$, and $P_s[f]$ and $Q_t[f]$ are irreducible polynomials in f.

In 2012, Beardon [4] studied entire solutions of the generalized functional equation

$$f(qz) = q f(z) f'(z), \quad f(0) = 0, \tag{4}$$

where q is a non-zero complex number. Beardon [4] obtained the main theorem as follows.

Theorem 1.7 *[4]. Any transcendental solution f of equation (4) is of the form*

$$f(z) = z + z(bz^p + \cdots),$$

where p is a positive integer, $b \neq 0$ and $q \in \mathcal{K}_p$. In particular, if $q \notin \mathcal{K}$, then the only formal solutions of (4) are $\mathcal{O}$ and $\mathcal{I}$, where $\mathcal{K}, \mathcal{K}_p, \mathcal{O}$ and $\mathcal{I}$ were stated as in [4].

In 2013, Zhang [35] further the growth of solutions of equation (4) and obtained the following theorem

Theorem 1.8 *[35, Theorem 1.1]. Suppose that f is a transcendental solution of (4) for $q \in \mathcal{K}$, then we have*

$$\rho(f) \leq \frac{\log 2}{\log |q|},$$

where

$$\rho(f) = \limsup_{r \to +\infty} \frac{\log T(r,f)}{\log r},$$

where $\mathcal{K}$ is stated as in Theorem 1.7.

Inspired by the ideas of Xu [27, 30] and Beardon [4], we investigate the growth of solutions of some q-shift difference differential equations and obtain the following results.

Theorem 1.9 *Suppose that f is a solution of*

$$f(qz+c) = \eta f(z)f'(z), \tag{5}$$

where $q, c, \eta \in \mathbb{C}\setminus\{0\}$ and $|q| > 1$. If f is a transcendental entire function, then we have

$$\rho(f) \leq \frac{\log 2}{\log |q|}.$$

Furthermore, if f is a polynomial, then f is a polynomial of degree 1, that is, $f(z) = a_1 z + a_0$, where

$$a_1 = \frac{q}{\eta}, \qquad a_0 = \frac{qc}{\eta(1+q)}.$$

The following example shows that equation (5) had a transcendental entire solution.

Example 1.1 *Let $q = 2, c = 2\pi$ and $\eta = 2$. Then $f(z) = \sin z$ satisfies equation*

$$f(2z+2\pi) = 2f(z)f'(z),$$

and

$$\rho(f) = 1 = \frac{\log 2}{\log 2}.$$

We also investigate the existence and growth of solutions of equation (5) when the constant η in equation (5) is replaced by a function, and obtain the following result.

Theorem 1.10 *Let f be a transcendental solution of equation*

$$f(qz+c)^n = R(z)f(z)[f^{(j)}(z)]^s, \tag{6}$$

where $q, c, \in \mathbb{C}$ and $|q| > 1$, n, j, s are positive integers and $R(z)$ is rational function in z. If f is an entire function, then $n \leq s+1$ and

$$\rho(f) \leq \frac{\log(s+1) - \log n}{\log |q|}.$$

Furthermore, if $n = 1$ and f is a meromorphic function with infinitely many poles, then we have

$$\frac{\log(s+1)}{\log |q|} \leq \mu(f) \leq \rho(f) \leq \frac{\log(sj+s+1)}{\log |q|}.$$

The following example shows that equation (6) has transcendental entire and meromorphic solutions.

Example 1.2 *Let $q = 2, c = 2\pi i$, $n = 1$ and $s = 1$, then $f(z) = ze^z$ satisfies system*

$$f(2z+2\pi i) = \frac{2z+2\pi i}{z(z+1)} f(z)f'(z).$$

and

$$\rho(f) = 1 \leq \frac{\log 2}{\log 2}.$$

Example 1.3 *Let $q = 2, c = \pi i$, $n = 1$ and $s = 1$, then $f(z) = \frac{e^{2z}}{z^2}$ satisfies equation*

$$f(2z + 2\pi i) = \frac{z^5}{(2z-2)(2z+2\pi i)^2} f(z)f'(z),$$

and

$$\frac{\log 2}{\log 2} = 1 \le \mu(f) = \rho(f) = 1 \le \frac{\log 3}{\log 2}.$$

Theorem 1.11 *Let f be a transcendental solution of the equation*

$$f(qz+c)^n = \varphi(z)f(z)[f^{(j)}(z)]^s, \tag{7}$$

where $q, c, \in \mathbb{C}$ and $|q| > 1$, n, j, s are positive integers and $\varphi(z)$ is a small function with respect of f. If f is a meromorphic function with $\overline{N}(r, f) = S(r, f)$, then $n < s + 1$ and f satisfies

$$\rho(f) \le \frac{\log(s+1) - \log n}{\log |q|}.$$

Furthermore, if $n = 1$ and f has infinitely many poles, and the number of distinct common poles of f and $\frac{1}{\varphi}$ is finite, then we have

$$\rho(f) = \frac{\log(s+1)}{\log |q|}.$$

The following example shows that equation (7) has transcendental meromorphic solution f with the order $\rho(f) = \frac{\log(s+1)}{\log|q|}$.

Example 1.4 *Let $n = j = s = 1$ and $q = \sqrt{2}, c = \frac{1}{2\sqrt{2}}$, then $f(z) = e^{z^2}$ satisfies equation*

$$f(2z + \frac{1}{2\sqrt{2}}) = \frac{1}{2z} e^{\frac{1}{8}} e^z f(z)f'(z).$$

Thus, $\varphi(z) = \frac{1}{2z} e^{\frac{1}{8}} e^z$ with $T(r, \varphi) = S(r, f)$ and the order of $f(z)$ satisfies

$$\rho(f) = 2 = \frac{\log 2 - \log 1}{\frac{1}{2}\log 2}.$$

2 Some Lemmas

Lemma 2.1 *(Valiron-Mohon'ko). [18] Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z)f(z)^i}{\sum_{j=0}^{n} b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = \max\{m, n\}$ and $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

Lemma 2.2 *(see [23]). Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz+\eta)}{f(z)}\right) = S_1(r, f).$$

Lemma 2.3 *(see [28]). Let $f(z)$ be a transcendental meromorphic function of zero order and q, η be two nonzero complex constants. Then*

$$T(r, f(qz+\eta)) = T(r, f(z)) + S_1(r, f), \quad N(r, f(qz+\eta)) \leq N(r, f) + S_1(r, f).$$

Lemma 2.4 *(see [34, p.37] or [33]). Let $f(z)$ be a nonconstant meromorphic function in the complex plane and l be a positive integer. Then*

$$N(r, f^{(l)}) = N(r, f) + l\overline{N}(r, f), \quad T(r, f^{(l)}) \leq T(r, f) + l\overline{N}(r, f) + S(r, f).$$

Lemma 2.5 *Let $q, c \in \mathbb{C} \setminus \{0\}$ and $f(z)$ be a nonconstant meromorphic function with zero order. Then for any positive finite integer k, we have*

$$m\left(r, \frac{f^{(k)}(qz+c)}{f(z)}\right) = S_1(r, f),$$

and

$$m\left(r, f^{(k)}(qz+c)\right) \leq m(r, f) + S_1(r, f).$$

Proof: It follows from Lemma 2.2 that

$$m\left(r, \frac{f^{(k)}(qz+c)}{f(z)}\right) \leq m\left(r, \frac{f^{(k)}(qz+c)}{f(qz+c)}\right) + m\left(r, \frac{f(qz+c)}{f(z)}\right) = S_1(r, f).$$

Moreover, we have

$$m\left(r, f^{(k)}(qz+c)\right) = m\left(r, \frac{f^{(k)}(qz+c)}{f(z)} f(z)\right) \leq m(r, f) + S_1(r, f).$$

This completes the proof of Lemma 2.5. □

Lemma 2.6 *(see [11]). Let $\Phi : (1, \infty) \to (0, \infty)$ be a monotone increasing function, and let f be a nonconstant meromorphic function. If for some real constant $\alpha \in (0, 1)$, there exist real constants $K_1 > 0$ and $K_2 \geq 1$ such that*

$$T(r, f) \leq K_1 \Phi(\alpha r) + K_2 T(\alpha r, f) + S(\alpha r, f),$$

then the order of growth of f satisfies

$$\rho(f) \leq \frac{\log K_2}{-\log \alpha} + \limsup_{r \to +\infty} \frac{\log \Phi(r)}{\log r}.$$

Lemma 2.7 *(see [9]). Let $f(z)$ be a transcendental meromorphic function and $p(z) = p_k z^k + p_{k-1} z^{k-1} + \cdots + p_1 z + p_0$ be a complex polynomial of degree $k > 0$. For given $0 < \delta < |p_k|$, let $\lambda = |p_k| + \delta, \mu = |p_k| - \delta$, then for given $\varepsilon > 0$ and for r large enough,*

$$(1 - \varepsilon) T(\mu r^k, f) \leq T(r, f \circ p) \leq (1 + \varepsilon) T(\lambda r^k, f).$$

Lemma 2.8 *(see [2, 10] or [6]). Let $g:(0,+\infty)\to R, h:(0,+\infty)\to R$ be monotone increasing functions such that $g(r)\le h(r)$ outside of an exceptional set E with finite linear measure, or $g(r)\le h(r)$, $r\notin H\cup(0,1]$, where $H\subset(1,\infty)$ is a set of finite logarithmic measure. Then, for any $\alpha>1$, there exists r_0 such that $g(r)\le h(\alpha r)$ for all $r\ge r_0$.*

3 Proofs of Theorems 1.5 and 1.6

3.1 The proof of Theorem 1.5

Suppose that w be non-constant entire solution of equation (3) with zero order. Let $E_1=\{z:|w(z)|>1\}$ and $E_2=\{z:|w(z)|\le 1\}$, then we have

$$|\Omega(z,w)|=\left|\sum_J a_J(z)(w(z))^{\lambda_i}\left(\frac{w'(q_1z+c_1)}{w(z)}\right)^{i_1}\cdots\left(\frac{w'(q_{n_1}z+c_{n_1})}{w(z)}\right)^{i_{n_1}}\right|$$
$$\le\begin{cases}|w(z)|^{\lambda}\sum_J|a_J(z)|\left|\frac{w'(q_1z+c_1)}{w(z)}\right|^{i_1}\cdots\left|\frac{w'(q_{n_1}z+c_{n_1})}{w(z)}\right|^{i_{n_1}}, & if\ \ z\in E_1,\\ \sum_J|a_J(z)|\left|\frac{w'(q_1z+c_1)}{w(z)}\right|^{i_1}\cdots\left|\frac{w'(q_{n_1}z+c_{n_1})}{w(z)}\right|^{i_{n_1}}, & if\ \ z\in E_2,\end{cases}$$

where $\lambda=\max\{\lambda_i\},\lambda_i=i_1+\cdots+i_{n_1}$. It follows from Lemma 2.2 and Lemma 2.5 that

$$m(r,\Omega(z,w))=\frac{1}{2\pi}\left(\int_{E_1}+\int_{E_2}\right)\log^+|\Omega(z,w)|d\theta\le\lambda m(r,w)+S_1(r,w).$$

And since $w(z)$ is a non-constant entire function, we have $N(r,w)=0$. Thus, we have $N(r,\Omega(z,w))=0$ and

$$T(r,\Omega)=m(r,\Omega)\le\lambda m(r,w)+S_1(r,w)=\lambda T(r,w)+S_1(r,w). \tag{8}$$

Since $P_s[f(w)]$ is a polynomial of $f(w)$, we can take a complex constant α such that

$$P_s[f(w)]-\alpha=[f(w)-\alpha_1]\cdots[f(w)-\alpha_s],$$

where $\alpha_1,\dots,\alpha_s$ are complex constants, and there at least exists a constant $\beta\in\{\alpha_1,\dots,\alpha_s\}$, which is not a Picard exceptional value of $f(w)$. Let $\xi_j,j=1,2,\dots,p$ be the zeros of $f(w)-\beta$, where p is an any positive integer with $p\ge1$. Then it follows

$$\sum_{j=1}^{p}N(r,\frac{1}{w-\xi_j})\le N(r,\frac{1}{f(w)-\beta})\le N(r,\frac{1}{P_s[f(w)]-\alpha}). \tag{9}$$

Thus, by using the second main theorem and (8), (9), we can get that

$$\begin{aligned}(p-2)T(r,w) &\leq \sum_{j=1}^{p} N(r,\frac{1}{w-\xi_j}) + S(r,w)\\ &\leq N(r,\frac{1}{P_s[f(w)]-\alpha}) + S(r,w)\\ &\leq T(r,P_s[f(w)]) + S(r,w)\\ &\leq T(r,\Omega(z,w)) + S(r,w)\\ &\leq \lambda T(r,w) + S_1(r,w). \end{aligned} \tag{10}$$

It follows from (8) and (10) that

$$(p-2-\lambda)T(r,w) \leq S_1(r,w). \tag{11}$$

Since w is transcendental and p is arbitrary, we can get a contradiction with (11). Hence, we complete the proof of Theorem 1.5.

3.2 The proof of Theorem 1.6

By using the same argument as in Theorem 1.5, and applying Lemma 2.1, we can prove the conclusion of Theorem 1.6 easily.

4 The proof of Theorem 1.9

Suppose that f is a solution of (5). If f is a polynomial of degree $m \geq 1$, let

$$f(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0,$$

where $a_m, \ldots, a_0$ are complex constants. From (5), we have

$$\begin{aligned}&a_m(qz+c)^m + a_{m-1}(qz+c)^{m-1} + \cdots + a_0\\ =&\eta(a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0)[ma_m z^{m-1} + (m-1)a_{m-1}z^{m-1} + \cdots + a_1].\end{aligned} \tag{12}$$

By computing the degree of two sides in z in (12), we can get that $m = 2m-1$, that is, $m = 1$. Thus, $f(z)$ can be rewritten as $f(z) = a_1 z + a_0$. It follows

$$a_1(qz+c) + a_0 = \eta(a_1 z + a_0)a_1,$$

that is,

$$a_1 q = \eta a_1^2, \qquad a_1 c + a_0 = \eta a_1 a_0.$$

Thus, we have $a_1 = \frac{q}{\eta}, \quad a_0 = \frac{qc}{\eta(1+q)}$.

If f is a transcendental entire function, from Lemma 2.4, we have

$$T(r,f(qz+c)) \leq 2T(r,f) + S(r,f) \leq 2(1+\varepsilon)T(\beta r,f), \tag{13}$$

for sufficiently large r and any given $\beta > 1, \varepsilon > 0$. By Lemma 2.7 and (13), for $\theta = |q| - \delta(0 < \delta < |q|, 0 < \theta < 1), i = 1, 2$ and sufficiently larger r, we get

$$(1-\varepsilon)T(\theta r, f) \leq 2(1+\varepsilon)T(\beta r, f),$$

outside of a possible exceptional set E of finite linear measure. From Lemma 2.8, for any given $\gamma > 1$ and sufficiently large r, we obtain

$$(1-\varepsilon)T(\theta r, f) \leq 2(1+\varepsilon)T(\gamma\beta r, f). \tag{14}$$

that is,

$$\frac{(1-\varepsilon)}{2(1+\varepsilon)}T(r, f) \leq T\left(\frac{\beta\gamma}{\theta}r, f\right). \tag{15}$$

Since $|q| > 1$, we can choose $\delta > 0$ such that $\theta > 1$, and let $\varepsilon \to 0$, $\delta \to 0, \beta \to 1$, $\gamma \to 1$, and for sufficiently large r, by Lemma 2.6, we have

$$\rho(f) \leq \frac{\log 2}{\log |q|}.$$

Thus, this completes the proof of Theorem 1.9.

5 Proofs of Theorems 1.10 and 1.11

5.1 The Proof of Theorem 1.10

Since $R(z)$ is a rational function, then we have $T(r, R(z)) = O(\log r)$. If f is a transcendental entire function, similar to the argument as in Theorem 1.9, we can get $\rho(f) \leq \frac{\log(s+1)-\log n}{\log |q|}$ easily.

If f is a meromorphic function, by Lemma 2.1 and Lemma 2.4, it follows from (6) that

$$T(r, f(qz+c)) \leq \frac{sj+s+1}{n}T(r, f(z)) + S(r, f).$$

Since $|q| > 1$, by Lemma 2.7 and using the same argument as in Theorem 1.9, we have $\rho(f) \leq \frac{\log(sj+s+1)-\log n}{\log |q|}$.

Suppose that $n = 1$. Since $R(z)$ is a rational function, we can choose a sufficiently large constant $R(> 0)$ such that $R(z)$ has no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$. Since f has infinitely many poles, we can choose a pole z_0 of f of multiplicity $\tau \geq 1$ satisfying $|z_0| > R$. Thus, it follows that the right side of the equation (6) has a pole of multiplicity $\tau_1 = (s+1)\tau + sj$ at z_0, and f has a pole of multiplicity τ_1 at $qz_0 + c$. Replacing z by $qz_0 + c$ in equation (6), we have that f has a pole of multiplicity $\tau_2 = (s+1)\tau_1 + sj$ at $q^2z_0 + qc + c$. We proceed to follow the step above. Since $R(z)$ has no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$ and f has infinitely many poles again, we may construct poles $\zeta_k = q^kz_0 + q^{k-1}c + \cdots + c, k \in \mathbb{N}_+$ of f of multiplicity τ_k satisfying

$$\tau_k = (s+1)\tau_{k-1} + sj = (s+1)^k\tau + sj[(s+1)^{k-1} + \cdots + 1],$$

as $k \to \infty, k \in \mathbb{N}$. Since $|q| > 1$, then $|\zeta_k| \to \infty$ as $k \to \infty$, for sufficiently large k, we have

$$\begin{aligned} \tau(s+1)^k &\leq (\tau + j)(s+1)^k - j = \tau_k \leq \tau + \tau_1 + \cdots + \tau_k \leq n(|\zeta_k|, f) \\ &\leq n(|q|^k|z_0| + |C|(|q|^{k-1} + \cdots + |q| + 1), f). \end{aligned} \tag{16}$$

Thus, for each sufficiently large r, there exists a $k \in \mathbb{N}_+$ such that

$$r \in [|q|^k|z_0| + |C|\sum_{i=0}^{k-1}|q|^i, |q|^{(k+1)}|z_0| + |C|\sum_{i=0}^{k}|q|^i),$$

that is,

$$k > \frac{\log r - \log(|z_0| + \frac{|c|}{|q|-1}) - \log\frac{|c|}{|q|-1} - \log|q|}{\log|q|}. \tag{17}$$

Thus, it follows from (17) that

$$n(r, f) \geq \tau(s+1)^k \geq K_1(s+1)^{\frac{\log r}{\log|q|}}, \tag{18}$$

where

$$K_1 = \tau(s+1)^{\frac{-\log(|z_0| + \frac{|c|}{|q|-1}) - \log\frac{|c|}{|q|-1} - \log|q|}{\log|q|}}.$$

Since for all $r \geq r_0$,

$$K_1(s+1)^{\frac{\log r}{\log|q|}} \leq n(r, f) \leq \frac{1}{\log 2}N(2r, f) \leq \frac{1}{\log 2}T(2r, f),$$

it follows from (18) that

$$\rho(f) \geq \mu(f) \geq \frac{\log(s+1)}{\log|q|}.$$

Thus, this completes the proof of Theorem 1.10.

5.2 The proof of Theorem 1.11

By using the same argument as in Theorem 1.10, we can prove the conclusion of Theorem 1.11 easily.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

HW and HYX completed the main part of this article. All authors read and approved the final manuscript.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.5, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Numerical method for solving inequality constrained matrix operator minimization problem☆

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Abstract

In this paper, we considered a matrix inequality constrained linear matrix operator minimization problems with a particular structure, some of whose reduced versions can be applicable to image restoration. We present an efficient iteration method to solve this problem. The approach belongs to the category of Powell-Hestense-Rockafellar augmented Lagrangian method, and combines a nonmonotone projected gradient type method to minimize the augmented Lagrangian function at each iteration. Several propositions and one theorem on the convergence of the proposed algorithm were established. Numerical experiments are performed to illustrate the feasibility and efficiency of the proposed algorithm, including when the algorithm is tested with randomly generated data and on image restoration problems with some special symmetry pattern images.

Key words:
matrix equation, matrix minimization problem, matrix inequality, augmented lagrangian method, image restoration.
2000 MSC: 65F30, 65H15, 15A24

1. Introduction

Let m, n, l_1, s_1, l_2, s_2 be positive integers. Let $\mathcal{A}(X; A_1, \cdots, A_p)$ be a linear mapping from $\mathbb{R}^{m\times n}$ onto $\mathbb{R}^{l_1\times s_1}$ and $\mathcal{G}(X; E_1, \cdots, E_q)$ be a linear mapping from $\mathbb{R}^{m\times n}$ onto $\mathbb{R}^{l_2\times s_2}$, where A_i $(i = 1, \ldots, p)$ and E_j $(j = 1, \ldots, q)$ with suitable sizes are the parameter matrices. In this paper we are interested in solving the following constrained matrix minimization problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\left\|\mathcal{A}\Big(X; A_1, \cdots, A_p\Big) - C\right\|^2 \\ \text{subject to} & X \in \mathcal{S} \\ & L \leq \mathcal{G}\Big(X; E_1, \cdots, E_q\Big) \leq U. \end{array} \tag{1.1}$$

where $\|\cdot\|$ denotes the Frobenius norm, the symbol $\geq$ means nonnegative, the set $\mathcal{S} \subseteq \mathbb{R}^{m\times n}$ shows the constraint, $C \in \mathbb{R}^{l_1\times s_1}$ and $L, U \in \mathbb{R}^{l_2\times s_2}$ are given matrices. In general, $\mathcal{S} \subseteq \mathbb{R}^{m\times n}$ is a linear space

☆Research supported by National Natural Science Foundation of China(11301107,11261014,11561015,51268006), Natural Science Foundation of Guangxi Province (2016GXNSFAA380011,2016GXNSFFA380003).
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Preprint submitted to Journal of Computational Analysis and Applications *September 23, 2016*

possessing special structures, such as symmetry/skew-symmetry, centrosymmetry/centro skew-symmetry, mirror-symmetry/mirror-skew-symmetry, P-commuting symmetry/ skew-symmetry with respect to a given symmetric matrix P, Toeplitz matrix and so on. It is obvious that the linear operator equation in (1.1) is quite general and includes several linear matrix equations such as the Lyapunov and Sylvester matrix equations which are shown in Table 1. For an instant, the Lyapunov matrix equation

$$A_1^T X A_2 + A_2^T X A_1 = -C$$

is equivalent to the linear operator equation in (1.1), if we define the operator $\mathcal{A}$ as:

$$\mathcal{A} : X \rightarrow A_1^T X A_2 + A_2^T X A_1.$$

Table 1: *One-sided and two-sided Lyapunov and Sylvester matrix equations.*

Name	Matrix equation
Continuous-time (CT) Lyapunov	$A_1 X + X A_1^T + BB^T = 0$
Generalized continuous-time (CT) Lyapunov	$A_1^T X A_2 + A_2^T X A_1 = -C$
Generalized discrete-time (CT) Lyapunov	$A_1^T X A_1 + A_2^T X A_2 = -C$
Continuous-time (CT) Sylvester	$A_1 X + X A_2 = C$
Discrete-time (DT) Sylvester	$A_1 X A_2^T + X = C$
Generalized Sylvester	$A_1 X A_2^T + A_3 X A_4^T = C$

Throughout we always assume that the matrix operator inequality in model (1.1) is consistent with these given matrices E_j, L, U and unknown $X \in \mathcal{S}$, then we known that the solution set of Problem (1.1) is nonempty.

The interest that we have in this problem stems from the following reasons. Firstly, by using the vec operator $vec(.)$ and the Kronecker produc $\otimes$, the model (1.1) can be equivalently rewritten as the convex linearly constrained quadratic programming(LCQP) in the vector-form

$$\begin{aligned} &\text{minimize } f(x) = \frac{1}{2} x^T Q x + g^T x + c \\ &\text{subject to } \ l \leq Gx \leq u, \end{aligned} \tag{1.2}$$

where

$$Q = P^T M^T M P, \ \ g = -P^T M^T vec(C), \ \ c = \frac{1}{2} vec(C)^T vec(C) \tag{1.3}$$

and

$$Px = vec(X), \ \ l = vec(L), \ \ u = vec(U). \tag{1.4}$$

The matrices M and G are the Kronecker product of the parameter matrices $\{A_i\}_{i=1}^p$ and $\{E_j\}_{j=1}^q$ which satisfies $vec(\mathcal{A}(X; A_1, \ldots, A_p)) = Mvec(X)$ and $vec(\mathcal{G}(X; E_1, \ldots, E_q)) = Gvec(X)$, respectively. Specifically, in (1.3)-(1.4), P is the matrix that characterizes the elements $X \in \mathcal{S}$ by $vec(X) = Px$ in terms of its independent parameter vector x of X[18]. In theory, the model (1.2) can be solved by some classical optimization methods, such as interior point method, active set method, trust region method, Newton method, and other available methods. In particular, Delbos F. in [2] considered the vector LCQP(1.2) by using an augmented Lagrangian method and given a global linear convergence of the proposed algorithm. However, using this transformation will on the one hand destroy the original structure of the unknown matrix $X \in \mathcal{S}$ if the linear

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subspace $\mathcal{S}$ has some special symmetrical structure. On the other hand, using this transformation will result in a coefficient matrix in large scale, and then increase computational complexity and storage requirement. Indeed, taking $l = m = n = s = p = q = 200$ in (1.1), then the matrices Q and G in the transformed model (1.2) have sizes of about 40000×40000. For these reasons, it cannot be a practicable method for solving Problem (1.1) by the vec operator and the Kronecker produc if the system scale is large. In this paper we will consider directly from the perspective of matrices.

Secondly, various simplified versions of Problem (1.1) have been studied extensively. If we drop the matrix inequality constraint, then Problem (1.1) is reduced to the minimization problem with special structures. Methods proposed for solving such problems can be broadly classified into two classes, including factorization techniques for small size problems, based on the special structure of the linear subspace $\mathcal{S}$ that produce a low-dimensional problems that are then solved using direct methods[3, 4, 5, 6, 7, 8, 9, 10, 11], and iterative schemes, for large-scale problems, based on Krylov subspace-type methods, such as the well-known Jacobi and Gauss-Seidel iterations[12, 13], the conjugate gradient-type methods[14, 15] and the least squares QR(LSQR) methods[16, 17, 18] and so on. On the other hand, if we simplify the general matrix inequality constraint in (1.1) into the nonnegative constraint $X \geq \mathbf{0}$ or the bound constraint $L \leq X \leq U$, then the similar problem has been studied with Dykstra's alternating projection algorithm[19, 20] and spectral projection gradient method[21]. In particular, Problem (1.1) can be regarded as a natural generalization of the problems in [21, 22, 23]. The authors in [21] considered the following constrained minimization problem

$$\text{Minimize } \left\| \sum_{i=1}^{q} A_i X B_i - C \right\|^2 \quad \text{subject to } X \in \Omega = \{X \in \mathbb{R}^{m\times n} : L \leq X \leq U\}. \tag{1.5}$$

They propose a globalized variants projected gradient method and apply the left and right preconditioning strategies to solve (1.5). While the authors in [22, 23] devoted to solve the matrix equation $AX = B$ or minimize $\|AX - B\|$ with special structures under the constraint $CXD \geq E$, respectively. The problems considered in [22] and [23] can be transformed into least nonnegative correction problems based on the fact that close-form optimal solutions of $AX = B$ or minimizing $\|AX - B\|$ with special structures can be readily derived, and then some fixed point-like algorithms can be applied to solve these transformed problems. However, all these previous ideas show difficulties when dealing with the Problem (1.1), due to the generalization of the objection function and the matrix operator inequality, so that either the projection onto the set $\{X \in \mathbb{R}^{m\times n} | L \leq \mathcal{G}(X) \leq U\}$ is not available, or a close-form optimal solution of minimizing the objection function in (1.1) with $X \in \mathcal{S}$ is not tractable.

Thirdly, we consider the application of the model (1.1) in image restoration. In fact, the authors in [21, 24] consider the problem of image restoration, combined with a Tikhonov regularization term, as a convex constrained minimization problem by use a Kronecker decomposition of the blurring matrix and the Tikhonov regularization matrix. And then they propose and show the effectiveness of their approaches, a globalized variants projected gradient method [21] and a conditional gradient-type method[21], to restore some blurred and highly noisy images. However, in this paper, we are only concerned with the restoration problems with some special symmetric pattern images, which have not yet studied in [21, 24]. Moreover, to the best of our knowledge, this class of image restoration problems have received little attention in the other literature. The main difficult is due to the fact that the restore image should preserve the same special symmetric structure with the original images. In this paper we undertake some significant attempts in this field.

In this paper, we will propose and study an algorithm in the framework of the classic Powell-Hestenes-Rockafellar augmented Lagrangian method, first suggested by Hestenes [25] and Powell [26], and developed by E.G. Birgin [27, 28] for solving Problem (1.1). The classic PHR-AL method is a fundamental and

effective approach in inequality-constrained optimization. The algorithm effectively combines a nonmonotone projected gradient type method to minimize the augmented Lagrangian function at each iteration. We will give several propositions and one theorem on the convergence of the proposed algorithm, and apply it to solving Problem (1.1) with randomly generated data and comparing it with existing methods. We also apply our approach, combined with a Tikhonov regularization term, to restore some blurred and highly noisy symmetric pattern images.

Throughout this paper, we use the following notations. Let e_i be the ith column of the identity matrix I_k and $S_k = (e_k, e_{k-1}, \ldots, e_1)$, i.e., the kth backward identity matrix. Let $\mathbf{0}$ be the zero matrix of suitable size and $P_{\mathcal{S}}$ be the Euclidean projection onto set $\mathcal{S}$.We write $\varepsilon_k \downarrow 0$ to indicate that ε_k is a (not necessarily decreasing) sequence of non-negative numbers that tends to zero. We denote $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, A_+(or A_-) be the matrix with the (i, j)-entry equals to $\max\{0, a_{ij}\}$(or $\min\{0, a_{ij}\}$), respectively. For $A, B \in \mathbb{R}^{m\times n}$, $\{A, B\}_-$ denotes a matrix with the ijth entry being equal to $\min\{a_{ij}, b_{ij}\}$, $\langle A, B\rangle = trace(B^T A)$ denotes the inner product of matrices A and B. Then $\mathbb{R}^{m\times n}$ is a Hilbert inner product space and the norm generated is the Frobenius norm $\|\cdot\|$. For any linear operator $\mathcal{L}$ form $\mathbb{R}^{m\times n}$ onto $\mathbb{R}^{l_1\times s_1}$, there is another operator called the adjoint of $\mathcal{L}$, written $\mathcal{L}^T$: $\mathbb{R}^{l_1\times s_1} \to \mathbb{R}^{m\times n}$. What defines the adjoint is that for any two matrices $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{l_1\times s_1}$,

$$\langle \mathcal{L}(X), Y\rangle = \langle X, \mathcal{L}^T(Y)\rangle.$$

The rest of this paper is organized as follows. In section 2, we will briefly characterize the application of model (1.1) in image restoration. Based on the classic augmented Lagrangian method, in section 3 we propose, analyze and test an algorithm for solving the inequality-constrained matrix minimization problem (1.1). Some numerical results are reported in section 4 to verify the efficiency of the proposed algorithm. Numerical tests on the proposed algorithm applied to some special image restoration problems are also reported in this section.

2. The application of model (1.1) in image restoration

For completeness, in this section we briefly characterize how to apply the model (1.1) into image restoration and we refer to [21, 24] for detailed description. Consider solving the following model in image restoration with Tikhonov regularization:

$$\min_{l\le x\le u} \frac{1}{2}\|Hx - g\|_2^2 + \frac{\lambda^2}{2}\|Tx\|_2^2, \tag{2.6}$$

where $\|\cdot\|_2$ is the 2-norm. In image restoration, H will be the blurring operator, g the observed image, T the regularization operator, λ the regularization parameter, and x the restored image to be sought. The constraints represent the dynamic range of the image.

The minimizer of (2.6) can be computed by the following linear system

$$H_\lambda x = H^T g, \quad \text{where } H_\lambda = H^T H + \lambda^2 T^T T. \tag{2.7}$$

In some practical problems in image restoration, often the system (2.7) may not be consistent due to measurement errors in the data matrices, and hence it is useful to consider the following minimization problem with constraints

$$\min_{l\le x\le u} \frac{1}{2}\left\|H_\lambda x - H^T g\right\|_2^2. \tag{2.8}$$

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Here we assume that the matrices H and T can be separated as Kronecker product of matrices with a smaller size, i.e., $H = H_1 \otimes H_2$ and $T = T_1 \otimes T_2$. In the case of nonseparable, one can still obtain an approximation solution of H_1 and H_2 by solving the Kronecker product approximation problem (KPA) of the form $(H_1, H_2) = \text{argmin}_{\hat{H}_1,\hat{H}_2}\|H - \hat{H}_1 \otimes \hat{H}_2\|$[29]. Then, (2.8) can be written as

$$\min_{L\leq X\leq U} \frac{1}{2}\left\|\left\{(H_1{}^T H_1) \otimes (H_2{}^T H_2) + \lambda^2 (T_1{}^T T_1) \otimes (T_2{}^T T_2)\right\} vec(X) - (H_1 \otimes H_2)^T vec(G)\right\|^2, \tag{2.9}$$

where X, G, L and U are the matrices such that $vec(X) = x$, $vec(G) = g$, $vec(L) = l$ and $vec(U) = u$. If some special symmetry pattern images are considered, by using some properties of the Kronecker product, (2.9) is then written as

$$\begin{array}{rl} \min & \frac{1}{2}\left\|A_1 X B_1 + \lambda^2 A_2 X B_2 - C\right\|^2 \\ \text{subject to} & L \leq X \leq U, \quad X \in \mathcal{S}, \end{array} \tag{2.10}$$

with $A_1 = H_2{}^T H_2$, $B_1 = H_1{}^T H_1$, $A_2 = T_2{}^T T_2$, $B_2 = T_1{}^T T_1$ and $C = H_2{}^T G H_1$ and $\mathcal{S}$ is the matrix set whose elements have the same symmetry structure with the original images. The parameter λ in (2.10) is a scalar need to be determined, and its optimal value can be obtained by the classical Generalized cross-validation (GCV) method[21, 24], which is chosen to minimize the GCV function defined by

$$GCV(\lambda) = \frac{\|H\hat{x}_\lambda - g\|_2^2}{\{trace(I - HH_\lambda^{-1}H^T)\}^2} = \frac{\|(I - HH_\lambda^{-1}H^T)g\|_2^2}{\{trace(I - HH_\lambda^{-1}H^T)\}^2},$$

where $H_\lambda = H^T H + \lambda^2 T^T T$. Then, the method proposed for solving Problem (1.1) could be applied directly to the model (2.10) by considering the linear matrix operators $\mathcal{A}(X) = A_1 X B_1 + \lambda^2 A_2 X B_2$ and $\mathcal{G}(X) = X$.

3. Augmented Lagrangian method for solving Problem (1.1)

In this section we propose a matrix-form iteration method, in the framework of the classic Powell-Hestense-Rockafellar augmented Lagrangian(PHR-AL) method, to compute the solution of Problem (1.1). We then prove some convergence results for the proposed algorithm at the end of this section. For convenience, the two linear matrix operators will be simply denote by $\mathcal{A}(X)$ and $\mathcal{G}(X)$ in the following discussion.

Lemma 1. *Assume x^* is a local minimizer of the quadratic program*

$$\min_{x\in R^s} f(x) = \frac{1}{2}x^T M x + g^T x + c \ \text{ subject to } \ Gx \geq b,$$

then there exists a vector y^* such that

$$Mx^* + g - G^T y^* = 0, \ \ Gx^* \geq b, \ \ \langle y^*, Gx^* - b\rangle = 0, \ \ y^* \geq \mathbf{0}.$$

Theorem 1. *Matrix $X^* \in \mathbb{R}^{m\times n}$ is a solution of Problem (1.1) if and only if there exists nonnegative matrices $Y_1^*, Y_2^* \in \mathbb{R}^{l_2\times s_2}$ such that the following conditions are satisfied:*

$$\begin{cases} P_{\mathcal{S}}\left\{\mathcal{A}^T(\mathcal{A}(X^*) - C) - \mathcal{G}^T(Y_1^* - Y_2^*)\right\} = \mathbf{0} \\ \mathcal{G}(X^*) - L \geq \mathbf{0} \\ U - \mathcal{G}(X^*) \geq \mathbf{0} \\ \langle Y_1^*, \mathcal{G}(X^*) - L\rangle = 0 \\ \langle Y_2^*, U - \mathcal{G}(X^*)\rangle = 0. \end{cases} \tag{3.11}$$

Proof. Assume that there are nonnegative matrices $Y_1^*, Y_2^* \in \mathbb{R}^{l_2 \times s_2}$ such that the conditions (3.11) are satisfied. Let

$$f(X) = \frac{1}{2}\left\|\mathcal{A}\left(X\right) - C\right\|^2$$

and

$$\widetilde{f}(X) = f(X) + \langle Y_1^*, L - \mathcal{G}(X)\rangle + \langle Y_2^*, \mathcal{G}(X) - U\rangle.$$

Then for any $\widetilde{W} \in \mathcal{S}$, we have

$$\begin{aligned}
&\widetilde{f}(X^* + \widetilde{W}) \\
&= \tfrac{1}{2}\|\mathcal{A}(X^* + \widetilde{W}) - C\|^2 + \left\langle Y_1^*, L - \mathcal{G}(X^* + \widetilde{W})\right\rangle + \left\langle Y_2^*, \mathcal{G}(X^* + \widetilde{W}) - U\right\rangle \\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2 + \left\langle \mathcal{A}(\widetilde{W}), \mathcal{A}(X^*) - C\right\rangle - \left\langle Y_1^* - Y_2^*, \mathcal{G}(\widetilde{W})\right\rangle \\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2 + \left\langle \widetilde{W}, \mathcal{A}^T(\mathcal{A}(X^*) - C) - \mathcal{G}^T(Y_1^* - Y_2^*)\right\rangle \\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2 + \tfrac{1}{2}\left\langle \widetilde{W}, P_{\mathcal{S}}\left(\mathcal{A}^T(\mathcal{A}(X^*) - C) - \mathcal{G}^T(Y_1^* - Y_2^*)\right)\right\rangle \\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2 \\
&\geq \widetilde{f}(X^*).
\end{aligned}$$

This implies that X^* is a global minimizer of the function $\widetilde{f}(X)$. Since $\langle Y_1^*, \mathcal{G}(X^*) - L\rangle = 0$, $\langle Y_2^*, U - \mathcal{G}(X^*)\rangle = 0$ and $\widetilde{f}(X) \geq \widetilde{f}(X^*)$ for all $X \in \mathcal{S}$, we have

$$\begin{aligned}
f(X) &\geq f(X^*) + \left\langle Y_1^*, L - \mathcal{G}(X^*)\right\rangle + \left\langle Y_2^*, \mathcal{G}(X^*) - U\right\rangle - \left\langle Y_1^*, L - \mathcal{G}(X)\right\rangle - \left\langle Y_2^*, \mathcal{G}(X) - U\right\rangle \\
&= f(X^*) - \left\langle Y_1^*, L - \mathcal{G}(X)\right\rangle - \left\langle Y_2^*, \mathcal{G}(X) - U\right\rangle.
\end{aligned}$$

Hence, we have from $Y_1^* \geq \mathbf{0}$ and $Y_2^* \geq \mathbf{0}$ that $f(X) \geq f(X^*)$ for all $X \in \mathcal{S}$ with $\mathcal{G}(X) - L \geq \mathbf{0}$ and $U - \mathcal{G}(X) \geq \mathbf{0}$. Hence X^* is a solution to Problem (1.1).

Conversely, assuming that X^* is a solution to Problem (1.1), then X^* certainly satisfies the Karush-Kuhn-Tucker conditions of Problem (1.1). That is, there exists a nonnegative matrix Y^* such that satisfies conditions (3.11).

We now define the following Powell-Hestenes-Rockafellar(PHR) Augmented Lagrangian function

$$L_\rho(X, Z_1, Z_2) = \frac{1}{2}\|\mathcal{A}(X) - C\|^2 + \frac{\rho}{2}\left\|\left(L - \mathcal{G}(X) + \frac{Z_1}{\rho}\right)_+\right\|^2 + \frac{\rho}{2}\left\|\left(\mathcal{G}(X) - U + \frac{Z_2}{\rho}\right)_+\right\|^2, \tag{3.12}$$

where $Z_1 \geq \mathbf{0}$ and $Z_2 \geq \mathbf{0}$ are the Lagrangian multiplier matrices and $\rho > 0$ is the penalty parameter. Clearly, the partial derivative of function $L_\rho(X, Z_1, Z_2)$ with respect to X is given by

$$\nabla_X L_\rho(X, Z_1, Z_2) = \mathcal{A}^T\left(\mathcal{A}(X) - C\right) - \rho\mathcal{G}^T\left(\left(L - \mathcal{G}(X) + \frac{Z_1}{\rho}\right)_+ - \left(\mathcal{G}(X) - U + \frac{Z_2}{\rho}\right)_+\right).$$

The augmented Lagrangian method proposed by E.G. Birgin et al in in [27, 28] (with necessary modifications) to solve Problem (1.1) can be described as follows:

Algorithm PHR-AL. *(The PHR-AL method for solving Problem (1.1).)*

1. Input coefficient matrices $A_i, B_i (i = 1, \ldots p)$ in the linear operator $\mathcal{A}$ and matrices $E_i, E_j (i = 1, \ldots q)$ in the linear operator $\mathcal{G}$. Input matrices C, L, U and a large parameter matrix $Z_{max} > \mathbf{0}$. Input $\gamma > 1$, $r \in (0, 1)$, $\rho_1 > 0$, a small tolerance $\varepsilon > 0$ and tolerance $\varepsilon_k \downarrow 0$. Choose initial matrices $\overline{Z}_1^1$ and $\overline{Z}_2^1$ with $\mathbf{0} \leq \overline{Z}_1^1, \overline{Z}_2^1 \leq Z_{max}$. Set $k \leftarrow 1$.

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2. Compute X^k as an approximate stationary point of
$$\text{minimize} \quad L_{\rho_k}(X, \overline{Z}_1^k,\ \overline{Z}_2^k) \quad \text{subject to} \ \ X \in \mathcal{S}. \tag{3.13}$$
That is, compute X^k such that $\left\| P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\right\}\right\| < \varepsilon_k$.
3. Define
$$Z_1^k = \left(\overline{Z}_1^k + \rho_k(L - \mathcal{G}(X^k))\right)_+, \quad Z_2^k = \left(\overline{Z}_2^k + \rho_k(\mathcal{G}(X^k) - U)\right)_+.$$
4. If $k = 1$ or
$$\begin{aligned}\left(\left\|\{\mathcal{G}(X^k) - L, Z_1^k\}_-\right\|^2 + \left\|\{U - \mathcal{G}(X^k),\ Z_2^k\}_-\right\|^2\right)^{1/2} \\ \le r\left(\left\|\{\mathcal{G}(X^{k-1}) - L, Z_1^{k-1}\}_-\right\|^2 + \left\|\{U - \mathcal{G}(X^{k-1}),\ Z_2^{k-1}\}_-\right\|^2\right)^{1/2},\end{aligned} \tag{3.14}$$
define $\rho_{k+1} = \rho_k$. Else, define $\rho_{k+1} = \gamma\rho_k$.
5. If
$$\left(\left\|P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\right\}\right\|^2 + \left\|\{\mathcal{G}(X^k) - L, Z_1^k\}_-\right\|^2 + \left\|\{U - \mathcal{G}(X^k),\ Z_2^k\}_-\right\|^2\right)^{1/2} < \varepsilon,$$
then stop.
6. Update $\overline{Z}_1^{k+1}$ and $\overline{Z}_2^{k+1}$ with $\mathbf{0} \le \overline{Z}_1^{k+1}, \overline{Z}_2^{k+1} \le Z_{max}$ in such a way that $(\overline{Z}_1^{k+1})_{ij} = (Z_1^k)^{ij}$ and $(\overline{Z}_2^{k+1})_{ij} = (Z_2^k)^{ij}$ if $0 \le (Z_1^k)_{ij}, (Z_2^k)_{ij} \le (Z_{max})_{ij}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$.
7. Set $k \leftarrow k + 1$ and go to step 2.

Problem (3.13) in Algorithm PHR-AL is a linear constrained matrix minimization problem. It is certainly solvable for all the known matrices and the scalar ρ_k. Here we will use the spectral projected gradient (SPG) method to compute the approximation stationary point X^k of problem (3.13). The SPG method is a nonmonotone projected gradient type method for minimizing general smooth functions on convex sets[27]. The SPG method is simple, easy to code, and does not require matrix factorizations. Moreover, it overcomes the traditional slowness of the gradient method by incorporating a spectral step length and a nonmonotone globalization strategy. The main steps of SPG algorithm (with necessary modifications) to compute an approximate stationary point of problem (3.13) can be described as follows:

Algorithm SPG. *(Compute an approximate stationary point of problem* (3.13)*)*

1. Input matrices $\overline{Z}_1^k$ and $\overline{Z}_2^k$; an integer $M > 1$, parameters $\alpha_{\min} > 0$, $\alpha_{\max} > \alpha_{\min}$, $\widetilde{\gamma} \in (0, 1)$, $0 < \sigma_1 < \sigma_2 < 1$ and $\alpha_1 \in [\alpha_{\min}, \alpha_{\max}]$. Choose an initial matrix $X_1 \in \mathcal{S}$ and let $i \leftarrow 1$.
2. If $\left\|P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)\right\}\right\| < \varepsilon_k$, stop. (In this case, X_i is an approximate stationary point of problem (3.13).)
3. Compute $dX_i = -\alpha_i P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)\right\}$. Let $\lambda = 1$.
4. Compute $\check{X} = X_i + \lambda dX_i$.
5. If
$$L_{\rho_k}(\check{X}, \overline{Z}_1^k, \overline{Z}_2^k) \le \max_{1 \le j \le \min\{i, M\}} L_{\rho_k}(X_{i-j}, \overline{Z}_1^k, \overline{Z}_2^k) + \widetilde{\gamma}\lambda\left\langle dX_i, \nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)\right\rangle, \tag{3.15}$$
define $\lambda_i = \lambda$, $X_{i+1} = \check{X}$, $s_i = X_{i+1} - X_i$, $y_i = \nabla L_{\rho_k}(X_{i+1}, \overline{Z}_1^k, \overline{Z}_2^k) - \nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)$. Then goto step 6. If (3.15) does not hold, define $\lambda_{new} \in [\sigma_1\lambda, \sigma_2\lambda]$, Let $\lambda = \lambda_{new}$ and goto step 4.

6. Compute $b_i = \langle s_i, y_i\rangle$. If $b_i \le 0$, let $\alpha_i = \alpha_{\max}$, otherwise, compute

$$a_i = \langle s_i, s_i\rangle, \quad \alpha_i = \min\{\alpha_{\max}, \max\{\alpha_{\min}, a_i/b_i\}\}.$$

7. Let $i \leftarrow i + 1$ and goto step2.

In the practical implementation of Algorithm PHR-AL, similarly to [27], we take the parameters $\gamma = 5$, $r = 0.5$, $\rho_1 = 1$, and the large matrix $Z_{\max}$ with all elements equal to 10^{10}. The initial matrices $\overline{Z}_1^1$ and $\overline{Z}_2^1$ are chosen as $\overline{Z}_1^1 = \overline{Z}_2^1 = \mathbf{0}$. For the implementation of Algorithm SPG, similarly to [30], we take the parameters $M = 10$, $\gamma = 10^{-4}$, $\alpha_{min} = 10^{-30}$, $\alpha_{max} = 10^{30}$, $\sigma_1 = 0.1$, $\sigma_2 = 0.9$, $\lambda_{new} = (\sigma_1\lambda + \sigma_2\lambda)/2$ and $\alpha_0 = 1$. The initial matrix X_1 is chosen as the $(k-1)$th approximate solution of Algorithm PHR-AL.

Lemma 2. *Assume that X^* is limit point of a sequence generated by Algorithm PHR-AL and the sequence ρ_k is bounded, then we have*

$$L \le \mathcal{G}(X^*) \le U.$$

Proof. Let $\mathbb{K}$ be an infinite subset of $\mathbb{N}$ such that $\lim_{k\in\mathbb{K}} X^k = X^*$. Since $\lim_{k\to\infty} \rho_k = \infty$ when (3.14) does not hold, the boundedness of ρ_k implies that there exists $k_0 \in \mathbb{N}$ such that (3.14) takes place for all $k \ge k_0$. Therefore,

$$\lim_{k\in\mathbb{K}} \left\|\{\mathcal{G}(X^k) - L, Z_1^k\}_-\right\| = 0 \quad \text{and} \quad \lim_{k\in\mathbb{K}} \left\|\{U - \mathcal{G}(X^k), Z_2^k\}_-\right\| = 0.$$

Note that $Z_1^k \ge \mathbf{0}$ and $Z_2^k \ge \mathbf{0}$ for all $k \in \mathbb{N}$, we have

$$\lim_{k\in\mathbb{K}} (L - \mathcal{G}(X^k))_+ = \mathbf{0} \quad \text{and} \quad \lim_{k\in\mathbb{K}} (\mathcal{G}(X^k) - U)_+ = \mathbf{0},$$

that is, $\mathcal{G}(X^*) - L \ge \mathbf{0}$ and $U - \mathcal{G}(X^*) \ge \mathbf{0}$.

Lemma 3. *Assume that X^* is limit point of a sequence generated by Algorithm PHR-AL, then X^* is a first-order stationary point of the problem*

$$\text{minimize} \quad \frac{1}{2}\left\{\left\|(L - \mathcal{G}(X^*))_+\right\|^2 + \left\|(\mathcal{G}(X^*) - U)_+\right\|^2\right\} \quad \text{subject to } \mathrm{X} \in \mathcal{S}. \tag{3.16}$$

In other words, $X^* \in \mathcal{S}$ satisfies

$$P_{\mathcal{S}}\left\{\mathcal{G}^T\left((L - \mathcal{G}(X^*))_+ - (\mathcal{G}(X^*) - U)_+\right)\right\} = \mathbf{0}.$$

Proof. Let $\mathbb{K}$ be an infinite subset of $\mathbb{N}$ such that $\lim_{k\in\mathbb{K}} X^k = X^*$. Consider first the case in which the sequence ρ_k is bounded. By the proof of Lemma 2, we have that

$$\lim_{k\in\mathbb{K}} \left\|(L - \mathcal{G}(X^k))_+\right\| = 0 \quad \text{and} \quad \lim_{k\in\mathbb{K}} \left\|(\mathcal{G}(X^k) - U)_+\right\| = 0.$$

Note that

$$\left\|\mathcal{G}^T\left(\left(L - \mathcal{G}(X^*)\right)_+\right)\right\| \le \left\|\left(L - \mathcal{G}(X^*)\right)_+\right\| \quad \text{and} \quad \left\|\mathcal{G}^T\left(\left(\mathcal{G}(X^*) - U\right)_+\right)\right\| \le \left\|\left(\mathcal{G}(X^*) - U\right)_+\right\|,$$

we have that

$$\lim_{k\in\mathbb{K}} \left\|\mathcal{G}^T\left((L - \mathcal{G}(X^k))_+ - (\mathcal{G}(X^k) - U)_+\right)\right\| = 0.$$

Since $X^k \in \mathcal{S}$ for all k, this implies the desired result in the case that $\{\rho_k\}$ is bounded.

Assume now that $\{\rho_k\}$ is not bounded. Therefore there exists an infinite sequence of indices $\mathbb{K}' \subset \mathbb{K}$ such that $\lim_{k\in\mathbb{K}'} \rho_k = \infty$. Note that $\varepsilon_k \downarrow 0$ and $\left\| P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\right\}\right\| < \varepsilon_k$, we have

$$\lim_{k\in\mathbb{K}} \left\| P_{\mathcal{S}} \left\{ \mathcal{A}^T\left(\mathcal{A}(X^k) - C\right) - \rho_k \mathcal{G}^T \left(\left(L - \mathcal{G}(X^k) + \frac{\overline{Z}_1^k}{\rho_k}\right)_+ - \left(\mathcal{G}(X^k) - U + \frac{\overline{Z}_2^k}{\rho_k}\right)_+ \right) \right\} \right\| = 0.$$

Therefore we have

$$\lim_{k\in\mathbb{K}} \left\| P_{\mathcal{S}} \left\{ \mathcal{A}^T\left(\mathcal{A}(X^k) - C\right)\big/\rho_k - \mathcal{G}^T \left(\left(L - \mathcal{G}(X^k) + \frac{\overline{Z}_1^k}{\rho_k}\right)_+ - \left(\mathcal{G}(X^k) - U + \frac{\overline{Z}_2^k}{\rho_k}\right)_+ \right) \right\} \right\| = 0.$$

Since $\{X^k\}$, $\{\overline{Z}_1^k\}$ and $\{\overline{Z}_2^k\}$ are bounded, we obtain

$$\left\| P_{\mathcal{S}}\left\{\mathcal{G}^T \left((L - \mathcal{G}(X^*))_+ - (\mathcal{G}(X^*) - U)_+ \right)\right\}\right\| = 0.$$

This implies that X^* is a stationary point of (3.16).

Theorem 2. *Assume that X^* is limit point of a sequence generated by Algorithm PHR-AL and the sequence $\{\rho_k\}$ is bounded, then X^* is a solution to Problem* (1.1).

Proof. Let $\mathbb{K}$ be an infinite subset of $\mathbb{N}$ such that

$$\lim_{k\in\mathbb{K}} X^k = X^*, \quad \lim_{k\in\mathbb{K}} \rho_k = \rho^*, \quad \lim_{k\in\mathbb{K}} \overline{Z}_1^k = \overline{Z}_1^* \quad \text{and} \quad \lim_{k\in\mathbb{K}} \overline{Z}_2^k = \overline{Z}_2^*.$$

By Lemma 2, we have $L \leq \mathcal{G}(X^*) \leq U$. Since

$$\left\| P_{\mathcal{S}}\left\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\right\}\right\| < \varepsilon_k$$

holds for all $\varepsilon_k \downarrow 0$, we have

$$\left\| P_{\mathcal{S}}\left\{\nabla L_{\rho^*}(X^*, \overline{Z}_1^*, \overline{Z}_2^*)\right\}\right\| = 0. \tag{3.17}$$

Let

$$Y_1^* = \rho^* \left(L - \mathcal{G}(X^*) + \overline{Z}_1^*/\rho^*\right)_+ \quad \text{and} \quad Y_2^* = \rho^* \left(\mathcal{G}(X^*) - U + \overline{Z}_2^*/\rho^*\right)_+,$$

then $Y_1^* \geq \mathbf{0}$ and $Y_2^* \geq \mathbf{0}$, and, from (3.17), we have

$$P_{\mathcal{S}}\left\{\mathcal{A}^T\left(\mathcal{A}(X^*) - C\right) - \mathcal{G}^T(Y^* - Z^*)\right\} = \mathbf{0}.$$

Since $\{\rho_k\}$ is bounded, then there exists $k_0 \in \mathbb{N}$ such that (3.14) takes place for all $k \geq k_0$. Hence, we have

$$\lim_{k\to\infty} \{\mathcal{G}(X^k) - L,\ Z_1^k\}_- = \{\mathcal{G}(X^*) - L,\ Z_1^*\}_- = \mathbf{0}$$

and

$$\lim_{k\to\infty} \{U - \mathcal{G}(X^k),\ Z_2^k\}_- = \{U - \mathcal{G}(X^*),\ Z_2^*\}_- = \mathbf{0},$$

which imply that $\langle \mathcal{G}(X^*) - L,\ Z_1^* \rangle = 0$ and $\langle U - \mathcal{G}(X^*),\ Z_2^* \rangle = 0$. By the definition of Z_1^k, Z_2^k and Y_1^*, Y_2^* we know that $(Z_1^*)_{ij} > 0$ if and only if $(Y_1^*)_{ij} > 0$ and $(Z_2^*)_{ij} > 0$ if and only if $(Y_2^*)_{ij} > 0$ ($i = 1, 2, \ldots, l_2$, $j = 1, 2, \ldots, s_2$). So we have $\langle \mathcal{G}(X^*) - L, Y_1^* \rangle = 0$ and $\langle U - \mathcal{G}(X^*), Y_2^* \rangle = 0$. Hence X^* satisfies conditions (3.11). By Theorem 1, we know that X^* is a solution to Problem (1.1).

4. Numerical examples

In this section, we first report some numerical results when Algorithm PHR-AL is implemented to solve Problem (1.1) with random data, and then we illustrate the applicability when the algorithm is applied to solve the model (2.10) in image restoration. All the tested algorithms were coded by MATLAB 7.8 (R2009a) and all our computational experiments were run on a personal computer with an Intel(R) Core i3 processor at 2.13 GHz with 2.00 GB of memory.

4.1. Tested with random data

In this example, we test the two linear operators as $\mathcal{A}(X) = A_1XB_1 + A_2XB_2$ and $\mathcal{G}(X) = E_1XF_1$, and $\mathcal{S}$ as the set of all real $m \times n$ rectangular centrosymmetric matrices[31].

Example 1. Given the matrices $A_1, B_1, A_2, B_2, E_1, F_1, C, L$ and U in Matlab style as follows:

$$\begin{aligned}
&A_1 = randn(l_1, m), \;\; B_1 = randn(n, s_1), \;\; A_2 = randn(l_1, m), \;\; B_2 = randn(n, s_1),\\
&E_1 = rand(l_2, m), \;\; F_1 = rand(n, s_2), \;\; C = A_1\overline{X}B_1 + A_2\overline{X}B_2,\\
&L = E_1\overline{X}F_1 - 10 * ones(l_2, s_2), \;\; U = E_1\overline{X}F_1 + 10 * ones(l_2, s_2),
\end{aligned}$$

where $\overline{X} = Z + S_mZS_n$ with $Z = rand(m, n)$. Matrices L, U and C are chosen in this way to guarantee that Problem (1.1) is solvable.

Note that the Algorithm PHR-AL involve an outer iteration and an inner iteration, the convergence stopping criterion of the outer iterations are all set to be $\varepsilon = 10^{-8}$, and the small tolerance ε_k in the inner iterations is set to

$$\varepsilon_0 = 10^0 \quad \text{and} \quad \varepsilon_k = \begin{cases} 0.1\varepsilon_{k-1} & \text{if } \varepsilon_{k-1} > \varepsilon, \\ \varepsilon_{k-1} & \text{if } \varepsilon_{k-1} < \varepsilon. \end{cases} \tag{4.18}$$

The largest number of the inner iteration is set to be 200. We consider the following two cases to be tested: (a) $l_1 \geq m$ and $s_1 \geq n$ and (b) $l_1 < m$ and $s_1 < n$.

Table 2: Numerical results for the case (a) $l_1 \geq m$ and $s_1 \geq n$ in Example 1.

l_1, m, n, s_1, l_2, s_2	CPU	$\frac{\|X^*-\overline{X}\|}{\|\overline{X}\|}$
10,10,10,10,10,10	0.1248	5.1294×10^{-11}
30,18,20,30,25,30	0.3588	3.4006×10^{-13}
50,50,50,50,50,50	3.4476	1.2540×10^{-12}
80,60,70,100,80,80	4.0404	6.7827×10^{-14}
100,100,100,100,100,100	13.3537	6.7580×10^{-14}
150,100,100,150,120,120	10.1401	4.8226×10^{-15}
150,150,150,150,150,150	44.2263	4.7307×10^{-14}
200,180,180,200,150,150	53.3367	1.2976×10^{-14}
250,250,250,250,200,200	161.7106	1.1052×10^{-13}

For case $l_1 \geq m$ and $s_1 \geq n$, Problem (1.1) has unique solution and the true solution is $\overline{X}$. Therefore in Table 2, we report the mean computing time in seconds and the mean relative error based on their average values of 10 repeated tests with randomly generated matrices A_1, B_1, A_2, B_2, E_1 and F_1 for each problem size. Here the relative error is defined as $Re = \frac{\|X^*-\overline{X}\|}{\|\overline{X}\|}$, where X^* is the estimated solution.

For case $l < n$ and $s < n$, as Problem(1.1) has multiple solutions, the algorithm is not guaranteed to converge to the solution $\overline{X}$, it is not meaningful to record the relative errors. In this case, we report the mean

Table 3: Numerical results for the case (b)$l_1 < m$ and $s_1 < n$ in Example 1.

l_1, m, n, s_1, l_2, s_2	CPU	$\|A_1XB_1 + A_2XB_2 - C\|$
6,10,10,6,10,10	0.1560	9.3373×10^{-9}
15,30,25,15,20,20	0.7644	1.4437×10^{-9}
30,60,75,35,50,50	4.2432	3.2598×10^{-10}
50,120,125,65,80,80	17.6749	2.6637×10^{-10}
50,200,200,50,100,100	43.9299	7.1933×10^{-11}
70,150,150,70,120,120	44.9595	1.7993×10^{-10}
100,200,200,100,150,150	132.8817	1.7718×10^{-10}
100,300,300,100,180,180	348.3970	5.1966×10^{-11}

computing time in seconds and the mean residual $\|A_1XB_1 + A_2XB_2 - C\|$ (see Table 3) based on 10 repeated tests with randomly generated matrices A, B, E and F for each problem size in each test.

4.2. Application to image restoration with some special symmetry pattern images

In this subsection, we test the efficiency when Algorithm PHR-AL is applied to solve the model (2.10) in image restoration. We only focus on some special symmetry pattern images. The original image is denoted by $\widehat{X}$ in each example and it consists of $m \times n$ grayscale pixel values in the range $[0, d]$ with $d = 255$ is the maximum possible pixel value of the image. Let $\hat{x} = vec(\widehat{X})$ denotes the vector obtained by stacking the columns of $\widehat{X}$ and H represents the blurring matrix. The vector $\hat{g} = H\hat{x}$ represents the associated blurred and noise-free image. In our tests, similarly to [24], we generated a blurred and noisy image g by

$$g = \hat{g} + \mathbf{n}_0 \times \sigma_{\hat{x}} \times 10^{-\frac{SNR}{20}},$$

where $\mathbf{n}_0$ is a random vector noise with a zero mean and a variance equal to one, and SNR is the signal to noise ratio defined by

$$SNR = 10\log_{10}\Big(\frac{\sigma_{\hat{x}}^2}{\sigma_{\mathbf{n}}^2}\Big),$$

where $\sigma_{\hat{x}}^2$ and $\sigma_{\mathbf{n}}^2$ are the variance of the noise and the original image, respectively. The performance of the Algorithm PHR-AL and its comparison are evaluated by the peak signal-to-noise ratio (PSNR) in decible (dB):

$$PSNR(X) = 10\log_{10}\Big(\frac{d^2mn}{\|\hat{x} - x\|_2^2}\Big) = 10\log_{10}\Big(\frac{d^2mn}{\|\widehat{X} - X\|^2}\Big).$$

In all the tests, the largest number of the involved inner iteration(Algorithm SPG) in the Algorithm PHR-AL is set to be 20. The algorithm started with the degraded images and terminated when the relative difference between the successive iterates of the restored image satisfy

$$R_{error} = \frac{\|X^{k+1} - X^k\|}{\|X^k\|} \leq 0.5 \times 10^{-2}.$$

Example 2. In the first example, we consider the *"butterfly"* original image of size 192×254 and is shown on the left side of Figure 1. The original image has perfectly mirror-symmetry[32], that is, the pixel value

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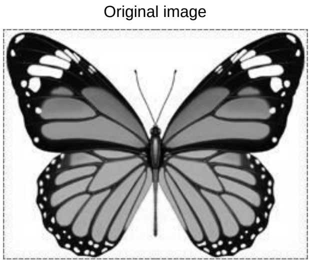

Figure 1: Original images. Left: "*Butterfly*"(mirror-symmetric). Right: "*PlayCard-K-Heart*" (centro-symmetric).

matrix $\widehat{X}$ can be expressed as $\widehat{X} = (X_L, X_L S_n)$, where X_L is the left half of the matrix $\widehat{X}$. Actually, we have $\|\widehat{X} - \mathcal{P}_{\mathcal{S}}(\widehat{X})\| = 0$, where $\mathcal{S}$ is the set of all real 192×254 column mirror-symmetry matrices and

$$\mathcal{P}_{\mathcal{S}}(X) = \left(\frac{X_L + X_R S_n}{2}, \ \frac{X_L S_n + X_R}{2}\right), \quad \forall X \in \mathbb{R}^{192\times 254}$$

where X_R is the left half and the right half of X. The blurring matrix H is chosen to be $H = H_1 \otimes H_2 \in \mathbb{R}^{192^2\times 254^2}$, where $H_1 = [h^{(1)}_{ij}] \in \mathbb{R}^{192\times 192}$ and $H_2 = [h^{(2)}_{ij}] \in \mathbb{R}^{254\times 254}$ are the Toeplitz matrices whose entries are given by

$$h^{(1)}_{ij} = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(i-j)^2}{2\sigma^2}\right), & |i-j| \le r, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad h^{(2)}_{ij} = \begin{cases} \frac{1}{2r-1}, & |i-j| \le r, \\ 0, & \text{otherwise.} \end{cases}$$

In this example we choose the band $r = 3$ and the variance $\sigma = 0.4$. A random Gaussian noise, with $SNR = 15dB$, was added to produce a blurred and noisy image G with $PSNR(G) = 8.1411$. The blurred and noisy image is shown on the left side of Figure 4. The restoration of the image from the degraded image is obtained by solving the minimization problem (2.10) using the PHR-AL algorithm. The regularization matrix T is chosen to be $T = T_1 \otimes T_2 \in \mathbb{R}^{192^2\times 254^2}$, where $T_1 = I_{192}$ and T_2 is the tridiagonal matrix, of size 254×254, generated by vector $(1, 2, 1)$. The optimal value of the parameter $\lambda = 0.015$ was obtained by using the GCV method. The corresponding GCV curve is plotted on the right side of Figure 2.

The restored image obtained by using Algorithm PHR-AL is given on the left of Figure 4, the relative error was $Re(X) = 1.2521 \times 10^{-1}$ with $PSNR(X) = 21.0231$, and the iterations are terminated after 3 iterations with a cpu time of 13.9309 s. Table 1 reports on more results for three levels of noise corresponding to different $SNR = 5, 10, 15$ and to different values of $\sigma = 0.35, 0.55, 0.85$ given in the definition of the blurring matrices H_1 and H_2 in Example 2.

Example 3. In the second example, the original image is the "*PlayCard-K-Heart*" image of size 628×423 and is shown on the right side of Figure 1. The original image is centrosymmetric, that is, the pixel value

Table 4: *Results for Example 3.*

σ	$SNR(dB)$	λ_{opt}	$PSNR(G)(dB)$	$PSNR(X)$(dB)	$Re(X)$	CPU-times(s)
	5	0.036	5.3075	19.6357	1.4690×10^{-1}	23.4002
	10	0.025	6.0042	20.9344	1.2650×10^{-1}	17.8621
0.35	15	0.017	6.4097	21.3394	1.2073×10^{-1}	18.0337
	20	0.011	6.6397	21.6077	1.1706×10^{-1}	18.3145
	25	0.007	6.7709	21.9395	1.1267×10^{-1}	19.5781
	5	0.036	8.3142	18.7410	1.6284×10^{-1}	29.6090
	10	0.025	9.3290	21.1153	1.2389×10^{-1}	40.3419
0.55	15	0.018	9.9286	21.8547	1.1378×10^{-1}	38.4386
	20	0.012	10.2655	21.9397	1.1267×10^{-1}	28.2830
	25	0.008	10.4569	21.1417	1.2351×10^{-1}	18.8137
	5	0.035	8.4387	18.5387	1.6667×10^{-1}	38.4542
	10	0.026	9.4712	20.7428	1.2932×10^{-1}	39.1875
0.85	15	0.019	10.0763	20.9952	1.2561×10^{-1}	27.9086
	20	0.014	10.4170	20.5296	1.3253×10^{-1}	12.9949
	25	0.010	10.6154	20.7946	1.2855×10^{-1}	18.8137

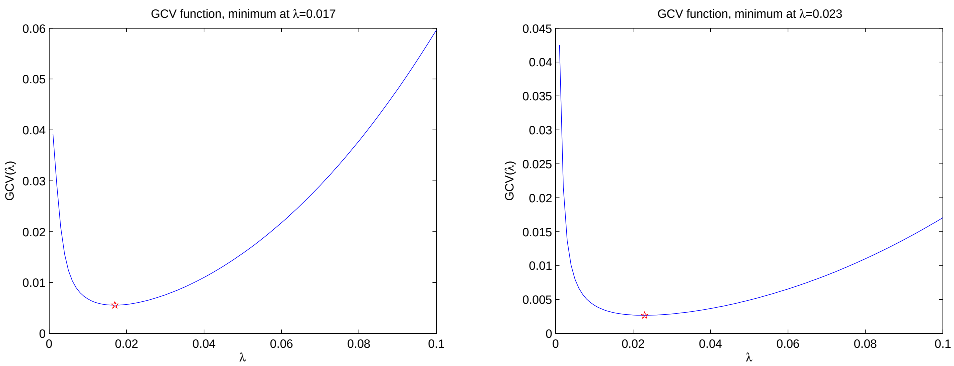

Figure 2: The GCV curve for the Example 2 with the optimal value of $\lambda = 0.017$ (left) and the GCV curve for the Example 3 with the optimal value of $\lambda = 0.023$.

Figure 3: The blurred and noisy image (left) with $PSNR(G) = 8.1411$, $r = 3$ and $\sigma = 0.45$ and the restored image (right) with $PSNR(X) = 21.0231$ and $Re(X) = 1.2521 \times 10^{-1}$.

matrix $\widehat{X}$ satisfies $\widehat{X} = S_{628}\widehat{X}S_{423}$. Actually, we have $\|\widehat{X}-P_{\mathcal{S}}(\widehat{X})\| = 0$, where $\mathcal{S}$ is the set of all real 628×423 rectangle centrosymmetry matrices and $P_{\mathcal{S}} = \frac{1}{2}(X + S_{628}XS_{423})$ for any $X \in \mathbb{R}^{628\times423}$. The blurring matrix H is chosen to be $H = H_1 \otimes H_2 \in \mathbb{R}^{256^2\times256^2}$, where $H_1 = I_{628}$ is the identity matrix and $H_2 = [h_{ij}^{(2)}]$ is the Toeplitz matrices of dimension 423×423 given by

$$h_{ij}^{(2)} = \begin{cases} \frac{1}{2r-1}, & |i-j| \le r, \\ 0, & \text{otherwise.} \end{cases}$$

The blurring matrix H models a uniform blur. The regularization matrix T is chosen to be $T = T_1 \otimes T_2 \in$

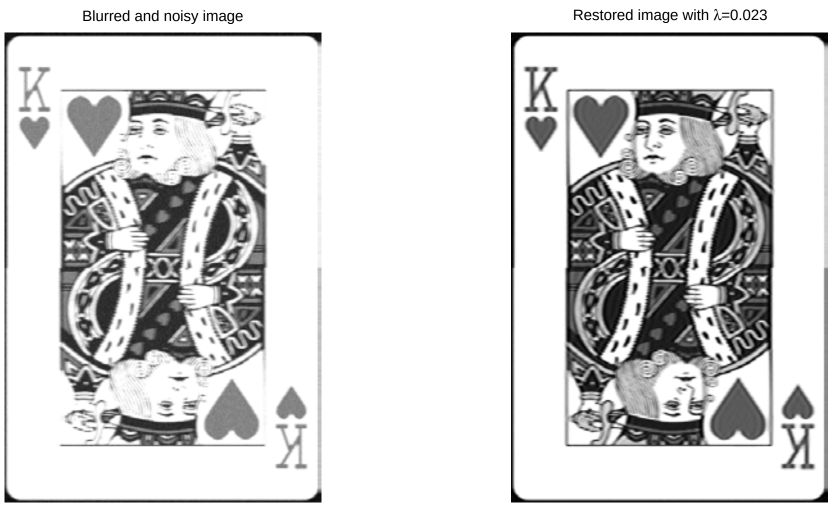

Figure 4: The blurred and noisy image (left) with $PSNR(G) = 8.0481$, $r = 3$ and $\sigma = 0.45$ and the restored image (right) with $PSNR(X) = 20.1459$ and $Re(X) = 1.5784 \times 10^{-1}$.

$\mathbb{R}^{256^2\times256^2}$, where T_1 and T_2 are similar to the ones given in Example 2. In this example we set $r = 3$ and a random Gaussian noise, with $SNR = 15dB$, was added to produce a blurred and noisy image G with $PSNR(G) = 8.0481$. The obtained image is shown on the middle of Figure 2. The optimal value of the parameter $\lambda = 0.023$ was obtained by using the GCV method. The corresponding GCV curve is plotted on the right side of Figure 2.

The restored image obtained by using our proposed Algorithm PHR-AL is also denoted by X and it is given on the right side of Figure 4. The relative error was $Re(X) = 1.5784 \times 10^{-1}$ with the $PSNR(X) = 20.1459$. The iterations are terminated after 5 iterations with a cpu time of 86.9699s.

5. Conclusion

In this paper, we consider solving a class of inequality constrained matrix-form minimization problems, whose various simplified versions have been studied extensively. These matrix-form minimization problems problem can be transformed into the convex linearly constrained quadratic programming in the vector-form by using the vec operator $vec(.)$ and the Kronecker produc $\otimes$. However, using this transformation will destroy the preindicated linear structure of the unknown matrix and will increase computational complexity and storage requirement. In this paper we will consider the problem from a general point of view and

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directly from the perspective of matrices. We propose, analyze and test a matrix-form iteration algorithm framework with the augmented Lagrangian method for solving this problem and its reduced versions which are applicable in image restoration. The numerical results, including when the algorithm is tested with some randomly generated data and on some image restoration problems with special symmetry pattern images, illustrate the effectiveness of the proposed algorithm.

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Volume 23, Number 6 November 15, 2017
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

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JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

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Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
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Memphis, TN 38152-3240, U.S.A.

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Some properties on non-admissible and admissible functions sharing some sets in the unit disc *

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Abstract

In this paper, we deal with the uniqueness problem of two non-admissible functions sharing some values and sets in the unit disc, and also investigate the problem on an admissible function and a non-admissible function sharing some values and sets. Some theorems of this paper improve the results given by Fang. In addition, the results in this paper analogous version of the uniqueness theorems of meromorphic functions sharing some sets on the whole complex plane which given by Yi and Cao.

Key words: uniqueness; meromorphic function; admissible; non-admissible.
Mathematical Subject Classification (2010): Primary 30D 35.

1 Introduction and main results

We should assume that reader is familiar with the basic results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see Hayman [6] , Yang [14] and Yi and Yang [18]). For a meromorphic function f, we use $S(r,f)$ to denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside a possible exceptional set of finite logarithmic measure, and use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}}:=\mathbb{C}\bigcup\{\infty\}$ to denote the extended complex plane, and $\mathbb{D}=\{z:|z|<1\}$ to denote the unit disc.

R. Nevanlinna [10] proved the following well-known theorems.

Theorem 1.1 (see [10]) *If f and g are two non-constant meromorphic functions that share five distinct values a_1,a_2,a_3,a_4,a_5 IM in $\mathbb{C}$, then $f(z)\equiv g(z)$.*

After this work, the uniqueness of meromorphic functions with shared sets and values attracted many investigations (see [18]). Moreover, the uniqueness theory of meromorphic functions is an important subject in the value distribution theory. In this paper, we mainly investigate the uniqueness of meromorphic functions with slow growth sharing some sets in the unit disc.

We firstly introduce the following basic notations and definitions of meromorphic functions in $\mathbb{D}$(see [2, 4, 7, 12, 8, 13, 22]).

Definition 1.1 *(see [12]). Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to1^-}T(r,f)=\infty$. Then*

$$D(f):=\limsup_{r\to1^-}\frac{T(r,f)}{-\log(1-r)}$$

is called the (upper) index of inadmissibility of f. If $D(f)=\infty$, f is called admissible.

*This work was supported by the NSF of China (11561033), the Natural Science Foundation of Jiangxi Province in China (20151BAB201008), and the Foundation of Education Department of Jiangxi of China (GJJ150902).

Definition 1.2 *(see [12]). Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r,f)=\infty$. Then*

$$\rho(f) := \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}$$

is called the order (of growth) of f.

The Second Main Theorem for admissible functions (see [12, Theorem 3]) is very important in studying the uniqueness of two admissible functions in the unit disc $\mathbb{D}$, which was proved by in 2005.

Theorem 1.2 (see [12, Theorem 3]). *Let f be an admissible meromorphic function in $\mathbb{D}$, q be a positive integer and $a_1, a_2, \ldots, a_q$ be pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q-2)T(r,f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

where $E \subset (0,1)$ is a possibly occurring exceptional set with $\int_E \frac{dr}{1-r} < \infty$. If the order of f is finite, the remainder $S(r,f)$ is a $O\left(\log \frac{1}{1-r}\right)$ without any exceptional set.

In 2005, Titzhoff [12] also obtained the five values theorem for admissible functions in the unit disc $\mathbb{D}$ as follows.

Theorem 1.3 (see [5, 12]). *If two admissible functions f, g share five distinct values, then $f \equiv g$.*

From Theorem 1.2(see [12, Theorem 3]), we can easily obtain a lot of theorems similar to meromorphic functions in the complex plane. In 1999, Fang [5] investigated the uniqueness of admissible functions sharing two sets and three sets and obtained a series of theorems. In 2015, Xu, Yang and Cao [15] investigated the problem on shared values of admissible function and non-admissible function, and obtained some interesting results. Inspired by Xu, Yang and Cao [15] and Fang[5], we further study the problem on shared-sets of admissible function and non-admissible function in the unit disc.

The following theorem also plays a very important role in studies non-admissible functions sharing some sets in this paper.

Theorem 1.4 (see [12, Theorem 2]). *Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r,f) = \infty$, q be a positive integer and $a_1, a_2, \ldots, a_q$ be pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q-2)T(r,f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + \log\frac{1}{1-r} + S(r,f).$$

Remark 1.1 *In contrast to admissible functions, the term $\log \frac{1}{1-r}$ in Theorem 1.4 does not necessarily enter the remainder $S(r,f)$ because the non-admissible function f may have $T(r,f) = O\left(\log \frac{1}{1-r}\right)$.*

Remark 1.2 *We can see that $S(r,f) = o\left(\log \frac{1}{1-r}\right)$ holds in Theorem 1.4 without a possible exception set when $0 < D(f) < \infty$.*

The following lemma for non-admissible functions in the unit disc is used in this paper.

Lemma 1.1 *(see [15]). Let $f(z)$ be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r,f)=\infty$, $a_j(j=1,2,\ldots,q)$ be q distinct complex numbers, and $k_j(j=1,2,\ldots,q)$ be positive integers or ∞. If f is a non-admissible function, then*

$$(q-2)T(r,f) < \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f-a_j}\right)+\sum_{j=1}^{q}\frac{1}{k_j+1}N\left(r,\frac{1}{f-a_j}\right)$$
$$+\log\frac{1}{1-r}+S(r,f),$$

and

$$\left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)T(r,f)\le\sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f-a_j}\right)+\log\frac{1}{1-r}+S(r,f),$$

where $\overline{n}_{k)}(r,\frac{1}{f-a})$ is used to denote the zeros of $f-a$ in $|z|\le r$, whose multiplicities are no greater than k and are counted only once, $\overline{N}_{k)}(r,\frac{1}{f-a})$ is the corresponding counting functions, and $\frac{k_j}{k_j+1}=1, \overline{N}_{k_j)}(r,\frac{1}{f-a_j})=\overline{N}(r,\frac{1}{f-a_j})$ and $\frac{1}{k_j+1}=0$ if $k_j=\infty$, $S(r,f)$ is stated as in Theorem 1.2.

The main purpose of this paper is to deal with the problem of two non-admissible functions sharing some sets, and an admissible function sharing some sets with an non-admissible function. *Section 2, the uniqueness of two non-admissible functions sharing some sets in $\mathbb{D}$ are investigated and some results showed that the number and weight of sharing sets is related with the index of inadmissibility of functions in $\mathbb{D}$. In section 3, the problem of an admissible function and a non-admissible function sharing some sets is studied, and one of those results shows that admissible function and non-admissible function can share at most five distinct values with reduced weighted 1.*

2 The uniqueness and sharing sets of non-admissible functions in the unit disc

Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\mathbb{X}\subseteq\mathbb{C}$. Define

$$E(S,\mathbb{D},f)=\bigcup_{a\in S}\{z\in\mathbb{D}|f_a(z)=0,\quad \textit{counting}\quad \textit{multiplicities}\},$$

$$\overline{E}(S,\mathbb{D},f)=\bigcup_{a\in S}\{z\in\mathbb{D}|f_a(z)=0,\quad \textit{ignoring}\quad \textit{multiplicities}\},$$

where $f_a(z)=f(z)-a$ if $a\in\mathbb{C}$ and $f_\infty(z)=1/f(z)$.

For two non-constant meromorphic functions f,g, we say f and g share the set S CM(counting multiplicities) in $\mathbb{D}$ if $E(S,\mathbb{D},f)=E(S,\mathbb{D},g)$; we say f and g share the set S IM(ignoring multiplicities) in $\mathbb{D}$ if $\overline{E}(S,\mathbb{D},f)=\overline{E}(S,\mathbb{D},g)$. In particular, as $S=\{a\}$ and $a\in\widehat{\mathbb{C}}$, we say f and g share the value a CM in $\mathbb{D}$ if $E(a,\mathbb{D},f)=E(a,\mathbb{D},g)$, and we say f and g share the value a IM in $\mathbb{D}$ if $\overline{E}(a,\mathbb{D},f)=\overline{E}(a,\mathbb{D},g)$. We use $\overline{E}_{k)}(a,\mathbb{D},f)$ to denote the set of zeros of $f-a$ in $\mathbb{D}$, with multiplicities no greater than k, in which each zero counted only once. We say that $f(z)$ and $g(z)$ share the value a with reduced weight k in $\mathbb{D}$, if $\overline{E}_{k)}(a,\mathbb{D},f)=\overline{E}_{k)}(a,\mathbb{D},g)$. If $\mathbb{D}=\mathbb{C}$, we have the simple notation as before, $E(S,f),\overline{E}(S,f),\overline{E}_{k)}(a,f)$ and so on(see [18]).

The deficiency of $a\in\widehat{\mathbb{C}}$ with respect to a meromorphic function f on the unit disc $\mathbb{D}$ is defined by

$$\delta(a,f)=\delta(0,f-a)=\liminf_{r\to 1^-}\frac{m(r,\frac{1}{f-a})}{T(r,f)}=1-\limsup_{r\to 1^-}\frac{N(r,\frac{1}{f-a})}{T(r,f)},$$

and the reduced deficiency by

$$\Theta(a,f)=\Theta(0,f-a)=1-\limsup_{r\to 1^-}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}.$$

We now show our main theorems. The first theorem can be called five values theorem of non-admissible functions.

Theorem 2.1 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $1<D(f_1),D(f_2)<\infty$, and f_1,f_2 share $a_j (j=1,2,3,4,5)$ IM. Then $f_1(z)\equiv f_2(z)$.*

Remark 2.1 *From Theorem 2.1, we can get that $f_1(z)\equiv f_2(z)$ if f_1,f_2 share five distinct values and $D(f_1),D(f_2)>1$. However, the conclusion holds in Theorem 1.3 under the condition which f_1,f_2 are admissible functions, that is, $D(f_1)=\infty$, and $D(f_2)=\infty$. Thus, we can see that Theorem 2.1 is a greatly improvement of Theorem 1.3.*

In order to prove Theorem 2.1, we will prove the following general results of two non-admissible functions sharing some sets.

Theorem 2.2 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0<D(f_1),D(f_2)<\infty$. Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b\neq 0$, $S_i\cap S_j=\emptyset$, $(i\neq j)$ and $q>2+\max\left\{\left[\frac{1}{D(f_1)}\right],\left[\frac{1}{D(f_2)}\right]\right\}$, where $[x]$ denotes the largest integer less than or equal to x. Let k_j $(j=1,2,\ldots,q)$ be positive integers or ∞ satisfying

$$k_1\geq k_2\geq\cdots\geq k_q \tag{1}$$

and

$$\overline{E}_{k_j)}(S_j,\mathbb{D},f_1)=\overline{E}_{k_j)}(S_j,\mathbb{D},f_2),\quad (j=1,2,\ldots,q). \tag{2}$$

Furthermore, let

$$\Theta(f_i)=\sum_a\Theta(0,f_i-a)-\sum_{j=1}^{q}\sum_{s=0}^{l-1}\Theta(0,f_i-(a_j+sb)),(i=1,2),$$

and

$$\begin{aligned}A_1 &= \frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0,f_1-(a_j+sb))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_1-(a_j+sb))}{k_j+1}\\ &\quad+\frac{(lm-3l+1)k_m}{k_m+1}-\frac{(2l-1)k_n}{k_n+1}+\Theta(f_1)-2,\end{aligned}$$

$$\begin{aligned}A_2 &= \frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0,f_2-(a_j+sb))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_2-(a_j+sb))}{k_j+1}\\ &\quad+\frac{(ln-3l+1)k_n}{k_n+1}-\frac{(2l-1)k_m}{k_m+1}+\Theta(f_2)-2,\end{aligned}$$

where m and n are positive integers in $\{1,2,\ldots,q\}$ and a is an arbitrary complex number or ∞. If

$$\min\{A_1,A_2\}\geq\frac{2}{D(f_1)+D(f_2)},\quad and\quad \max\{A_1,A_2\}>\frac{2}{D(f_1)+D(f_2)}. \tag{3}$$

Then $f_1(z)\equiv f_2(z)$.

By letting $l = 1$, $q = 5$ and $k_1 = k_2 = \cdots = k_5 = \infty$ in Theorem 2.2, we can get Theorem 2.1 easily. Now, we start to prove Theorem 2.2 as follows.

Proof of Theorem 2.2: Suppose that $f_1(z) \not\equiv f_2(z)$. From the second fundamental theorem in the unit disc (Theorem 1.4) we have

$$\begin{aligned}(ql+p-2)T(r,f_1) \quad &< \quad \sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}\left(r,\frac{1}{f_1-(a_j+sb)}\right)+\sum_{k=1}^{p}\overline{N}\left(r,\frac{1}{f_1-d_k}\right)\\ &\quad +\log\frac{1}{1-r}+S(r,f_1).\end{aligned}$$

By definition we have

$$\overline{N}\left(r,\frac{1}{f_1-d_k}\right) < (1-\Theta(0,f_1-d_k))\,T(r,f_1)+S(r,f_1).$$

From Lemma 1.1 and the definition of deficiency, it follows that for $s \in \{0,1,\ldots,l-1\}$

$$\begin{aligned}&\overline{N}\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\ \leq\ &\frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)+\frac{1}{k_j+1}N\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\ <\ &\frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)+\frac{1}{k_j+1}\left(1-\delta(0,f_1-(a_j+sb))\right)T(r,f_1)\\ &+S(r,f_1).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&(ql+p-2)T(r,f_1)\\ <\ &\left\{\sum_{k=1}^{p}(1-\Theta(0,f_1-d_k))\right\}T(r,f_1)+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_j}{k_j+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(a_j+sb)})\\ &+\left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1}{k_j+1}\left(1-\delta(0,f_1-(a_j+sb))\right)\right\}T(r,f_1)+\log\frac{1}{1-r}+S(r,f_1).\end{aligned}$$

Since $\Theta(0, f-a) \geq 0$ for any meromorphic function f and any complex number $a \in \widehat{\mathbb{C}}$. Without loss of generality, we assume that there exist infinitely many d such that $\Theta(0, f_1 - d) > 0$ and $d \notin \{a_j + sb : j = 1, 2, \ldots, q \text{ and } s = 0, 1, \ldots, l-1\}$. We denote them by d_k $(k = 1, 2, \ldots, \infty)$. Obviously, $\Theta(f_1) = \Sigma_{k=1}^{\infty}\Theta(0, f_1-d_k)$. Thus there exits a p such that $\Sigma_{k=1}^{p}\Theta(0, f_1-d_k) > \Theta(f_1)-\varepsilon$ holds for any given ε (> 0). Noting that

$$1 \geq \frac{k_1}{k_1+1} \geq \frac{k_2}{k_2+1} \geq \cdots \geq \frac{k_q}{k_q+1} \geq \frac{1}{2},$$

we can deduce that

$$\begin{aligned}&(ql+p-2)T(r,f_1)\\ <\ &(p-\Theta(f_1)+\varepsilon)\,T(r,f_1)+\frac{k_m}{k_m+1}\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\ &+\left\{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\left(\frac{k_j}{k_j+1}-\frac{k_m}{k_m+1}\right)\left(1-\delta(0,f_1-(a_j+sb))\right)\right\}T(r,f_1)\\ &+\left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1-\delta(0,f_1-(a_j+sb))}{k_j+1}\right\}T(r,f_1)+\log\frac{1}{1-r},\end{aligned}$$

namely,

$$\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(a_j+sb)})$$
$$+\log\frac{1}{1-r},$$

where

$$B_1=\frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0,f_1-(a_j+sb))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_1-(a_j+sb))}{k_j+1}+\Theta(f_1)-2.$$

By a similar discussion as above, we also have

$$\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T(r,f_2)<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{k_j)}\left(r,\frac{1}{f_2-(a_j+sb)}\right)+\log\frac{1}{1-r},$$

where

$$B_2=\frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0,f_2-(a_j+sb))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_2-(a_j+sb))}{k_j+1}+\Theta(f_2)-2.$$

Hence

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T(r,f_2)\\
<\;&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(a_j+sb)})+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{k_j)}(r,\frac{1}{f_2-(a_j+sb)})\\
&+2\log\frac{1}{1-r}.\end{aligned}$$

We now assert that $f_1(z)-f_2(z)\not\equiv sb$, $s=1,2,\ldots,l-1$. Otherwise, we get that a_j $(j=1,2,\ldots,q)$ are the Picard exceptional values of f_1, and that $a_j+(l-1)b$ $(j=1,2,\ldots,q)$ are the Picard exceptional values of f_2. By $q>2+\frac{1}{D(f_1)}$ and Theorem 1.4, we get a contradiction. Similarly, we have $f_2(z)-f_1(z)\not\equiv sb$, $s=1,2,\ldots,l-1$.

By condition (2) and the first fundamental theorem, we have

$$\begin{aligned}&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\
&\le\overline{N}\left(r,\frac{1}{f_1-f_2}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_1-f_2-sb}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_2-f_1-sb}\right)\\
&\le(2l-1)(T(r,f_1)+T(r,f_2))+O(1).\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}\left(r,\frac{1}{f_2-(a_j+sb)}\right)\\
&\le\overline{N}\left(r,\frac{1}{f_1-f_2}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_1-f_2-sb}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_2-f_1-sb}\right)\\
&\le(2l-1)(T(r,f_1)+T(r,f_2))+O(1).\end{aligned}$$

Therefore, from the above discussion we obtain

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T(r,f_2)\\ <\ &(2l-1)\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(T(r,f_1)+T(r,f_2))+2\log\frac{1}{1-r},\end{aligned}$$

namely,

$$(A_1-\varepsilon)\,T(r,f_1)+(A_2-\varepsilon)\,T(r,f_2)\le 2\log\frac{1}{1-r}. \tag{4}$$

Since $0<D(f_1),D(f_2)<\infty$, we have $S(r,f_1)=o\left(\log\frac{1}{1-r}\right), S(r,f_2)=o\left(\log\frac{1}{1-r}\right)$. And from the definition of index, for any ε satisfying

$$0<2\varepsilon<\min\left\{D(f_1),D(f_2),\max\{A_1,A_2\}-\frac{2}{D(f_1)+D(f_2)}\right\}, \tag{5}$$

there exists a sequence $\{r_t\}\to 1^-$ such that

$$T(r_t,f_1)>(D(f_1)-\varepsilon)\log\frac{1}{1-r_t},\quad T(r_t,f_2)>(D(f_2)-\varepsilon)\log\frac{1}{1-r_t}, \tag{6}$$

for all $t\to\infty$. From (4)-(6), we have

$$[(D(f_1)-\varepsilon)(A_1-\varepsilon)+(D(f_2)-\varepsilon)(A_2-\varepsilon)-2]\log\frac{1}{1-r_t}<o\left(\log\frac{1}{1-r_t}\right). \tag{7}$$

From (7) and ε being arbitrary, the above inequality contradicts to (3). Therefore, the proof of Theorem 2.2 is completed.

We can get the following corollaries from Theorem 2.2.

Corollary 2.1 *Let k_j $(j=1,2,\ldots,q)$ be positive integers or ∞ satisfying (1), and let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0<D(f_1),D(f_2)<\infty$ and (2). Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b\neq 0$, $S_i\cap S_j=\emptyset$, $(i\neq j)$ and $q>2+\max\left\{\left[\frac{1}{D(f_1)}\right],\left[\frac{1}{D(f_2)}\right]\right\}$, where $[x]$ denotes the largest integer less than or equal to x. If

$$\sum_{j=3}^{q}\sum_{s=0}^{l-1}\frac{k_j}{k_j+1}+\frac{(2-2l)k_3}{k_3+1}>2+\frac{2}{D(f_1)+D(f_2)}.$$

Then $f_1(z)\equiv f_2(z)$.

Proof: Let $m=n=3$. Noting that $\Theta(f_i)\ge 0$ and $\delta(0,f_i-(a_j+sb))\ge 0$ for $j=1,2,\ldots,q$ and $i=1,2$, one can deduce from Theorem 2.2 that Corollary 2.1 follows. □

The following corollary is an analog of a result due to H.-X. Yi (Theorem 10.7 in [18], see also [21]) on $\mathbb{C}$.

Corollary 2.2 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0<D(f_1),D(f_2)<\infty$. Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b\neq 0$, $S_i\cap S_j=\emptyset$, $(i\neq j)$ and

$$q>\max\left\{4+\frac{2}{(D(f_1)+D(f_2))l},2+\max\left\{\left[\frac{1}{D(f_1)}\right],\left[\frac{1}{D(f_2)}\right]\right\}\right\}.$$

If $\overline{E}(S_j,\mathbb{D},f_1)=\overline{E}(S_j,\mathbb{D},f_2),(j=1,2,\ldots,q)$. Then $f_1(z)\equiv f_2(z)$.

Proof: Let $k_1 = k_2 = \ldots = k_q = \infty$. One can deduce from Corollary 2.1 that Corollary 2.2 follows immediately. □

Let $l = 1$. Then it is easily derived the following corollary from Corollary 2.1, which is an analog of the Corollary of Theorem 3.15 in [18].

Corollary 2.3 *Let a_j $(j = 1, 2, \ldots, q)$ be q distinct complex numbers in $\widehat{\mathbb{C}}$, and k_j $(j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1), and let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0 < D(f_1), D(f_2) < \infty$ and $\overline{E}_{k_j)}(a_j, \mathbb{D}, f_1) = \overline{E}_{k_j)}(a_j, \mathbb{D}, f_2)$. Set $D := \min\{D(f_1), D(f_2)\}$. Then*

(i) if $D > 1$, $q = 7$ and $k_7 \geq 2$, then $f_1(z) \equiv f_2(z)$;
(ii) if $D > 1$, $q = 6$ and $k_6 \geq 4$, then $f_1(z) \equiv f_2(z)$;
(iii) if $D > 2$ and $q = 7$, then $f_1(z) \equiv f_2(z)$;
(iv) if $D > 3$, $q = 6$ and $k_3 \geq 2$, then $f_1(z) \equiv f_2(z)$;
(v) if $D > 6$, $q = 5$, $k_3 \geq 3$ and $k_5 \geq 2$, then $f_1(z) \equiv f_2(z)$;
(vi) if $D > 10$, $q = 5$ and $k_4 \geq 4$, then $f_1(z) \equiv f_2(z)$;
(vii) if $D > 12$, $q = 5$, $k_3 \geq 5$ and $k_4 \geq 3$, then $f_1(z) \equiv f_2(z)$;
$(viii)$ if $D > 42$, $q = 5$, $k_3 \geq 6$ and $k_4 \geq 2$, then $f_1(z) \equiv f_2(z)$.

We now state another main theorem.

Theorem 2.3 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0 < D(f_1), D(f_2) < \infty$. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \ldots, c + a_j w^{l-1}\}, \quad j = 1, 2, \ldots, q,$$

with $a_j \neq 0$, $(j = 1, 2, \ldots, q)$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$ and $q > 2 + \max\left\{\left[\frac{1}{D(f_1)}\right], \left[\frac{1}{D(f_2)}\right]\right\}$. Let k_j $(j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1), and

$$\overline{E}_{k_j)}(S_j, \mathbb{D}, f_1) = \overline{E}_{k_j)}(S_j, \mathbb{D}, f_2), \quad (j = 1, 2, \ldots, q). \tag{8}$$

Furthermore, let

$$\Theta(f_i) = \sum_a \Theta(0, f_i - a) - \sum_{j=1}^{q} \sum_{s=0}^{l-1} \Theta(0, f_i - (c + a_j w^s)), (i = 1, 2),$$

and

$$\begin{aligned} A_3 &= \frac{\sum_{j=1}^{m-1} \sum_{s=0}^{l-1} \delta(0, f_1 - (c + a_j w^s))}{k_m + 1} + \sum_{j=m}^{q} \sum_{s=0}^{l-1} \frac{k_j + \delta(0, f_1 - (c + a_j w^s))}{k_j + 1} \\ &\quad + \frac{l(m-2)k_m}{k_m + 1} - \frac{l k_n}{k_n + 1} + \Theta(f_1) - 2, \end{aligned}$$

$$\begin{aligned} A_4 &= \frac{\sum_{j=1}^{n-1} \sum_{s=0}^{l-1} \delta(0, f_2 - (c + a_j w^s))}{k_n + 1} + \sum_{j=n}^{q} \sum_{s=0}^{l-1} \frac{k_j + \delta(0, f_2 - (c + a_j w^s))}{k_j + 1} \\ &\quad + \frac{l(n-2)k_n}{k_n + 1} - \frac{l k_m}{k_m + 1} + \Theta(f_2) - 2, \end{aligned}$$

where m and n are positive integers in $\{1, 2, \ldots, q\}$ and a is an arbitrary complex number or ∞. If

$$\min\{A_3, A_4\} \geq \frac{2}{D(f_1) + D(f_2)}, \quad \textit{and} \quad \max\{A_3, A_4\} > \frac{2}{D(f_1) + D(f_2)}. \tag{9}$$

Then $(f_1(z) - c)^l \equiv (f_1(z) - c)^l$.

Proof: We assume that $(f_1(z)-c)^l \not\equiv (f_2(z)-c)^l$. Without loss of generality, we assume that there exist infinitely many d such that $\Theta(0, f_1-d)>0$ and $d \notin \{c+a_jw^s : j=1,2,\ldots,q \text{ and } s=0,1,\ldots,l-1\}$. We denote them by d_k $(k=1,2,\ldots,\infty)$. Obviously, $\Theta(f_1)=\sum_{k=1}^{\infty}\Theta(0,f_1-d_k)$. Thus there exits a p such that $\sum_{k=1}^{p}\Theta(0,f_1-d_k)>\Theta(f_1)-\varepsilon$ holds for any given ε (>0).

Using a similar discussion as in the proof of Theorem 2.2, we obtain

$$\begin{aligned}
&\left(\frac{l(m-1)k_m}{k_m+1}+B_3-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_4-\varepsilon\right)T(r,f_2)\\
<\ &\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(c+a_jw^s)})+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{k_j)}(r,\frac{1}{f_2-(c+a_jw^s)})\\
&+2\log\frac{1}{1-r},
\end{aligned}$$

where

$$B_3=\frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0,f_1-(c+a_jw^s))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_1-(c+a_jw^s))}{k_j+1}+\Theta(f_1)-2.$$

$$B_4=\frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0,f_2-(c+a_jw^s))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_2-(c+a_jw^s))}{k_j+1}+\Theta(f_2)-2.$$

Furthermore, from condition (8) and the first fundamental theorem, we have

$$\begin{aligned}
\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(c+a_jw^s)}) &< \overline{N}(r,\frac{1}{(f_1-c)^l-(f_2-c)^l})\\
&\le l(T(r,f_1)+T(r,f_2))+O(1).
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}(r,\frac{1}{f_2-(c+a_jw^s)}) &< \overline{N}(r,\frac{1}{(f_1-c)^l-(f_2-c)^l})\\
&\le l(T(r,f_1)+T(r,f_2))+O(1).
\end{aligned}$$

Therefore, from the above discussion we obtain

$$\begin{aligned}
&\left(\frac{l(m-1)k_m}{k_m+1}+B_3-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_4-\varepsilon\right)T(r,f_2)\\
<\ & l\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(T(r,f_1)+T(r,f_2))+2\log\frac{1}{1-r},
\end{aligned}$$

namely,

$$(A_3-\varepsilon)\,T(r,f_1)+(A_4-\varepsilon)\,T(r,f_2)<2\log\frac{1}{1-r}. \tag{10}$$

Since $0<D(f_1),D(f_2)<\infty$, we have $S(r,f_1)=o\left(\log\frac{1}{1-r}\right), S(r,f_2)=o\left(\log\frac{1}{1-r}\right)$. And from the definition of index, for any ε satisfying

$$0<2\varepsilon<\min\left\{D(f_1),D(f_2),\max\{A_3,A_4\}-\frac{2}{D(f_1)+D(f_2)}\right\}, \tag{11}$$

there exists a sequence $\{r_t\} \to 1^-$ such that

$$T(r_t, f_1) > (D(f_1) - \varepsilon) \log \frac{1}{1 - r_t}, \quad T(r_t, f_2) > (D(f_2) - \varepsilon) \log \frac{1}{1 - r_t}, \tag{12}$$

for all $t \to \infty$. From (10)-(12), we have

$$[(D(f_1) - \varepsilon)(A_3 - \varepsilon) + (D(f_2) - \varepsilon)(A_4 - \varepsilon) - 2] \log \frac{1}{1 - r_t} < o\left(\log \frac{1}{1 - r_t}\right). \tag{13}$$

From (13) and ε being arbitrary, the above inequality contradicts to (9).

Therefore, the proof of Theorem 2.3 is completed. □

We have an analog of a result due to H.-X. Yi (Theorem 10.8 in [18], see also [21]).

Corollary 2.4 *let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0 < D(f_1), D(f_2) < \infty$. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \dots, c + a_j w^{l-1}\}, \quad j = 1, 2, \dots, q,$$

with $a_j \neq 0$, $(j = 1, 2, \dots, q)$, $q > 2 + \frac{2}{l} + \frac{2}{D(f_1)+D(f_2)}$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. If $\overline{E}(S_j, \mathbb{D}, f_1) = \overline{E}(S_j, \mathbb{D}, f_2)$ for $j = 1, 2, \dots, q$, then $(f_1(z) - c)^l \equiv (f_2(z) - c)^l$.

Proof: Let $m = n = 1$ and $k_1 = k_2 = \ldots = \infty$. Noting that $\Theta(f_i) \geq 0$ and $\delta(0, f_i - (a_j + sb)) \geq 0$ for $j = 1, 2, \dots, q$ and $i = 1, 2$, Then Corollary 2.4 follows immediately from Theorem 2.2. □

3 The problem of sharing sets of admissible function and non-admissible function in the unit disc

We now show that an admissible function can share sufficiently many sets concerning multiple values with another non-admissible function as follows.

Theorem 3.1 *If f_1 is admissible and f_2 is a non-admissible satisfying $\lim_{r\to 1^-} T(r, f_2) = \infty$, $a_j (j = 1, 2, \dots, q)$ be q distinct complex numbers, and let $k_j (j = 1, 2, \dots, q)$ be positive integers or ∞ satisfying (1). Then*

$$\overline{E}_{k_j)}(a_j, \mathbb{D}, f_1) = \overline{E}_{k_j)}(a_j, \mathbb{D}, f_2), \quad (j = 1, 2, \dots, q).$$

and

$$\sum_{j=m+1}^{q} \frac{k_j}{k_j + 1} + \frac{(m-1)k_m}{k_m + 1} - 2 > 0$$

do not hold at same time.

Theorem 3.2 *If f_1 is admissible and f_2 is a non-admissible satisfying $\lim_{r\to 1^-} T(r, f_2) = \infty$. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \dots, c + a_j w^{l-1}\}, \quad j = 1, 2, \dots, q,$$

with $a_j \neq 0$, $(j = 1, 2, \dots, q)$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. Then $\overline{E}(S_j, \mathbb{D}, f_1) = \overline{E}(S_j, \mathbb{D}, f_2)$ for $j = 1, 2, \dots, q$, and $q > 1 + \frac{2}{l}$ can not hold at the same time.

To prove the above theorems, we require the following lemmas.

Lemma 3.1 (see [12, Lemma 1]). *Let $f(z), g(z)$ satisfy $\lim_{r\to 1^-} T(r, f) = \infty$ and $\lim_{r\to 1^-} T(r, g) = \infty$. If there is a $K \in (0, \infty)$ with*

$$T(r, f) \leq KT(r, g) + S(r, f) + S(r, g),$$

then each $S(r, f)$ is also an $S(r, g)$.

Lemma 3.2 *If f_1 is admissible and f_2 is a non-admissible satisfying* $\lim_{r\to 1^-} T(r, f_2) = \infty$, $a_j(j = 1, 2, \dots, q)$ *be q distinct complex numbers, and let $k_j(j = 1, 2, \dots, q)$ be positive integers or ∞ satisfying (1). Set $A_5 = B_1 + \frac{[(m-3)l+1]k_m}{k_m+1}$. Then (2) and $A_5 > 0$ do not hold at same time, where $B_1, S_j(j = 1, 2, \dots, q)$ are stated as in Theorem 2.1.*

Proof: Suppose that (2) and $A_5 > 0$ can hold at the same time. Since $f_1(z)$ is an admissible function, using the same argument as in Theorem 2.2 and from Theorem 1.2 and Lemma 1.1, for any $\varepsilon(0 < 2\varepsilon < A_5)$, we have

$$\left(\frac{(m-1)lk_m}{k_m+1} + B_1 - \varepsilon\right) T(r, f_1) < \sum_{j=1}^{q}\sum_{s=0}^{l-1} \frac{k_m}{k_m+1} \overline{N}_{k_j)}(r, \frac{1}{f_1 - (a_j + sb)}) + S(r, f_1),$$

where B_1 is stated as in Section 2.

Since f_1 is admissible and f_2 is non-admissible, we can get that $f_1(z) \not\equiv f_2(z)$. Thus, by condition (2) and the first fundamental theorem, we have

$$\begin{aligned}\sum_{j=1}^{q}\sum_{s=0}^{l-1} \overline{N}_{k_j)}\left(r, \frac{1}{f_1 - (a_j + sb)}\right) \leq& \overline{N}\left(r, \frac{1}{f_1 - f_2}\right) + \sum_{s=1}^{l-1} \overline{N}\left(r, \frac{1}{f_1 - f_2 - sb}\right)\\ &+ \sum_{s=1}^{l-1} \overline{N}\left(r, \frac{1}{f_2 - f_1 - sb}\right)\\ \leq& (2l-1)(T(r, f_1) + T(r, f_2)) + O(1).\end{aligned}$$

From the two above inequality, we get

$$\left(\frac{[(m-3)l+1]k_m}{k_m+1} + B_1 - \varepsilon\right) T(r, f_1) \leq \frac{(2l-1)k_m}{k_m+1} T(r, f_2). \tag{14}$$

Since $0 < \varepsilon < A_5$, we have $\frac{[(m-3)l+1]k_m}{k_m+1} + B_1 - \varepsilon > 0$. From (14), we have

$$T(r, f_1) \leq \frac{1}{A_5 - \varepsilon} \frac{(2l-1)k_m}{k_m+1} T(r, f_2). \tag{15}$$

From Lemma 3.1, (15) and $\frac{1}{A_5-\varepsilon}\frac{(2l-1)k_m}{k_m+1} > 0$, we can get that each $S(r, f_1)$ is also an $S(r, f_2)$. Since $f_1(z)$ is admissible and $f_2(z)$ is non-admissible, we can get $T(r, f_2) = S(r, f_1)$. Thus, we have

$$T(r, f_2) = S(r, f_1) = S(r, f_2) = o(T(r, f_2)).$$

This is a contradiction. Hence, we can get that (2) and $A_5 > 0$ do not hold at the same time. □

Lemma 3.3 *If f_1 is admissible and f_2 is a non-admissible satisfying* $\lim_{r\to 1^-} T(r, f_2) = \infty$, $a_j(j = 1, 2, \dots, q)$ *be q distinct complex numbers, and let $k_j(j = 1, 2, \dots, q)$ be positive integers or ∞ satisfying (1). Set $A_6 = B_3 + \frac{(m-2)lk_m}{k_m+1}$. Then (8) and $A_6 > 0$ do not hold at same time, where $B_3, S_j(j = 1, 2, \dots, q)$ are stated as in Theorem 2.3.*

Proof: Suppose that (8) and $A_6 > 0$ can hold at the same time. Since $f_1(z)$ is an admissible function, using the same argument as in Theorem 2.3 and from Theorem 1.1 and Lemma 1.1, for any $\varepsilon(0 < \varepsilon < A_6)$, we have

$$\left(\frac{(m-1)lk_m}{k_m+1} + B_3 - \varepsilon\right) T(r, f_1) < \sum_{j=1}^{q}\sum_{s=0}^{l-1} \frac{k_m}{k_m+1} \overline{N}_{k_j)}(r, \frac{1}{f_1 - (c + a_j w^s)}) + S(r, f_1),$$

where B_3 is stated as in Section 2.

From the assumptions of Lemma 3.3, we can get that $(f_1(z)-c)^l \not\equiv (f_2(z)-c)^l$. Thus, by condition (8) and the first fundamental theorem, we have

$$\begin{aligned}\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(c+a_jw^s)}) &< \overline{N}(r,\frac{1}{(f_1-c)^l-(f_2-c)^l})\\ &\le l(T(r,f_1)+T(r,f_2))+O(1).\end{aligned}$$

From the two above inequality, we get

$$\left(\frac{(m-2)lk_m}{k_m+1}+B_3-\varepsilon\right)T(r,f_1)\le\frac{lk_m}{k_m+1}T(r,f_2). \tag{16}$$

Since $0<\varepsilon<A_6$, we have $\frac{(m-2)lk_m}{k_m+1}+B_3-\varepsilon>0$. From (16), we have

$$T(r,f_1)\le\frac{1}{A_5-\varepsilon}\frac{(2l-1)k_m}{k_m+1}T(r,f_2). \tag{17}$$

From Lemma 3.1, (17) and $\frac{1}{A_6-\varepsilon}\frac{lk_m}{k_m+1}>0$, we can get that each $S(r,f_1)$ is also an $S(r,f_2)$. Since $f_1(z)$ is admissible and $f_2(z)$ is non-admissible, we can get $T(r,f_2)=S(r,f_1)$. Thus, we have

$$T(r,f_2)=S(r,f_1)=S(r,f_2)=o(T(r,f_2)).$$

This is a contradiction. Hence, we can get that (8) and $A_6>0$ do not hold at the same time.

Thus, the proof of Lemma 3.3 is completed. □

Proof of Theorem 3.1: Let $l=1$, and since $\Theta(f_i)\ge 0$ $(i=1,2)$ and $\delta(0,f_1-a_j)\ge 0$ $(j=1,2,\ldots,q)$, the assertion follows from Lemma 3.2.

Proof of Theorem 3.2: Let $k_1=k_2=\cdots=k_q=\infty$, and since $\Theta(f_i)\ge 0$ $(i=1,2)$ and $\delta(0,f_1-a_j)\ge 0$ $(j=1,2,\ldots,q)$, the assertion follows from Lemma 3.3.

It is very interesting to consider distinct small functions instead of distinct complex numbers (see [9, 11, 17],etc). *Thus it may be interesting to consider the following questions:*

Question 3.1 *What condition on two non-admissible functions in the unit disc $\mathbb{D}$ sharing small functions will guarantee that the two non-admissible functions are identical?*

Question 3.2 *How many small functions can an admissible function and non-admissible function in the unit disc $\mathbb{D}$ share at most?*

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THE FIXED POINT ALTERNATIVE TO THE STABILITY OF AN ADDITIVE (α, β)-FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we solve the additive (α, β)-functional equation

$$f(x) + f(y) + 2f(z) = \alpha f(\beta(x + y + 2z)), \tag{0.1}$$

where α, β are fixed real or complex numbers with $\alpha \neq 4$ and $\alpha\beta = 1$.

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [24] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [18] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. See [5, 7, 14, 15, 20, 21, 19, 22, 23, 19, 25] for more information on functional equations.

We recall a fundamental result in fixed point theory.

Theorem 1.1. [2, 6] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several

2010 *Mathematics Subject Classification.* Primary 39B52, 39B62, 47H10.

Key words and phrases. Hyers-Ulam stability; additive (α, β)-functional equation; fixed point method; direct method; Banach space.

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functional equations have been extensively investigated by a number of authors (see [3, 4, 12, 13, 16, 17]).

In Section 2, we solve the additive (α, β)-functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in Banach spaces by using the direct method.

Throughout this paper, assume that X is a normed space and that Y is a Banach space. Let α, β be fixed real or complex numbers with $\alpha \neq 4$ and $\alpha\beta = 1$.

2. Additive (α, β)-functional equation (0.1) in Banach spaces I

We solve the additive (α, β)-functional equation (0.1) in vector spaces.

Lemma 2.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$f(x) + f(y) + 2f(z) = \alpha f(\beta(x + y + 2z)) \tag{2.1}$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = z = 0$ in (2.1), we get $4f(0) = \alpha f(0)$. So $f(0) = 0$.

Letting $y = -x$ and $z = 0$ in (2.1), we get $f(x) + f(-x) = 0$ and so $f(-x) = -f(x)$ for all $x \in X$.

Letting $x = -2z$ and $y = 0$ in (2.1), we get $f(-2z)+2f(z) = 0$ and so $f(2z) = 2f(z)$ for all $z \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x)$$

for all $x \in X$.

Letting $z = -\frac{x+y}{2}$ in (2.1), we get

$$f(x) + f(y) - f(x + y) = f(x) + f(y) + 2f\left(-\frac{x + y}{2}\right) = 0$$

and so

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (2.1) in Banach spaces.

Theorem 2.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2} \varphi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|f(x) + f(y) + 2f(z) - \alpha f(\beta(x + y + 2z))\| \leq \varphi(x, y, z) \tag{2.3}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{L}{2(1 - L)} (\varphi(x, x, -x) + \varphi(2x, 0, -x)) \tag{2.4}$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = -x$ in (2.3), we get

$$\|2f(x) + 2f(-x)\| \leq \varphi(x, x, -x) \tag{2.5}$$

for all $x \in X$.

Replacing x by $2x$ and letting $y = 0$ and $z = -x$ in (2.3), we get

$$\|f(2x) + 2f(-x)\| \leq \varphi(2x, 0, -x) \tag{2.6}$$

for all $x \in X$.

It follows from (2.5) and (2.6) that

$$\|f(2x) - 2f(x)\| \leq \varphi(x, x, -x) + \varphi(2x, 0, -x) \tag{2.7}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf \left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu(\varphi\left(x, x, -x\right) + \varphi\left(2x, 0, -x\right)), \ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [11]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varepsilon(\varphi\left(x, x, -x\right) + \varphi\left(2x, 0, -x\right))$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \leq 2\varepsilon\left(\varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right) + \varphi\left(x, 0, -\frac{x}{2}\right)\right) \\ &\leq 2\varepsilon\frac{L}{2}(\varphi\left(x, x, -x\right) + \varphi\left(2x, 0, -x\right)) = L\varepsilon(\varphi\left(x, x, -x\right) + \varphi\left(2x, 0, -x\right)) \end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.7) that

$$\begin{aligned} \left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| &\leq \varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right) + \varphi\left(x, 0, -\frac{x}{2}\right) \\ &\leq \frac{L}{2}(\varphi(x, x, -x) + \varphi(2x, 0, -x)) \end{aligned}$$

for all $x \in X$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(x\right) = 2A\left(\frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - A(x)\| \quad \le \quad \mu(\varphi(x,x,-x) + \varphi(2x,0,-x))$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f,A) \le \frac{1}{1-L} d(f,Jf)$, which implies

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)}(\varphi(x,x,-x) + \varphi(2x,0,-x))$$

for all $x \in X$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\|A(x) + A(y) + 2A(z) - \alpha A(\beta(x+y+2z))\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + 2f\left(\frac{z}{2^n}\right) - \alpha f\left(\beta\left(\frac{x+y+2z}{2^n}\right)\right)\right\| \\
&\le \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0
\end{aligned}$$

for all $x, y, z \in X$. So

$$A(x) + A(y) + 2A(z) - \alpha A(\beta(x+y+2z)) = 0$$

for all $x, y, z \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. □

Corollary 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x) + f(y) + 2f(z) - \alpha f(\beta(x+y+2z))\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{2.9}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2^r + 4}{2^r - 2}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2(1-L)}(\varphi(x,x,-x) + \varphi(2x,0,-x))$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \le \frac{1}{2}(\varphi(x,x,-x)+\varphi(2x,0,-x))$$

for all $x \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.9). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \le \frac{4+2^r}{2-2^r}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y,z)=\theta(\|x\|^r+\|y\|^r+\|z\|^r)$ for all $x,y,z \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. □

3. Additive (α,β)-functional equation (0.1) in Banach spaces II

In this section, using the direct method, we prove the Hyers-Ulam stability of the additive (α,β)-functional equation (2.1) in Banach spaces.

Theorem 3.1. *Let $\varphi : X^3 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x,y,z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right) < \infty,$$

$$\|f(x)+f(y)+2f(z)-\alpha f(\beta(x+y+2z))\| \le \varphi(x,y,z) \tag{3.1}$$

for all $x,y,z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{1}{2}(\Psi(x,x,-x)+\Psi(2x,0,-x)) \tag{3.2}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2},\frac{x}{2},-\frac{x}{2}\right)+\varphi\left(x,0,-\frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\| \le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\|$$

$$\le \sum_{j=l}^{m-1}\left(2^j\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},-\frac{x}{2^{j+1}}\right)+2^j\varphi\left(\frac{x}{2^j},0,-\frac{x}{2^{j+1}}\right)\right) \tag{3.3}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.3) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2).

Now, let $T : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned} \|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}, -\frac{x}{2^q}\right) + 2^q \Psi\left(\frac{2x}{2^q}, 0, -\frac{x}{2^q}\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^r + 4}{2^r - 2}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. □

Theorem 3.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$,* (3.1) *and*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, 2^j z) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}(\Psi(x, x, -x) + \Psi(2x, 0, -x)) \tag{3.4}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}(\varphi\left(x, x, -x\right) + \varphi\left(2x, 0, -x\right))$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x)\right\| &\leq \sum_{j=l}^{m-1} \left\|\frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right)\right\| \\ &\leq \sum_{j=l}^{m-1} \left(\frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, -2^j x) + \frac{1}{2^{j+1}} \varphi(2^{j+1} x, 0, -2^j x)\right) \qquad (3.5)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.5) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get (3.4).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.1. □

Corollary 3.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{4 + 2^r}{2 - 2^r} \theta \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.3 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. □

Acknowledgments

This research was supported by Hanshin University Research Grant.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.6, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

The approximation problem of Dirichlet series with regular growth *

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Abstract

By introducing the concept of β_U-order functions, we study the error in approximating Dirichlet series of infinite order in the half plane by Dirichlet polynomials. Some necessary and sufficient conditions on the error and regular growth of finite β_U-order of these functions have been obtained.

Key words: β-order, β_U-order, Regular growth, Dirichlet series.
2010 Mathematics Subject Classification: 30B50, 30D15.

1 Introduction and basic notes

Consider Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}a_ne^{\lambda_n s},\qquad s=\sigma+it, \tag{1}$$

where

$$0\leq\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots,\lambda_n\to\infty\ \ as\ \ n\to\infty; \tag{2}$$

$s=\sigma+it$ (σ,t are real variables); a_n are nonzero complex numbers and

$$\limsup_{n\to+\infty}(\lambda_{n+1}-\lambda_n)=h<+\infty, \tag{3}$$

$$\limsup_{n\to+\infty}\frac{\log^+|a_n|}{\lambda_n}=0, \tag{4}$$

*The first author was supported by The Natural Science Foundation of China(11561033, 11301233), the Natural Science Foundation of Jiangxi Province in China (20151BAB201008), and the Foundation of Education Department of Jiangxi of China (GJJ150902). The second author holds the Project Supported by Guangdong Natural Science Foundation(2015A030313628) and The Training plan for Outstanding Young Teachers in Higher Education of Guangdong(Yqgdufe1405).

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then from (2) and (3), by using the similar method in [19] or [15], we can get

$$\limsup_{n\to\infty}\frac{n}{\lambda_n}=E<+\infty,\qquad \limsup_{n\to\infty}\frac{\log n}{\lambda_n}=0. \tag{5}$$

Then the abscissas of convergence and absolutely convergence is 0, that is, $f(s)$ is an analytic function in the left half plane $H=\{s=\sigma+it:\sigma<0,t\in\mathbb{R}\}$.

We denote D to be the class of all functions $f(s)$ satisfying (2)-(4) and analytic in $Res<0$, denote $\overline{D}_\alpha$ to be the class of all functions $f(s)$ satisfying (2)-(3) and analytic in $Re\le\alpha$ where $-\infty<\alpha<+\infty$. Thus, if $-\infty<\alpha<0$ and $f(s)\in D$, then $f(s)\in\overline{D}_\alpha$; if $0<\alpha<+\infty$ and $f(s)\in\overline{D}_\alpha$, then $f(s)\in D$. We denote Π_k to be the class of all exponential polynomial of degree almost k, that is,

$$\Pi_k=\left\{\sum_{j=1}^{k}b_je^{\lambda_js}:(b_1,b_2,\dots,b_k)\in\mathbb{C}^k\right\}.$$

For $f(s)\in D$,

$$M(\sigma,f)=\max_{-\infty<t<\infty}|f(\sigma+it)|,\qquad m(\sigma,f)=\max_{n\ge1}\{|a_n|e^{\sigma\lambda_n}\}$$

are called, respectively, the maximum modulus, the maximum term of $f(s)$ for $Res=\sigma<0$.

Definition 1.1 *Let $f(s)\in D$, the order of $f(s)$ can be defined by*

$$\rho=\limsup_{\sigma\to0^-}\frac{\log\log^+M(\sigma,f)}{-\log(-\sigma)},$$

where $\log^+x=\begin{cases}\log x & x\ge1\\ 0 & x<1\end{cases}$

For $\rho=0,0<\rho<\infty,\rho=\infty$, $f(s)$ can be called, respectively, zero order, finite order, infinite order Dirichlet series. Considerable attention has been paid to the growth and the value distribution of analytic functions defined by Dirichlet series; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18] for some results.

For $f(s)\in\overline{D}_\alpha,-\infty<\alpha<+\infty$, we denote $E_n(f,\alpha)$ by the error in approximating the function $f(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(f,\alpha)=\inf_{p\in\Pi_n}\parallel f-p\parallel_\alpha,\quad n=1,2,\dots,$$

where

$$\parallel f-p\parallel_\alpha=\max_{-\infty<t<+\infty}|f(\alpha+it)-p(\alpha+it)|.$$

In 2010, the authors [17] investigated the relations between the error $E_n(f,\alpha)$ and the growth order of $f(s)$, and obtained some equivalence relation between $E_n(f,\alpha)$ and the regular growth of $f(s)$ with finite order as follows:

Theorem 1.1 (see [17]). *Let $f(s)\in D$ be of finite order ρ, then for any real number $-\infty<\alpha<0$, we have*

$$\lim_{\sigma\to0^-}\frac{\log^+M(\sigma,f)}{U_1(-\frac{1}{\sigma})}=1\Longleftrightarrow\limsup_{n\to+\infty}\frac{\log^+\left[E_n(f,\alpha)e^{-\alpha\lambda_{n+1}}\right]}{BU_1\left(\frac{\lambda_{n+1}}{\log^+\left[E_n(f,\alpha)e^{-\alpha\lambda_{n+1}}\right]}\right)}=1;$$

and there exists a increasing, positive integer sequence $\{n_\nu\}$ satisfying

$$\lim_{\nu\to+\infty}\frac{\log^+\left[E_{n_\nu}(f,\alpha)e^{-\alpha\lambda_{n_\nu+1}}\right]}{BU_1\left(\frac{\lambda_{n_\nu+1}}{\log^+\left[E_{n_\nu}(f,\alpha)e^{-\alpha\lambda_{n_\nu+1}}\right]}\right)}=1,\qquad\lim_{\nu\to+\infty}\frac{\lambda_{n_{\nu+1}}}{\lambda_{n_\nu}}=1,$$

where $B = \frac{(1+\rho)^{1+\rho}}{\rho^\rho}$ *and* $U_1(r) = r^{\rho(r)}$, $\rho(r)$ *satisfies the following conditions:*

(i) there exists a real number $r_0 > 0$, $\rho(r)$ *is nonnegative, continuous, monotone on* $[r_0, +\infty)$, *and tends to* ρ *as* $r \to +\infty$*;*

(ii) $\lim_{r\to+\infty} \rho'(r) r \log r = 0,$*;*

(iii) $U_1(kr) = [k^\rho + o(1)]U_1(r)(r \to +\infty)$ *for every positive integer* k*, and* $U_1(r)$ *is an increasing function on* $r \geq r_0' > r_0$*.*

Recently, the authors [18] further investigated the relations between the error $E_n(f, \alpha)$ and the growth order of $f(s)$ when $f(s)$ has infinite order, by introducing the concept of β-order.

Theorem 1.2 (see [18]). *Let* $f(s) \in D$ *be of finite* β*-order* ρ_β*, then for any real number* $-\infty < \alpha < 0$*, we have*

$$\limsup_{n\to\infty} \frac{\beta(\lambda_n)}{\log \lambda_n - \log\log^+(E_{n-1}(f,\alpha)e^{-\alpha\lambda_n})} = \rho_\beta.$$

Remark 1.1 *In Theorem 1.2, the definitions of* β*-order and the function* $\beta(x)$ *will be introduced in Section 2.*

Thus, a question arises naturally: what will happen when $\rho_\beta = \infty$ *in Theorem1.2?*

In this paper, we will investigate the above question by using the type functions $U_2(x)$ to enlarge the growth of the denominator $-\log(-\sigma)$ and obtain the main results as follows.

Theorem 1.3 *If Dirichlet series* $f(s) \in D$ *of infinite* β*-order, then we have*

$$\limsup_{\sigma\to 0^-} \frac{\beta(\log^+ M(\sigma, F))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T \Longleftrightarrow \limsup_{\sigma\to 0^+} \frac{\beta(\log^+ m(\sigma, F))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T,$$

where $0 < T < \infty$ *and* $U_2(x) = x^{\rho(x)}$ *satisfies the following conditions*

(i) $\rho(x)$ *is monotone and* $\lim_{x\to\infty} \rho(x) = \infty$*;*

(ii) $\lim_{x\to\infty} \frac{\log U_2(x')}{\log U_2(x)} = 1$*, where* $x' = x\left(1 + \frac{1}{\log U_2(x)}\right)$*.*

Remark 1.2 *From Lemma 2.1 and Lemma 1.1 in Section 2, we can prove the conclusion of Theorem 1.3 easily.*

Remark 1.3 *This type function* $U_2(x)$ *is different from the type function* $U_1(x)$ *in Theorem 1.1.*

Remark 1.4 *If Dirichlet series* $f(s)$ *of infinite order has infinite* β*-order and satisfies*

$$\limsup_{\sigma\to 0^-} \frac{\beta(\log^+ M(\sigma, f))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T, \tag{6}$$

then T *is called the* β_U*-order of Dirichlet series* $f(s)$*.*

Theorem 1.4 *If Dirichlet series* $f(s) \in D$ *with infinite* β*-order, then for any fixed real number* $-\infty < \alpha < 0$*, we have*

$$\limsup_{\sigma\to 0^-} \frac{\beta(\log^+ M(\sigma, f))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T \Longleftrightarrow \limsup_{n\to\infty} \Psi_n(f, \alpha, \lambda_n) = T; \tag{7}$$

where

$$\Psi_n(f, \alpha, \lambda_n) = \frac{\beta(\lambda_n)}{\log U_2\left(\frac{\lambda_n}{\log^+[E_{n-1}(f,\alpha)e^{-\alpha\lambda_n}]}\right)}.$$

Remark 1.5 *From Theorem 1.4, we can see that the type function $U_2(x)$ is more simple then the type function of Wang [16].*

Theorem 1.5 *Under the assumptions of Theorem 1.4, we have*

$$\lim_{\sigma\to 0^-}\frac{\beta(\log^+ M(\sigma,f))}{\log U_2\left(\frac{1}{-\sigma}\right)}=T \Longleftrightarrow \textit{ the right hand of } (7) \textit{ is verified},$$

and there exists a subsequence $\{\lambda_{n(p)}\}\subseteq\{\lambda_n\}$ satisfying

$$\lim_{p\to\infty}\Psi_{n(p)}(f,\alpha,\lambda_{n(p)})=T, \quad \textit{and} \quad \lim_{p\to\infty}\frac{\beta(\lambda_{n(p)})}{\beta(\lambda_{n(p+1)})}=1, \tag{8}$$

where

$$\Psi_{n(p)}(f,\alpha,\lambda_{n(p)})=\frac{\beta(\lambda_{n(p)})}{\log U_2\left(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}(f,\alpha)e^{-\alpha\lambda_{n(p)}}]}\right)}.$$

Remark 1.6 *From Theorem 1.5, we get the necessary and sufficient conditions for the limit about the regular growth of $f(s)$, however, Wang [16] only gave the necessary and sufficient conditions for the superior limit. Thus, our results of this paper are more accurate than the previous form [16].*

2 Some Lemmas and the concept of β-order

According to observations, we find that to study the growth of Dirichlet series better, many mathematicians proposed the type functions $U(x)$ to enlarge the growth of the denominator $\log\frac{1}{-\sigma}$ or $-\sigma$ (see [13, 4, 12]), or use some function to control the molecular $M(\sigma,f)$ or $m(\sigma,f)$ in the definition of order. In this paper, we will deal with the growth of Dirichlet series of infinite order by using a class of functions to reduce $M(\sigma,f)$ or $m(\sigma,f)$ which is better than the previous form. So, we firstly give the definition of β-order of Dirichlet series as follows, which is an extension of [10].

Let $\mathfrak{F}$ be the class of all functions $\beta(x)$ satisfies the following conditions:

(i) $\beta(x)$ is defined on $[a,+\infty), a>0$, and positive, strictly increasing, differential and tends to $+\infty$ as $x\to+\infty$;

(ii) $x\beta'(x)=o(1)$ as $x\to+\infty$.

Definition 2.1 *([18]). If Dirichlet series $f(s)$ of infinite order satisfies*

$$\limsup_{\sigma\to 0^+}\frac{\beta(\log^+ M(\sigma,f))}{\log\frac{1}{-\sigma}}=\rho^*,$$

where $\beta(x)\in\mathfrak{F}$, then ρ^* is called the β-order of $f(s)$.

Remark 2.1 *Obviously, the functions $h(x)=\log_p x, p\geq 2, p\in N_+$ satisfy the conditions (i) and (ii), where p is a positive integer, and $\log_1 x=\log x$ and $\log_p x=\log(\log_{p-1}x)$. Thus, p-order is regard as a special case of β-order of Dirichlet series.*

Remark 2.2 *Furthermore, β-order is more precise than p-order to some extent. In fact, for $p(\geq 2)$ is a positive integer, we can find function $\beta(x)\in\mathfrak{F}$ and a positive real function $M(x)$ satisfying*

$$\limsup_{x\to\infty}\frac{\beta(\log M(x))}{\log x}=t, \quad (0<t<\infty),$$

and

$$\limsup_{x\to\infty}\frac{\log_p(\log M(x))}{\log x}=\infty,\quad and\quad \limsup_{x\to\infty}\frac{\log_{p+1}(\log M(x))}{\log x}=0.$$

For example, let

$$M(x)=\exp_{p+1}\{(t\log x)^{1/d}\},\qquad \beta(x)=(\log_{p+1}x)^d,$$

where t is a finite positive real constant and $0<d<1$, we can get that $\rho_p(M)=\infty,\rho_{p+1}(M)=0$ and $\rho_\beta(M)=t$, where $\rho_p(f)$ denote the p-order of f, and $\rho_\beta(f)$ the β-order of f.

Remark 2.3 *If $\rho^*=\infty$ in Definition 2.1, then $f(s)$ is called a Dirichlet series of infinite β-order.*

Lemma 2.1 *(see [16]). Let $\beta(x)\in\mathfrak{F}$ and $\varphi(x)$ be the function satisfying*

$$\limsup_{x\to\infty}\frac{\log^+\varphi(x)}{\log x}=\varrho,\quad (0\le\varrho<\infty),$$

if $M(x)$ satisfies $\limsup_{x\to\infty}\frac{\beta(\log M(x))}{\log x}=\nu(>0)$. Then we have

$$\limsup_{x\to\infty}\frac{\beta(\varphi(x)\log M(x))}{\log x}=\nu.$$

Proof: To prove this lemma, two cases will be considered as follows.

Case 1. If $\varphi(x)$ is not a constant. From the assumptions of Lemma 2.1, we can get that $\varphi(x)\to\infty$ as $x\to\infty$. Then, for sufficiently large x, we have $\varphi(x)>1$. From $\beta(x)\in\mathfrak{F}$, we have $\lim_{x\to\infty}\log M(x)=\infty$. Then from the Cauchy mean value theorem, there exists $\xi(\log M(x)<\xi<\beta(x)\log M(x))$ satisfying

$$\frac{\beta(\varphi(x)\log M(x))-\beta(\log M(x))}{\log(\varphi(x)\log M(x))-\log\log M(x)}=\frac{\beta'(\xi)}{(\log\xi)'}=\xi\beta'(\xi),$$

that is,

$$\beta(\varphi(x)\log M(x))=\beta(\log M(x))+\log\varphi(x)\xi\beta'(\xi).\tag{9}$$

Since $x\beta'(x)=o(1)$ as $x\to+\infty$ and $\limsup_{x\to\infty}\frac{\log\varphi(x)}{\log x}=\varrho,(0\le\varrho<\infty)$, by (9), we can get the conclusion of Lemma 2.1.

Case 2. If $\varphi(x)$ is a constant. By using the same argument as in Case 1, we can prove that Lemma 2.1 is true.

Thus, this completes the proof of Lemma 2.1. □

The following lemma plays an important role to deal with the growth of Dirichlet series, which shows the relation between $M(\sigma,f)$ and $m(\sigma,f)$ of such functions.

Lemma 2.2 ([19]). If Dirichlet series (1) satisfies (2) (3), then for any given $\varepsilon\in(0,1)$ and for $\sigma(<0)$ sufficiently reaching 0, we have

$$m(\sigma,f)\le M(\sigma,f)\le K(\varepsilon)\frac{1}{-\sigma}m((1-\varepsilon)\sigma,f),$$

where $K(\varepsilon)$ is a constant depending on ε and (3).

Lemma 2.3 *If $f(s)\in\overline{D}_\alpha(-\infty<\alpha<+\infty)$, then for any positive integer $n\in\mathbb{N}_+:=\mathbb{N}\backslash\{0\}$, we have*

$$|a_n|e^{\alpha\lambda_n}\le K_2E_{n-1}(f,\alpha),$$

where $K_2>1$ is a real constant.

Proof: From the definition of $E_n(f,\alpha)$, there exists $p(s)\in\Pi_{n-1}$ such that

$$||f-p||_\alpha \le K_2 E_{n-1}(f,\alpha). \tag{10}$$

Since $f(s)\in\overline{D}_\alpha$ and from [19, P.16], for any real numbers $t_0,\vartheta(\neq 0)$, we have

$$\lim_{R\to+\infty}\frac{1}{R}\int_{t_0}^{R} e^{\vartheta it}dt = 0 \tag{11}$$

and

$$a_n e^{\alpha\lambda_n} = \lim_{R\to\infty}\frac{1}{R}\int_{t_0}^{R} f(\alpha+it)e^{-\lambda_n it}dt. \tag{12}$$

From (11), for any real number $x\neq 0$, we have

$$\lim_{R\to\infty}\frac{1}{R}\int_{t_0}^{R} e^{x(\alpha+it)}dt = 0. \tag{13}$$

Thus, from (12) and (13), for any $p_1(s)\in\Pi_{n-1}$, we have

$$a_n e^{\alpha\lambda_n} = \lim_{R\to\infty}\frac{1}{R}\int_{t_0}^{R}[f(\alpha+it)-p_1(\alpha+it)]e^{-\lambda_n it}dt,$$

that is,

$$|a_n|e^{\alpha\lambda_n} \le ||f-p_1||_\alpha. \tag{14}$$

From (10) and (14), we can prove the conclusion of Lemma 2.3. □

3 The proof of Theorem 1.4

We prove the conclusions of Theorem 1.4 by using the properties of two functions $\beta(x)$ and $U_2(x)$, this method is different from the previous method to some extent.

We first prove " $\Longleftarrow$ " of Theorem 1.4. Suppose that

$$\limsup_{n\to\infty}\Psi_n(f,\alpha,\lambda_n) = \limsup_{n\to\infty}\frac{\beta(\lambda_n)}{\log U_2\left(\frac{\lambda_n}{\log^+[E_{n-1}(f,\alpha)e^{-\alpha\lambda_n}]}\right)} = T. \tag{15}$$

Let

$$A_n = E_{n-1}(f,\alpha)e^{-\alpha\lambda_n},\quad n=1,2,\ldots,$$

then for any positive real number $\tau>0$, for sufficiently large n, we have

$$\lambda_n < \gamma\left((T+\tau)\log U_2\left(\frac{\lambda_n}{\log^+ A_n}\right)\right),$$

where $\gamma(x)$ is the inverse functions of $\beta(x)$. Let $V_2(x)$ and $U_2(x)$ be two reciprocally inverse functions, then we have

$$V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right) < \frac{\lambda_n}{\log^+ A_n},\quad \log^+ A_n \le \lambda_n\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right)\right)^{-1}.$$

Thus, we have

$$\log^+(A_n e^{\lambda_n\sigma}) \le \lambda_n\left(\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right)\right)^{-1}+\sigma\right). \tag{16}$$

For any fixed and sufficiently small $\sigma < 0$, set

$$G = \gamma\left((T+\tau)\log U_2\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right),$$

that is,

$$\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)} = V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(G)\right\}\right). \tag{17}$$

If $\lambda_n \le G$, for sufficiently large n, let $V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right) \ge 1$, from $\sigma < 0$,(16),(17) and the definition of $U_2(x)$, we have

$$\begin{aligned}
\log^+ A_n e^{\lambda_n\sigma} &\le G\left(\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right)\right)^{-1}+\sigma\right)\\
&\le G = \gamma\left((T+\tau)\log U_2\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right)\\
&\le \gamma\left((T+\tau)\log\left[(1+o(1))U_2\left(\frac{1}{-\sigma}\right)\right]\right).
\end{aligned} \tag{18}$$

If $\lambda_n > G$, from (16) and (17), we have

$$\begin{aligned}
\log^+ A_n e^{\lambda_n\sigma} &\le \lambda_n\left(\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(G)\right\}\right)\right)^{-1}+\sigma\right)\\
&\le \lambda_n\left(\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)^{-1}+\sigma\right) < 0.
\end{aligned} \tag{19}$$

For sufficiently large n, from (18) and (19), we have

$$\log^+ A_n e^{\lambda_n\sigma} \le \gamma\left((T+\tau)\log\left[(1+o(1))U_2\left(\frac{1}{-\sigma}\right)\right]\right)$$

Since $A_n = E_{n-1}e^{-\alpha\lambda_n}$ and τ is arbitrary, by Lemma 2.1,Lemma 2.3 and Theorem 1.3, we can get

$$\limsup_{\sigma\to 0^-}\frac{\beta(\log^+ M(\sigma,f))}{\log U_2(\frac{1}{-\sigma})} \le T.$$

Suppose that

$$\limsup_{\sigma\to 0^+}\frac{\beta(\log^+ M(\sigma,f))}{\log U_2(\frac{1}{-\sigma})} = \eta < T.$$

Thus, there exists any real number $\varepsilon(0<\varepsilon<\frac{\eta}{2})$, for any positive integer n and any sufficient small $\sigma < 0$, from Lemma 2.2, we have

$$\log^+ |a_n| e^{\lambda_n\sigma} \le \log M(\sigma,f) \le \gamma\left((T-2\varepsilon)\log U_2(\frac{1}{-\sigma})\right). \tag{20}$$

From (15), there exists a subsequence $\{\lambda_{n(p)}\}$, for sufficiently large p, we have

$$\beta(\lambda_{n(p)}) > (T-\varepsilon)\log U_2\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\right). \tag{21}$$

Take a sequence $\{\sigma_p\}$ satisfying

$$\gamma\left((\eta-2\varepsilon)\log U_2(\frac{1}{-\sigma_p})\right)=\frac{\log^+ A_{n(p)}}{1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}. \tag{22}$$

From (20) and (22), we get

$$\log^+ A_{n(p)}+\lambda_{n(p)}\sigma_p\leq\gamma\left((\eta-2\varepsilon)\log U_2(\frac{1}{-\sigma_p})\right)=\frac{\log^+ A_{n(p)}}{1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})},$$

that is,

$$\frac{1}{-\sigma_p}\leq\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\frac{1}{\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}\right).$$

Thus, we have

$$U_2(\frac{1}{-\sigma_p})\leq U_2\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\frac{1}{\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}\right)\right)\leq U_2^{1+o(1)}\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\right). \tag{23}$$

From (22) and (23), we have

$$\begin{aligned}\lambda_{n(p)}&=\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\gamma\left((T-2\varepsilon)\log U_2(\frac{1}{\sigma_p})\right)\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\\&=\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\gamma\left((\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right).\end{aligned}$$

Thus, from the Cauchy mean value theorem, there exists a real number ξ between $\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\gamma(\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})$ and $\gamma(\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})$ such that

$$\begin{aligned}\beta\left(\lambda_{n(p)}\right)&=\beta\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\gamma\left((\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)\\&=\beta\left(\gamma\left((T-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)\\&\quad+\log\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)\xi\beta'(\xi),\end{aligned}$$

Since

$$\lim_{p\to\infty}\frac{\log\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)}{\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}=0,$$

then for sufficiently large p, we have

$$\beta\left(\lambda_{n(p)}\right)=(\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})+K_2\xi\beta'(\xi)\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}), \tag{24}$$

where K_2 is a constant.

From (21),(24) and $\eta < T$, we can get a contradiction. Thus, we can get

$$\limsup_{\sigma\to 0^-}\frac{\beta(\log^+ M(\sigma,f))}{\log U_2(\frac{1}{-\sigma})}=T.$$

Hence, the sufficiency of Theorem 1.4 is completed.

We can prove the necessity of Theorem 1.4 by using the similar argument as in the proof of the sufficiency of Theorem 1.4.

Thus, the proof of Theorem 1.4 is completed.

4 The Proof of Theorem 1.5

We will consider two steps as follows:

Step one: We first prove the sufficiency of Theorem 1.5. From the conditions of Theorem 1.5, for any $\varepsilon(>0)$, there exists a subsequence $\{\lambda_{n(p)}\}$ such that

$$\lambda_{n(p)}\geq \gamma\left((T-\varepsilon)\log U_2\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\right)\right),\qquad \lim_{p\to\infty}\frac{\beta(\lambda_{n(p)})}{\beta(\lambda_{n(p+1)})}=1, \tag{25}$$

that is,

$$\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\leq V_2\left(\exp\left\{\frac{1}{T-\varepsilon}\beta(\lambda_{n(p)})\right\}\right),\quad \log^+ A_{n(p)}\geq \lambda_{n(p)}V_2\left(\exp\left\{\frac{1}{T-\varepsilon}\beta(\lambda_{n(p)})\right\}\right)^{-1}.$$

Take the sequence $\{\sigma_p\}$ satisfying

$$\begin{aligned}\lambda_{n(p)}&=\gamma\left((T-\varepsilon)\log U_2\left(\frac{1}{-\sigma_p}+\frac{1}{\sigma_p\log U_2(\frac{1}{-\sigma_p})}\right)\right),\\ \frac{1}{-\sigma_p}+\frac{1}{\sigma_p\log U_2(\frac{1}{-\sigma_p})}&=V_2\left(\exp\left\{\frac{1}{T-\varepsilon}\beta(\lambda_{n(p)})\right\}\right).\end{aligned} \tag{26}$$

For any sufficiently small $\sigma<0$ and $-\infty<\alpha<\sigma<0$, we have

$$E_{n-1}(f,\alpha)\leq ||f-p_{n-1}||_\alpha\leq\sum_{k=n}^{\infty}|a_k|e^{\lambda_k\alpha}\leq M(\sigma,f)\sum_{k=n}^{\infty}e^{\lambda_n(\alpha-\sigma)}, \tag{27}$$

where $p_{n-1}(s)=\sum_{k=1}^{n-1}a_ke^{\lambda_k s}$. From (3), we take $0<h'<h$ satisfying $\lambda_{n+1}-\lambda_n\geq h'$ for any integer $n\geq 1$. Thus, for sufficiently small $\sigma<0$ such that $\sigma\geq\frac{\alpha}{2}$, from (27) we have

$$\begin{aligned}E_{n-1}(f,\alpha)&\leq M(\sigma,f)e^{\lambda_n(\alpha-\sigma)}\sum_{k=n}^{\infty}e^{(\lambda_k-\lambda_n)(\alpha-\sigma)}\\ &\leq M(\sigma,f)e^{\lambda_n(\alpha-\sigma)}e^{-\frac{\alpha}{2}h'n}\sum_{k=n}^{\infty}e^{\frac{\alpha}{2}h'k}\\ &=M(\sigma,f)e^{\lambda_n(\alpha-\sigma)}\left(1-e^{\frac{\alpha}{2}h'}\right)^{-1}.\end{aligned}$$

Then for sufficiently small $\sigma<0$ and $-\infty<\alpha<\sigma<0$, we have

$$M(\sigma,f)\geq K_3E_{n-1}(f,\alpha)e^{-\lambda_n(\alpha-\sigma)}=K_3A_ne^{\lambda_n\sigma}, \tag{28}$$

where $K_3 = 1 - e^{\frac{\alpha}{2}h'}$. For sufficiently small $\sigma < 0$, we take $\sigma_p \le \sigma < \sigma_{p+1}$, from (25),(26) and (28), we have

$$\begin{aligned}
\log^+ M(\sigma, f) &\ge \log^+ A_{n(p)} + \lambda_{n(p)}\sigma_p + O(1) \qquad (29)\\
&\ge \lambda_{n(p)} \left(V_2 \left(\exp\left\{ \frac{1}{T-\varepsilon}\beta(\lambda_{n(p)}) \right\} \right)^{-1} + \sigma_p \right) + O(1)\\
&\ge \gamma \left((T-\varepsilon)\log U_2 \left(\frac{1}{-\sigma_p} + \frac{1}{\sigma_p \log U_2(\frac{1}{-\sigma_p})} \right) \right) \frac{-\sigma_p}{\log U_2(\frac{1}{-\sigma_p}) - 1} + O(1)\\
&\ge (1+o(1))\gamma \left((T-\varepsilon)\log U_2 \left(\frac{1}{-\sigma_{p+1}} + \frac{1}{\sigma_{p+1} \log U_2(\frac{1}{-\sigma_{p+1}})} \right) \right) \frac{-\sigma_p}{\log U_2(\frac{1}{-\sigma_p}) - 1}\\
&\ge (1+o(1))\gamma \left((T-\varepsilon)\log U_2 \left(\frac{1}{-\sigma} + \frac{1}{\sigma \log U_2(\frac{1}{-\sigma})} \right) \right) \frac{-\sigma}{\log U_2(\frac{1}{-\sigma}) - 1}.
\end{aligned}$$

Set

$$\frac{1}{-\sigma} + \frac{1}{\sigma \log U_2(\frac{1}{-\sigma})} = r, \quad r\left(1 + \frac{1}{\log U_2(r)}\right) = R, \quad R\left(1 + \frac{1}{\log U_2(R)}\right) = R',$$

by using a simple calculation, we can get $R' \ge \frac{1}{-\sigma}$. Thus, from the definitions of $U_2(x)$ (ii), we can get

$$\limsup_{\sigma\to 0^-} \frac{\log U_2(r)}{\log U_2(\frac{1}{-\sigma})} = 1. \qquad (30)$$

Since

$$\limsup_{\sigma\to 0^-} \frac{\log \frac{-\sigma}{\log U_2(\frac{1}{-\sigma})-1}}{\log U_2(\frac{1}{-\sigma})} = 0,$$

and from Lemma 2.1, (29) and (30), we have

$$\limsup_{\sigma\to 0^-} \frac{\beta(\log^+ M(\sigma, f))}{\log U_2(\frac{1}{-\sigma})} = T.$$

Step two: The necessity of the Theorem 1.5 will be proved as follows. From Theorem 1.4, we can get that the right hand of (7) is verified. Next, we will prove that (8) also holds. We take a positive decreasing sequence $\{\varepsilon_i\}(0 < \varepsilon_i < T)$,$\varepsilon_i \to 0(i\to\infty)$.

Set

$$F_i = \left\{ n : \Psi_n(f, \alpha, \lambda_n) = \frac{\beta(\lambda_n)}{\log U_2\left(\frac{\lambda_n}{\log^+ A_n}\right)} > T - \varepsilon_i \right\}, \qquad (31)$$

it follows that $\forall i, F_i \ne \Phi$ and $F_i \subset F_{i-1}$. For each i, we arrange the $n(\in F_i)$ in an increasing sequence $\{n^{(i)}(p)\}_{p=1}^{\infty}$, then we consider the two cases in the following.

Case 1. Suppose that $\lim_{\nu\to+\infty} \frac{\beta(\lambda_{n^{(i)}(p+1)})}{\beta(\lambda_{n^{(i)}(p)})} = 1$ for any i. Then there exists $N_i \in F_i (i \in N_+)$, when $n^{(i)}(p) \ge N_i$, we have

$$\frac{\beta\left(\lambda_{n^{(i)}(p+1)}\right)}{\beta\left(\lambda_{n^{(i)}(p)}\right)} \le 1 + \varepsilon_i. \qquad (32)$$

Note $F_{i+1} \subset F_i$, take $N_{i+1} > N_i$, denote F_i' the subset of F_i

$$F_i' = \{n \in F_i : N_i \le n \le N_{i+1}\},$$

thus the elements of F_i' satisfy (31) and (32).

Therefore let $F=\bigcup_{i=1}^{\infty} F_i'$ and arrange the $n(\in E_i')$ in an increasing sequence $\{n_\nu\}$. Thus, the necessity of Theorem 1.5 is proved.

Case 2. If there exists $i\in N_+$ satisfying $\lim_{\nu\to+\infty}\frac{\beta\left(\lambda_{n^{(i)}(p+1)}\right)}{\beta\left(\lambda_{n^{(i)}(p)}\right)}\neq 1$, then since $\lambda_{n^{(i)}(p+1)}>\lambda_{n^{(i)}(p)}$, we get $\lim_{\nu\to+\infty}\frac{\beta(\lambda_{n^{(i)}(p+1)})}{\beta(\lambda_{n^{(i)}(p)})}>1$. Hence there exists $\{n^{(i)}(p_k)\}\subseteq\{n^{(i)}(p)\}$ (still marked with $\{n^{(i)}(p)\}$) and positive real constant $\tau>0$, it follows that

$$\frac{\beta\left(\lambda_{n^{(i)}(p+1)}\right)}{\beta\left(\lambda_{n^{(i)}(p)}\right)}\geq 1+\tau.$$

Let

$$n'(1)=n^{(i)}(1),n'(2)=n^{(i)}(3),\cdots,n'(p)=n^{(i)}(2p-1),\cdots$$
$$n''(1)=n^{(i)}(1),n''(2)=n^{(i)}(4),\cdots,n''(p)=n^{(i)}(2p),\cdots$$

where $\{n'(p)\},\{n''(p)\}$ are two increasing positive integer sequences, and

$$n''(p)<n'(p+1),\quad \beta(\lambda_{n''(p)})>(1+\tau)\beta(\lambda_{n'(p)}),\quad \nu=1,2,\cdots.$$

From (31), for any sufficiently large p, when $n\notin F_i$ satisfies $n'(p)<n<n''(p)$, there exists a positive real number $\delta>0$ such that

$$\lambda_n\leq\gamma\left((T-\delta)\log U_2(\frac{\lambda_n}{\log^+ A_n})\right),\quad \frac{\lambda_n}{\log^+ A_n}\geq V_2\left(\exp\{\frac{1}{T-\delta}\beta(\lambda_n)\}\right). \tag{33}$$

Thus we have

$$\log^+ A_n e^{\sigma\lambda_n}<\lambda_n\left(\frac{1}{V_2\left(\exp\{\frac{1}{T-\delta}\beta(\lambda_n)\}\right)}+\sigma\right). \tag{34}$$

Set

$$G=\gamma\left((T-\delta)\log U_2\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right),$$

that is,

$$\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}=V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(G)\right\}\right). \tag{35}$$

If $\lambda_n\geq G$, from (34) and (35), we have

$$\log^+ A_n e^{\sigma\lambda_n}\leq\lambda_n\left(\frac{1}{V_2\left(\exp\{\frac{1}{T-\delta}\beta(\lambda_n)\}\right)}+\sigma\right)<0. \tag{36}$$

If $\lambda_n<G$, from (34) and (35), we have

$$\log^+|a_n|e^{\sigma\lambda_n}<G=\gamma\left((T-\delta)\log U_2\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right). \tag{37}$$

Choose the sequence $\{\sigma_p\}$ satisfying

$$\sigma_p=-\left[V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(\lambda_{n''(p)})\right\}\right)\right]^{-1}, \tag{38}$$

from the assumptions of the necessity of Theorem 1.5, there exists an integer $N_2 \in N_+$ such that $V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(\lambda_n)\right\}\right) \geq 1$. Then for $n \geq N_2$, we have

$$\log^+ A_n e^{\sigma_p \lambda_n} < \lambda_n \left(V_2 \left(\exp\left\{ \frac{1}{T-\delta}\beta(\lambda_n) \right\} \right)^{-1} + \sigma_p \right).$$

When $n \geq n''(p)$, it follows $\lambda_n \geq \lambda_{n''(p)}$, and from (38), we have

$$\log^+ A_n e^{\sigma_p \lambda_n} < \lambda_n \left(V_2 \left(\exp\left\{ \frac{1}{T-\delta}\beta(\lambda_{n''(p)}) \right\} \right)^{-1} + \sigma_p \right) = 0. \tag{39}$$

For sufficiently large ν, we have $\lambda_{n'(p)} \geq \lambda_n$ as $N_2 \leq n \leq n'(p)$, and

$$\log^+ A_n e^{\sigma_p \lambda_n} \leq \lambda_{n'(p)} \left(V_2 \left(\exp\{ \frac{1}{T-\delta}\beta(\lambda_n)\} \right)^{-1} + \sigma_p \right).$$

Since $\lambda_{n'(p)} < \gamma\left(\frac{1}{1+\tau}\beta(\lambda_{n''(p)})\right)$ and $\sigma_p < 0$, from the definition of σ_p, N_2, we can get

$$\log^+ A_n e^{\sigma_\nu \lambda_n} \leq \gamma\left(\frac{1}{1+\tau}\beta(\lambda_{n''(p)})\right) \leq \gamma\left(\frac{T-\delta}{1+\tau}\log U_2\left(\frac{1}{-\sigma_p}\right)\right). \tag{40}$$

Thus, from (36), (37), (39) and (40), we have

$$\log^+ A_n e^{\sigma_p \lambda_n} \leq \gamma\left((T-\delta)\log U_2\left(\frac{1}{-\sigma} + \frac{1}{-\sigma \log U_2\left(\frac{1}{-\sigma}\right)}\right)\right), \quad as \quad n > N_2.$$

By Lemma 2.2, we have

$$\lim_{\sigma_p \to 0^-} \frac{\beta(\log^+ m(\sigma_p, f))}{\log U_2\left(\frac{1}{-\sigma_p}\right)} \leq T-\delta < T. \tag{41}$$

From (41), Theorem 1.3, we can get a contradiction with the following equality

$$\lim_{\sigma \to 0^-} \frac{\beta(\log^+ M(\sigma, f))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T.$$

Thus, the proof of Theorem 1.5 is completed by Step one and Step two.

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On special fuzzy differential subordinations using multiplier transformation

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Abstract

In the present paper we establish several fuzzy differential subordinations regardind the operator $I(m,\lambda,l)$, given by $I(m,\lambda,l):\mathcal{A}\to\mathcal{A}$, $I(m,\lambda,l)f(z)=z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j z^j$ and $\mathcal{A}=\{f\in\mathcal{H}(U),\ f(z)=z+a_2z^2+\dots,\ z\in U\}$ is the class of normalized analytic functions. A certain fuzzy class, denoted by $SI_{\mathcal{F}}^{\delta}(m,\lambda,l)$, of analytic functions in the open unit disc is introduced by means of this operator. By making use of the concept of fuzzy differential subordination we will derive various properties and characteristics of the class $SI_{\mathcal{F}}^{\delta}(m,\lambda,l)$. Also, several fuzzy differential subordinations are established regarding the operator $I(m,\lambda,l)$.

Keywords: fuzzy differential subordination, convex function, fuzzy best dominant, differential operator.
2000 Mathematical Subject Classification: 30C45, 30A20.

1 Introduction

S.S. Miller and P.T. Mocanu have introduced [10], [11] and developed [12] in the one complex variable functions theory the admissible functions method known as "the differential subordination method" . The application of this method allows to one obtain some special results and to prove easily some classical results from this domain.

G.I. Oros and Gh.Oros [13], [14] wanted to launch a new research direction in mathematics that combines the notions from the complex functions domain with the fuzzy sets theory.

In the same way as mentioned, we can justify that by knowing the properties of a differential expression on a fuzzy set for a function one can be determined the properties of that function on a given fuzzy set. We have analyzed the case of one complex functions, leaving as "open problem" the case of real functions. We are aware that this new research alternative can be realized only through the joint effort of researchers from both domains. The "open problem" statement leaves open the interpretation of some notions from the fuzzy sets theory such that each one interpret them personally according to their scientific concerns, making this theory more attractive.

The notion of fuzzy subordination was introduced in [13]. In [14] the authors have defined the notion of fuzzy differential subordination. In this paper we will study fuzzy differential subordinations obtained with the differential operator studied in [3] using the methods from [4], [5].

Denote by U the unit disc of the complex plane, $U=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\dots,\ z\in U\}$ with $\mathcal{A}_1=\mathcal{A}$ and $\mathcal{H}[a,n]=\{f\in\mathcal{H}(U):f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\dots,\ z\in U\}$ for $a\in\mathbb{C}$ and $n\in\mathbb{N}$.

Denote by $\mathcal{K}=\left\{f\in\mathcal{A}:\operatorname{Re}\frac{zf''(z)}{f'(z)}+1>0,\ z\in U\right\}$, the class of normalized convex functions in U.

In order to use the concept of fuzzy differential subordination, we remember the following definitions:

Definition 1.1 *[9] A pair (A,F_A), where $F_A:X\to[0,1]$ and $A=\{x\in X:0<F_A(x)\le 1\}$ is called fuzzy subset of X. The set A is called the support of the fuzzy set (A,F_A) and F_A is called the membership function of the fuzzy set (A,F_A). One can also denote $A=\operatorname{supp}(A,F_A)$.*

Remark 1.1 *In the development work we use the following notations for fuzzy sets:*
$F_{f(D)}(f(z)) = supp\big(f(D), F_{f(D)}\cdot\big) = \{z \in D : 0 < F_{f(D)}f(z) \le 1\}$,
$F_{g(D)}(g(z)) = supp\big(g(D), F_{g(D)}\cdot\big) = \{z \in D : 0 < F_{g(D)}g(z) \le 1\}$,
$p(U) = supp\big(p(U), F_{p(U)}\cdot\big) = \{z \in U : 0 < F_{p(U)}(p(z)) \le 1\}$,
$q(U) = supp\big(q(U), F_{q(U)}\cdot\big) = \{z \in U : 0 < F_{q(U)}(q(z)) \le 1\}$,
$h(U) = supp\big(h(U), F_{h(U)}\cdot\big) = \{z \in U : 0 < F_{h(U)}(h(z)) \le 1\}$.

We give a new definition of membership function on complex numbers set using the module notion of a complex number $z = x + iy$, $x, y \in \mathbb{R}$, $|z| = \sqrt{x^2 + y^2} \ge 0$.

Example 1.1 *Let $F : \mathbb{C} \to \mathbb{R}_+$ a function such that $F_{\mathbb{C}}(z) = |F(z)|$, $\forall\ z \in \mathbb{C}$. Denote by $F_{\mathbb{C}}(\mathbb{C}) = \{z \in \mathbb{C} : 0 < F(z) \le 1\} = \{z \in \mathbb{C} : 0 < |F(z)| \le 1\} = supp(\mathbb{C}, F_{\mathbb{C}})$ the fuzzy subset of the complex numbers set.*

Remark 1.2 *We call the subset $F_{\mathbb{C}}(\mathbb{C}) = \{z \in \mathbb{C} : 0 < |F(z)| \le 1\} = U_{\mathcal{F}}(0,1)$ the fuzzy unit disk.*

Example 1.2 *Let $F : \mathbb{C} \to \mathbb{R}_+$, $F(z) = \frac{2-|z|}{2+|z|}$, where $|z| = \sqrt{x^2+y^2} \ge 0$. A fuzzy subset of the complex numbers set is $A = \{z \in \mathbb{C} : 0 < F_A(z) \le 1\} = supp(A, F_A) = \{z \in \mathbb{C} : |z| < 2\}$, where $F_A(z) = \begin{cases} F(z), & z \in \{|z| \le 2\} \\ 0, & z \in \mathbb{C} - \{|z| \le 2\}. \end{cases}$*

We show that the fuzzy subset is nonempty. Indeed, for $z = 0$, $F_A(0) = F(0) = 1$, so $z = 0 \in A$. More we see that the fuzzy subset A contains all the complex numbers with the properties $|z| < 2$ and all the complex numbers for which $|z| > 2$ not belong to A, i.e. $supp(A, F_A) = \{z \in \mathbb{C} : x^2 + y^2 < 4\}$.

Remark 1.3 *The membership functions can be defined otherwise and we propose that each choose how to define according to their research.*

Definition 1.2 *([13]) Let $D \subset \mathbb{C}$, $z_0 \in D$ be a fixed point and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z)$, if are satisfied the conditions:*
1) $f(z_0) = g(z_0)$,
2) $F_{f(D)}f(z) \le F_{g(D)}g(z)$, $z \in D$.

Definition 1.3 *([14, Definition 2.2]) Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h univalent in U, with $\psi(a, 0; 0) = h(0) = a$. If p is analytic in U, with $p(0) = a$ and satisfies the (second-order) fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3\times U)}\psi(p(z), zp'(z), z^2p''(z); z) \le F_{h(U)}h(z), \quad z \in U, \tag{1.1}$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $F_{p(U)}p(z) \le F_{q(U)}q(z)$, $z \in U$, for all p satisfying (1.1). A fuzzy dominant $\widetilde{q}$ that satisfies $F_{\widetilde{q}(U)}\widetilde{q}(z) \le F_{q(U)}q(z)$, $z \in U$, for all fuzzy dominants q of (1.1) is said to be the fuzzy best dominant of (1.1).

Lemma 1.1 *([12, Corollary 2.6g.2, p. 66]) Let $h \in \mathcal{A}$ and $L[f](z) = G(z) = \frac{1}{z}\int_0^z h(t)\,dt$, $z \in U$. If $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$, $z \in U$, then $L(f) = G \in \mathcal{K}$.*

Lemma 1.2 *([15]) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\,\gamma \ge 0$. If $p \in \mathcal{H}[a, n]$ with $p(0) = a$, $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = p(z) + \frac{1}{\gamma}zp'(z)$ an analytic function in U and $F_{\psi(\mathbb{C}^2\times U)}\left(p(z) + \frac{1}{\gamma}zp'(z)\right) \le F_{h(U)}h(z)$, i.e. $p(z) + \frac{1}{\gamma}zp'(z) \prec_{\mathcal{F}} h(z)$, $z \in U$, then $F_{p(U)}p(z) \le F_{g(U)}g(z) \le F_{h(U)}h(z)$, i.e. $p(z) \prec_{\mathcal{F}} g(z) \prec_{\mathcal{F}} h(z)$, $z \in U$, where $g(z) = \frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{\gamma/n-1}dt$, $z \in U$. The function q is convex and is the fuzzy best dominant.*

Lemma 1.3 *([15]) Let g be a convex function in U and let $h(z) = g(z) + n\alpha zg'(z)$, $z \in U$, where $\alpha > 0$ and n is a positive integer. If $p(z) = g(0) + p_nz^n + p_{n+1}z^{n+1} + \ldots$, $z \in U$, is holomorphic in U and $F_{p(U)}(p(z) + \alpha zp'(z)) \le F_{h(U)}h(z)$, i.e. $p(z) + \alpha zp'(z) \prec_{\mathcal{F}} h(z)$, $z \in U$, then $F_{p(U)}p(z) \le F_{g(U)}g(z)$, i.e. $p(z) \prec_{\mathcal{F}} g(z), z \in U$, and this result is sharp.*

We will study the following differential operator, known as multiplier transformation.

Definition 1.4 *For* $f \in \mathcal{A} = \{f \in \mathcal{H}(U) : f(z) = z + a_2 z^2 + \dots,\ z \in U\}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, *the operator* $I(m, \lambda, l) f(z)$ *is defined by the following infinite series* $I(m, \lambda, l) f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{\lambda(j-1)+l+1}{l+1}\right)^m a_j z^j$.

Remark 1.4 *It follows from the above definition that* $(l+1) I(m+1, \lambda, l) f(z) = [l+1-\lambda] I(m, \lambda, l) f(z) + \lambda z (I(m, \lambda, l) f(z))'$, $z \in U$.

Remark 1.5 *For* $l = 0$, $\lambda \geq 0$, *the operator* $D_\lambda^m = I(m, \lambda, 0)$ *was introduced and studied by Al-Oboudi [2], which is reduced to the Sălăgean differential operator [16] for* $\lambda = 1$. *The operator* $I(m, 1, l)$ *was studied by Cho and Srivastava [8] and Cho and Kim [7]. The operator* $I(m, 1, 1)$ *was studied by Uralegaddi and Somanatha [17] and the operator* $I(\alpha, \lambda, 0)$ *was introduced by Acu and Owa [1]. Cătaş [6] has studied the operator* $I_p(m, \lambda, l)$ *which generalizes the operator* $I(m, \lambda, l)$.

2 Main results

Using the operator $I(m, \lambda, l)$ we define the class $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ and we study fuzzy subordinations.

Definition 2.1 *Let* $f(D) = supp\big(f(D), F_{f(D)}\big) = \{z \in D : 0 < F_{f(D)} f(z) \leq 1\}$, *where* $F_{f(D)} \cdot$ *is the membership function of the fuzzy set* $f(D)$ *asociated to the function* f.

The membership function of the fuzzy set $(\mu f)(D)$ *asociated to the function* μf *coincide with the membership function of the fuzzy set* $f(D)$ *asociated to the fuction* f, *i.e.* $F_{(\mu f)(D)}((\mu f)(z)) = F_{f(D)} f(z)$, $z \in D$.

The membership function of the fuzzy set $(f+g)(D)$ *asociated to the function* $f+g$ *coincide with the half of the sum of the membership functions of the fuzzy sets* $f(D)$, *respectively* $g(D)$, *asociated to the function* f, *respectively* g, *i.e.* $F_{(f+g)(D)}((f+g)(z)) = \frac{F_{f(D)} f(z) + F_{g(D)} g(z)}{2}$, $z \in D$.

Remark 2.1 $F_{(f+g)(D)}((f+g)(z))$ *can be defined in other ways.*

Remark 2.2 *Since* $0 < F_{f(D)} f(z) \leq 1$ *and* $0 < F_{g(D)} g(z) \leq 1$, *it is evidently that* $0 < F_{(f+g)(D)}((f+g)(z)) \leq 1$, $z \in D$.

Definition 2.2 *Let* $\delta \in (0, 1]$, $\lambda, l \geq 0$ *and* $m \in \mathbb{N}$. *A function* $f \in \mathcal{A}$ *is said to be in the class* $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ *if it satisfies the inequality* $F_{(I(m,\lambda,l)f)'(U)} (I(m, \lambda, l) f(z))' > \delta$, $z \in U$.

Theorem 2.1 *The set* $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ *is convex.*

Proof. Let the functions $f_j(z) = z + \sum_{j=2}^{\infty} a_{jk} z^j$, $k = 1, 2$, $z \in U$, be in the class $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$. It is sufficient to show that the function $h(z) = \eta_1 f_1(z) + \eta_2 f_2(z)$ is in the class $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ with η_1 and η_2 nonnegative such that $\eta_1 + \eta_2 = 1$.

We have $h'(z) = (\mu_1 f_1 + \mu_2 f_2)'(z) = \mu_1 f_1'(z) + \mu_2 f_2'(z)$, $z \in U$, and
$(I(m, \lambda, l) h(z))' = (I(m, \lambda, l)(\mu_1 f_1 + \mu_2 f_2)(z))' = \mu_1 (I(m, \lambda, l) f_1(z))' + \mu_2 (I(m, \lambda, l) f_2(z))'$.

From Definition 2.1 we obtain that
$F_{(I(m,\lambda,l)h)'(U)} (I(m, \lambda, l) h(z))' = F_{(I(m,\lambda,l)(\mu_1 f_1 + \mu_2 f_2))'(U)} (I(m, \lambda, l)(\mu_1 f_1 + \mu_2 f_2)(z))' =$
$F_{(I(m,\lambda,l)(\mu_1 f_1 + \mu_2 f_2))'(U)} \big(\mu_1 (I(m, \lambda, l) f_1(z))' + \mu_2 (I(m, \lambda, l) f_2(z))'\big) =$
$\frac{F_{(\mu_1 I(m,\lambda,l) f_1)'(U)} \left(\mu_1 (I(m,\lambda,l) f_1(z))'\right) + F_{(\mu_2 I(m,\lambda,l) f_2)'(U)} \left(\mu_2 (I(m,\lambda,l) f_2(z))'\right)}{2} =$
$\frac{F_{(I(m,\lambda,l) f_1)'(U)} (I(m,\lambda,l) f_1(z))' + F_{(I(m,\lambda,l) f_2)'(U)} (I(m,\lambda,l) f_2(z))'}{2}$.

Since $f_1, f_2 \in SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ we have $\delta < F_{(I(m,\lambda,l)f_1)'(U)} (I(m, \lambda, l) f_1(z))' \leq 1$ and
$\delta < F_{(I(m,\lambda,l)f_2)'(U)} (I(m, \lambda, l) f_2(z))' \leq 1$, $z \in U$.

Therefore $\delta < \frac{F_{(I(m,\lambda,l) f_1)'(U)} (I(m,\lambda,l) f_1(z))' + F_{(I(m,\lambda,l) f_2)'(U)} (I(m,\lambda,l) f_2(z))'}{2} \leq 1$ and we obtain that
$\delta < F_{(I(m,\lambda,l)h)'(U)} (I(m, \lambda, l) h(z))' \leq 1$, which means that $h \in SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ and $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ is convex. ∎

We highlight a fuzzy subset obtained using a convex function. Let the function $h(z) = \frac{1+z}{1-z}$, $z \in U$. After a short calculation we obtain that $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\frac{1+z}{1-z} > 0$, so $h \in \mathcal{K}$ and $h(U) = \{z \in \mathbb{C} : Rez > 0\}$. We define the membership function for the set $h(U)$ as $F_{h(U)}(h(z)) = Reh(z)$, $z \in U$ and we have $F_{h(U)} h(z) = \text{supp}\big(h(U), F_{h(u)}\big) = \{z \in \mathbb{C} : 0 < F_{h(U)}(h(z)) \leq 1\} = \{z \in U : 0 < Rez \leq 1\}$.

Remark 2.3 *In this case the membership function can be defined otherwise too and we recommend that those interested to make it in accordance with their scientific concern.*

Theorem 2.2 *Let g be a convex function in U and let $h(z) = g(z) + \frac{1}{c+2} z g'(z)$, where $z \in U$, $c > 0$. If $f \in SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ and $G(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t)\,dt$, $z \in U$, then*

$$F_{(I(m,\lambda,l)f)'(U)} \left(I(m,\lambda,l) f(z)\right)' \leq F_{h(U)} h(z), \quad i.e. \quad \left(I(m,\lambda,l) f(z)\right)' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.1}$$

implies $F_{(I(m,\lambda,l)G)'(U)} \left(I(m,\lambda,l) G(z)\right)' \leq F_{g(U)} g(z)$, i.e. $\left(I(m,\lambda,l) G(z)\right)' \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

Proof. We obtain that

$$z^{c+1} G(z) = (c+2) \int_0^z t^c f(t)\,dt. \tag{2.2}$$

Differentiating (2.2), with respect to z, we have $(c+1) G(z) + z G'(z) = (c+2) f(z)$ and

$$(c+1) I(m,\lambda,l) G(z) + z \left(I(m,\lambda,l) G(z)\right)' = (c+2) I(m,\lambda,l) f(z), \quad z \in U. \tag{2.3}$$

Differentiating (2.3) we have

$$\left(I(m,\lambda,l) G(z)\right)' + \frac{1}{c+2} z \left(I(m,\lambda,l) G(z)\right)'' = \left(I(m,\lambda,l) f(z)\right)', \quad z \in U. \tag{2.4}$$

Using (2.4), the fuzzy differential subordination (2.1) becomes

$$F_{I(m,\lambda,l)G(U)} \left(\left(I(m,\lambda,l) G(z)\right)' + \frac{1}{c+2} z \left(I(m,\lambda,l) G(z)\right)'' \right) \leq F_{g(U)} \left(g(z) + \frac{1}{c+2} z g'(z) \right). \tag{2.5}$$

If we denote

$$p(z) = \left(I(m,\lambda,l) G(z)\right)', \quad z \in U, \tag{2.6}$$

then $p \in \mathcal{H}[1,1]$.

Replacing (2.6) in (2.5) we obtain $F_{p(U)} \left(p(z) + \frac{1}{c+2} z p'(z) \right) \leq F_{g(U)} \left(g(z) + \frac{1}{c+2} z g'(z) \right)$, $z \in U$.

Using Lemma 1.3 we have $F_{p(U)} p(z) \leq F_{g(U)} g(z)$, $z \in U$, i.e. $F_{(I(m,\lambda,l)G)'(U)} \left(I(m,\lambda,l) G(z)\right)' \leq F_{g(U)} g(z)$, $z \in U$, and g is the fuzzy best dominant. We have obtained that $\left(L_{\alpha}^m G(z)\right)' \prec_{\mathcal{F}} g(z)$, $z \in U$. ■

Example 2.1 *If $f \in SI_{\mathcal{F}}^{\frac{1}{2}}\left(1, \frac{1}{2}, \frac{1}{2}\right)$, then $f'(z) + \frac{1}{3} z f''(z) \prec_{\mathcal{F}} \frac{3-2z}{3(1-z)^2}$ implies $G'(z) + \frac{1}{3} z G''(z) \prec_{\mathcal{F}} \frac{1}{1-z}$, where $G(z) = \frac{3}{z^2} \int_0^z t f(t)\,dt$.*

Theorem 2.3 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$, $\beta \in [0,1)$ and $c > 0$. If $\lambda, l \geq 0$, $m \in \mathbb{N}$ and $I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t)\,dt$, $z \in U$, then*

$$I_c \left[SI_{\mathcal{F}}^{\beta}(m,\lambda,l) \right] \subset SI_{\mathcal{F}}^{\beta^*}(m,\lambda,l), \tag{2.7}$$

where $\beta^* = 2\beta - 1 + (c+2)(2-2\beta) \int_0^1 \frac{t^{c+1}}{t+1}\,dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2.2 we get from the hypothesis of Theorem 2.3 that $F_{p(U)} \left(p(z) + \frac{1}{c+2} z p'(z) \right) \leq f_{h(U)} h(z)$, where $p(z)$ is defined in (2.6). Using Lemma 1.2 we deduce that $F_{p(U)} p(z) \leq F_{g(U)} g(z) \leq F_{h(U)} h(z)$, i.e. $F_{(I(m,\lambda,l)G)'(U)} \left(I(m,\lambda,l) G(z)\right)' \leq F_{g(U)} g(z) \leq F_{h(U)} h(z)$, where $g(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} \frac{1+(2\beta-1)t}{1+t}\,dt = 2\beta - 1 + \frac{(c+2)(2-2\beta)}{z^{c+2}} \int_0^z \frac{t^{c+1}}{t+1}\,dt$. Since g is convex and $g(U)$ is symmetric with respect to the real axis, we deduce

$$F_{I(m,\lambda,l)G(U)} \left(I(m,\lambda,l) G(z)\right)' \geq \min_{|z|=1} F_{g(U)} g(z) = F_{g(U)} g(1) \tag{2.8}$$

and $\beta^* = g(1) = 2\beta - 1 + (c+2)(2-2\beta) \int_0^1 \frac{t^{c+1}}{t+1}\,dt$.

From (2.8) we deduce inclusion (2.7). ■

Theorem 2.4 *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + z g'(z)$, $z \in U$. If $\lambda, l \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the fuzzy differential subordination*

$$F_{(I(m,\lambda,l)f)'(U)} \left(I(m,\lambda,l) f(z)\right)' \leq F_{h(U)} h(z), \quad i.e. \quad \left(I(m,\lambda,l) f(z)\right)' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.9}$$

then $F_{I(m,\lambda,l)f(U)} \frac{I(m,\lambda,l) f(z)}{z} \leq F_{g(U)} g(z)$, i.e. $\frac{I(m,\lambda,l) f(z)}{z} \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

Proof. Consider $p(z) = \frac{I(m,\lambda,l)f(z)}{z} = \frac{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j z^j}{z} = 1 + p_1 z + p_2 z^2 + ...,\quad z \in U$. We deduce that $p \in \mathcal{H}[1,1]$.

Let $I(m,\lambda,l) f(z) = zp(z)$, for $z \in U$. Differentiating we obtain $(I(m,\lambda,l) f(z))' = p(z) + zp'(z)$, $z \in U$. Then (2.9) becomes $F_{p(U)}(p(z) + zp'(z)) \le F_{h(U)}h(z) = F_{g(U)}(g(z) + zg'(z))$, $z \in U$.

By using Lemma 1.3, we have $F_{p(U)}p(z) \le F_{g(U)}g(z)$, $z \in U$, i.e. $F_{(I(m,\lambda,l)f)'(U)}\frac{I(m,\lambda,l)f(z)}{z} \le F_{g(U)}g(z)$, $z \in U$.We obtain that $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp. ■

Theorem 2.5 *Let h be an holomorphic function which satisfies the inequality* $\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, *and $h(0) = 1$. If $\lambda, l \ge 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the fuzzy differential subordination*

$$F_{(I(m,\lambda,l)f)'(U)}(I(m,\lambda,l) f(z))' \le F_{h(U)}h(z), \quad i.e. \quad (I(m,\lambda,l) f(z))' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.10}$$

then $F_{I(m,\lambda,l)f(U)}\frac{I(m,\lambda,l)f(z)}{z} \le F_{q(U)}q(z)$, i.e. $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$, where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the fuzzy best dominant.

Proof. Let $p(z) = \frac{I(m,\lambda,l)f(z)}{z}$, $z \in U$, $p \in \mathcal{H}[1,1]$. Since $\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, from Lemma 1.1, we obtain that $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.10) $q(z) + zq'(z) = h(z)$, therefore it is the fuzzy best dominant.

Differentiating, we obtain $(I(m,\lambda,l) f(z))' = p(z)+zp'(z)$, $z \in U$ and (2.10) becomes $F_{p(U)}(p(z) + zp'(z)) \le F_{h(U)}h(z)$, $z \in U$.

Using Lemma 1.3, we have $F_{p(U)}p(z) \le F_{q(U)}q(z)$, $z \in U$, i.e. $F_{I(m,\lambda,l)f(U)}\frac{I(m,\lambda,l)f(z)}{z} \le F_{q(U)}q(z)$, $z \in U$. We have obtained that $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$. ■

Corollary 2.6 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ a convex function in U, $0 \le \beta < 1$. If $\lambda, l \ge 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and verifies the fuzzy differential subordination*

$$F_{(I(m,\lambda,l)f)'(U)}(I(m,\lambda,l) f(z))' \le F_{h(U)}h(z), \ i.e. \ (I(m,\lambda,l) f(z))' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.11}$$

then $F_{I(m,\lambda,l)f(U)}\frac{I(m,\lambda,l)f(z)}{z} \le F_{q(U)}q(z)$, i.e. $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$, where q is given by $q(z) = 2\beta - 1 + \frac{2(1-\beta)}{z}\ln(1+z)$, $z \in U$. The function q is convex and it is the fuzzy best dominant.

Proof. We have $h(z) = \frac{1+(2\beta-1)z}{1+z}$ with $h(0) = 1$, $h'(z) = \frac{-2(1-\beta)}{(1+z)^2}$ and $h''(z) = \frac{4(1-\beta)}{(1+z)^3}$, therefore $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-\rho\cos\theta - i\rho\sin\theta}{1+\rho\cos\theta+i\rho\sin\theta}\right) = \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} > 0 > -\frac{1}{2}$.

Following the same steps as in the proof of Theorem 2.5 and considering $p(z) = \frac{I(m,\lambda,l)f(z)}{z}$, the fuzzy differential subordination (2.11) becomes $F_{I(m,\lambda,l)f(U)}(p(z) + zp'(z)) \le F_{h(U)}h(z)$, $z \in U$.

By using Lemma 1.2 for $\gamma = 1$ and $n = 1$, we have $F_{p(U)}p(z) \le F_{q(U)}q(z)$, i.e., $F_{I(m,\lambda,l)f(U)}\frac{I(m,\lambda,l)f(z)}{z} \le F_{q(U)}q(z)$ and $q(z) = \frac{1}{z}\int_0^z h(t)\,dt = \frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t}dt = 2\beta - 1 + \frac{2(1-\beta)}{z}\ln(1+z)$, $z \in U$. ■

Example 2.2 *Let $h(z) = \frac{1-z}{1+z}$ with $h(0) = 1$, $h'(z) = \frac{-2}{(1+z)^2}$ and $h''(z) = \frac{4}{(1+z)^3}$.*

Since $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-\rho\cos\theta-i\rho\sin\theta}{1+\rho\cos\theta+i\rho\sin\theta}\right) = \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} > 0 > -\frac{1}{2}$, the function h is convex in U.

Let $f(z) = z + z^2$, $z \in U$. For $n = 1$, $m = 1$, $l = 2$, $\lambda = 1$, we obtain $I(1,1,2) f(z) = \frac{2}{3}f(z) + \frac{1}{3}zf'(z) = z + \frac{4}{3}z^2$. Then $(I(1,1,2) f(z))' = 1 + \frac{8}{3}z$ and $\frac{I(1,1,2)f(z)}{z} = 1 + \frac{4}{3}z$. We have $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = -1 + \frac{2\ln(1+z)}{z}$.

Using Theorem 2.5 we obtain $1 + \frac{8}{3}z \prec_{\mathcal{F}} \frac{1-z}{1+z}$, $z \in U$, induce $1 + \frac{4}{3}z \prec_{\mathcal{F}} -1 + \frac{2\ln(1+z)}{z}$, $z \in U$.

Theorem 2.7 *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$. If $\lambda, l \ge 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and the fuzzy differential subordination*

$$F_{I(m,\lambda,l)f(U)}\left(\frac{zI(m+1,\lambda,l) f(z)}{I(m,\lambda,l) f(z)}\right)' \le F_{h(U)}h(z), \quad i.e. \quad \left(\frac{zI(m+1,\lambda,l) f(z)}{I(m,\lambda,l) f(z)}\right)' \prec_{\mathcal{F}} h(z), \quad z \in U \tag{2.12}$$

holds, then $F_{I(m,\lambda,l)f(U)}\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} \le F_{g(U)}g(z)$, i.e. $\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

Proof. Consider $p(z)=\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)}$. We have $p'(z)=\frac{(I(m+1,\lambda,l)f(z))'}{I(m,\lambda,l)f(z)}-p(z)\cdot\frac{(I(m+1,\lambda,l)f(z))'}{I(m,\lambda,l)f(z)}$ and we obtain $p(z)+z\cdot p'(z)=\left(\frac{zI(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)}\right)'$.

Relation (2.12) becomes $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)=F_{g(U)}\left(g(z)+zg'(z)\right)$, $z\in U$. By using Lemma 1.3, we have $F_{p(U)}p(z)\leq F_{g(U)}g(z)$, $z\in U$, i.e. $F_{I(m,\lambda,l)f(U)}\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)}\leq F_{g(U)}g(z)$, $z\in U$. We obtain that $\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)}\prec_{\mathcal{F}} g(z)$, $z\in U$. ■

Theorem 2.8 *Let g be a convex function such that $g(0)=1$ and let h be the function $h(z)=g(z)+zg'(z)$, $z\in U$. If $\lambda,l\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}$ and the fuzzy differential subordination $F_{I(m,\lambda,l)f(U)}\left(\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\right)\leq F_{h(U)}h(z)$, i.e.*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\prec_{\mathcal{F}} h(z),\quad z\in U \tag{2.13}$$

holds, then $F_{I(m,\lambda,l)f(U)}[I(m,\lambda,l)f(z)]'\leq F_{g(U)}g(z)$, i.e. $[I(m,\lambda,l)f(z)]'\prec_{\mathcal{F}} g(z)$, $z\in U$. This result is sharp.

Proof. Let $p(z)=(I(m,\lambda,l)f(z))'$. We deduce that $p\in\mathcal{H}[1,1]$. We obtain $p(z)+z\cdot p'(z)=I(m,\lambda,l)f(z)+z(I(m,\lambda,l)f(z))'=I(m,\lambda,l)f(z)+\frac{(l+1)I(m+1,\lambda,l)f(z)-(l+1-\lambda)I(m,\lambda,l)f(z)}{\lambda}=\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)$.

The fuzzy differential subordination becomes $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)=F_{g(U)}\left(g(z)+zg'(z)\right)$. By using Lemma 1.3, we have $F_{p(U)}p(z)\leq F_{g(U)}g(z)$, $z\in U$, i.e. $F_{I(m,\lambda,l)f(U)}\left(I(m,\lambda,l)f(z)\right)'\leq F_{g(U)}g(z)$, $z\in U$, and this result is sharp. ■

Theorem 2.9 *Let h be an holomorphic function which satisfies the inequality* Re $\left[1+\frac{zh''(z)}{h'(z)}\right]>-\frac{1}{2}$, *$z\in U$, and $h(0)=1$. If $\lambda,l\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}$ and satisfies the fuzzy differential subordination $F_{I(m,\lambda,l)f(U)}\left(\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\right)\leq F_{h(U)}h(z)$, i.e.*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\prec_{\mathcal{F}} h(z),\quad z\in U, \tag{2.14}$$

then $F_{I(m,\lambda,l)f(U)}\left(I(m,\lambda,l)f(z)\right)'\leq F_{q(U)}q(z)$, i.e. $\left(I(m,\lambda,l)f(z)\right)'\prec_{\mathcal{F}} q(z)$, $z\in U$, where q is given by $q(z)=\frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the fuzzy best dominant.

Proof. Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we obtain that $q(z)=\frac{1}{z}\int_0^z h(t)dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.14) $q(z)+zq'(z)=h(z)$, therefore it is the fuzzy best dominant.
Considering $p(z)=\left(I(m,\lambda,l)f(z)\right)'$, we obtain $p(z)+zp'(z)=\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)$, $z\in U$. Then (2.14) becomes $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$.
Since $p\in\mathcal{H}[1,1]$, using Lemma 1.3, we deduce $F_{p(U)}p(z)\leq F_{q(U)}q(z)$, $z\in U$, i.e. $F_{I(m,\lambda,l)f(U)}\left(I(m,\lambda,l)f(z)\right)'\leq F_{q(U)}q(z)$, $z\in U$. We have obtained that $\left(I(m,\lambda,l)f(z)\right)'\prec_{\mathcal{F}} q(z)$, $z\in U$. ■

Corollary 2.10 *Let $h(z)=\frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0\leq\beta<1$.If $\lambda,l\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}$ and satisfies the differential subordination $F_{I(m,\lambda,l)f(U)}\left(\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\right)\leq F_{h(U)}h(z)$, i.e.*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\prec_{\mathcal{F}} h(z),\qquad z\in U, \tag{2.15}$$

then $F_{I(m,\lambda,l)f(U)}\left(I(m,\lambda,l)f(z)\right)'\leq F_{q(U)}q(z)$, i.e. $\left(I(m,\lambda,l)f(z)\right)'\prec_{\mathcal{F}} q(z)$, $z\in U$, where q is given by $q(z)=2\beta-1+2(1-\beta)\frac{\ln(1+z)}{z}$, for $z\in U$. The function q is convex and it is the fuzzy best dominant.

Proof. Following the same steps as in the proof of Theorem 2.8 and considering $p(z)=\left(I(m,\lambda,l)f(z)\right)'$, the fuzzy differential subordination (2.15) becomes $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$.

By using Lemma 1.2 for $\gamma=1$ and $n=1$, we have $F_{p(U)}p(z)\leq F_{q(U)}q(z)$, i.e., $F_{I(m,\lambda,l)f(U)}\left(I(m,\lambda,l)f(z)\right)'\leq F_{q(U)}q(z)$, i.e. $\left(I(m,\lambda,l)f(z)\right)'\prec_{\mathcal{F}} q(z)$, $z\in U$, and $q(z)=\frac{1}{z}\int_0^z h(t)dt=\frac{1}{z}\int_0^z\frac{1+(2\beta-1)t}{1+t}dt=2\beta-1+2(1-\beta)\frac{1}{z}\ln(z+1)$, $z\in U$. ■

Example 2.3 *Let* $h(z) = \frac{1-z}{1+z}$ *a convex function in* U *with* $h(0) = 1$ *and* $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$ *(see Example 2.2).*

Let $f(z) = z + z^2$, $z \in U$. *For* $n = 1$, $m = 1$, $l = 2$, $\lambda = 1$, *we obtain* $I(1,1,2) f(z) = \frac{2}{3} f(z) + \frac{1}{3} z f'(z) = z + \frac{4}{3} z^2$ *and* $(I(1,1,2) f(z))' = 1 + \frac{8}{3} z$. *We obtain also* $\frac{l+1}{\lambda} I(m+1,\lambda,l) f(z) + \left(2 - \frac{l+1}{\lambda}\right) I(m,\lambda,l) f(z) = 3I(2,1,2) f(z) - I(1,1,2) f(z) = 2z + 4z^2$, *where* $I(2,1,2) f(z) = \frac{2}{3} I(1,1,2) f(z) + \frac{z}{3}(I(1,1,2) f(z))' = 3z + \frac{16}{3} z^2$. *We have* $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t} dt = -1 + \frac{2\ln(1+z)}{z}$.

Using Theorem 2.9 we obtain $2z + 4z^2 \prec_{\mathcal{F}} \frac{1-z}{1+z}$, $z \in U$, *induce* $1 + \frac{8}{3} z \prec_{\mathcal{F}} -1 + \frac{2\ln(1+z)}{z}$, $z \in U$.

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On some differential sandwich theorems involving a multiplier transformation and Ruscheweyh derivative

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Abstract

In this paper we obtain some subordination and superordination results for the operator $IR_{\lambda,l}^{m,n}$ and we establish differential sandwich-type theorems. The operator $IR_{\lambda,l}^{m,n}$ is defined as the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and Ruscheweyh derivative R^n.

Keywords: analytic functions, differential operator, differential subordination, differential superordination.
2010 Mathematical Subject Classification: 30C45.

1 Introduction

Consider $\mathcal{H}(U)$ the class of analytic function in the open unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\mathcal{H}(a,n)$ the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ with $\mathcal{A} = \mathcal{A}_1$.

Next we remind the definition of differential subordination and superordination.

Let the functions f and g be analytic in U. The function f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad \text{for } z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \qquad z \in U, \tag{1.2}$$

then p is a solution of the differential superordination (1.2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Miller and Mocanu [6] obtained conditions h, q and ψ for which the following implication holds $h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z)$.

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc U, the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by $f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j$.

We need the following differential operators.

Definition 1.1 *[5] For $f \in \mathcal{A}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the multiplier transformation $I(m,\lambda,l) f(z)$ is defined by the following infinite series $I(m,\lambda,l) f(z) := z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{1+l}\right)^m a_j z^j$.*

Remark 1.1 *We have* $(l+1)\, I(m+1,\lambda,l)\, f(z) = (l+1-\lambda)\, I(m,\lambda,l)\, f(z) + \lambda z\,(I(m,\lambda,l)\, f(z))'$, $\quad z \in U$.

Remark 1.2 *For* $l=0$, $\lambda \geq 0$, *the operator* $D_\lambda^m = I(m,\lambda,0)$ *was introduced and studied by Al-Oboudi , which reduced to the Sălăgean differential operator* $S^m = I(m,1,0)$ *for* $\lambda = 1$.

Definition 1.2 *(Ruscheweyh [8]) For* $f \in \mathcal{A}$ *and* $n \in \mathbb{N}$, *the Ruscheweyh derivative* R^n *is defined by* $R^n : \mathcal{A} \to \mathcal{A}$,

$$\begin{aligned} R^0 f(z) &= f(z),\ R^1 f(z) = z f'(z),\ ... \\ (n+1)\, R^{n+1} f(z) &= z\,(R^n f(z))' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.3 *If* $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ *for* $z \in U$.

Definition 1.3 *([2]) Let* $\lambda, l \geq 0$ *and* $n, m \in \mathbb{N}$. *Denote by* $IR_{\lambda,l}^{m,n} : \mathcal{A} \to \mathcal{A}$ *the operator given by the Hadamard product of the multiplier transformation* $I(m,\lambda,l)$ *and the Ruscheweyh derivative* R^n, $IR_{\lambda,l}^{m,n} f(z) = (I(m,\lambda,l) * R^n)\, f(z)$, *for any* $z \in U$ *and each nonnegative integers* m, n.

Remark 1.4 *If* $f \in \mathcal{A}$ *and* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $IR_{\lambda,l}^{m,n} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.

Using simple computation we obtain the following relation.

Proposition 1.1 *[1]For* $m, n \in \mathbb{N}$ *and* $\lambda \geq 0$ *we have*

$$IR_{\lambda,l}^{m+1,n} f(z) = \frac{1+l-\lambda}{l+1} IR_{\lambda,l}^{m,n} f(z) + \frac{\lambda}{l+1} z \left(IR_{\lambda,l}^{m,n} f(z)\right)' \tag{1.3}$$

Definition 1.4 *[7] Denote by* Q *the set of all functions* f *that are analytic and injective on* $\overline{U} \backslash E(f)$, *where* $E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$, *and are such that* $f'(\zeta) \neq 0$ *for* $\zeta \in \partial U \backslash E(f)$.

Lemma 1.1 *[7] Let the function* q *be univalent in the unit disc* U *and* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$ *with* $\phi(w) \neq 0$ *when* $w \in q(U)$. *Set* $Q(z) = z q'(z)\, \phi(q(z))$ *and* $h(z) = \theta(q(z)) + Q(z)$. *Suppose that* Q *is starlike univalent in* U *and* $Re\left(\frac{z h'(z)}{Q(z)}\right) > 0$ *for* $z \in U$. *If* p *is analytic with* $p(0) = q(0)$, $p(U) \subseteq D$ *and* $\theta(p(z)) + z p'(z)\, \phi(p(z)) \prec \theta(q(z)) + z q'(z)\, \phi(q(z))$, *then* $p(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 1.2 *[4] Let the function* q *be convex univalent in the open unit disc* U *and* ν *and* ϕ *be analytic in a domain* D *containing* $q(U)$. *Suppose that* $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) > 0$ *for* $z \in U$ *and 2.* $\psi(z) = z q'(z)\, \phi(q(z))$ *is starlike univalent in* U. *If* $p(z) \in \mathcal{H}[q(0),1] \cap Q$, *with* $p(U) \subseteq D$ *and* $\nu(p(z)) + z p'(z)\, \phi(p(z))$ *is univalent in* U *and* $\nu(q(z)) + z q'(z)\, \phi(q(z)) \prec \nu(p(z)) + z p'(z)\, \phi(p(z))$, *then* $q(z) \prec p(z)$ *and* q *is the best subordinant.*

2 Main results

We intend to find sufficient conditions for certain normalized analytic functions f such that $q_1(z) \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z)$, $z \in U$, $0 < \delta \leq 1$, where q_1 and q_2 are given univalent functions.

Theorem 2.1 *Let* $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}(U)$ *and let the function* $q(z)$ *be analytic and univalent in* U *such that* $q(z) \neq 0$, *for all* $z \in U$. *Suppose that* $\frac{z q'(z)}{q(z)}$ *is starlike univalent in* U. *Let*

$$Re\left(\frac{\xi}{\beta} q(z) + \frac{2\mu}{\beta} q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right) > 0, \tag{2.1}$$

for $\alpha, \xi, \beta, \mu \in \mathbb{C}$, $\beta \neq 0$, $z \in U$ *and*

$$\psi_{\lambda,l}^{m,n}(\alpha,\xi,\mu,\beta; z) := \alpha + \beta\frac{(l+1)}{\lambda} + \beta\frac{(l+1)}{\lambda}\frac{IR_{\lambda,l}^{m+2,n} f(z)}{IR_{\lambda,l}^{m+1,n} f(z)} - \tag{2.2}$$

$$\beta\frac{(l+1)(1+\delta)}{\lambda}\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}+\xi\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\mu\frac{z^{2\delta}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+2\delta}}.$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi q(z)+\mu(q(z))^{2}+\beta\frac{zq'(z)}{q(z)}, \tag{2.3}$$

for $\alpha,\xi,\beta,\mu\in\mathbb{C}$, $\beta\neq0$, then $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q(z)$, and q is the best dominant.

Proof. Consider $p(z):=\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $z\in U$, $z\neq0$, $f\in\mathcal{A}$. Differentiating we obtain $p'(z)=\frac{\delta(1+l)}{\lambda}\frac{z^{\delta-1}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\frac{l+1}{\lambda}\frac{z^{\delta-1}IR_{\lambda,l}^{m+2,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\frac{(l+1)(1+\delta)}{\lambda}\frac{z^{\delta-1}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}$.

By using the identity (1.3), we obtain

$$\frac{zp'(z)}{p(z)}=\frac{\delta(l+1)}{\lambda}+\frac{l+1}{\lambda}\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}-\frac{(l+1)(1+\delta)}{\lambda}\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}. \tag{2.4}$$

By setting $\theta(w):=\alpha+\xi w+\mu w^{2}$ and $\phi(w):=\frac{\beta}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z)=zq'(z)\phi(q(z))=\beta\frac{zq'(z)}{q(z)}$ and $h(z)=\theta(q(z))+Q(z)=\alpha+\xi q(z)+\mu(q(z))^{2}+\beta\frac{zq'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in U.

We get $h'(z)=\xi q'(z)+2\mu q(z)q'(z)+\beta\frac{q'(z)}{q(z)}+\beta z\frac{q''(z)}{q(z)}-\beta z\left(\frac{q'(z)}{q(z)}\right)^{2}$ and $\frac{zh'(z)}{Q(z)}=\frac{\xi}{\beta}q(z)+\frac{2\mu}{\beta}q^{2}(z)+1+z\frac{q''(z)}{q(z)}-z\frac{q'(z)}{q(z)}$.

So we deduce that $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\xi}{\beta}q(z)+\frac{2\mu}{\beta}q^{2}(z)+1+z\frac{q''(z)}{q(z)}-z\frac{q'(z)}{q(z)}\right)>0$.

By using (2.4), we obtain $\alpha+\xi p(z)+\mu(p(z))^{2}+\beta\frac{zp'(z)}{p(z)}=$
$\alpha+\beta\frac{(l+1)}{\lambda}+\beta\frac{(l+1)}{\lambda}\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}-\beta\frac{(l+1)(1+\delta)}{\lambda}\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}+\xi\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\mu\frac{z^{2\delta}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+2\delta}}$.

By using (2.3), we have $\alpha+\xi p(z)+\mu(p(z))^{2}+\beta\frac{zp'(z)}{p(z)}\prec\alpha+\xi q(z)+\mu(q(z))^{2}+\beta\frac{zq'(z)}{q(z)}$.

Appying Lemma 1.1, we obtain $p(z)\prec q(z)$, $z\in U$, i.e. $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q(z)$, $z\in U$ and q is the best dominant. ■

Corollary 2.2 *Let $m,n\in\mathbb{N}$, $\lambda,l\geq0$. Assume that (2.1) holds. If $f\in\mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^{2}+\frac{\beta(A-B)z}{(1+Az)(1+Bz)}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\neq0$, $-1\leq B<A\leq1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec\frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq1$ in Theorem 2.1 we get the corollary. ■

Corollary 2.3 *Let $m,n\in\mathbb{N}$, $\lambda,l\geq0$. Assume that (2.1) holds. If $f\in\mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{1-z^{2}}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $0<\gamma\leq1$, $\beta\neq0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$, and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.*

Proof. Corollary follows by using Theorem 2.1 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq1$. ■

Theorem 2.4 *Let q be analytic and univalent in U such that $q(z)\neq0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that*

$$Re\left(\frac{\xi}{\beta}q(z)q'(z)+\frac{2\mu}{\beta}q^{2}(z)q'(z)\right)>0,\ \textit{for}\ \xi,\beta,\mu\in\mathbb{C},\ \beta\neq0. \tag{2.5}$$

If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}\left[q(0), 1\right] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is univalent in* U, *where* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (2.2), then*

$$\alpha + \xi q(z) + \mu (q(z))^2 + \frac{\beta z q'(z)}{q(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \tag{2.6}$$

implies $q(z) \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, *and* q *is the best subordinant.*

Proof. Consider $p(z) := \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)q(z)[\xi+2\mu q(z)]}{\beta}$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)\right) > 0$, for $\alpha,\beta,\mu \in \mathbb{C}$, $\mu \neq 0$.

By using (2.4) and (2.6) we get $\alpha + \xi q(z) + \mu (q(z))^2 + \frac{\beta z q'(z)}{q(z)} \prec \alpha + \xi p(z) + \mu (p(z))^2 + \frac{\beta z p'(z)}{p(z)}$. Applying Lemma 1.2, we obtain $q(z) \prec p(z) = \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, and q is the best subordinant. ■

Corollary 2.5 *Let* $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.5) holds. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}\left[q(0), 1\right] \cap Q$ *and* $\alpha + \xi \frac{1+Az}{1+Bz} + \mu \left(\frac{1+Az}{1+Bz}\right)^2 + \frac{\beta(A-B)z}{(1+Az)(1+Bz)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$, *for* $\alpha,\beta,\xi,\mu \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\frac{1+Az}{1+Bz} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, *and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.4 we get the corollary. ■

Corollary 2.6 *Let* $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.5) holds. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}\left[q(0), 1\right] \cap Q$ *and* $\alpha + \xi \left(\frac{1+z}{1-z}\right)^{\gamma} + \mu \left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\beta\gamma z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$, *for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.*

Proof. For $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$ in Theorem 2.4 we get the corollary. ■

Combining Theorem 2.1 and Theorem 2.4, we state the following sandwich theorem.

Theorem 2.7 *Let* q_1 *and* q_2 *be analytic and univalent in* U *such that* $q_1(z) \neq 0$ *and* $q_2(z) \neq 0$, *for all* $z \in U$, *with* $\frac{zq_1'(z)}{q_1(z)}$ *and* $\frac{zq_2'(z)}{q_2(z)}$ *being starlike univalent. Suppose that* q_1 *satisfies (2.1) and* q_2 *satisfies (2.5). If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}\left[q(0), 1\right] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (2.2) univalent in* U, *then* $\alpha + \xi q_1(z) + \mu (q_1(z))^2 + \frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha + \xi q_2(z) + \mu (q_2(z))^2 + \frac{\beta z q_2'(z)}{q_2(z)}$, *for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, *implies* $q_1(z) \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z)$, *and* q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.8 *Let* $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (2.1) and (2.5) hold. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}\left[q(0), 1\right] \cap Q$ *and* $\alpha + \xi \frac{1+A_1 z}{1+B_1 z} + \mu \left(\frac{1+A_1 z}{1+B_1 z}\right)^2 + \frac{\beta(A_1-B_1)z}{(1+A_1 z)(1+B_1 z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha + \xi \frac{1+A_2 z}{1+B_2 z} + \mu \left(\frac{1+A_2 z}{1+B_2 z}\right)^2 + \frac{\beta(A_2-B_2)z}{(1+A_2 z)(1+B_2 z)}$, *for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\frac{1+A_1 z}{1+B_1 z} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \frac{1+A_2 z}{1+B_2 z}$, *hence* $\frac{1+A_1 z}{1+B_1 z}$ *and* $\frac{1+A_2 z}{1+B_2 z}$ *are the best subordinant and the best dominant, respectively.*

For $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0<\gamma_1<\gamma_2\le 1$, we have the following corollary.

Corollary 2.9 *Let $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. Assume that (2.1) and (2.5) hold. If $f\in\mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\in\mathcal{H}[q(0),1]\cap Q$ and $\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_1}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_1}+\frac{2\beta\gamma_1 z}{1-z^2}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_2}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_2}+\frac{2\beta\gamma_2 z}{1-z^2}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\ne 0$, $0<\gamma_1<\gamma_2\le 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

Changing the functions θ and ϕ we obtain the following results.

Theorem 2.10 *Let $\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\in\mathcal{H}(U)$, $f\in\mathcal{A}$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda,l\ge 0$ and let the function $q(z)$ be convex and univalent in U such that $q(0)=1$, $z\in U$. Assume that*

$$Re\left(\frac{\alpha+\beta}{\beta}+z\frac{q''(z)}{q'(z)}\right)>0, \tag{2.7}$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$, $z\in U$, and

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta;z):=\frac{\beta(l+1)}{\lambda}\frac{z^\delta IR_{\lambda,l}^{m+2,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\left(\alpha+\frac{\beta\delta(l+1)}{\lambda}\right)\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}} \tag{2.8}$$

$$-\frac{\beta(1+\delta)(l+1)}{\lambda}\frac{z^\delta\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}.$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)\prec\alpha q(z)+\beta zq'(z), \tag{2.9}$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$, $z\in U$, then $\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q(z)$, $z\in U$, and q is the best dominant.

Proof. Consider $p(z):=\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $z\in U$, $z\ne 0$, $f\in\mathcal{A}$. The function p is analytic in U and $p(0)=1$. Differentiating we get $p'(z)=\frac{\delta(1+l)}{\lambda}\frac{z^{\delta-1}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\frac{l+1}{\lambda}\frac{z^{\delta-1}IR_{\lambda,l}^{m+2,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\frac{(l+1)(1+\delta)}{\lambda}\frac{z^{\delta-1}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}$.
By using the identity (1.3), we get

$$zp'(z)=\frac{l+1}{\lambda}\frac{z^\delta IR_{\lambda,l}^{m+2,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\frac{\delta(1+l)}{\lambda}\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\frac{(l+1)(1+\delta)}{\lambda}\frac{z^\delta\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}. \tag{2.10}$$

By setting $\theta(w):=\alpha w$ and $\phi(w):=\beta$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\ne 0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z)=zq'(z)\phi(q(z))=\beta zq'(z)$, we find that $Q(z)$ is starlike univalent in U.

Let $h(z)=\theta(q(z))+Q(z)=\alpha q(z)+\beta zq'(z)$. We have $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\alpha+\beta}{\beta}+z\frac{q''(z)}{q'(z)}\right)>0$.

By using (2.10), we obtain $\alpha p(z)+\beta zp'(z)=\frac{\beta(l+1)}{\lambda}\frac{z^\delta IR_{\lambda,l}^{m+2,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\left(\alpha+\frac{\beta\delta(l+1)}{\lambda}\right)\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\frac{\beta(1+\delta)(l+1)}{\lambda}\frac{z^\delta\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}$. By using (2.9), we have $\alpha p(z)+\beta zp'(z)\prec\alpha q(z)+\beta zq'(z)$. From Lemma 1.1, we have $p(z)\prec q(z)$, $z\in U$, i.e. $\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q(z)$, $z\in U$, and q is the best dominant. ■

Corollary 2.11 *Let $q(z) = \frac{1+Az}{1+Bz}$, $z \in U$, $-1 \le B < A \le 1$, $m, n \in \mathbb{N}$, $\lambda, l \ge 0$. Assume that (2.7) holds. If $f \in \mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \prec \alpha\frac{1+Az}{1+Bz} + \frac{\beta(A-B)z}{(1+Bz)^2}$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \le B < A \le 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.8), then $\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}} \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, in Theorem 2.10 we get the corollary. ■

Corollary 2.12 *Let $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $m, n \in \mathbb{N}$, $\lambda, l \ge 0$. Assume that (2.7) holds. If $f \in \mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \prec \alpha\left(\frac{1+z}{1-z}\right)^\gamma + \frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^\gamma$, for $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \le 1$, $\beta \neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.8), then $\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}} \prec \left(\frac{1+z}{1-z}\right)^\gamma$, and $\left(\frac{1+z}{1-z}\right)^\gamma$ is the best dominant.*

Proof. Corollary follows by using Theorem 2.10 for $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \le 1$. ■

Theorem 2.13 *Let q be convex and univalent in U such that $q(0) = 1$. Assume that*

$$Re\left(\frac{\alpha}{\beta}q'(z)\right) > 0, \text{ for } \alpha, \beta \in \mathbb{C},\ \beta \neq 0. \tag{2.11}$$

If $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ is univalent in U, where $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ is as defined in (2.8), then

$$\alpha q(z) + \beta z q'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \tag{2.12}$$

implies $q(z) \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $\delta \in \mathbb{C}$, $\delta \neq 0$, $z \in U$, and q is the best subordinant.

Proof. Consider $p(z) := \frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \beta$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{\alpha}{\beta}q'(z)$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\alpha}{\beta}q'(z)\right) > 0$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$.

Now, by using (2.12) we obtain $\alpha q(z) + \beta z q'(z) \prec \alpha p(z) + \beta z p'(z)$, $z \in U$. From Lemma 1.2, we have $q(z) \prec p(z) = \frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $z \in U$, and q is the best subordinant. ■

Corollary 2.14 *Let $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, $z \in U$, $m, n \in \mathbb{N}$, $\lambda, l \ge 0$. Assume that (2.11) holds. If $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$, and $\alpha\frac{1+Az}{1+Bz} + \frac{\beta(A-B)z}{(1+Bz)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \le B < A \le 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.8), then $\frac{1+Az}{1+Bz} \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, and $\frac{1+Az}{1+Bz}$ is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, in Theorem 2.13 we get the corollary. ■

Corollary 2.15 *Let $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $m, n \in \mathbb{N}$, $\lambda, l \ge 0$. Assume that (2.11) holds. If $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha\left(\frac{1+z}{1-z}\right)^\gamma + \frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^\gamma \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$, for $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \le 1$, $\beta \neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.8), then $\left(\frac{1+z}{1-z}\right)^\gamma \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, and $\left(\frac{1+z}{1-z}\right)^\gamma$ is the best subordinant.*

Proof. Corollary follows by using Theorem 2.13 for $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \le 1$. ■

Combining Theorem 2.10 and Theorem 2.13, we state the following sandwich theorem.

Theorem 2.16 *Let q_1 and q_2 be convex and univalent in U such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$. Suppose that q_1 satisfies (2.7) and q_2 satisfies (2.11). If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$, and $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ is as defined in (2.8) univalent in U, then $\alpha q_1(z) + \beta z q_1'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \prec \alpha q_2(z) + \beta z q_2'(z)$, for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, implies $q_1(z) \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z)$, $z \in U$, and q_1 and q_2 are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.17 *Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.7) and (2.11) hold for $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ and $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, respectively. If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha \frac{1+A_1 z}{1+B_1 z} + \frac{\beta(A_1 - B_1) z}{(1+B_1 z)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ $\prec \alpha \frac{1+A_2 z}{1+B_2 z} + \frac{\beta(A_2 - B_2) z}{(1+B_2 z)^2}$, $z \in U$, for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{1+A_1 z}{1+B_1 z} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \frac{1+A_2 z}{1+B_2 z}$, $z \in U$, hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.*

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 2.18 *Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.7) and (2.11) hold for $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, respectively. If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha \left(\frac{1+z}{1-z}\right)^{\gamma_1} + \frac{2\beta\gamma_1 z}{1-z^2} \left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ $\prec \alpha \left(\frac{1+z}{1-z}\right)^{\gamma_2} + \frac{2\beta\gamma_2 z}{1-z^2} \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z \in U$, for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma_1 < \gamma_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z \in U$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

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FUZZY STABILITY OF A CLASS OF ADDITIVE-QUADRATIC FUNCTIONAL EQUATIONS

CHANG IL KIM AND GILJUN HAN*

Abstract. In this paper, we consider the following functional equation

$$\begin{aligned} &af(x+y)+bf(x-y)+cf(y-x) \\ &=(a+b)f(x)+cf(-x)+(a+c)f(y)+bf(-y) \end{aligned}$$

for a fixed real numbers a, b, c with $a = b + c$ and $a \neq 0$. We study the fuzzy version of the generalized Hyers-Ulam stability for it in the sense of Mirmostafaee and Moslehian.

1. Introduction and preliminaries

In 1940, Ulam proposed the following stability problem (cf. [20]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exists a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [11] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki [1] for additive mappings, and by Rassias [19] for linear mappings, to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians ([5], [6], [7], [10], [18]).

Recently, the stability in fuzzy spaces has been extensively studied ([3], [12], [15], [16], [17]). The concept of fuzzy norm on a linear space was introduced by Katsaras [14] in 1984. Later, Cheng and Mordeson [4] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13]. In 2008, for the first time, Mirmostafaee and Moslehian [16], [17] used the definition of a fuzzy norm in [2] to obtain a fuzzy version of stability for the Cauchy functional equation

$$f(x+y)=f(x)+f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.2}$$

2010 *Mathematics Subject Classification.* 39B52, 39B72, 46S40.

Key words and phrases. additive-quadratic mapping, fuzzy almost quadratic-additive mapping, fuzzy normed space.

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We call a solution of (1.1) *an additive mapping* and a solution of (1.2) is called *a quadratic mapping*. Also,

$$f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)=0$$

is called *Drygas functional equation*(see [8], [9] for detail.). It is easy to see that the function $f(x)=px^2+qx$ is a solution of Drygas functional equation and so we can expect that a solution of Drygas functional equation is an additive-quadratic mapping.

Now, we consider the following functional equation

$$\begin{aligned} &af(x+y)+bf(x-y)+cf(y-x) \\ &=(a+b)f(x)+cf(-x)+(a+c)f(y)+bf(-y) \end{aligned} \tag{1.3}$$

for fixed real numbers a,b,c with $a=b+c$ and $a\neq 0$ and show the generalized Hyers-Ulam stability of (1.3) in a fuzzy sense [18].

Definition 1.1. Let X be a real vector space. A function $N: X\times\mathbb{R}\longrightarrow[0,1]$ is called *a fuzzy norm on* X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

(N1) $N(x,t)=0$ for $t\leq 0$;
(N2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
(N3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;
(N4) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$;
(N6) for any $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X,N) is called *a fuzzy normed space*.

Let (X,N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* in (X,N) if there exists an $x\in X$ such that $\lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* (X,N) and one denotes it by $N-\lim_{n\to\infty}x_n=x$. A sequence $\{x_n\}$ in X is said to be *Cauchy* if for any $\epsilon>0$, there is an $m\in N$ such that for any $n\geq m$ and any positive integer p, $N(x_{n+p}-x_n,t)>1-\epsilon$ for all $t>0$.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space*.

2. Solutions and the Generalized Hyers-Ulam stability of (1.3)

In this section, we investigate solutions of (1.3) and prove the generalized Hyers-Ulam stability of (1.3) in fuzzy Banach spaces. Throughout this section, we assume that (X,N) is a fuzzy normed space and (Y,N') is a fuzzy Banach space. In Theorem 2.3, it can be concluded that any solution of (1.3) is additive-quadratic. We start with the following lemma.

Lemma 2.1. *Let $f: X\longrightarrow Y$ be an odd mapping satisfying (1.3). Then f is an additive mapping.*

Proof. Since $a\neq 0$, $f(0)=0$. Since f is an odd mapping, the functional equation (1.3) can be written by

$$af(x+y)+(b-c)f(x-y)=(a+b-c)f(x)+(a-b+c)f(y) \tag{2.1}$$

for all $x, y \in X$. Interchanging x and y in (2.1), we have

$$af(x+y) - (b-c)f(x-y) = (a+b-c)f(y) + (a-b+c)f(x) \tag{2.2}$$

for all $x, y \in X$. By (2.1) and (2.2),

$$af(x+y) = af(x) + af(y)$$

for all $x, y \in X$ and since $a \neq 0$, f is additive. □

Lemma 2.2. *Let $f : X \longrightarrow Y$ be an even mapping satisfying (1.3). Then f is a quadratic mapping.*

Proof. Since $a \neq 0$, $f(0) = 0$. Since f is an even mapping, the functional equation (1.3) can be written by

$$af(x+y) + (b+c)f(x-y) = (a+b+c)f(x) + (a+b+c)f(y) \tag{2.3}$$

for all $x, y \in X$. Letting $y = -y$ in (2.3), we have

$$af(x-y) + (b+c)f(x+y) = (a+b+c)f(x) + (a+b+c)f(y) \tag{2.4}$$

for all $x, y \in X$. Since $a = b + c$, by (2.3) and (2.4), we have

$$2af(x-y) + 2af(x+y) = 4af(x) + 4af(y)$$

for all $x, y \in X$ and since $a \neq 0$, f is a quadratic mapping. □

Combining Lemma 2.1 and Lemma 2.2, we have the following theorem.

Theorem 2.3. *Let $f : X \longrightarrow Y$ be a mapping. If f satisfies (1.3), then f is an additive-quadratic mapping.*

For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$Df(x,y) = af(x+y)+bf(x-y)+cf(y-x)-(a+b)f(x)-cf(-x)-(a+c)f(y)-bf(-y)$$

for all $x, y \in X$. For a given $q > 0$, the mapping f is said to be *a fuzzy q-almost additive-quadratic mapping if*

$$N'(Df(x,y), t+s) \geq \min\{N(x, t^q), N(y, s^q)\} \tag{2.5}$$

for all $x, y \in X$ and all positive real numbers t, s.

Theorem 2.4. *Let q be a positive real number with $q \neq 1$, $\frac{1}{2}$ and $f : X \longrightarrow Y$ a fuzzy q-almost additive-quadratic mapping. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that*

(2.6)

$$N(F(x)-f(x),t) \geq \begin{cases} \sup_{s<t}\{N(x,(1-2^{p-1})^q|a|^q s^q)\}, & \text{if } q > 1 \\ \sup_{s<t}\{N(x,(2^{p-1}-1)^q(2-2^{(p-1)})^q|a|^q s^q)\}, & \text{if } \frac{1}{2} < q < 1 \\ \sup_{s<t}\{N(x,(2^{p-1}-2)^q|a|^q s^q\}, & \text{if } 0 < q < \frac{1}{2} \end{cases}$$

holds for all $x \in X$ and all $t > 0$, where $p = \frac{1}{q}$.

Proof. By (2.5), (N2), and (N4), since $a = b + c$, we have

$$N'(Df(0,0),\ t) = N'(f(0),\ \frac{t}{2|a|}) \geq N'(0,\ t^q) = 1$$

for all $t > 0$ and by (N2), $f(0) = 0$.

Case 1. Let $q > 1$ and define a mapping $J_n f : X \longrightarrow Y$ by

$$J_n f(x) = \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2 \cdot 2^n}$$

for all $x \in X$ and all positive integer n. Then we have

$$\begin{aligned} & J_n f(x) - J_{n+1} f(x) \\ = & \frac{2^{n+1} - 1}{a \cdot 2 \cdot 4^{n+1}} Df(-2^n x, -2^n x) - \frac{2^{n+1} + 1}{a \cdot 2 \cdot 4^{n+1}} Df(2^n x, 2^n x) \end{aligned} \tag{2.7}$$

for all $x \in X$ and all positive integer n. By (2.5), (2.7), (N3), and (N4), we have

(2.8)

$$\begin{aligned} & N'(J_m f(x) - J_{m+n} f(x), \sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a| \cdot 2^i} t^p) \\ = & N'(\sum_{i=m}^{m+n-1} [J_i f(x) - J_{i+1} f(x)], \sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a| \cdot 2^i} t^p) \\ \geq & \min\{N'(J_i f(x) - J_{i+1} f(x), \frac{2^{pi}}{|a| \cdot 2^i} t^p) \mid m \leq i \leq m + n - 1\} \\ \geq & \min\{N'(\frac{2^{i+1} - 1}{a \cdot 2 \cdot 4^{i+1}} Df(-2^i x, -2^i x) - \frac{2^{i+1} + 1}{a \cdot 2 \cdot 4^{i+1}} Df(2^i x, 2^i x), \frac{2^{pi}}{|a| \cdot 2^i} t^p) \mid \\ & m \leq i \leq m + n - 1\} \\ \geq & \min\{\min\{N'(\frac{2^{i+1} + 1}{a \cdot 2 \cdot 4^{i+1}} Df(2^i x, 2^i x), \frac{(2^{i+1} + 1)2^{pi}}{|a| \cdot 4^{i+1}} t^p), \\ & N'(\frac{2^{i+1} - 1}{a \cdot 2 \cdot 4^{i+1}} Df(-2^i x, -2^i x), \frac{(2^{i+1} - 1)2^{pi}}{|a| \cdot 4^{i+1}} t^p)\} \mid m \leq i \leq m + n - 1\} \\ \geq & \min\{\min\{N'(Df(2^i x, 2^i x), 2^{pi+1} t^p), N'(Df(-2^i x, -2^i x), 2^{pi+1} t^p)\} | m \leq i \leq m + n - 1\} \\ \geq & \min\{\min\{N(2^i x, 2^i t), N(-2^i x, 2^i t)\} \mid m \leq i \leq m + n - 1\} \\ = & N(x, t) \end{aligned}$$

for all $x \in X$, all $t > 0$, and all positive integers m, n. Let $\epsilon > 0$ be given. Since $\lim_{t \longrightarrow \infty} N(x, t) = 1$, there is a t_1 such that $N(x, t_1) > 1 - \epsilon$. Let $t_2 > t_1$. Since $p < 1$, $\sum_{n=0}^{\infty} \frac{2^{pn}}{|a| \cdot 2^n} t_2^p$ is convergent. Let $s > 0$. Then there is a positive integer k such that $\sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a| \cdot 2^i} t_2^p < s$ for $m, n > k$ and so by (2.8), we have

$$\begin{aligned}
&N'(J_m f(x) - J_{m+n} f(x), s) \\
&\geq N'(J_m f(x) - J_{m+n} f(x), \sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a| \cdot 2^i} t_2^p) \\
&\geq N(x, t_2) \\
&\geq 1 - \epsilon
\end{aligned}$$

for all $x \in X$. Hence $\{J_n f(x)\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a fuzzy Banach space, we can define a mapping $F : X \longrightarrow Y$ by

$$F(x) = N' - \lim_{n\to\infty} J_n f(x)$$

for all $x \in X$. Letting $m = 0$ in (2.8), we have

$$N'(f(x) - J_n f(x), t) \geq N(x, \frac{t^q}{[\sum_{i=0}^{n-1} \frac{2^{pi}}{|a| \cdot 2^i}]^q}) \tag{2.9}$$

for all $x \in X$, all positive integer n, and all $t > 0$. By (N4), we have

$$\begin{aligned}
&N'(DF(x, y), t) \\
&\geq \min\{N'(a[F - J_n f](x + y), \frac{t}{14}), N'(b[F - J_n f](x - y), \frac{t}{14}), \\
&N'(c[F - J_n f](y - x), \frac{t}{14}), N'((a + b)[F - J_n f](x), \frac{t}{14}) \\
&- N'(c[F - J_n f](-x), \frac{t}{14}), N'((a + c)[F - J_n f](y), \frac{t}{14}) \\
&- N'(b[F - J_n f](-y), \frac{t}{14}), N'(J_n Df(x, y), \frac{t}{2})\}
\end{aligned} \tag{2.10}$$

for all $x, y \in X$ and all positive integer n. The first seven terms on the right-hand of (2.10) tend to 1 as $n \to \infty$ and by (N4), we have

$$\begin{aligned}
&N'(J_n Df(x, y), \frac{t}{2}) \\
&\geq \min\{N'(\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}, \frac{t}{8}), N'(\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}, \frac{t}{8}), \\
&N'(\frac{Df(-2^n x, -2^n y)}{2 \cdot 2^n}, \frac{t}{8}), N'(\frac{Df(2^n x, 2^n y)}{2 \cdot 2^n}, \frac{t}{8})\}
\end{aligned} \tag{2.11}$$

for all $x, y \in X$, all positive integer n and all $t > 0$. By (N3) and (2.5), we have

$$\begin{aligned}
&N'(\frac{Df(\pm 2^n x, \pm 2^n y)}{2 \cdot 4^n}, \frac{t}{8}) \\
&= N'(Df(\pm 2^n x, \pm 2^n y, 4^{n-1} t)) \\
&\geq \min\{N(2^n x, 2^{q(2n-3)} t^q), N(2^n y, 2^{q(2n-3)} t^q)\} \\
&\geq \min\{N(x, 2^{(2q-1)n-3q} t^q), N(y, 2^{(2q-1)n-3q} t^q)\}
\end{aligned} \tag{2.12}$$

for all $x, y \in X$, all positive integer n, and all $t > 0$. Since $q > 1$, by (2.11) and (2.12), we have

$$\lim_{n\to\infty} N'(J_n Df(x, y), \frac{t}{2}) = 1$$

and so by (2.10), $N'(DF(x,y),t) = 0$ for all $x, y \in X$ and all $t > 0$. By (N2), $DF(x,y) = 0$ for all $x, y \in X$ and by Theorem 2.3, F is additive-quadraic.

Now we will show that (2.6) holds. Let $x \in X$, $t > 0$, $s > 0$ with $0 < s < t$ and $0 < \epsilon < 1$. Since $F(x) = N' - \lim_{n\to\infty} J_n f(x)$, there is a positive integer n such that

$$N'(F(x) - J_n f(x), t - s) \geq 1 - \epsilon$$

and so by (2.9),

$$\begin{aligned}
&N'(F(x) - f(x), t)\\
&\geq \min\{N'(F(x) - J_n f(x), t - s), N'(J_n f(x) - f(x), s)\}\\
&\geq \min\{1 - \epsilon, N(x, \frac{s^q}{[\sum_{i=0}^{n-1} \frac{2^{pi}}{|a|\cdot 2^i}]^q})\}\\
&\geq \min\{1 - \epsilon, N(x, (1 - 2^{p-1})^q s^q |a|^q)\}.
\end{aligned}$$

and so we have (2.6).

To prove the uniqueness of F, let $F_1 : X \longrightarrow Y$ be another additive-quadratic mapping satisfying (2.6). Then

$$F(x) - F_1(x) = J_n F(x) - J_n F_1(x)$$

for all $x \in X$ and all positive integer n. Hence by (N4), (N5), and (2.6), we have

$$\begin{aligned}
&N'(F(x) - F_1(x), t)\\
&= N'(J_n F(x) - J_n F_1(x), t)\\
&\geq \min\{N'(J_n F(x) - J_n f(x), \frac{t}{2}), N'(J_n F_1(x) - J_n f(x), \frac{t}{2})\}\\
&\geq \min\{N'(\frac{F(2^n x) - f(2^n x)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F(-2^n x) - f(-2^n x)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(\frac{F(2^n x) - f(2^n x)}{2\cdot 2^n}, \frac{t}{8}), N'(\frac{F(-2^n x) - f(-2^n x)}{2\cdot 2^n}, \frac{t}{8}),\\
&\quad N'(\frac{F_1(2^n x) - f(2^n x)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F_1(-2^n x) - f(-2^n x)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(\frac{F_1(2^n x) - f(2^n x)}{2\cdot 2^n}, \frac{t}{8}), N'(\frac{F_1(-2^n x) - f(-2^n x)}{2\cdot 2^n}, \frac{t}{8})\}\\
&\geq \sup_{s<t}\{N(2^n x, (1 - 2^{p-1})^q 2^{(n-3)q} s^q |a|^q)\}\\
&\geq \sup_{s<t}\{N(x, (1 - 2^{p-1})^q |a|^q s^q 2^{(q-1)n-3q})\}
\end{aligned}$$

for all $x, y \in X$, all positive integer n and all $0 < s < t$. Since $q > 1$,

$$\lim_{n\to\infty} \sup_{s<t}\{N(x, (1 - 2^{p-1})^q |a|^q s^q 2^{(q-1)n-3q}\} = 1$$

and so $N'(F(x) - F_1(x), t) = 1$ for all $t > 0$. Hence $F = F_1$.

Case 2. Let $\frac{1}{2} < q < 1$ and define a mapping $J_n f : X \longrightarrow Y$ by

$$J_n f(x) = \frac{f(2^n x) + f(-2^n x)}{2\cdot 4^n} + \frac{2^n}{2}[f(2^{-n} x) - f(-2^{-n} x)]$$

for all $x \in X$ and all positive integer n. Then we have

$$\begin{aligned}
&J_nf(x) - J_{n+1}f(x)\\
&= \frac{2^n}{2\cdot a}Df(2^{-(n+1)}x, 2^{-(n+1)}x) - \frac{2^n}{2\cdot a}Df(-2^{-(n+1)}x, -2^{-(n+1)}x)\\
&\quad - \frac{1}{a\cdot 2\cdot 4^{n+1}}Df(2^nx, 2^nx) - \frac{1}{a\cdot 2\cdot 4^{n+1}}Df(-2^nx, -2^nx)
\end{aligned} \tag{2.13}$$

for all $x \in X$ and all positive integer n. By (2.5), (2.13), (N3), and (N4), we have

(2.14)

$$\begin{aligned}
&N'(J_mf(x) - J_{m+n}f(x), \sum_{i=m}^{m+n-1}[\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t^p)\\
&= N'(\sum_{i=m}^{m+n-1}[J_if(x) - J_{i+1}f(x)], \sum_{i=m}^{m+n-1}[\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t^p)\\
&\geq \min\{N'(J_if(x) - J_{i+1}f(x), [\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t^p) \mid m \leq i \leq m+n-1\}\\
&\geq \min\{N'(\frac{1}{a\cdot 2\cdot 4^{i+1}}Df(2^ix, 2^ix) + \frac{1}{a\cdot 2\cdot 4^{i+1}}Df(-2^ix, -2^ix)\\
&\quad - \frac{2^i}{2\cdot a}Df(2^{-(i+1)}x, 2^{-(i+1)}x) + \frac{2^i}{2\cdot a}Df(-2^{-(i+1)}x, -2^{-(i+1)}x),\\
&\quad \frac{2^{pi+1}}{|a|\cdot 4^{i+1}}t^p + \frac{2^{1-p(i+1)+i}}{|a|}t^p) \mid m \leq i \leq m+n-1\}\\
&\geq \min\{\min\{N'(\frac{1}{a\cdot 2\cdot 4^{i+1}}Df(2^ix, 2^ix), \frac{2^{pi+1}}{|a|\cdot 2\cdot 4^{i+1}}t^p),\\
&\quad N'(\frac{1}{a\cdot 2\cdot 4^{i+1}}Df(-2^ix, -2^ix), \frac{2^{pi+1}}{|a|\cdot 2\cdot 4^{i+1}}t^p),\\
&\quad N'(\frac{2^i}{2\cdot a}Df(2^{-(i+1)}x, 2^{-(i+1)}x), \frac{2^{1-p(i+1)+i}}{2\cdot |a|}t^p),\\
&\quad N'(\frac{2^i}{2\cdot a}Df(-2^{-(i+1)}x, -2^{-(i+1)}x), \frac{2^{1-p(i+1)+i}}{2\cdot |a|}t^p)\} \mid m \leq i \leq m+n-1\}\\
&\quad \geq \min\{\min\{N'(Df(2^ix, 2^ix), 2^{pi+1}t^p), N'(Df(-2^ix, -2^ix), 2^{pi+1}t^p),\\
&\quad N'(Df(2^{-(i+1)}x, 2^{-(i+1)}x), 2^{1-p(i+1)}t^p), N'(Df(-2^{-(i+1)}x, -2^{-(i+1)}x), 2^{1-p(i+1)}t^p)\} \mid\\
&\quad m \leq i \leq m+n-1\}]\\
&\geq \min\{\min\{N(2^ix, 2^it), N(-2^ix, 2^it), N(2^{-(i+1)}x, 2^{-(i+1)}t),\\
&\quad N(-2^{-(i+1)}x, 2^{-(i+1)}t)\} \mid m \leq i \leq m+n-1\}\\
&= N(x,t)
\end{aligned}$$

for all $x \in X$, all $t > 0$, and all positive integers m, n. Let $\epsilon > 0$ be given. Since $\lim_{t\longrightarrow\infty} N(x,t) = 1$, there is a t_1 such that $N(x, t_1) > 1 - \epsilon$. Let $t_2 > t_1$. Since $1 < p < 2$, $\sum_{n=0}^{\infty}[\frac{2^{pn+1}}{|a|\cdot 4^{n+1}} + \frac{2^{1-p(n+1)+n}}{|a|}]t_2^p$ is convergent. Let $s > 0$. Then there is a positive integer n such that $\sum_{i=m}^{m+n-1}[\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t_2^p < s$ for $m, n > k$ and

so by (2.14), we have

$$
\begin{aligned}
&N'(J_mf(x) - J_{m+n}f(x), s)\\
&\geq N'(J_mf(x) - J_{m+n}f(x), \sum_{i=m}^{m+n-1}[\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t_2^p)\\
&\geq N(x, t_2)\\
&\geq 1-\epsilon
\end{aligned}
$$

for all $x \in X$. Hence $\{J_nf(x)\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a fuzzy Banach space, we can define a mapping $F : X \longrightarrow Y$ by

$$F(x) = N' - \lim_{n\to\infty} J_nf(x)$$

for all $x \in X$. Letting $m = 0$ in (2.14), we have

$$N'(f(x) - J_nf(x), t) \geq N(x, \frac{t^q}{[\sum_{i=0}^{n-1}(\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|})]^q}) \tag{2.15}$$

for all $x \in X$, all positive integer n, and all $t > 0$. By (N4), we have

$$
\begin{aligned}
&N'(DF(x, y), t)\\
&\geq \min\{N'(a[F - J_nf](x + y), \frac{t}{14}), N'(b[F - J_nf](x - y), \frac{t}{14}),\\
&N'(c[F - J_nf](y - x), \frac{t}{14}), N'((a + b)[F - J_nf](x), \frac{t}{14})\\
&- N'(c[F - J_nf](-x), \frac{t}{14}), N'((a + c)[F - J_nf](y), \frac{t}{14})\\
&- N'(b[F - J_nf](-y), \frac{t}{14}), N'(J_nDf(x, y), \frac{t}{2})\}
\end{aligned}
\tag{2.16}
$$

for all $x, y \in X$ and all positive integer n. The first seven terms on the right-hand of (2.16) tend to 1 as $n \to \infty$ and by (N4), we have

$$
\begin{aligned}
&N'(J_nDf(x, y), \frac{t}{2})\\
&\geq \min\{N'(\frac{Df(-2^nx, -2^ny)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{Df(2^nx, 2^ny)}{2\cdot 4^n}, \frac{t}{8}),\\
&N'(2^{n-1}Df(2^{-n}x, 2^{-n}y), \frac{t}{8}), N'(2^{n-1}Df(-2^{-n}x, -2^{-n}y), \frac{t}{8})\}
\end{aligned}
\tag{2.17}
$$

for all $x, y \in X$, all positive integer n and all $t > 0$. By (N3) and (2.5), we have

$$
\begin{aligned}
&N'(\frac{Df(\pm 2^nx, \pm 2^ny)}{2\cdot 4^n}, \frac{t}{8})\\
&\geq \min\{N(x, 2^{(2q-1)n-3q}t^q), N(y, 2^{(2q-1)n-3q}t^q)\}
\end{aligned}
\tag{2.18}
$$

and

$$
\begin{aligned}
&N'(2^{n-1}Df(\pm 2^{-n}x, \pm 2^{-n}y), \frac{t}{8})\\
&\geq \min\{N(x, 2^{(1-q)n-3q)}t^q), N(y, 2^{(1-q)n-3q)}t^q)\}
\end{aligned}
\tag{2.19}
$$

for all $x, y \in X$, all positive integer n, and all $t > 0$. Since $\frac{1}{2} < q < 1$, by (2.17), (2.18), and (2.19), we have

$$\lim_{n\to\infty} N'(J_nDf(x, y), \frac{t}{2}) = 1$$

and so by (2.16), $N'(DF(x,y),t) = 0$ for all $x, y \in X$ and all $t > 0$. By (N2), $DF(x,y) = 0$ for all $x, y \in X$ and by Theorem 2.3, F is additive-quadratic.

Now we will show that (2.6) holds. Let $x \in X$, $t > 0$, $s > 0$ with $0 < s < t$ and $0 < \epsilon < 1$. Since $F(x) = N' - \lim_{n\to\infty} J_nf(x)$, there is a positive integer n such that

$$N'(F(x) - J_nf(x), t-s) \geq 1-\epsilon$$

and so by (2.15),

$$\begin{aligned}
&N'(F(x) - f(x), t)\\
&\geq \min\{N'(F(x) - J_nf(x), t-s), N'(J_nf(x) - f(x), s)\}\\
&\geq \min\{1-\epsilon, N(x, \frac{s^q}{[\sum_{i=0}^{n-1}(\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|})]^q})\}\\
&\geq \min\{1-\epsilon, N(x, (2^{p-1}-1)^q(2-2^{p-1})^q|a|^q s^q)\}.
\end{aligned}$$

and so we have (2.6).

To prove the uniqueness of F, let $F_1 : X \longrightarrow Y$ be another additive-quadratic mapping satisfying (2.6). Then

$$F(x) - J_nF(x) = F_1(x) - J_nF_1(x)$$

for all $x \in X$ and all positive integer n. Hence by (N4), (N5), and (2.6), we have

$$\begin{aligned}
&N'(F(x) - F_1(x), t)\\
&= N'(J_nF(x) - J_nF_1(x), t)\\
&\geq \min\{N'(J_nF(x) - J_nf(x), \frac{t}{2}), N'(J_nF_1(x) - J_nf(x), \frac{t}{2})\}\\
&\geq \min\{N'(\frac{F(2^nx) - f(2^nx)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F(-2^nx) - f(-2^nx)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(2^{n-1}[F(2^{-n}x) - f(2^{-n}x)], \frac{t}{8}), N'(2^{n-1}[F(-2^{-n}x) - f(-2^{-n}x)], \frac{t}{8}),\\
&\quad N'(\frac{F_1(2^nx) - f(2^nx)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F_1(-2^nx) - f(-2^nx)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(2^{n-1}[F_1(2^{-n}x) - f(2^{-n}x)], \frac{t}{8}), N'(2^{n-1}[F_1(-2^{-n}x) - f(-2^{-n}x)], \frac{t}{8})\}\\
&\geq \sup_{s<t}\{N(\pm 2^nx, (2^{p-1}-1)^q(2-2^{p-1})^q 4^{(n-1)q}|a|^q s^q)\}\\
&\geq \sup_{s<t}\{N(x, (2^{p-1}-1)^q(2-2^{p-1})^q 2^{(2q-1)n-2q}|a|^q s^q)\}
\end{aligned}$$

for all $x, y \in X$, all positive integer n and all $t > 0$. Since $\frac{1}{2} < q < 1$, $N'(F(x) - F_1(x), t) = 1$ for all $t > 0$. Hence $F = F_1$.

Case 3. Let $0 < q < \frac{1}{2}$ and define a mapping $J_nf : X \longrightarrow Y$ by

$$J_nf(x) = 2^{2n-1}[f(2^{-n}x) + f(-2^{-n}x)] + 2^{n-1}[f(2^{-n}x) - f(-2^{-n}x)]$$

for all $x \in X$ and all positive integer n. Then we have

(2.20)

$$\begin{aligned}
&J_nf(x) - J_{n+1}f(x)\\
&= \frac{2^{2n-1} + 2^{n-1}}{a}Df(2^{-(n+1)}x, 2^{-(n+1)}x) + \frac{2^{2n-1} - 2^{n-1}}{a}Df(-2^{-(n+1)}x, -2^{-(n+1)}x)
\end{aligned}$$

for all $x \in X$ and all positive integer n. By (2.5), (2.20), (N3), and (N4), we have

$$\begin{aligned}
&N'(J_m f(x) - J_{m+n} f(x), \sum_{i=m}^{m+n-1} \frac{2^{1-p(i+1)+2i}}{|a|} t^p) \\
&= N'(\sum_{i=m}^{m+n-1} [J_i f(x) - J_{i+1} f(x)], \sum_{i=m}^{m+n-1} \frac{2^{1-p(i+1)+2i}}{|a|} t^p) \\
&\geq \min\{N'(J_i f(x) - J_{i+1} f(x), \frac{2^{1-p(i+1)+2i}}{|a|} t^p) \mid m \leq i \leq m+n-1\} \\
&\geq \min\{N'(\frac{2^{2i-1}+2^{i-1}}{a} Df(2^{-(i+1)}x, 2^{-(i+1)}x) \\
&\quad + \frac{2^{2i-1}-2^{i-1}}{a} Df(-2^{-(i+1)}x, -2^{-(i+1)}x), \frac{2^{1-p(i+1)+2i}}{|a|} t^p)\} \mid m \leq i \leq m+n-1\} \\
&\geq \min\{\min\{N'(\frac{2^{2i-1}+2^{i-1}}{a} Df(2^{-(i+1)}x, 2^{-(i+1)}x), \frac{2^{2i-1}+2^{i-1}}{|a|} 2^{1-p(i+1)} t^p), \\
&\quad N'(\frac{2^{2i-1}-2^{i-1}}{a} Df(-2^{-(i+1)}x, -2^{-(i+1)}x), \frac{2^{2i-1}-2^{i-1}}{|a|} 2^{1-p(i+1)} t^p)\} \\
&\quad \mid m \leq i \leq m+n-1\} \\
&\geq \min\{\min\{N'(Df(2^{-(i+1)}x, 2^{-(i+1)}x), 2^{1-p(i+1)} t^p), \\
&\quad N'(Df(-2^{-(i+1)}x, -2^{-(i+1)}x), 2^{1-p(i+1)} t^p)\} \mid m \leq i \leq m+n-1\} \\
&\geq \min\{\min\{N(2^{-(i+1)}x, 2^{-(i+1)}t), N(-2^{-(i+1)}x, 2^{-(i+1)}t)\} \mid m \leq i \leq m+n-1\} \\
&= N(x,t)
\end{aligned}$$

for all $x \in X$, all $t > 0$, and all positive integers m, n. Similar to **Case 1.** and **Case 2.**, there is a unique cubic mapping $C : X \longrightarrow Y$ with (2.6). □

We can use Theorem 2.4 to get a classical result in the framework of normed spaces. For example, it is well known that for any normed space $(X, ||\cdot||)$, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0,1]$, defined by

$$N_X(x,t) = \begin{cases} 0, & \text{if } t < \|x\| \\ 1, & \text{if } t \geq \|x\| \end{cases}$$

a fuzzy norm on X. In [15], [16] *and* [17], *some examples are provided for the fuzzy norm N_X. Here using the fuzzy norm N_X, we have the following corollary.*

Corollary 2.5. *Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\|Df(x,y)\| \leq \|x\|^p + \|y\|^p \tag{2.21}$$

for a fixed positive number p such that $p \neq 1, 2$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that the inequality

$$\|F(x) - f(x)\| \leq \begin{cases} \frac{1}{(1-2^{p-1})|a|}\|x\|^p, & \text{if } 1 < p \\ \frac{1}{(2^{p-1}-1)(2-2^{(p-1)})|a|}\|x\|^p, & \text{if } 1 < p < 2 \\ \frac{1}{(2^{p-1}-2)|a|}\|x\|^p, & \text{if } 2 < p \end{cases}$$

holds for all $x \in X$.

Proof. By the definition of N_Y, we have

$$N_Y(Df(x,y), s+t) = \begin{cases} 0, & \text{if } s+t \le \|Df(x,y)\| \\ 1, & \text{if } s+t \ge \|Df(x,y)\|. \end{cases}$$

for all $x, y \in X$ and all $s, t \in \mathbb{R}$. Now, we claim that

$$N_Y(Df(x,y), s+t) \ge \min\{N_X(x, s^q), N_X(y, t^q)\}$$

for all $x, y \in X$ and $s, t > 0$. If $N_Y(Df(x,y), s+t) = 1$, then it is trivial. Suppose that $N_Y(Df(x,y), s+t) = 0$. Then $s+t \le \|Df(x,y)\|$ and by (2.21), either $s \le \|x\|^p$ or $t \le \|y\|^p$. Hence either $N_X(x, s^q) = 0$ or $N_X(y, t^q) = 0$ and thus f is a fuzzy q-almost additive-quadratic mapping. By Theorem 2.4, we have the results. □

The condition $p \neq 1, 2$ in Corollary 2.5 is indispensable. The following example shows that the inequality (2.21) is not stable for $p = 1$, 2, especially in the case of $b = 2$ and $c = -1$. We will give the proof when $p = 1$, and the proof when $p = 2$ is similar. For any $f : X \longrightarrow Y$, let $f_o(x) = \frac{f(x)-f(-x)}{2}$ and $f_e(x) = \frac{f(x)+f(-x)}{2}$.

Example 2.6. Define mappings $t, s : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$t(x) = \begin{cases} x, & \text{if } |x| < 1 \\ -1, & \text{if } x \le -1 \\ 1, & \text{if } 1 \le x, \end{cases}$$

$$s(x) = \begin{cases} x^2, & \text{if } |x| < 1 \\ 1, & \text{ortherwise} \end{cases}$$

and a mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} [\frac{t(2^n x)}{2^n} + \frac{s(2^n x)}{4^n}]$$

We will show that there is a positive integer M such that

$$|D_2 f(x,y)| \le M(|x| + |y|) \tag{2.22}$$

for all $x, y \in \mathbb{R}$, where

$$D_2 g(x,y) = g(x+y) + 2g(x-y) - g(y-x) - 3g(x) + g(-x) - 2g(-y).$$

But there do not exist an additive-quadratic mapping $F : \mathbb{R} \longrightarrow \mathbb{R}$ and a non-negative constant K such that

$$|F(x) - f(x)| \le K|x|^2 \tag{2.23}$$

for all $x \in \mathbb{R}$.

Proof. Note that $s_o(x) = 0$, $t_o(x) = t(x)$, and $|f_o(x)| \le 2$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{2} \le |x| + |y|$. Then $|D_2 f_o(x,y)| \le 40(|x| + |y|)$. Now suppose that $\frac{1}{2} > |x| + |y|$. Then there is a non-negative integer m such that

$$\frac{1}{2^{m+2}} \le |x| + |y| < \frac{1}{2^{m+1}}$$

and so $2^m|x| < \frac{1}{2}$, $2^m|y| < \frac{1}{2}$. Hence $\{2^m(x \pm y),\ 2^m x,\ 2^m y\} \subseteq (-1,\ 1)$ and so for any $n = 0, 1, 2, \cdots, m$, $D_2 t_0(2^n x, 2^n y) = 0$ for all $x, y \in X$. Thus

$$D_2 f_o(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} D_2 t(2^n x, 2^n y) = \sum_{n=m+1}^{\infty} \frac{1}{2^n} D_2 t(2^n x, 2^n y) \le \frac{40}{2^{m+2}} \le 40(|x|+|y|).$$

Note that $t_e(x) = 0$, $s_e(x) = s(x)$, and $|f_e(x)| \le \frac{4}{3}$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{4} \le |x| + |y|$. Then $|D_2 f_e(x, y)| \le \frac{128}{3}(|x| + |y|)$ for all $x, y \in \mathbb{R}$. Now suppose that $\frac{1}{4} > |x| + |y|$. Then there is a non-negative integer k such that

$$\frac{1}{2^{k+2}} \le \Big(|x| + |y|\Big)^{\frac{1}{2}} < \frac{1}{2^{k+1}}.$$

Hence $\{2^k(x \pm y),\ 2^k x,\ 2^k y\} \subseteq (-1,\ 1)$ and so for any $n = 0, 1, 2, \cdots, m$, $D_2 s_e(2^n x, 2^n y) = 0$. Hence

$$D_2 f_e(x, y) = \sum_{n=0}^{\infty} \frac{1}{4^n} D_2 s_e(2^n x, 2^n y) = \sum_{n=k+1}^{\infty} \frac{1}{4^n} D_2 s_e(2^n x, 2^n y) \le \frac{8}{3} \cdot \frac{1}{2^{2k}}.$$

and so we have

$$\Big(D_2 f_e(x, y)\Big)^{\frac{1}{2}} \le 4\Big(\frac{8}{3}\Big)^{\frac{1}{2}} \Big(|x| + |y|\Big)^{\frac{1}{2}}.$$

Thus we have

$$D_2 f_e(x, y) \le \frac{128}{3}(|x| + |y|).$$

and so we have (2.22).

Suppose that there exist an additive mapping $A : \mathbb{R} \longrightarrow \mathbb{R}$, a quadratic mapping $Q : \mathbb{R} \longrightarrow \mathbb{R}$, and a non-negative constant K such that $A + Q$ satisfies (2.23). Since $|f(x)| \le \frac{10}{3}$, by (2.23), we have

$$\frac{10}{3n} - K|x|^2 \le \frac{A(x)}{n} + Q(x) \le \frac{10}{3n} + K|x|^2$$

for all $x \in X$ and all positive integers n and so

$$|Q(x)| \le K|x|^2$$

for all $x \in X$. Hence by (2.23), we have

$$|f - A(x)| \le 2K|x|^2$$

for all $x \in X$.

Since f_o, A are odd and f_e is even,

$$(2.24)\quad |f_e(x)| \le \frac{1}{2}\Big[|f_e(x) + f_o(x) - A(x)| + |f_e(-x) + f_o(-x) - A(-x)|\Big] \le 4K|x|^2$$

for all $x \in X$. Take a positive integer l such that $l > 4K$, and pick $x \in \mathbb{R}$ with $0 < 2^l x < 1$. Then

$$f_e(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{4^n} \ge \sum_{n=0}^{l-1} \frac{s(2^n x)}{4^n} \ge l x^2 > 4K x^2$$

which contradicts to (2.24). $\square$

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.6, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Exact controllability for fuzzy differential equations using extremal solutions

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Abstract

In this paper, we devoted study exact controllability for fuzzy differential equations with the control function in credibility spaces. Moreover we study exact controllability for every solutions of fuzzy differential equations. The result is obtained by using extremal solutions.

1 Introduction

The theory of controlled processes is one of the most recent mathematical concepts to enable very important applications in modern engineering. However, actual systems subject to control do not admit a strictly deterministic analysis in view of various random factors that influence their behavior. The theory of controlled processes takes the random nature of a systems behavior into account. Many researchers have studied controlled processes in a credibility space. Arapostathis et al. [1] studied the controllability properties of the class of stochastic differential systems characterized by a linear controlled diffusion perturbed by a

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smooth, bounded, and uniformly Lipschitz nonlinearity. Kwun et al. [8] proved the approximate controllability for fuzzy differential equations driven by Liu process. Lee et al. [10] examined the exact controllability for abstract fuzzy differential equations in a credibility space.

Recently, Kwun et al. [14] studied the existence of extremal solutions for fuzzy differential equations driven by Liu process. Kwun et al. [6, 7] have studied the existence of extremal solutions for fuzzy differential equations in a n-dimensional fuzzy vector space. In this paper, using the extremal solutions, we study the exact controllability for every solutions of fuzzy differential equations in credibility space. We consider the following fuzzy differential equation:

$$\begin{cases} dx(t,\theta) = f(t, x(t,\theta))dC_t + Bu(t), \ t \in [0,T], \\ x(0) = x_0 \in E_N, \end{cases} \tag{1}$$

where the state function $x(t,\theta)$ takes values in $X(\subset E_N)$ and another bounded space $Y(\subset E_N)$. E_N is the set of all upper semi-continuously convex fuzzy numbers on R, $(\Theta, \mathcal{P}, Cr)$ is credibility space, the state function $x : [0,T] \times (\Theta, \mathcal{P}, Cr) \to X$ is a fuzzy process, $f : [0,T] \times X \to X$ is a regular fuzzy function, $u : [0,T] \times (\Theta, \mathcal{P}, Cr) \to Y$ is a control function, B is a linear bounded operator from Y to X. C_t is a standard Liu process, $x_0 \in E_N$ is an initial value.

2 Preliminaries

In this section, we give basic definitions, terminologies, notations and lemmas which are most relevant to our investigated and are needed in later section. All undefined concepts and notions used here are standard.

A fuzzy set of R^n is a function $u : R^n \to [0,1]$. For each fuzzy set u, we denote by $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$ for any $\alpha \in [0,1]$, its α-level set. Let u, v be fuzzy sets of R^n. It is well known that $[u]^\alpha = [v]^\alpha$ for each $\alpha \in [0,1]$ implies $u = v$. Let E^n denote the collection of all fuzzy sets of R^n that satisfies the following conditions:

(1) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(2) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$, $0 \leq \lambda \leq 1$;

(3) $u(x)$ is upper semi-continuous, i.e., $u(x_0) \geq \overline{\lim}_{k\to\infty} u(x_k)$ for any $x_k \in R^n$ $(k = 0, 1, 2, \ldots)$, $x_k \to x_0$;

(4) $[u]^0$ is compact.

Definition 2.1. [17] The complete metric D_L on E_N is defined by

$$\begin{aligned} D_L(u,v) &= \sup_{0<\alpha\leq 1} d_L([u]^\alpha, [v]^\alpha) \\ &= \sup_{0<\alpha\leq 1} \max\{|u_l^\alpha - v_l^\alpha|, |u_r^\alpha - v_r^\alpha|\}, \end{aligned}$$

for any $u, v \in E_N$, which satisfies $d_L(u+w, v+w) = d_L(u,v)$.

Definition 2.2. [5] Let $u, v \in C([0,T], E_N)$. The metric H_1 on $C([0,T], E_N)$ is defined by

$$H_1(u,v) = \sup_{0<t\leq T} D_L(u(t), v(t)).$$

Let Θ be a nonempty set, and let $\mathcal{P}$ the power set of Θ. Each element in $\mathcal{P}$ is called an event. In order to present an axiomatic definition of credibility, it is necessary to assign to each event A a number $Cr\{A\}$ which indicates the credibility that A will occur. In order to ensure that the number $Cr\{A\}$ has certain mathematical properties which we intuitively expect a credibility to have, we accept the following four axioms:

1. (Normality) $Cr\{A\} = 1$.
2. (Monotonicity) Cr is increasing, i.e., $Cr\{A\} \leq Cr\{B\}$ whenever $A \subset B$.
3. (Self-Duality) Cr is self-dual, i.e., $Cr\{A\}+Cr\{A^c\} = 1$ for any $A \in \mathcal{P}(\Theta)$.
4. (Maximality) $Cr\{\cup_i A_i\} = \sup_i Cr\{A_i\}$ for any $\{A_i\}$ with $Cr\{A_i\} \leq 0.5$.

Definition 2.3. [11] Let ξ be a fuzzy variable with the possibility distribution function $\mu : R \to [0,1]$. A fuzzy variable ξ is said to be normal if there exists a real number r such that $\mu(r) = 1$. It is well known that the possibility of $\{\xi \leq r\}$ is defined by

$$\text{Pos}\{\xi \leq r\} = \sup_{u\leq r} \mu(u)$$

while the necessity of $\{\xi \leq r\}$ is defined by

$$\text{Nec}\{\xi \leq r\} = 1 - \text{Pos}\{\xi < r\} = 1 - \sup_{u<r} \mu(u).$$

Definition 2.4. [11] The set function Cr is called a credibility measure if it satisfies above four axioms, and defined as follows:

$$Cr\{A\} = \frac{1}{2}(\text{Pos}\{A\} + \text{Nec}\{A\}),$$

where $\text{Pos}\{A\} = 1 - \text{Nec}\{A^c\}$ with A^c is the complement of A.

Definition 2.5. [12] Let Θ be a nonempty set, $\mathcal{P}$ be the power set of Θ, and let Cr be a credibility measure. Then the triplet $(\Theta, \mathcal{P}, C_r)$ is called a credibility space.

Definition 2.6. [13] A fuzzy variable is a function from a credibility space $(\Theta, \mathcal{P}, C_r)$ to the set of real numbers.

Definition 2.7. [13] Let T be an index set and let $(\Theta, \mathcal{P}, C_r)$ be a credibility space. A fuzzy process is a function from $T \times (\Theta, \mathcal{P}, C_r)$ to the set of real numbers.

That is, a fuzzy process $x(t,\theta)$ is a function of two variables such that the function $x(t^*,\theta)$ is a fuzzy variable for each t^*. For each fixed θ^*, the function $x(t,\theta^*)$ is called a sample path of the fuzzy process. A fuzzy process $x(t,\theta)$ is said to be sample-continuous if the sample path is continuous for almost all θ.

Definition 2.8. Let $(\Theta,\mathcal{P},C_r)$ be a credibility space. For fuzzy random variable $x(t,\theta)$ in a credibility space, for each $\alpha\in(0,1]$, the α-level set $[x(t,\theta)]^\alpha=[x_l^\alpha(t,\theta),x_r^\alpha(t,\theta)]$ is defined by

$$x_l^\alpha(t,\theta)=\inf x^\alpha(t,\theta)=\inf\{a\in R|x(t,\theta)(a)\geq\alpha\},$$
$$x_r^\alpha(t,\theta)=\sup x^\alpha(t,\theta)=\sup\{a\in R|x(t,\theta)(a)\geq\alpha\}.$$

Definition 2.9. [11] Let ξ be a fuzzy variable and r is a real number. Then the expected value of ξ is defined by

$$E\xi=\int_0^{+\infty}Cr\{\xi\geq r\}dr-\int_{-\infty}^0 Cr\{\xi\leq r\}dr$$

provided that at least one of the integrals is finite.

Definition 2.10. [13] A fuzzy process C_t is said to be a Liu process if
(1) $C_0=0$;
(2) C_t has stationary and independent increments;
(3) every increment $C_{t+s}-C_s$ is a normally distributed fuzzy variable with expected value et and variance σ^2t^2, whose membership function is

$$\mu(x)=2\Big(1+\exp\Big(\frac{\pi|x-et|}{\sqrt{6}\sigma t}\Big)\Big)^{-1},\ x\in R.$$

The parameters e and σ are called the *drift* and *diffusion* coefficients, respectively. Liu process is said to be standard if $e=0$ and $\sigma=1$.

Definition 2.11. [3] Let $x(t)$ be a fuzzy process and let C_t be a standard Liu process. For any partition of closed interval $[c,d]$ with $c=t_0<\cdots<t_n=d$, the mesh is written as $\triangle=\max_{1\leq i\leq n}(t_i-t_{i-1})$. Then the fuzzy integral of $x(t)$ with respect to C_t is

$$\int_c^d x(t)dC_t=\lim_{\triangle\to 0}\sum_{i=1}^n x(t_{i-1})(C_{t_i}-C_{t_{i-1}})$$

provided that the limit exists almost surely and is a fuzzy variable.

Lemma 2.1. [3] Let C_t be a standard Liu process. For any given θ with $Cr\{\theta\}>0$, the path C_t is Lipschitz continuous, that is, the following inequality holds

$$|C_{t_1}-C_{t_2}|<K(\theta)|t_1-t_2|,$$

where K is a fuzzy variable called the Lipschitz constant of a Liu process with

$$K(\theta)=\begin{cases}\sup_{0\leq s<t}\frac{|C_t-C_s|}{t-s}, & Cr\{\theta\}>0,\\ \infty, & \text{otherwise},\end{cases}$$

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and $E[K^p] < \infty$, $\forall p > 0$.

Lemma 2.2. [3] Let C_t be a standard Liu process, and let $h(t; c)$ be a continuously differentiable function. Define $x_t = h(t; C_t)$. Then we have the following chain rule

$$dx_t = \frac{\partial h(t; C_t)}{\partial t} dt + \frac{\partial h(t; C_t)}{\partial C} dC_t.$$

Lemma 2.3. [3] Let $f(t)$ be continuous fuzzy process, the following inequality of fuzzy integral holds

$$\Big| \int_c^d f(t) dC_t \Big| \leq K \int_c^d |f(t)| dt,$$

where $K = K(\theta)$ is defined in Lemma 2.1.

Definition 2.12. [14] For the partial ordering $\leq_T$, a function $a \in C([0, T] \times (\Theta, \mathcal{P}, C_r), E_N)$ is a $\leq_T$-lower solution for equation (1)($u \equiv 0$) if

$$\begin{cases} a(t, \theta) \leq_T U(t)x_0 + \int_0^t U(t-s) G(s, a(s, \theta)) dC(s), \ t \in [0, T], \\ a(0) \leq_T x_0 \in E_N \end{cases} \tag{2}$$

and a function $b \in C([0, T] \times (\Theta, \mathcal{P}, C_r), E_N)$ is a $\leq_T$-upper solution for equation (1)($u \equiv 0$) if

$$\begin{cases} b(t, \theta) \geq_T S(t)x_0 + \int_0^t S(t-s) F(s, b(s, \theta)) dC(s), \ t \in [0, T], \\ b(0) \geq_T x_0 \in E_N. \end{cases} \tag{3}$$

Theorem 2.1. [14] Let $a, b \in C([0, T] \times (\Theta, \mathcal{P}, C_r), E_N)$ be, respectively, $\leq_T$-lower and $\leq_T$-upper solutions for equation (1)($u \equiv 0$) on $[0, T]$. Then, there exist monotone sequences $\{a_n\} \uparrow \rho$, $\{b_n\} \downarrow \gamma$ in $C([0, T] \times (\Theta, \mathcal{P}, C_r), E_N)$, where ρ, γ are extremal solutions to equation (1) in the stochastic fuzzy functional interval $[a, b] := \{x \in C([0, T] \times (\Theta, \mathcal{P}, C_r), E_N) | a \leq_T x \leq_T b \text{ on } [0, T]\}$.

3 Exact controllability for fuzzy differential equation using extremal solutions

In this section, we study exact controllability for fuzzy differential equation using extremal solutions (1). In [14], Park et al. proved the existence of extremal solutions for the equation (1). Hence we consider extremal solutions for the equation (1), for each u in Y.

$$\begin{cases} x_t = U(t)x_0 + \int_0^t U(t-s) G(s, x_s) dC_s + \int_0^t U(t-s) B u_s ds, \\ x(0) = x_0 \in E_N, \end{cases} \tag{4}$$

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where $U(t) = e^{-Mt}$ is continuous with $U(0) = I$, $|U(t)| \le c$, $c > 0$, for all $t \in [0,T]$. And

$$\begin{cases} x_t = S(t)x_0 + \int_0^t S(t-s)F(s,x_s)dC_s + \int_0^t S(t-s)Bu_s ds, \\ x(0) = x_0 \in E_N, \end{cases} \tag{5}$$

where $S(t) = e^{Mt}$ is continuous with $S(0) = I$, $|S(t)| \le d$, $d > 0$, for all $t \in [0,T]$.

Now we assume the following hypotheses:

(**H1**) For $L_1, L_2 > 0$, $x_0 \in E_N$,

$$d_L\Big([U(t)x_0]^\alpha, [x_0]^\alpha\Big) \le L_1, \quad d_L\Big([S(t)x_0]^\alpha, [x_0]^\alpha\Big) \le L_2.$$

(**H2**) For $x(\cdot), y(\cdot) \in C([0,T]\times(\Theta,\mathcal{P},C_r),E_N)$, $t \in [0,T]$, there exist positive numbers m_1, m_2 such that

$$d_L\Big([G(t,x)]^\alpha, [G(t,y)]^\alpha\Big) \le m_1 d_L([x]^\alpha,[y]^\alpha),$$
$$d_L\Big([F(t,x)]^\alpha, [F(t,y)]^\alpha\Big) \le m_2 d_L([x]^\alpha,[y]^\alpha)$$

and $F(0,\mathcal{X}_{\{0\}}(0)) \equiv 0$, $G(0,\mathcal{X}_{\{0\}}(0)) \equiv 0$.

(**H3**) For $L_3 > 0$, $x_0 \in E_N$, $d_L\Big([x_0]^\alpha, [\mathcal{X}_{\{0\}}(0)]^\alpha\Big) \le L_3$.

(**H4**) For $\varepsilon > 0$, $(L_1 + cm_1KL_3T)e^{cm_1KT} \le \varepsilon$.

(**H5**) For $\varepsilon > 0$, $(L_2 + dm_2KL_3T)e^{dm_2KT} \le \varepsilon$.

(**H6**) Let a, b be, respectively, lower solution and upper solution of equation (1)($u \equiv 0$), then $[a,b]$ is convex.

We define the controllability concept for a fuzzy differential equation.

Definition 3.1. The equation (1) is said to be controllable on [0,T], if for every $x_0 \in E_N$ there exists a control $u_t \in Y$ such that every solutions $x(\cdot)$ of (1) satisfies a.s. θ, $x_T = x^1 \in X$ (i.e., $[x_T]^\alpha = [x^1]^\alpha$).

Definition 3.2. Define the fuzzy mappings $P_1 : \widetilde{P}(R) \to X$ and $P_2 : \widetilde{P}(R) \to X$ by

$$P_1^\alpha(v) = \begin{cases} \int_0^T U^\alpha(T-s)Bv_s ds, & v \subset \overline{\Gamma}_u, \\ 0, & \text{otherwise}, \end{cases}$$
$$P_2^\alpha(v) = \begin{cases} \int_0^T S^\alpha(T-s)Bv_s ds, & v \subset \overline{\Gamma}_u, \\ 0, & \text{otherwise}, \end{cases}$$

where $\widetilde{P}(R)$ is a nonempty fuzzy subset of R and $\overline{\Gamma}_u$ is the closure of support u. Then there exist $P_{1i}^\alpha, P_{2i}^\alpha (i = l, r)$ such that

$$P_{1l}^\alpha(v_l) = \int_0^T U_l^\alpha(T-s)B(v_s)_l ds, \ (v_s)_l \in [(u_s)_l^\alpha, (u_s)^1],$$
$$P_{1r}^\alpha(v_r) = \int_0^T U_r^\alpha(T-s)B(v_s)_r ds, \ (v_s)_r \in [(u_s)^1, (u_s)_r^\alpha],$$

$$
\begin{aligned}
P_{2l}^{\alpha}(v_l) &= \int_0^T S_l^{\alpha}(T-s)B(v_s)_l ds, \ (v_s)_l \in [(u_s)_l^{\alpha}, (u_s)^1], \\
P_{2r}^{\alpha}(v_r) &= \int_0^T S_r^{\alpha}(T-s)B(v_s)_r ds, \ (v_s)_r \in [(u_s)^1, (u_s)_r^{\alpha}].
\end{aligned}
$$

We assume that $\widetilde{P}_{1l}^{\alpha}, \widetilde{P}_{1r}^{\alpha}, \widetilde{P}_{2l}^{\alpha}$ and $\widetilde{P}_{2r}^{\alpha}$ are bijective mappings.

By Definition 3.2, we can introduce α-level set of u_s is

$$
\begin{aligned}
[u_s]^{\alpha} &= [(u_s)_l^{\alpha}, (u_s)_r^{\alpha}] \\
&= \frac{1}{2}\Big[(\widetilde{P}_{1l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - U_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\} \\
&\quad +(\widetilde{P}_{2l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - S_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T S_l^{\alpha}(T-s)F_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}, \\
&\quad (\widetilde{P}_{1r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - U_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\} \\
&\quad +(\widetilde{P}_{2r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - S_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T S_r^{\alpha}(T-s)F_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\Big].
\end{aligned}
$$

Theorem 3.1. If Lemma 2.3 and hypotheses (H1)-(H5) are satisfied, then the equation (4) is controllable on $[0,T]$.

Proof By Definition 3.2 and above u_s, substitute the control into the equation (4) yields α-level of $\underline{x}_T$.

$$
\begin{aligned}
[\underline{x}_T]^{\alpha} &= \Big[U(T)x_0 + \int_0^T U(T-s)G(s,x_s)dC_s + \int_0^T U(T-s)Bu_s ds\Big]^{\alpha} \\
&= \Big[U_l^{\alpha}(T)(x_0)_l^{\alpha} + \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s + \int_0^T U_l^{\alpha}(T-s)B \\
&\quad \times\frac{1}{2}\Big[(\widetilde{P}_{1l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - U_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - S_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T S_l^{\alpha}(T-s)F_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}\Big]ds, \\
&\quad U_r^{\alpha}(T)(x_0)_r^{\alpha} + \int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s + \int_0^T U_r^{\alpha}(T-s)B \\
&\quad \times\frac{1}{2}\Big[(\widetilde{P}_{1r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - U_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - S_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T S_r^{\alpha}(T-s)F_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\Big]ds\Big] \\
&= \Big[U_l^{\alpha}(T)(x_0)_l^{\alpha} + \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}P_{1l}^{\alpha}\Big[(\widetilde{P}_{1l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha}-U_l^{\alpha}(T)(x_0)_l^{\alpha}-\int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}\\
&\quad+(\widetilde{P}_{2l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha}-S_l^{\alpha}(T)(x_0)_l^{\alpha}-\int_0^T S_l^{\alpha}(T-s)F_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}\Big],\\
&U_r^{\alpha}(T)(x_0)_r^{\alpha}+\int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\\
&+\frac{1}{2}P_{1r}^{\alpha}\Big[(\widetilde{P}_{1r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha}-U_r^{\alpha}(T)(x_0)_r^{\alpha}-\int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\\
&\quad+(\widetilde{P}_{2r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha}-S_r^{\alpha}(T)(x_0)_r^{\alpha}-\int_0^T S_r^{\alpha}(T-s)F_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\Big]\Big]\\
=\;&[(x^1)_l^{\alpha},(x^1)_r^{\alpha}]=[x^1]^{\alpha}.
\end{aligned}
$$

Hence this control u_t satisfy a.s. θ, $x_T=x^1$.

Also, using this control, we shall show that the nonlinear operator Φ_1 defined by

$$
\begin{aligned}
(\Phi_1x)_t=\;&U(t)x_0+\int_0^t U(t-s)G(s,x_s)dC_s+\int_0^t U(t-s)B\\
&\times\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds,
\end{aligned}
$$

where the fuzzy mappings $(\widetilde{P}_1)^{-1}$ satisfy above statements.

Form hypothesis (H2) and Lemma 2.3, for any given θ with $Cr\{\theta\}>0$, $x(\cdot),\ y(\cdot)\in C([0,T]\times(\Theta,\mathcal{P},Cr),E_N)$, we have

$$
\begin{aligned}
&d_L\Big([(\Phi_1x)_t]^{\alpha},[(\Phi_1y)_t]^{\alpha}\Big)\\
&=d_L\Big(\Big[U(t)x_0+\int_0^t U(t-s)G(s,x_s)dC_s\\
&\quad+\int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\qquad+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^{\alpha},\\
&\quad\Big[U(t)x_0+\int_0^t U(t-s)G(s,y_s)dC_s\\
&\quad+\int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}\\
&\qquad+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^{\alpha}\Big)\\
&\le d_L\Big(\Big[\int_0^t U(t-s)G(s,x_s)dC_s\Big]^{\alpha},\Big[\int_0^t U(t-s)G(s,y_s)dC_s\Big]^{\alpha}\Big)
\end{aligned}
$$

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$$
\begin{aligned}
&+d_L\Big(\Big[\int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\qquad\qquad +\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha,\\
&\quad \int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}\\
&\qquad\qquad +\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha\Big)\\
&\le d_L\Big(\Big[\int_0^t U(t-s)G(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^t U(t-s)G(s,y_s)dC_s\Big]^\alpha\Big)\\
&\quad +d_L\Big(\Big[\frac{1}{2}P_1\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\qquad\qquad +\frac{1}{2}P_1\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]^\alpha,\\
&\quad \Big[\frac{1}{2}P_1\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}\\
&\qquad\qquad +\frac{1}{2}P_1\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]^\alpha\Big)\\
&\le d_L\Big(\Big[\int_0^t U(t-s)G(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^T U(t-s)G(s,y_s)dC_s\Big]^\alpha\Big)\\
&\quad +d_L\Big(\Big[\int_0^T U(T-s)G(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^t U(T-s)G(s,y_s)dC_s\Big]^\alpha\Big)\\
&\le cm_1K\int_0^t d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds+cm_1K\int_0^T d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds.
\end{aligned}
$$

Therefore, by Lemma 2.1, we get

$$
\begin{aligned}
&E\Big(H_1(\Phi_1x,\Phi_1y)\Big)\\
&\quad =E\Big(\sup_{t\in[0,T]}D_L\Big((\Phi_1x)_t,(\Phi_1y)_t\Big)\Big)\\
&\quad =E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\le1}d_L\Big([(\Phi_1x)_t]^\alpha,[(\Phi_1y)_t]^\alpha\Big)\Big)\\
&\quad \le E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\le1}cm_1K\Big(\int_0^T d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds+\int_0^T d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds\Big)\Big)\\
&\quad \le E\Big(\sup_{t\in[0,T]}cm_1K\Big(\int_0^t D_L(x_s,y_s)ds+\int_0^T D_L(x_s,y_s)ds\Big)\Big)\\
&\quad \le 2cm_1KTE\Big(H_1(x,y)\Big).
\end{aligned}
$$

We take sufficiently small T, $2cm_1KT<1$. Hence Φ_1 is contraction mapping. By the Banach fixed point theorem, (4) has a unique fixed point. Thus

the equation (1) is controllable in $[0, T]$.

Theorem 3.2. If Lemma 2.3 and hypotheses (H1)-(H5) are satisfied, then the equation (5) is controllable on $[0, T]$.

Proof By Definition 3.2 and above u_s, substitute the control into the equation (5) yields α-level of $\overline{x}_T$.

$$
\begin{aligned}
[\overline{x}_T]^\alpha &= \Big[S(T)x_0 + \int_0^T S(T-s)F(s,x_s)dC_s + \int_0^T S(T-s)Bu_s ds\Big]^\alpha \\
&= \Big[S_l^\alpha(T)(x_0)_l^\alpha + \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s + \int_0^T S_l^\alpha(T-s)B \\
&\quad \times\frac{1}{2}\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\quad +(\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big]ds, \\
&\quad S_r^\alpha(T)(x_0)_r^\alpha + \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s + \int_0^T S_r^\alpha(T-s)B \\
&\quad \times\frac{1}{2}\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\quad +(\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]ds\Big] \\
&= \Big[S_l^\alpha(T)(x_0)_l^\alpha + \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{2l}^\alpha\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\quad +(\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big], \\
&\quad S_r^\alpha(T)(x_0)_r^\alpha + \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{2r}^\alpha\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\quad +(\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]\Big] \\
&= [(x^1)_l^\alpha, (x^1)_r^\alpha] = [x^1]^\alpha.
\end{aligned}
$$

Hence this control u_t satisfy a.s. θ, $x_T = x^1$.

Also, using this control, we shall show that the nonlinear operator Φ_2 defined by

$$
(\Phi_2 x)_t = S(t)x_0 + \int_0^t S(t-s)F(s,x_s)dC_s + \int_0^t S(t-s)B
$$

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$$\times\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}$$
$$+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds,$$

where the fuzzy mappings $(\widetilde{P}_2)^{-1}$ satisfy above statements.

Form hypothesis (H2) and Lemma 2.3, for any given θ with $Cr\{\theta\}>0$, $x(\cdot),\ y(\cdot)\in C([0,T]\times(\Theta,\mathcal{P},Cr),E_N)$, we have

$$\begin{aligned}
&d_L\Big([(\Phi_2x)_t]^\alpha,[(\Phi_2y)_t]^\alpha\Big)\\
&=d_L\Big(\Big[S(t)x_0+\int_0^t S(t-s)F(s,x_s)dC_s\\
&\quad+\int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\qquad+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha,\\
&\quad\Big[S(t)x_0+\int_0^t S(t-s)F(s,y_s)dC_s\\
&\quad+\int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}\\
&\qquad+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha\Big)\\
&\le d_L\Big(\Big[\int_0^t S(t-s)F(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^t S(t-s)F(s,y_s)dC_s\Big]^\alpha\Big)\\
&\quad+d_L\Big(\Big[\int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\qquad+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha,\\
&\qquad\int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}\\
&\qquad+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha\Big)\\
&\le d_L\Big(\Big[\int_0^t S(t-s)F(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^t S(t-s)F(s,y_s)dC_s\Big]^\alpha\Big)\\
&\quad+d_L\Big(\Big[\frac{1}{2}P_2\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\qquad+\frac{1}{2}P_2\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]^\alpha,\\
&\qquad\Big[\frac{1}{2}P_2\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}
\end{aligned}$$

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$$+\frac{1}{2}P_2\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau, y_\tau)dC_\tau\Big\}\Big]^\alpha\Big)$$
$$\le d_L\Big(\Big[\int_0^t S(t-s)F(s,x_s)dC_s\Big]^\alpha, \Big[\int_0^T S(t-s)F(s,y_s)dC_s\Big]^\alpha\Big)$$
$$+d_L\Big(\Big[\int_0^T S(T-s)F(s,x_s)dC_s\Big]^\alpha, \Big[\int_0^t S(T-s)F(s,y_s)dC_s\Big]^\alpha\Big)$$
$$\le dm_2K\int_0^t d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds + dm_2K\int_0^T d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds.$$

Therefore, by Lemma 2.1, we get

$$E\Big(H_1(\Phi_2x, \Phi_2y)\Big)$$
$$= E\Big(\sup_{t\in[0,T]} D_L\Big((\Phi_2x)_t, (\Phi_2y)_t\Big)\Big)$$
$$= E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\le 1} d_L\Big([(\Phi_2x)_t]^\alpha, [(\Phi_2y)_t]^\alpha\Big)\Big)$$
$$\le E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\le 1} dm_2K\Big(\int_0^T d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds + \int_0^T d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds\Big)\Big)$$
$$\le E\Big(\sup_{t\in[0,T]} 3m_2K\Big(\int_0^t D_L(x_s, y_s)ds + \int_0^T D_L(x_s, y_s)ds\Big)\Big)$$
$$\le 2dm_2KTE\Big(H_1(x,y)\Big).$$

We take sufficiently small T and $2dm_2KT < 1$. Hence Φ_2 is contraction mapping. By the Banach fixed point theorem, (5) has a unique fixed point. Thus the equation (1) is controllable in $[0,T]$.

Theorem 3.3. If Theorems 3.1 and 3.2 and hypotheses (H1)-(H6) are satisfied, then the equation (1) is controllable on $[0,T]$.

Proof For $x_T \in [\underline{x}_T, \overline{x}_T]$, if $[\underline{x}_T, \overline{x}_T]$ is convex, then $x_T = \lambda\underline{x}_T + (1-\lambda)\overline{x}_T, 0 \le \lambda \le 1$, we can obtain the following result.

$$[x_T]^\alpha = [\lambda\underline{x}_T + (1-\lambda)\overline{x}_T]^\alpha$$
$$= \Big[\lambda\Big\{U(T)x_0 + \int_0^T U(T-s)G(s,x_s)dC_s + \int_0^T U(T-s)Bu_sds\Big\}$$
$$+(1-\lambda)\Big\{S(T)x_0 + \int_0^T S(T-s)F(s,x_s)dC_s + \int_0^T S(T-s)Bu_sds\Big\}\Big]^\alpha$$
$$= \lambda\Big[U(T)x_0 + \int_0^T U(T-s)G(s,x_s)dC_s + \int_0^T U(T-s)Bu_sds\Big]^\alpha$$
$$+(1-\lambda)\Big[S(T)x_0 + \int_0^T S(T-s)F(s,x_s)dC_s + \int_0^T S(T-s)Bu_sds\Big]^\alpha$$

$$
\begin{aligned}
&= \lambda\Big[U_l^\alpha(T)(x_0)_l^\alpha + \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{1l}^\alpha\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big], \\
&\quad U_r^\alpha(T)(x_0)_r^\alpha + \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{1r}^\alpha\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]\Big] \\
&\ +(1-\lambda)\Big[S_l^\alpha(T)(x_0)_l^\alpha + \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{2l}^\alpha\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big], \\
&\quad S_r^\alpha(T)(x_0)_r^\alpha + \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{2r}^\alpha\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]\Big] \\
&= [(x^1)_l^\alpha, (x^1)_r^\alpha] = [x^1]^\alpha.
\end{aligned}
$$

Hence this control u_t satisfy a.s. θ, $x_T = x^1, x_T \in [\underline{x}_T, \overline{x}_T]$. Therefore every solutions of the equation (1) are controllable in $[0,T]$.

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Generalized interval-valued intuitionistic fuzzy soft rough set and its application

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Abstract

In this paper, by integrating interval-valued intuitionistic fuzzy soft set with rough set theory, the concept of generalized interval-valued intuitionistic fuzzy soft rough sets is proposed, which is an extension of generalized intuitionistic fuzzy soft rough sets. Then the properties of this model are investigated. Furthermore, classical representations of generalized interval-valued intuitionistic fuzzy soft rough approximation operators are also introduced. Finally, an approach based on generalized interval-valued intuitionistic fuzzy soft rough sets in decision making is developed, and we provide a practical example to illustrate the validity of this approach.

Key words: Interval-valued intuitionistic fuzzy soft set; Rough set; Generalized interval-valued intuitionistic fuzzy soft rough set; Decision making

1 Introduction

As a framework for the construction of approximations of concepts, rough sets proposed by Pawlak [21,22], is a formal tool for modeling and processing insufficient and incomplete information. In Pawlak's rough set model, the equivalence relation plays an important role, which seems very stringent in daily life. Therefore many researchers have generalized the notion of Pawlak rough set by replacing the equivalence relation with other binary relations. Since the appearance of Pawlak rough set, lots of fruitful results have been achieved [5, 10–12, 15, 16, 25, 28, 29, 31–40, 42, 44–46].

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Soft set theory is presented by Molodtsov [17], which is different from the existing uncertainty theories, such as fuzzy set theory [43], intuitionistic fuzzy set theory [1, 2], interval-valued fuzzy set theory [9, 13, 24], interval-valued intuitionistic fuzzy set theory [3,4], rough set theory [21,22], and so on. In [17], the author pointed out that these theories mentioned above have their inherent difficulties, but soft set has enough parameters so that it is free from inherent difficulties. Therefore, in recent years more and more researchers have joined the ranks of soft set research. For example, Maji et al. [18] initiated the study on hybrid structures involving fuzzy sets and soft sets, and introduced the concept of fuzzy soft sets, which can be viewed as a generalization of soft sets. Subsequently, Maji et al [19] modified the concept of fuzzy soft sets, and proposed a generalized fuzzy soft set theory. Furthermore, Yang et al. [30] extended soft sets to interval-valued fuzzy environment, and first presented the concept of interval-valued fuzzy soft sets by combining interval-valued fuzzy set and soft set. By integrating the intuitionistic fuzzy set with soft set theory, Maji et al. [20] presented the concept of the intuitionistic fuzzy soft set theory. Jiang et al. [14] initiated the concept of interval-valued intuitionistic fuzzy soft sets by the combination of the interval-valued intuitionistic fuzzy sets and soft sets. On the basis of [14], Zhang [46] presented an adjustable approach to interval-valued intuitionistic fuzzy soft sets based decision making by mean of level soft sets of interval-valued intuitionistic fuzzy soft sets. Recently, soft set theory has been developed into hesitant fuzzy environment, and the result is called hesitant fuzzy soft sets [6, 26, 27]. Because it is unreasonable to use hesitant fuzzy soft sets to handle some decision making problems, Zhang et al. [41] extended hesitant fuzzy soft sets to interval-valued hesitant fuzzy environment, and introduced the concept of interval-valued hesitant fuzzy soft sets by combining the interval-valued hesitant fuzzy set and soft set theory. More recently, by combining intuitionistic fuzzy soft set and rough set theory, Zhang et al. [38] introduced the concept of intuitionistic fuzzy soft rough sets, and gave an approach to decision making based on this model. Furthermore, in [42], they pointed out the drawback of the intuitionistic fuzzy soft rough sets, proposed a generalized intuitionistic fuzzy soft rough set model, and then illustrated the validity of this model by a practical example.

As a generalization of fuzzy soft sets, interval-valued fuzzy soft sets and intuitionistic fuzzy soft sets, interval-valued intuitionistic fuzzy soft set is more flexible and effective than other soft set theories to cope with imperfect and imprecise information. Meanwhile, we can note that the final decision results for the decision approach presented by Zhang [46] may be different based on different types of thresholds. That is to say, there actually does not exist a unique or uniform criterion for the evaluation of decision alternatives. That is its limitations and disadvantages. In order to overcome these limitations, we need to define a new interval-valued intuitionistic fuzzy soft set model such that the decision approach based on this model is less affected by subjective factors. In this paper, we mainly devote to the generalization of intuitionistic fuzzy soft rough sets [42] and propose the concept of generalized interval-valued intuitionistic fuzzy soft rough sets by integrating interval-

valued intuitionistic fuzzy soft set with rough set. Also its decision making method is given. The most advantage of the decision making method is that it will only use the data information provided by the decision making problem without any additional available information provided by decision makers. Thus it can avoid the effect of subjective factors provided by different experts.

The rest of this paper is organized as follows. Section 2 briefly reviews some preliminaries. In Section 3, an interval-valued intuitionistic fuzzy soft relation is first defined by us. By combining the interval-valued intuitionistic fuzzy soft set and rough sets, then the concept of generalized interval-valued intuitionistic fuzzy soft rough approximation operators is presented and the properties of generalized upper and lower interval-valued intuitionistic fuzzy soft rough approximation operators are examined. Furthermore, classical representations of generalized interval-valued intuitionistic fuzzy soft rough approximation operators are presented. Section 4 is devoted to studying the application of generalized interval-valued intuitionistic fuzzy soft rough sets. Some conclusions and outlooks for further research are given in Section 5.

2 Preliminaries

In this section, we shall briefly recall some basic notions being used in the study.

Before introducing the notion of interval-valued intuitionistic fuzzy soft relation, we first give the concept of soft sets [17] and fuzzy soft sets [18].

Definition 2.1 ([17]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called a soft set over U if $F : E \to P(U)$, where $P(U)$ is the set of all subsets of U.*

Definition 2.2 ([18]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called a fuzzy soft set over U if $F : E \to F(U)$, where $F(U)$ is the set of all fuzzy subsets of U.*

By using the concepts of soft set and fuzzy soft set, Cagman et al. [7,8] introduced the definitions of crisp soft relation and fuzzy soft relation, respectively.

Definition 2.3 ([7]) *Let (F, E) be a soft set over U. Then a subset of $U \times E$ called a crisp soft relation from U to E is uniquely defined by*

$R = \{< (u, x), \mu_R(u, x) > |(u, x) \in U \times E\}$,

where $\mu_R : U \times E \to \{0, 1\}$, $\mu_R(u, x) = \begin{cases} 1, & (u, x) \in R \\ 0, & (u, x) \notin R. \end{cases}$

Definition 2.4 ([8]) *Let (F, E) be a fuzzy soft set over U. Then a fuzzy subset of $U \times E$ called a fuzzy soft relation from U to E is uniquely defined by*

$R = \{< (u,x), \mu_R(u,x) > |(u,x) \in U \times E\}$,
where $\mu_R : U \times E \to [0,1]$, $\mu_R(u,x) = \mu_{F(x)}(u)$.

Based on the crisp soft relation proposed by Cagman, Zhang et al. [42] constructed the following crisp soft rough sets.

Definition 2.5 ([42]) *Let U be an initial universe set and E be a universe set of parameters. For an arbitrary crisp soft relation R over $U \times E$, we can define a set-valued function $R_s : U \to P(E)$ by $R_s(u) = \{x \in E|(u,x) \in R\}, u \in U$.*

R is referred to as serial if for all $u \in U, R_s(u) \neq \emptyset$. The pair (U,E,R) is called a crisp soft approximation space. For any $A \subseteq E$, the upper and lower soft approximations of A with respect to (U,E,R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, are defined, respectively, as follows:

$$\overline{R}(A) = \{u \in U|R_s(u) \cap A \neq \emptyset\},\ \underline{R}(A) = \{u \in U|R_s(u) \subseteq A\}.$$

The pair $(\overline{R}(A), \underline{R}(A))$ is referred to as a crisp soft rough set, and $\overline{R}, \underline{R} : P(E) \to P(U)$ are, referred to as upper and lower crisp soft rough approximation operators, respectively.

Definition 2.6 ([3,4]) *Denote $L = \{(\alpha,\beta)|\alpha = [\alpha_1,\alpha_2] \in Int[0,1], \beta = [\beta_1,\beta_2] \in Int[0,1], \alpha_2 + \beta_2 \leq 1\}$, where $Int[0,1]$ denotes the set of all closed subintervals of $[0,1]$. We define a relation $\leq_L$ on L as follows: $\forall(\alpha,\beta),(\xi,\eta) \in L$,*

$$\begin{aligned}(\alpha,\beta) \leq_L (\xi,\eta) &\Leftrightarrow [\alpha_1,\alpha_2] \leq_{L^I} [\xi_1,\xi_2] \text{ and } [\beta_1,\beta_2] \geq_{L^I} [\eta_1,\eta_2]\\ &\Leftrightarrow \alpha_1 \leq \xi_1, \alpha_2 \leq \xi_2, \beta_1 \geq \eta_1, \text{ and } \beta_2 \geq \eta_2.\end{aligned}$$

Then the relation $\leq_L$ is a partial ordering on L and the pair $(L, \leq_L)$ is a complete lattice with the smallest element $0_L = ([0,0],[1,1])$ and the greatest element $1_L = ([1,1],[0,0])$. The meet operator $\wedge$ and the join operator $\vee$ on $(L, \leq_L)$ which are linked to the ordering $\leq_L$ are, respectively, defined as follows: $\forall(\alpha,\beta),(\xi,\eta) \in L$,
$(\alpha,\beta) \wedge (\xi,\eta) = ([\alpha_1 \wedge \xi_1, \alpha_2 \wedge \xi_2],[\beta_1 \vee \eta_1, \beta_2 \vee \eta_2])$,
$(\alpha,\beta) \vee (\xi,\eta) = ([\alpha_1 \vee \xi_1, \alpha_2 \vee \xi_2],[\beta_1 \wedge \eta_1, \beta_2 \wedge \eta_2])$.

Definition 2.7 ([3,4]) *Let a set U be fixed. The mapping $A : U \to L$ is called an interval-valued intuitionistic fuzzy (IVIF, for short) set on U. An interval-valued intuitionistic fuzzy set A on U can also be denoted by*
$A = \{< x, \mu_A(x), \gamma_A(x) > |x \in U\} = \{< x, [\mu_A^-(x), \mu_A^+(x)], [\gamma_A^-(x), \gamma_A^+(x)] > |x \in U\}$,
where $\mu_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ and $\gamma_A(x) = [\gamma_A^-(x), \gamma_A^+(x)]$ satisfy $0 \leq \mu_A^+(x) + \gamma_A^+(x) \leq 1$ for all $x \in U$, and are, respectively, called the degree of membership and the degree of non-membership of the element $x \in U$ to A.

Let $IVIF(U)$ denotes the family of all interval-valued intuitionistic fuzzy sets on U.

3 Construction of generalized interval-valued intuitionistic fuzzy soft rough sets

In this section, we will present the concept of generalized IVIF soft rough sets by using the IVIF soft relation defined by us.

Definition 3.1 ([14]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called an IVIF soft set over U if $F : E \to IVIF(U)$, where $IVIF(U)$ is the set of all IVIF subsets of U.*

In the following, an IVIF soft relation will be presented, which is important for us to construct generalized IVIF soft rough sets.

Definition 3.2 *Let (F, E) be an IVIF soft set over U. Then an IVIF subset of $U \times E$ called an IVIF soft relation from U to E is uniquely defined by*

$R = \{< (u, x), \mu_R(u, x), \gamma_R(u, x) > |(u, x) \in U \times E\}$,

where $\mu_R : U \times E \to Int[0, 1]$ and $\gamma_R : U \times E \to Int[0, 1]$, for all $(u, x) \in U \times E$ such that $\mu_R(u, x) = [\mu_R^-(u, x), \mu_R^+(u, x)]$ and $\gamma_R(u, x) = [\gamma_R^-(u, x), \gamma_R^+(u, x)]$, which satisfy the condition $0 \leq \mu_R^+(u, x) + \gamma_R^+(u, x) \leq 1$.

Remark 3.3 *In Definition 3.2, if $\mu_R^-(u, x) = \mu_R^+(u, x)$ and $\gamma_R^-(u, x) = \gamma_R^+(u, x)$, namely, $\mu_R : U \times E \to [0, 1]$ and $\gamma_R : U \times E \to [0, 1]$, for all $(u, x) \in U \times E$ such that $0 \leq \mu_R(u, x) + \gamma_R(u, x) \leq 1$, then R is referred to as an intuitionistic fuzzy soft relation on $U \times E$. If R is an intuitionistic fuzzy soft relation on $U \times E$ and $\mu_R(u, x) + \gamma_R(u, x) = 1$, then R is degenerated to a fuzzy soft relation [8] in Definition 2.4. Hence, among fuzzy soft relation, intuitionistic fuzzy soft relation [42] and IVIF soft relation, the IVIF soft relation is the most generalized one. That is, the IVIF soft relation has included fuzzy soft relation and intuitionistic fuzzy soft relation.*

Let $U = \{u_1, u_2, \cdots, u_m\}$ and $E = \{x_1, x_2, \cdots, x_n\}$. Then the IVIF soft relation R from U to E can be presented by a table as in the following form

R	x_1	x_2	$\cdots$	x_n
u_1	$(\mu_R(u_1, x_1), \gamma_R(u_1, x_1))$	$(\mu_R(u_1, x_2), \gamma_R(u_1, x_2))$	$\cdots$	$(\mu_R(u_1, x_n), \gamma_R(u_1, x_n))$
u_2	$(\mu_R(u_2, x_1), \gamma_R(u_2, x_1))$	$(\mu_R(u_2, x_2), \gamma_R(u_2, x_2))$	$\cdots$	$(\mu_R(u_2, x_n), \gamma_R(u_2, x_n))$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$(\mu_R(u_m, x_1), \gamma_R(u_m, x_1))$	$(\mu_R(u_m, x_2), \gamma_R(u_m, x_2))$	$\cdots$	$(\mu_R(u_m, x_n), \gamma_R(u_m, x_n))$

From the above form and the definition of IVIF soft set, we know that every IVIF soft set (F, E) is uniquely characterized by the IVIF soft relation, namely they are mutual determined. It means that an IVIF soft set (F, E) is formally equal to IVIF soft relation.

Therefore, we shall identify any IVIF soft set with IVIF soft relation and view these two concepts as interchangeable. Now, any discussion regard to IVIF soft set could be converted into analysis about IVIF soft relation, which will bring great convenience for our future researches.

In this case, according to the definition of IVIF soft relation, we can construct generalized IVIF soft rough sets as follows.

Definition 3.4 *Let U be an initial universe set and E be a universe set of parameters. For an arbitrary IVIF soft relation R over $U \times E$, the pair (U, E, R) is called an IVIF soft approximation space. For any $A \in IVIF(E)$, we define the upper and lower soft approximations of A with respect to (U, E, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, as follows:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) > | u \in U\}, \tag{1}$$

$$\underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\}. \tag{2}$$

where

$$\mu_{\overline{R}(A)}(u) = [\bigvee_{x \in E} (\mu^-_R(u,x) \wedge \mu^-_A(x)), \bigvee_{x \in E} (\mu^+_R(u,x) \wedge \mu^+_A(x))],$$
$$\gamma_{\overline{R}(A)}(u) = [\bigwedge_{x \in E} (\gamma^-_R(u,x) \vee \gamma^-_A(x)), \bigwedge_{x \in E} (\gamma^+_R(u,x) \vee \gamma^+_A(x))],$$
$$\mu_{\underline{R}(A)}(u) = [\bigwedge_{x \in E} (\gamma^-_R(u,x) \vee \mu^-_A(x)), \bigwedge_{x \in E} (\gamma^+_R(u,x) \vee \mu^+_A(x))],$$
$$\gamma_{\underline{R}(A)}(u) = [\bigvee_{x \in E} (\mu^-_R(u,x) \wedge \gamma^-_A(x)), \bigvee_{x \in E} (\mu^+_R(u,x) \wedge \gamma^+_A(x))].$$

The pair $(\overline{R}(A), \underline{R}(A))$ is referred to as a generalized IVIF soft rough set of A with respect to (U, E, R).

By $\mu^+_R(u,x) + \gamma^+_R(u,x) \leq 1$ and $\mu^+_A(x) + \gamma^+_A(x) \leq 1$, it can be easily verified that $\overline{R}(A)$ and $\underline{R}(A) \in IVIF(U)$. So we call $\overline{R}, \underline{R} : IVIF(E) \to IVIF(U)$ generalized upper and lower IVIF soft rough approximation operators, respectively.

Remark 3.5 *If R is an intuitionistic fuzzy soft relation on $U \times E$, then generalized IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ in Definition 3.4 degenerate to the following forms:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) > | u \in U\},$$
$$\underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\}.$$

where

$$\mu_{\overline{R}(A)}(u) = \bigvee_{x \in E} (\mu_R(u,x) \wedge \mu_A(x)), \ \gamma_{\overline{R}(A)}(u) = \bigwedge_{x \in E} (\gamma_R(u,x) \vee \gamma_A(x)),$$

$\mu_{\underline{R}(A)}(u) = \bigwedge_{x\in E}(\gamma_R(u,x) \vee \mu_A(x)),\ \gamma_{\underline{R}(A)}(u) = \bigvee_{x\in E}(\mu_R(u,x) \wedge \gamma_A(x)).$

In that case, the pair $(\overline{R}(A), \underline{R}(A))$ is generated into a generalized IF soft rough set of A with respect to (U, E, R) proposed by Zhang et al. [42]. That is, generalized IVIF soft rough set in Definition 4.4 includes generalized IF soft rough set [42] as a special case.

Remark 3.6 *If R is a fuzzy soft relation on $U \times E$ and $A \in F(E)$, then generalized IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ degenerate to the following forms:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u) > | u \in U\},\quad \underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u) > | u \in U\}.$$

where $\mu_{\overline{R}(A)}(u) = \bigvee_{x\in E}[\mu_R(u,x) \wedge \mu_A(x)]$, $\mu_{\underline{R}(A)}(u) = \bigwedge_{x\in E}[(1 - \mu_R(u,x)) \vee \mu_A(x)]$.

In that case, generalized IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ are identical with the soft fuzzy rough approximation operators defined by Sun [23]. That is, generalized IVIF soft rough approximation operators in Definition 4.4 are an extension of the soft fuzzy rough approximation operators defined by Sun [23].

In order to better understand the concept of generalized IVIF soft rough approximation operators, let us consider the following example.

Example 3.7 *Suppose that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is the set of five houses under consideration of a decision maker to purchase. Let E be a parameter set, where $E = \{e_1, e_2, e_3, e_4\}$={expensive; beautiful; size; location}. Mr. X wants to buy the house which qualifies with the parameters of E to the utmost extent from available houses in U. Assume that Mr. X describes the "attractiveness of the houses" by constructing an IVIF soft relation R from U to E. And it is presented by a table as in the following form.*

R	e_1	e_2	e_3	e_4
u_1	$([0.7, 0.8], [0.2, 0.2])$	$([0.3, 0.4], [0.2, 0.5])$	$([0.1, 0.1], [0.7, 0.8])$	$([0.3, 0.4], [0.1, 0.3])$
u_2	$([0.1, 0.2], [0.4, 0.6])$	$([0.6, 0.7], [0.1, 0.2])$	$([0.2, 0.3], [0.5, 0.7])$	$([0.3, 0.6], [0.2, 0.3])$
u_3	$([0.5, 0.6], [0.2, 0.4])$	$([0.3, 0.6], [0.2, 0.3])$	$([0.5, 0.7], [0.1, 0.3])$	$([0.1, 0.8], [0.1, 0.2])$
u_4	$([0.1, 0.3], [0.2, 0.6])$	$([0.5, 0.7], [0.1, 0.2])$	$([0.1, 0.4], [0.3, 0.5])$	$([0.2, 0.3], [0.5, 0.7])$
u_5	$([0.8, 0.9], [0.0, 0.1])$	$([0.3, 0.5], [0.4, 0.5])$	$([0.6, 0.8], [0.1, 0.2])$	$([0.4, 0.6], [0.1, 0.4])$

We can see that the precise evaluation for each object on each parameter is unknown while the lower and upper limits of such an evaluation are given. For example, we can not present the precise membership degree and non-membership degree of how beautiful house u_2 is, however, house u_2 is at least beautiful on the membership degree of 0.6 and it is at most beautiful on the membership degree of 0.7; house u_2 is not at least beautiful on

the non-membership degree of 0.1 and it is not at most beautiful on the non-membership degree of 0.2.

Now give an IVIF subset A over the parameter set E as follows:

$$A = \{ < e_1, [0.7, 0.8], [0.1, 0.2] >, < e_2, [0.5, 0.7], [0.2, 0.3] >,\\ < e_3, [0.4, 0.6], [0.1, 0.3] >, < e_4, [0.2, 0.6], [0.3, 0.4] > \}.$$

By Equations (1) and (2), we have

$\mu_{\overline{R}(A)}(u_1) = [0.7, 0.8],\ \gamma_{\overline{R}(A)}(u_1) = [0.2, 0.2],\ \mu_{\overline{R}(A)}(u_2) = [0.5, 0.7],$
$\gamma_{\overline{R}(A)}(u_2) = [0.2, 0.3],\ \mu_{\overline{R}(A)}(u_3) = [0.5, 0.6],\ \gamma_{\overline{R}(A)}(u_3) = [0.1, 0.3],$
$\mu_{\overline{R}(A)}(u_4) = [0.5, 0.7],\ \gamma_{\overline{R}(A)}(u_4) = [0.2, 0.3],\ \mu_{\overline{R}(A)}(u_5) = [0.7, 0.8],$
$\gamma_{\overline{R}(A)}(u_5) = [0.1, 0.2];\ \mu_{\underline{R}(A)}(u_1) = [0.2, 0.6],\ \gamma_{\underline{R}(A)}(u_1) = [0.3, 0.4],$
$\mu_{\underline{R}(A)}(u_2) = [0.2, 0.6],\ \gamma_{\underline{R}(A)}(u_2) = [0.3, 0.4],\ \mu_{\underline{R}(A)}(u_3) = [0.2, 0.6],$
$\gamma_{\underline{R}(A)}(u_3) = [0.2, 0.4],\ \mu_{\underline{R}(A)}(u_4) = [0.4, 0.6],\ \gamma_{\underline{R}(A)}(u_4) = [0.2, 0.3],$
$\mu_{\underline{R}(A)}(u_5) = [0.2, 0.6],\ \gamma_{\underline{R}(A)}(u_5) = [0.3, 0.4].$

Thus

$$\overline{R}(A) = \{ < u_1, [0.7, 0.8], [0.2, 0.2] >, < u_2, [0.5, 0.7], [0.2, 0.3] >, < u_3, [0.5, 0.6], [0.1, 0.3] >,\\ < u_4, [0.5, 0.7], [0.2, 0.3] >, < u_5, [0.7, 0.8], [0.1, 0.2] > \}$$

and

$$\underline{R}(A) = \{ < u_1, [0.2, 0.6], [0.3, 0.4] >, < u_2, [0.2, 0.6], [0.3, 0.4] >, < u_3, [0.2, 0.6], [0.2, 0.4] >,\\ < u_4, [0.4, 0.6], [0.2, 0.3] >, < u_5, [0.2, 0.6], [0.3, 0.4] > \}.$$

In what follows, we investigate the properties of generalized IVIF soft rough approximation operators.

Theorem 3.8 *Let (U, E, R) be an IVIF soft approximation space. Then the generalized upper and lower IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ satisfy the following properties: $\forall A, B \in IVIF(E)$,*

(IVIFSL1) $\underline{R}(A) =\sim \overline{R}(\sim A)$,
(IVIFSU1) $\overline{R}(A) =\sim \underline{R}(\sim A)$;
(IVIFSL2) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$,
(IVIFSU2) $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;
(IVIFSL3) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$,
(IVIFSU3) $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$;
(IVIFSL4) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$,
(IVIFSU4) $\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$;

Proof. We only prove the properties of the lower IVIF soft rough approximation operator $\underline{R}(A)$. The upper IVIF soft rough approximation operator $\overline{R}(A)$ can be proved similarly.
(IVIFSL1) By Definition 3.4, then we have

$$\begin{aligned}\sim \underline{R}(\sim A) &= \{< u, \gamma_{\underline{R}(\sim A)}(u), \mu_{\underline{R}(\sim A)}(u) > | u \in U\}\\
&= \{< u, [\bigvee_{x\in E} (\mu_R^-(u,x) \wedge \gamma_{\sim A}^-(x)), \bigvee_{x\in E} (\mu_R^+(u,x) \wedge \gamma_{\sim A}^+(x))],\\
&\quad [\bigwedge_{x\in E} (\gamma_R^-(u,x) \vee \mu_{\sim A}^-(x)), \bigwedge_{x\in E} (\gamma_R^+(u,x) \vee \mu_{\sim A}^+(x))] > | u \in U\}\\
&= \{< u, [\bigvee_{x\in E} (\mu_R^-(u,x) \wedge \mu_A^-(x)), \bigvee_{x\in E} (\mu_R^+(u,x) \wedge \mu_A^+(x))],\\
&\quad [\bigwedge_{x\in E} (\gamma_R^-(u,x) \vee \gamma_A^-(x)), \bigwedge_{x\in E} (\gamma_R^+(u,x) \vee \gamma_A^+(x))] > | u \in U\}\\
&= \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\} = \overline{R}(A).\end{aligned}$$

(IVIFSL2) By virtue of Equation (2), we have

$$\begin{aligned}\underline{R}(A \cap B) &= \{< u, \mu_{\underline{R}(A\cap B)}(u), \gamma_{\underline{R}(A\cap B)}(u) > | u \in U\}\\
&= \{< u, \bigwedge_{x\in E} (\gamma_R(u,x) \vee \mu_{A\cap B}(x)), \bigvee_{x\in E} (\mu_R(u,x) \wedge \gamma_{A\cap B}(x)) > | u \in U\}\\
&= \{< u, [\bigwedge_{x\in E} (\gamma_R^-(u,x) \vee (\mu_A^-(x) \wedge \mu_B^-(x))), \bigwedge_{x\in E} (\gamma_R^+(u,x) \vee (\mu_A^+(x) \wedge \mu_B^+(x)))],\\
&\quad [\bigvee_{x\in E} (\mu_R^-(u,x) \wedge (\gamma_A^-(x) \vee \gamma_B^-(x))), \bigvee_{x\in E} (\mu_R^+(u,x) \wedge (\gamma_A^+(x) \vee \gamma_B^+(x)))] > | u \in U\}\\
&= \{< u, [\mu_{\underline{R}(A)}^-(u) \wedge \mu_{\underline{R}(B)}^-(u), \mu_{\underline{R}(A)}^+(u) \wedge \mu_{\underline{R}(B)}^+(u)],\\
&\quad [\gamma_{\underline{R}(A)}^-(u) \vee \gamma_{\underline{R}(B)}^-(u), \gamma_{\underline{R}(A)}^+(u) \vee \gamma_{\underline{R}(B)}^+(u)] > | u \in U\}\\
&= \{< u, \mu_{\underline{R}(A)}(u) \wedge \mu_{\underline{R}(B)}(u), \gamma_{\underline{R}(A)}(u) \vee \gamma_{\underline{R}(B)}(u) > | u \in U\} = \underline{R}(A) \cap \underline{R}(B).\end{aligned}$$

(IVIFSL3) It can be easily verified by Definition 3.4.
(IVIFSL4) By (IVIFSL3), it is straightforward. □

In Theorem 3.8, properties (IVIFSL1) and (IVIFSU1) show that the generalized upper lower IVIF soft rough approximation operators $\overline{R}$ and $\underline{R}$ are dual to each other.

Inspired by the concept of cut sets of IF sets in [44, 45], we first present the concept of cut sets of IVIF sets before investigating the representing method of the generalized IVIF soft rough approximation operators.

Definition 3.9 *Let $A = \{< x, \mu_A(x), \gamma_A(x) > | x \in U\} \in IVIF(U)$, and $(\alpha, \beta) \in L$, where $\alpha = [\alpha_1, \alpha_2], \beta = [\beta_1, \beta_2] \in Int[0,1]$ with $\alpha_2 + \beta_2 \leq 1$. The (α, β)-level cut set of A,*

denoted by A_α^β *, is defined as follows:*

$$\begin{aligned} A_\alpha^\beta &= \{x \in U | \mu_A(x) \geq_{L^I} \alpha, \gamma_A(x) \leq_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2, \gamma_A^-(x) \leq \beta_1, \gamma_A^+(x) \leq \beta_2\}. \end{aligned}$$

$$A_\alpha = \{x \in U | \mu_A(x) \geq_{L^I} \alpha\} = \{x \in U | \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2\},$$

and

$$A_{\alpha+} = \{x \in U | \mu_A(x) >_{L^I} \alpha\} = \{x \in U | \mu_A^-(x) > \alpha_1, \mu_A^+(x) > \alpha_2\}$$

are, respectively, called the α*-level cut set and the strong* α*-level cut set of membership generated by* A*. Meanwhile,*

$$A^\beta = \{x \in U | \gamma_A(x) \leq_{L^I} \beta\} = \{x \in U | \gamma_A^-(x) \leq \beta_1, \gamma_A^+(x) \leq \beta_2\}$$

and

$$A^{\beta+} = \{x \in U | \gamma_A(x) <_{L^I} \beta\} = \{x \in U | \gamma_A^-(x) < \beta_1, \gamma_A^+(x) < \beta_2\}$$

are, respectively, referred to as the β*-level cut set and the strong* β*-level cut set of non-membership generated by* A*.*

At the same time, other types of cut sets of the IVIF set A *are denoted as follows:*

$$\begin{aligned} A_{\alpha+}^\beta &= \{x \in U | \mu_A(x) >_{L^I} \alpha, \gamma_A(x) \leq_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) > \alpha_1, \mu_A^+(x) > \alpha_2, \gamma_A^-(x) \leq \beta_1, \gamma_A^+(x) \leq \beta_2\}, \end{aligned}$$

which is called the $(\alpha+, \beta)$*-level cut set of* A*;*

$$\begin{aligned} A_\alpha^{\beta+} &= \{x \in U | \mu_A(x) \geq_{L^I} \alpha, \gamma_A(x) <_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2, \gamma_A^-(x) < \beta_1, \gamma_A^+(x) < \beta_2\}, \end{aligned}$$

which is called the $(\alpha, \beta+)$*-level cut set of* A*;*

$$\begin{aligned} A_{\alpha+}^{\beta+} &= \{x \in U | \mu_A(x) >_{L^I} \alpha, \gamma_A(x) <_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) > \alpha_1, \mu_A^+(x) > \alpha_2, \gamma_A^-(x) < \beta_1, \gamma_A^+(x) < \beta_2\}, \end{aligned}$$

which is called the $(\alpha+, \beta+)$*-level cut set of* A*.*

Theorem 3.10 *The cut sets of IVIF sets satisfy the following properties:* $\forall A \in IVIF(U)$, $\alpha = [\alpha_1, \alpha_2], \beta = [\beta_1, \beta_2] \in Int[0,1]$ *with* $\alpha_2 + \beta_2 \leq 1$,

(1) $A_\alpha^\beta = A_\alpha \cap A^\beta$,

(2) $A \subseteq B \Rightarrow A_\alpha^\beta \subseteq B_\alpha^\beta$,

(3) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha,\ (A \cap B)^\beta = A^\beta \cap B^\beta$,

(4) $\alpha \geq_{L^I} \beta, \xi \leq_{L^I} \eta \Rightarrow A_\alpha \subseteq A_\beta, A^\xi \subseteq A^\eta, A_\alpha^\xi \subseteq A_\beta^\eta$.

Proof. By Definition 3.9, (1), (2) and (4) are straightforward.

(3) Since

$$\begin{aligned} A\cap B &= \{< x, \mu_{A\cap B}(x), \gamma_{A\cap B}(x) > | x\in U\} \\ &= \{< x, [\mu_A^-(x)\wedge\mu_B^-(x), \mu_A^+(x)\wedge\mu_B^+(x)], \\ &\qquad [\gamma_A^-(x)\vee\gamma_B^-(x), \gamma_A^+(x)\vee\gamma_B^+(x)] > | x\in U\}, \end{aligned}$$

we have

$$\begin{aligned} (A\cap B)_\alpha &= \{x\in U|\mu_A^-(x)\wedge\mu_B^-(x)\geq\alpha_1, \mu_A^+(x)\wedge\mu_B^+(x)\geq\alpha_2\} \\ &= \{x\in U|\mu_A^-(x)\geq\alpha_1, \mu_B^-(x)\geq\alpha_1, \mu_A^+(x)\geq\alpha_2, \mu_B^+(x)\geq\alpha_2\} \\ &= \{x\in U|\mu_A(x)\geq_{L^I}\alpha, \mu_B(x)\geq_{L^I}\alpha\} = A_\alpha\cap B_\alpha, \end{aligned}$$

and

$$\begin{aligned} (A\cap B)^\beta &= \{x\in U|\gamma_A^-(x)\vee\gamma_B^-(x)\leq\beta_1, \gamma_A^+(x)\vee\gamma_B^+(x)\leq\beta_2\} \\ &= \{x\in U|\gamma_A^-(x)\leq\beta_1, \gamma_B^-(x)\leq\beta_1, \gamma_A^+(x)\leq\beta_2, \gamma_B^+(x)\leq\beta_2\} \\ &= \{x\in U|\gamma_A(x)\leq_{L^I}\beta, \gamma_B(x)\leq_{L^I}\beta\} = A^\beta\cap B^\beta. \end{aligned}$$

Meanwhile, according to (1), we can obtain

$$\begin{aligned} (A\cap B)_\alpha^\beta &= (A\cap B)_\alpha\cap(A\cap B)^\beta \\ &= (A_\alpha\cap A^\beta)\cap(B_\alpha\cap B^\beta) = A_\alpha^\beta\cap B_\alpha^\beta. \end{aligned}$$

□

Assume that R is an IVIF soft relation from U to E, denote

$$R_\alpha = \{(u,x)\in U\times E|\mu_R(u,x)\geq_{L^I}\alpha\} = \{(u,x)\in U\times E|\mu_R^-(u,x)\geq\alpha_1, \mu_R^+(u,x)\geq\alpha_2\},$$

$$R_\alpha(u) = \{x\in E|\mu_R(u,x)\geq_{L^I}\alpha\} = \{x\in E|\mu_R^-(u,x)\geq\alpha_1, \mu_R^+(u,x)\geq\alpha_2\}, \alpha_1,\alpha_2\in[0,1];$$

$$R_{\alpha+} = \{(u,x)\in U\times E|\mu_R(u,x)>_{L^I}\alpha\} = \{(u,x)\in U\times E|\mu_R^-(u,x)>\alpha_1, \mu_R^+(u,x)>\alpha_2\},$$

$$R_{\alpha+}(u) = \{x\in E|\mu_R(u,x)>_{L^I}\alpha\} = \{x\in E|\mu_R^-(u,x)>\alpha_1, \mu_R^+(u,x)>\alpha_2\}, \alpha_1,\alpha_2\in[0,1);$$

$$R^\beta = \{(u,x)\in U\times E|\gamma_R(u,x)\leq_{L^I}\beta\} = \{(u,x)\in U\times E|\gamma_R^-(u,x)\leq\beta_1, \gamma_R^+(u,x)\leq\beta_2\},$$

$$R^\beta(u) = \{x\in E|\gamma_R(u,x)\leq_{L^I}\beta\} = \{x\in E|\gamma_R^-(u,x)\leq\beta_1, \gamma_R^+(u,x)\leq\beta_2\}, \beta_1,\beta_2\in[0,1];$$

$$R^{\beta+} = \{(u,x)\in U\times E|\gamma_R(u,x)<_{L^I}\beta\} = \{(u,x)\in U\times E|\gamma_R^-(u,x)<\beta_1, \gamma_R^+(u,x)<\beta_2\},$$

$$R^{\beta+}(u) = \{x\in E|\gamma_R(u,x)<_{L^I}\beta\} = \{x\in E|\gamma_R^-(u,x)<\beta_1, \gamma_R^+(u,x)<\beta_2\}, \beta_1,\beta_2\in(0,1].$$

Then R_α, $R_{\alpha+}$, R^β and $R^{\beta+}$ are crisp soft relations on $U\times E$.

The following Theorems 3.12 and 3.13 show that the generalized IVIF soft rough approximation operators can be represented by crisp soft rough approximation operators proposed by Zhang et al. [42].

Theorem 3.11 *Let (U,E,R) be an IVIF soft approximation space, and $A \in IVIF(E)$. Then the generalized upper IVIF soft rough approximation operator can be represented as follows: $\forall u \in U$, $\overline{a}=[a,a] \in L^I$,*

(1)

$$\mu_{\overline{R}(A)}(u) = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_\alpha)(u)}] = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_{\alpha+})(u)}]$$
$$= \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_\alpha)(u)}] = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_{\alpha+})(u)}],$$

(2)

$$\gamma_{\overline{R}(A)}(u) = \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^\alpha}(A^\alpha)(u)}] = \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^\alpha}(A^{\alpha+})(u)}]$$
$$= \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^{\alpha+}}(A^\alpha)(u)}] = \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^{\alpha+}}(A^{\alpha+})(u)}]$$

and moreover, for any $\alpha \in L^I$,
(3) $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R_{\alpha+}}(A_{\alpha+}) \subseteq \overline{R_{\alpha+}}(A_\alpha) \subseteq \overline{R_\alpha}(A_\alpha) \subseteq [\overline{R}(A)]_\alpha$,
(4) $[\overline{R}(A)]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+}) \subseteq \overline{R^{\alpha+}}(A^\alpha) \subseteq \overline{R^\alpha}(A^\alpha) \subseteq [\overline{R}(A)]^\alpha$.

Proof. (1) For any $u \in U$, we have

$$\bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_\alpha)(u)}] = sup\{\alpha \in L^I | u \in \overline{R_\alpha}(A_\alpha)\} = sup\{\alpha \in L^I | R_\alpha(u) \cap A_\alpha \neq \emptyset\}$$
$$= sup\{\alpha \in L^I | \exists x \in E[x \in R_\alpha(u), x \in A_\alpha]\}$$
$$= sup\{\alpha \in L^I | \exists x \in E[\mu_R(u,x) \geq_{L^I} \alpha, \mu_A(x) \geq_{L^I} \alpha]\}$$
$$= sup\{[\alpha_1,\alpha_2] \in L^I | \exists x \in E[\mu_R^-(u,x) \geq \alpha_1, \mu_R^+(u,x) \geq \alpha_2, \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2]\}$$
$$= sup\{[\alpha_1,\alpha_2] \in L^I | \exists x \in E[\mu_R^-(u,x) \wedge \mu_A^-(x) \geq \alpha_1, \mu_R^+(u,x) \wedge \mu_A^+(x) \geq \alpha_2]\}$$
$$= [\bigvee_{x \in E} (\mu_R^-(u,x) \wedge \mu_A^-(x)), \bigvee_{x \in E} (\mu_R^+(u,x) \wedge \mu_A^+(x))] = \mu_{\overline{R}(A)}(u).$$

Likewise, we can conclude that

$$\mu_{\overline{R}(A)}(u) = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_{\alpha+})(u)}] = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_\alpha)(u)}]$$
$$= \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_{\alpha+})(u)}].$$

(2) In terms of Definition 2.5 and notations above, we have

$$\begin{aligned}
&\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^{\alpha}}(A^{\alpha})(u)}]=inf\{\alpha\in L^I]|u\in\overline{R^{\alpha}}(A^{\alpha})\}=inf\{\alpha\in L^I|R^{\alpha}(u)\cap A^{\alpha}\neq\emptyset\}\\
&=inf\{\alpha\in L^I|\exists x\in E[x\in R^{\alpha}(u),x\in A^{\alpha}]\}\\
&=inf\{\alpha\in L^I|\exists x\in E[\gamma_R(u,x)\leq_{L^I}\alpha,\gamma_A(x)\leq_{L^I}\alpha]\}\\
&=inf\{[\alpha_1,\alpha_2]\in L^I|\exists x\in E[\gamma_R^-(u,x)\leq\alpha_1,\gamma_R^+(u,x)\leq\alpha_2,\gamma_A^-(x)\leq\alpha_1,\gamma_A^+(x)\leq\alpha_2]\}\\
&=inf\{[\alpha_1,\alpha_2]\in L^I|\exists x\in E[\gamma_R^-(u,x)\vee\gamma_A^-(x)\leq\alpha_1,\gamma_R^+(u,x)\vee\gamma_A^+(x)\leq\alpha_2]\}\\
&=[\bigwedge_{x\in E}(\gamma_R^-(u,x)\vee\gamma_A^-(x)),\bigwedge_{x\in E}(\gamma_R^+(u,x)\vee\gamma_A^+(x))]=\gamma_{\overline{R}(A)}(u).
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
\gamma_{\overline{R}(A)}(u)&=\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^{\alpha}}(A^{\alpha+})(u)}]=\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^{\alpha+}}(A^{\alpha})(u)}]\\
&=\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^{\alpha+}}(A^{\alpha+})(u)}].
\end{aligned}$$

(3) It is easily verified that $\overline{R_{\alpha+}}(A_{\alpha+})\subseteq\overline{R_{\alpha+}}(A_{\alpha})\subseteq\overline{R_{\alpha}}(A_{\alpha})$. We only need to prove that $[\overline{R}(A)]_{\alpha+}\subseteq\overline{R_{\alpha+}}(A_{\alpha+})$ and $\overline{R_{\alpha}}(A_{\alpha})\subseteq[\overline{R}(A)]_{\alpha}$.

In fact, $\forall u\in[\overline{R}(A)]_{\alpha+}$, we have $\mu_{\overline{R}(A)}(u)>_{L^I}\alpha$. According to Definition 3.4, $\bigvee_{x\in E}[\mu_R^-(u,x)\wedge\mu_A^-(x)]>\alpha_1$ and $\bigvee_{x\in E}[\mu_R^+(u,x)\wedge\mu_A^+(x)]>\alpha_2$. Then $\exists x_0\in E$, such that $\mu_R^-(u,x_0)\wedge\mu_A^-(x_0)>\alpha_1$ and $\mu_R^+(u,x_0)\wedge\mu_A^+(x_0)>\alpha_2$, that is, $\mu_R^-(u,x_0)>\alpha_1,\mu_A^-(x_0)>\alpha_1,\mu_R^+(u,x_0)>\alpha_2$, and $\mu_A^+(x_0)>\alpha_2$. Thus $\mu_R(u,x_0)>_{L^I}\alpha$ and $\mu_A(x_0)>_{L^I}\alpha$, which imply that $x_0\in R_{\alpha+}(u)$ and $x_0\in A_{\alpha+}$. Namely, $R_{\alpha+}(u)\cap A_{\alpha+}\neq\emptyset$. By Definition 2.5, we have $u\in\overline{R_{\alpha+}}(A_{\alpha+})$. Hence $[\overline{R}(A)]_{\alpha+}\subseteq\overline{R_{\alpha+}}(A_{\alpha+})$.

On the other hand, for any $u\in\overline{R_{\alpha}}(A_{\alpha})$, we have $\overline{R_{\alpha}}(A_{\alpha})(u)=1$. Since $\mu_{\overline{R}(A)}(u)=\bigvee_{\beta\in L^I}[\beta\wedge\overline{\overline{R_{\beta}}(A_{\beta})(u)}]\geq_{L^I}\alpha\wedge\overline{\overline{R_{\alpha}}(A_{\alpha})(u)}=\alpha$, we obtain $u\in[\overline{R}(A)]_{\alpha}$. Hence, $\overline{R_{\alpha}}(A_{\alpha})\subseteq[\overline{R}(A)]_{\alpha}$.

(4) Similar to the proof of (3), it can be easily verified. □

Theorem 3.12 *Let (U,E,R) be an IVIF soft approximation space, and $A\in IVIF(E)$. Then the generalized lower IVIF soft rough approximation operator can be represented as follows: $\forall u\in U$*

(1)

$$\begin{aligned}
\mu_{\underline{R}(A)}(u)&=\bigwedge_{\alpha\in L^I}[\alpha\vee(\bar{1}-\overline{\underline{R^{\alpha}}(A_{\alpha+})(u)}]=\bigwedge_{\alpha\in L^I}[\alpha\vee(\bar{1}-\overline{\underline{R^{\alpha}}(A_{\alpha})(u)}]\\
&=\bigwedge_{\alpha\in L^I}[\alpha\vee(\bar{1}-\overline{\underline{R^{\alpha+}}(A_{\alpha+})(u)}]=\bigwedge_{\alpha\in L^I}[\alpha\vee(\bar{1}-\overline{\underline{R^{\alpha+}}(A_{\alpha})(u)}],
\end{aligned}$$

(2)

$$\begin{aligned}\gamma_{\underline{R}(A)}(u) &= \bigvee_{\alpha\in L^I}[\alpha\wedge(\bar{1}-\overline{\underline{R_\alpha}(A^{\alpha+})(u)}] = \bigvee_{\alpha\in L^I}[\alpha\wedge(\bar{1}-\overline{\underline{R_\alpha}(A^{\alpha})(u)}]\\ &= \bigvee_{\alpha\in L^I}[\alpha\wedge(\bar{1}-\overline{\underline{R_{\alpha+}}(A^{\alpha+})(u)}] = \bigvee_{\alpha\in L^I}[\alpha\wedge(\bar{1}-\overline{\underline{R_{\alpha+}}(A^{\alpha})(u)}]\end{aligned}$$

and moreover, for any $\alpha \in L^I$,
(3) $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R^\alpha}(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_\alpha) \subseteq [\underline{R}(A)]_\alpha$,
(4) $[\underline{R}(A)]^{\alpha+} \subseteq \underline{R_\alpha}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^\alpha) \subseteq [\underline{R}(A)]^\alpha$.

Proof. The proof is similar to Theorem 3.12. □

4 Application of IVIF soft rough sets in decision making

In [46], Zhang et al. gave a decision method based on IVIF soft set theory. However, we note that the decision method need to choose the thresholds in advance by decision makers. Thus the decision results will be depend on the threshold values at some degree. Since the thresholds have different kind of subjective preference information, different experts can obtain the different decision results for the same decision problem. So, in order to avoid the effect of the subjective information for the decision results, we only use the data information provided by the decision making problem and don't need any additional available information provided by decision makers. Thus the decision results are more objectively.

Next, we shall develop a new approach to decision making problem based on the generalized IVIF soft rough sets proposed in this paper.

Let (U, E, R) be an IVIF soft approximation space, where U is the universe of the discourse, E is the parameter set, and R is an IVIF soft relation on $U \times E$. Then we can give this decision-making approach based on generalized IVIF soft rough sets with five steps.

First, according to their own needs, the decision makers can construct an IVIF soft relation R from U to E, or IVIF soft set (F, E) over U.

Second, for a ceratin decision evaluation problem, we suppose that one wants to find out the decision alternative in universe with the evaluation value as larger as possible on every evaluate index. On the basis of the assumption, we construct an optimum normal decision object A which is an IVIF set on the evaluation universe E as follows:

$$A = \{< e_i, \max_{1\le j\le |U|}\mu_R(u_j, e_i), \min_{1\le j\le |U|}\gamma_R(u_j, e_i) >\},$$

where $|U|$ denotes the cardinality of the universe set U.

Third, by Equations (1) and (2), we can compute the generalized IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ of the optimum normal decision object A. Thus, we obtain two most close values $\overline{R}(A)$ and $\underline{R}(A)$ to the decision alternative u_i of the universe set U.

Fourth, Atanassov and Gargov [3,4] introduced the notion of IVIF sets, and gave two operations on two IVIF sets, shown as follows, for all $F, G \in IVIF(U)$,

• Union operation:

$$\begin{aligned} F \cup G = \{< u, & [\mu_F^-(u) \vee \mu_G^-(u), \mu_F^+(u) \vee \mu_G^+(u)], \\ & [\gamma_F^-(u) \wedge \gamma_G^-(u), \gamma_F^+(u) \wedge \gamma_G^+(u)] > | u \in U\}, \end{aligned}$$

• Intersection operation:

$$\begin{aligned} F \cap G = \{< u, & [\mu_F^-(u) \wedge \mu_G^-(u), \mu_F^+(u) \wedge \mu_G^+(u)], \\ & [\gamma_F^-(u) \vee \gamma_G^-(u), \gamma_F^+(u) \vee \gamma_G^+(u)] > | u \in U\}. \end{aligned}$$

In general, the union operation and intersection operation on IVIF sets may result in loss of information in practical decision making problem which affects the accuracy of decision making. Therefore, inspired by the concept of $\oplus$-union operation of intuitionistic fuzzy subset, we also introduce the concept of $\oplus$-union operation of IVIF subset.

Definition 4.1 *Let $F, G \in IVIF(U)$. The $\oplus$-union operation about IVIF sets F and G can be defined as follows:*

$$\begin{aligned} F \oplus G = \{< u, & [\mu_F^-(u) + \mu_G^-(u) - \mu_F^-(u) \cdot \mu_G^-(u), \mu_F^+(u) + \mu_G^+(u) - \mu_F^+(u) \cdot \mu_G^+(u)], \\ & [\gamma_F^-(u) \cdot \gamma_G^-(u), \gamma_F^+(u) \cdot \gamma_G^+(u)] > | u \in U\}. \end{aligned}$$

By using the $\oplus$-union operation rather than the union and intersection operations, we can obtain the choice set as follows

$$\begin{aligned} H = \overline{R}(A) \oplus \underline{R}(A) = \{< u, & [\mu_{\overline{R}(A)}^-(u) + \mu_{\underline{R}(A)}^-(u) - \mu_{\overline{R}(A)}^-(u) \cdot \mu_{\underline{R}(A)}^-(u), \\ & \mu_{\overline{R}(A)}^+(u) + \mu_{\underline{R}(A)}^+(u) - \mu_{\overline{R}(A)}^+(u) \cdot \mu_{\underline{R}(A)}^+(u)], \\ & [\gamma_{\overline{R}(A)}^-(u) \cdot \gamma_{\underline{R}(A)}^-(u), \gamma_{\overline{R}(A)}^+(u) \cdot \gamma_{\underline{R}(A)}^+(u)] > | u \in U\}. \end{aligned}$$

Denote $H = \{< u, \mu_H(u), \gamma_H(u) >\}$.

Finally, define an IVIF value $\lambda = (\mu, \gamma) \in L$, where $\mu = \sup_{1 \le j \le |U|} [\mu_H^-(u_j), \mu_H^+(u_j)]$, $\gamma = \inf_{1 \le j \le |U|} [\gamma_H^-(u_j), \gamma_H^+(u_j)]$. Obviously, IVIF value $\lambda = (\mu, \gamma)$ is the maximum choice value in the choice set H. Hence we take the object u_j in universe U with the maximum choice value as the optimum decision for the given decision making problem. That is to say, if $\mu_H(u_j) \ge_{L^I} \mu$ and $\gamma_H(u_j) \le_{L^I} \gamma$, the optimum decision is u_j.

In general, if there exist two or more objects with the same maximum choice value , then we can take one of them as the optimum decision for the given decision making problem.

To illustrate the new method given above, let us consider the example as follows.

Example 4.2 *Reconsider Example 3.7. Now all the available information on houses under consideration can be formulated as an IVIF soft relation describing attractiveness of house that Mr.X is going to buy. By using the second step of the algorithm for generalized IVIF soft rough sets in decision making presented in this section, we can obtain the optimum normal decision object A as follows*

$$\begin{aligned} A = \{ &< e_1, [0.8, 0.9], [0.0, 0.1] >, < e_2, [0.6, 0.7], [0.1, 0.2] >, \\ &< e_3, [0.6, 0.8], [0.1, 0.2] >, < e_4, [0.4, 0.8], [0.1, 0.2] >\}. \end{aligned}$$

According to Equations (1) and (2), we can conclude that

$$\begin{aligned} \overline{R}(A) = \{ &< u_1, [0.7, 0.8], [0.1, 0.2] >, < u_2, [0.6, 0.7], [0.1, 0.2] >, < u_3, [0.5, 0.8], [0.1, 0.2] >, \\ &< u_4, [0.5, 0.7], [0.1, 0.2] >, < u_5, [0.8, 0.9], [0.0, 0.1] >\} \end{aligned}$$

and

$$\begin{aligned} \underline{R}(A) = \{ &< u_1, [0.4, 0.8], [0.1, 0.2] >, < u_2, [0.4, 0.8], [0.1, 0.2] >, < u_3, [0.4, 0.8], [0.1, 0.2] >, \\ &< u_4, [0.5, 0.7], [0.1, 0.2] >, < u_5, [0.4, 0.8], [0.1, 0.2] >\}. \end{aligned}$$

Now by Definition 4.1, we have

$$\begin{aligned} H = \overline{R}(A) \oplus \underline{R}(A) = \{ &< u_1, [0.82, 0.96], [0.01, 0.04] >, < u_2, [0.76, 0.94], [0.01, 0.04] >, \\ &< u_3, [0.70, 0.96], [0.01, 0.04] >, < u_4, [0.75, 0.91], [0.01, 0.04] >, \\ &< u_5, [0.88, 0.98], [0.00, 0.02] >\}. \end{aligned}$$

Obviously, IVIF value $\lambda = ([0.88, 0.98], [0.00, 0.02])$ is the maximum choice value in the choice set H. Thus the optimal decision is u_5. Hence, Mr X will buy the house u_5.

5 Conclusion

Recently, there has been a growing interest in soft set theory. Some extensions of soft sets have been obtained by combining soft set theory with other mathematical models, including fuzzy soft sets, interval-valued fuzzy soft sets, intuitionistic fuzzy soft sets and interval-valued intuitionistic fuzzy soft sets. Among them, the interval-valued intuitionistic fuzzy soft set is the most generalized one. This paper is devoted to the discussion of the combinations of interval-valued intuitionistic fuzzy soft set and rough set. By using an

interval-valued intuitionistic fuzzy soft relation, we present a new soft rough set model, called generalized IVIF soft rough sets. Furthermore, the generalized upper and lower IVIF soft rough approximation operators are represented by crisp soft rough approximation operators. Finally, a practical application is provided to illustrate the validity of the generalized IVIF soft rough set.

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments and suggestions. This work is supported by the National Natural Science Foundation of China (No. 11461082) and by the Research Project Funds for Higher Education Institutions of Gansu Province (No. 2015B-006)

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GENERALIZATIONS OF HEINZ MEAN OPERATOR INEQUALITIES INVOLVING POSITIVE LINEAR MAP

CHANGSEN YANG AND YINGYA TAO

ABSTRACT. In this paper, we study the Heinz mean inequalities of two positive operators involving positive linear map. We obtain a generalized conclusion based on operator Diaz-Metcalf type inequality. The conclusion is presented as follows: Let Φ be a unital positive linear map, if $0 < {m_1}^2 \leq A \leq {M_1}^2$ and $0 < {m_2}^2 \leq B \leq {M_2}^2$ for some positive real numbers $m_1 \leq M_1$, $m_2 \leq M_2$, then for $\alpha \in [0,1]$ and $p \geq 2$, the following inequality holds :

$$\begin{aligned}&(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^p\\ \leq &2^{-(p+4)}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).\end{aligned}$$

1. INTRODUCTION AND PRELIMINARIES

We represent the set of all bounded operators on $\mathcal{H}$ by $B(\mathcal{H})$. If an operator A satisfies $\langle Ax, x\rangle \geq 0$ for any $x \in \mathcal{H}$, then A is called a positive operator. For two self-adjoint operators A and B, $A \geq B$ means $A - B \geq 0$. The notation $A > 0$ means A is an invertible positive operator.

A linear map Φ: $B(\mathcal{H}) \longrightarrow B(\mathcal{H})$ is called positive (strictly positive), if $\Phi(A) \geq 0$ ($\Phi(A) > 0$) whenever $A \geq 0$ ($A > 0$), and Φ is said to be unital if $\Phi(I) = I$. Take $A, B > 0$ and $\alpha \in [0,1]$, the weighted arithmetic operator mean $A\nabla_\alpha B$, geometric mean $A\sharp_\alpha B$ and harmonic mean $A!_\alpha B$ are defined as follows :

$$A\nabla_\alpha B = (1-\alpha)A + \alpha B,\ A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}},\ A!_\alpha B = [(1-\alpha)A^{-1} + \alpha B^{-1}]^{-1}$$

when $\alpha = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$ and $A!B$ for brevity, respectively. The Heniz mean is defined by $H_\alpha(A,B) = \frac{A\sharp_\alpha B + A\sharp_{1-\alpha}B}{2}$, where $A, B > 0$ and $\alpha \in [0,1]$. Recently, M. S. Moslehian, R. Nakamoto and Y. Seo [1, Theorem 2.1, part (ii)] showed that

Theorem 1.1 Let Φ be positive linear map, if $0 < {m_1}^2 \leq A \leq {M_1}^2$ and $0 < {m_2}^2 \leq B \leq {M_2}^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, we can get operator Diaz-Metcalf type inequality:

$$\frac{M_2m_2}{M_1m_1}\Phi(A) + \Phi(B) \leq (\frac{M_2}{m_1} + \frac{m_2}{M_1})\Phi(A\sharp B).$$

Thus $A\sharp B \leq H_\alpha(A,B)$ implies the following.

2010 *Mathematics Subject Classification.* Primary 47A63; Secondary 47B20.
Key words and phrases. Heinz mean; Heinz operator inequality; positive linear map.

Remark 1.2 Let Φ be positive linear map, if $0 < {m_1}^2 \leq A \leq {M_1}^2$ and $0 < {m_2}^2 \leq B \leq {M_2}^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, then for $\alpha \in [0, 1]$, the following inequality holds:

$$\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B) \leq (\frac{M_2}{m_1} + \frac{m_2}{M_1})\Phi(H_\alpha(A, B)).$$

In 2015, Mohammad Sal Moslehian and Xiaohui Fu obtained a second powering of the operator Diaz-Metcalf type inequality:

Theorem 1.3 [9] Let Φ be positive linear map, if $0 < {m_1}^2 \leq A \leq {M_1}^2$ and $0 < {m_2}^2 \leq B \leq {M_2}^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, then the following inequality holds:

$$(\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B))^2 \leq \left(\frac{(M_1 m_1({M_2}^2 + {m_2}^2) + M_2 m_2({M_1}^2 + {m_1}^2))^2}{8\sqrt{M_1 m_1 M_2 m_2} M_1^2 m_1^2 M_2 m_2}\right)^2 (\Phi(A \sharp B))^2.$$

In the paper we shall give further generalizations of Remark 1.2 in the following section, along with presenting p-th powering of some inequality for Heniz mean based on Remark 1.2 and the following consideration: It is easy to see that the Heniz operator mean interpolates the arithmetic-geometric operator mean inequality: $A!B \leq A\sharp B \leq H_\alpha(A, B) \leq A\nabla B$, and the geometric mean has so-called maximal characterization [2], which says that $\begin{bmatrix} A & A\sharp B \\ A\sharp B & B \end{bmatrix}$ is positive, and moreover, if the operator matrix $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$ is positive with X being self-adjoint, then $A\sharp B \geq X$.

2. Results and Proofs

In order to prove the first main theorem of the paper, first we give the following lemmas.

lemma 2.1. [3] Let Φ be a unital strictly positive linear map and $A > 0$, then $\Phi(A)^{-1} \leq \Phi(A^{-1})$.

lemma 2.2. [5] Let $A, B \geq 0$, then the following norm inequality holds : $\|AB\| \leq \frac{1}{4}\|A + B\|^2$.

lemma 2.3. [4] Let $A, B \geq 0$, then for $1 \leq r < +\infty$, $\|A^r + B^r\| \leq \|(A + B)^r\|$.

lemma 2.4. [7] (L-H inequality) If $0 \leq \alpha \leq 1$, $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$.

Theorem 2.5. Let Φ be a unital positive linear map, if $0 < {m_1}^2 \leq A \leq {M_1}^2$ and $0 < {m_2}^2 \leq B \leq {M_2}^2$ for some positive real numbers $m_1 \leq M_1$, $m_2 \leq M_2$, then for $\alpha \in [0, 1]$ and $p \geq 2$, the following inequality holds :

$$\begin{aligned}&(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^p\\ \leq&2^{-(p+4)}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).\end{aligned}\tag{2.1}$$

Proof. Obviously (2.1) is equivalent to

$$\begin{aligned}&\|(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\ \leq&2^{-(\frac{p}{2}+2)}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^{p}.\end{aligned}$$

Note that

$$({M_1}^2-A)({m_1}^2-A)A^{-1}\leq 0,$$

implies

$${M_1}^2{m_1}^2A^{-1}-{M_1}^2-{m_1}^2+A\leq 0,$$

therefore

$${M_1}^2{m_1}^2\Phi(A^{-1})+\Phi(A)\leq {M_1}^2+{m_1}^2,$$

which equals to

$$M_1m_1M_2m_2\Phi(A^{-1})+\tfrac{M_2m_2}{M_1m_1}\Phi(A)\leq\tfrac{M_2m_2}{M_1m_1}({M_1}^2+{m_1}^2).\tag{2.2}$$

Similarly, we have

$${M_2}^2{m_2}^2\Phi(B^{-1})+\Phi(B)\leq {M_2}^2+{m_2}^2.\tag{2.3}$$

Since

$$H_\alpha^{-1}(A,B)\leq(A!B)^{-1}=\frac{A^{-1}+B^{-1}}{2},$$

therefore

$$\begin{aligned}&H_\alpha(\frac{A}{M_2m_2M_1m_1},\frac{B}{{M_2}^2{m_2}^2})\\ =&\frac{(\frac{1}{M_2m_2M_1m_1})^{1-\alpha}(\frac{1}{{M_2}^2{m_2}^2})^\alpha(A\sharp_\alpha B)+(\frac{1}{M_2m_2M_1m_1})^\alpha(\frac{1}{{M_2}^2{m_2}^2})^{1-\alpha}(A\sharp_{1-\alpha}B)}{2}\\ \leq&\max\{(\frac{1}{M_2m_2M_1m_1})^{1-\alpha}(\frac{1}{M_2m_2})^{2\alpha},(\frac{1}{M_2m_2M_1m_1})^{\alpha}(\frac{1}{M_2m_2})^{2-2\alpha}\}H_\alpha(A,B)\\ =&\frac{H_\alpha(A,B)}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^{1+\alpha},(M_1m_1)^\alpha(M_2m_2)^{2-\alpha}\}}.\end{aligned}\tag{2.4}$$

If we put

$$\beta=\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^{1+\alpha},(M_1m_1)^\alpha(M_2m_2)^{2-\alpha}\},$$

then

$$\begin{aligned}
&\beta\Phi^{-1}(H_\alpha(A,B))\\
\le&\Phi^{-1}(H_\alpha(\frac{A}{M_2m_2M_1m_1},\frac{B}{{M_2}^2{m_2}^2}))\\
\le&\Phi({H^{-1}}_\alpha(\frac{A}{M_2m_2M_1m_1},\frac{B}{{M_2}^2{m_2}^2}))\\
\le&\frac{1}{2}\Phi(M_2m_2M_1m_1A^{-1}+{M_2}^2{m_2}^2B^{-1})\\
=&\frac{1}{2}(M_2m_2M_1m_1\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1})).
\end{aligned}$$

By (2.2) and (2.3), we have

$$\begin{aligned}
&\|(\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)))^{\frac{p}{2}}\beta^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\
\le&\frac{1}{4}\|(\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)))^{\frac{p}{2}}+\beta^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|^2\\
\le&\frac{1}{4}\|(\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))+\beta\Phi^{-1}(H_\alpha(A,B)))^{\frac{p}{2}}\|^2\\
=&\frac{1}{4}\|\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))+\beta\Phi^{-1}(H_\alpha(A,B))\|^p\\
\le&\frac{1}{4}\|\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)+M_2m_2M_1m_1\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1}))\|^p\\
\le&2^{-(p+2)}({M_2}^2+{m_2}^2+\frac{M_2m_2}{M_1m_1}({M_1}^2+{m_1}^2))^p.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\
\le&2^{-(\frac{p}{2}+2)}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^p.
\end{aligned}$$

Corollary 2.6. In Theorem 2.5, if $1\le p\le 2$, we get

$$\begin{aligned}
&(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^p\\
\le&2^{-3p}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).
\end{aligned}$$

Theorem 2.7. Let Φ be a unital positive linear map, if $0<{m_1}^2\le A\le {M_1}^2$ and $0<{m_2}^2\le B\le {M_2}^2$ for some positive real numbers $m_1\le M_1$, $m_2\le M_2$, then for $\alpha\in[0,1]$ and $p\ge 2$, the following inequality holds :

$$(\Phi(A)\nabla_\alpha\Phi(B))^p\le 2^{-(p+4)}\left[\frac{{M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)). \quad (2.5)$$

Proof. Obviously (2.5) is equivalent to

$$\begin{aligned}&\|(\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\ \leq&2^{-(\frac{p}{2}+2)}\left[\frac{{M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^p.\end{aligned}$$

Note that

$$({M_1}^2-(1-\alpha)A)({m_1}^2-A)A^{-1}\leq 0,$$

implies

$${M_1}^2{m_1}^2A^{-1}-{M_1}^2-(1-\alpha){m_1}^2+(1-\alpha)A\leq 0.$$

Therefore

$${M_1}^2{m_1}^2\Phi(A^{-1})+(1-\alpha)\Phi(A)\leq{M_1}^2+(1-\alpha){m_1}^2. \tag{2.6}$$

Similarly, we have

$${M_2}^2{m_2}^2\Phi(B^{-1})+\alpha\Phi(B)\leq{M_2}^2+\alpha{m_2}^2. \tag{2.7}$$

Since

$$H_\alpha^{-1}(A,B)\leq(A!B)^{-1}=\frac{A^{-1}+B^{-1}}{2},$$

and by analogy to (2.4)

$$\begin{aligned}&H_\alpha(\frac{A}{{M_1}^2{m_1}^2},\frac{B}{{M_2}^2{m_2}^2})\\ =&\frac{H_\alpha(A,B)}{\min\{(M_1m_1)^{2-2\alpha}(M_2m_2)^{2\alpha},(M_1m_1)^{2\alpha}(M_2m_2)^{2-2\alpha}\}}.\end{aligned}$$

By puting

$$h=\min\{(M_1m_1)^{2-2\alpha}(M_2m_2)^{2\alpha},(M_1m_1)^{2\alpha}(M_2m_2)^{2-2\alpha}\},$$

we have

$$\begin{aligned}&h\Phi^{-1}(H_\alpha(A,B))\\ \leq&h\Phi^{-1}(H_\alpha(\frac{A}{M_1^2m_1^2},\frac{B}{{M_2}^2{m_2}^2}))\\ \leq&h\Phi(H^{-1}{}_\alpha(\frac{A}{M_1^2m_1^2},\frac{B}{{M_2}^2{m_2}^2}))\\ \leq&\frac{1}{2}\Phi(M_1^2m_1^2A^{-1}+{M_2}^2{m_2}^2B^{-1})\\ =&\frac{1}{2}(M_1^2m_1^2\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1})).\end{aligned}$$

By (2.6) and (2.7), we have

$$
\begin{aligned}
&\|(\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}h^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\| \\
\le&\frac{1}{4}\|(\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}+h^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|^2 \\
\le&\frac{1}{4}\|(\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B)+h\Phi^{-1}(H_\alpha(A,B)))^{\frac{p}{2}}\|^2 \\
=&\frac{1}{4}\|\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B)+h\Phi^{-1}(H_\alpha(A,B))\|^p \\
\le&\frac{1}{4}\|\frac{1}{2}((1-\alpha)\Phi(A)+\alpha\Phi(B)+M_1^2m_1^2\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1}))\|^p \\
\le&2^{-(p+2)}({M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2)^p.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\|(\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\| \\
\le&2^{-(\frac{p}{2}+2)}\left[\frac{{M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^p.
\end{aligned}
$$

Theorem 2.8. Let Φ be a unital positive linear map, if $0<{m_1}^2\le A\le {M_1}^2$ and $0<{m_2}^2\le B\le {M_2}^2$ for some positive real numbers $m_1\le M_1$, $m_2\le M_2$, δ is a arbitrary mean less than or equal to arithmetic mean, then for $\alpha\in[0,1]$ and $p\ge 2$, the following inequality holds :

$$(\Phi(A)\delta\Phi(B))^p\le 2^{-(2p+4)}\left[\frac{{M_1}^2+{M_2}^2+{m_1}^2+{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).$$

Proof. By the similar method of proofing Theorem 2.7.

Corollary 2.9. In Theorem 2.8, we easily get

$${H_\alpha}^p(\Phi(A),\Phi(B))\le 2^{-(2p+4)}\left[\frac{{M_1}^2+{M_2}^2+{m_1}^2+{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).$$

Theorem 2.10. [8] Let $0<m\le A,\ B\le M$, with the scalars $m,M>0$ and σ,τ two arbitrary means beween harmonic and arithmetic means, then for every positive unital linear map Φ, $2\le p<\infty$,

$$\Phi^p(A\sigma B)\le(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm})^p(\Phi(A)\tau\Phi(B))^p.$$

By $A!B\le H_\alpha(A,B)\le A\nabla B$, we obtain the following inequality.

Remark 2.11. Let $0<m\le A,B\le M$, then for every positive unital linear map Φ and $0<\alpha<1$, $K(h)=\frac{(h+1)^2}{4h}$, $h=\frac{M}{m}$, $p\ge 2$, the following inequality holds :

$$\Phi^p(H_\alpha(A,B))\le 2^{2p-4}K^p(h){H_\alpha}^p(\Phi(A),\Phi(B)). \tag{2.8}$$

lemma 2.12. [6] For any bounded operator X,

$$|X| \leq tI \Longleftrightarrow \|X\| \leq t \Longleftrightarrow \begin{bmatrix} tI & X \\ X^* & tI \end{bmatrix} \geq 0 \ (t \geq 0).$$

Theorem 2.13. Let $0 < m \leq A, B \leq M$, then for every positive unital linear map Φ and $0 < \alpha < 1$, $K(h) = \frac{(h+1)^2}{4h}, h = \frac{M}{m}$, $p \geq 2$, the following inequality holds :

$$\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) + {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) \leq 2^{p-1}K^{\frac{p}{2}}(h). \quad (2.9)$$

Proof. By (2.8) we get

$$\|\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\| \leq 2^{p-2}K^{\frac{p}{2}}(h). \quad (2.10)$$

By (2.10) and Lemma 2.12, we obtain

$$\begin{bmatrix} 2^{p-2}K^{\frac{p}{2}}(h)I & \Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) \\ {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) & 2^{p-2}K^{\frac{p}{2}}(h)I \end{bmatrix} \geq 0,$$

and

$$\begin{bmatrix} 2^{p-2}K^{\frac{p}{2}}(h)I & {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) \\ \Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) & 2^{p-2}K^{\frac{p}{2}}(h)I \end{bmatrix} \geq 0.$$

Summing up these two operator matrices above, put

$$2^{p-2}K^{\frac{p}{2}}(h) = t,$$

$$\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) + {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) = X.$$

We have

$$\begin{bmatrix} 2tI & X \\ X^* & 2tI \end{bmatrix} \geq 0.$$

Since $\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) + {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B))$ is self-adioint, (2.9) follows from the maximal characterization of geometric mean.

Corollary 2.14. Let Φ be a unital positive linear map, if $0 < {m_1}^2 \leq A \leq {M_1}^2$ and $0 < {m_2}^2 \leq B \leq {M_2}^2$ for some positive real numbers $m_1 \leq M_1$, $m_2 \leq M_2$, then for $\alpha \in [0,1]$ and $p \geq 2$, the following inequality holds :

$$\begin{aligned} &{H_\alpha}^{\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{-\frac{p}{2}}(H_\alpha(A,B)) + \Phi^{-\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{\frac{p}{2}}(\Phi(A),\Phi(B)) \\ \leq& 2^{-(p+1)}\left[\frac{{M_1}^2 + {M_2}^2 + {m_1}^2 + {m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^{\alpha}, (M_1m_1)^{\alpha}(M_2m_2)^{1-\alpha}\}}\right]^{2p} \Phi^p(H_\alpha(A,B)). \end{aligned}$$

Proof. By Corollary 2.9 and the similar method of proofing Theorem 2.13, we can easily get.

Acknowledgement. The research is supported by National Natural Science Foundation of China with grant (no. 11271112,11201127) and Technology and the Innovation Team in Henan Province (NO.14IRTSTHN023).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.6, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Existence and uniqueness results of nonlocal fractional sum-difference boundary value problems for fractional difference equations involving sequential fractional difference operators.

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Abstract

In this article, we study some new existence results for a nonlinear fractional difference equation with fractional sum-difference boundary conditions. Our problem containing sequential fractional difference operators that have different orders. The existence and uniqueness results are based on Banach contraction mapping principle and Schaefer's fixed point theorem. Finally, we present some examples to show the importance of these results.

Keywords: Fractional difference equations; boundary value problems; existence.

(2010) Mathematics Subject Classifications: 39A05; 39A12.

1 Introduction

In this paper we consider a fractional sum-difference boundary value problem of a fractional difference equation of the form

$$\begin{cases} \Delta^{\alpha}u(t) = f(t+\alpha-1, u(t+\alpha-1), \Delta^{\mu}\Delta^{\nu}u(t+\alpha-\mu-\nu+1)), \\ u(\alpha-2) = \Delta^{\theta}u(\alpha-\theta-2) = p\,y(u), \\ u(T+\alpha) = q\,\Delta^{-\beta}u(\eta+\beta), \end{cases} \tag{1.1}$$

where $t \in \mathbb{N}_{0,T} := \{0,1,...,T\}$, $p,q > 0$, $2 < \alpha \leq 3$, $0 < \beta,\theta,\mu,\nu \leq 1$, $1 < \mu+\nu \leq 2$, $\eta \in \mathbb{N}_{\alpha-1,T+\alpha-1}$, $f \in (\mathbb{N}_{\alpha-3,T+\alpha} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a given function, and $y : C(\mathbb{N}_{\alpha-3,T+\alpha}, \mathbb{R}) \to \mathbb{R}$ is a given functional.

Mathematicians have used this fractional calculus in recent years to model and solve various related problems. In particular, fractional calculus is a powerful tool for the processes which appears in nature, e.g. biology, ecology and other areas.

Fractional difference equations have been interested many researchers since can use for describing many problems in the real-world phenomena such as physics, chemistry,

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mechanics, control systems, flow in porous media, and electrical networks can be found in [1] and [2] and the references therein. An excellent papers dealing with discrete fractional boundary value problems, which has helped to establish some of the basic theory of this field, one may see the papers [3]-[17], and references cited therein.

For example, Kang *et* al. [3] obtained sufficient conditions for the existence of solutions for the nonlocal boundary value problem as follows,

$$\begin{cases} -\Delta^{\mu} y(t) = \lambda h(t+\mu-1)\, f(y(t+\mu-1)), \quad t \in \mathbb{N}_{0,b} := \{0,1,...,b\}, \\ y(\mu-2) = \Psi(y), \quad y(\mu+b) = \Phi(y), \end{cases} \tag{1.2}$$

where $1 < \mu \leq 2$, $f \in C([0,\infty),[0,\infty))$ and $h \in C(\mathbb{N}_{\mu-1,\mu+b-1},[0,\infty))$ are given functions, and $\Psi, \Phi : \mathbb{R}^{b+3} \to \mathbb{R}$ are given functionals.

Presently, Chasreechai *et* al. [15] examined a Caputo fractional sum-difference equation with nonlocal fractional sum boundary value conditions of the form

$$\begin{cases} \Delta^{\alpha}_{C} u(t) = f(t+\alpha-1, u(t+\alpha-1), (\Psi^{\beta}u)(t+\alpha-2)), \quad t \in \mathbb{N}_{0,T}, \\ u(\alpha-2) = y(u), \\ u(T+\alpha) = \Delta^{-\gamma} g(T+\alpha+\gamma-3)\, u(T+\alpha+\gamma-3), \end{cases} \tag{1.3}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $2 < \gamma \leq 3$. For $U \subseteq \mathbb{R}$, $g \in C(\mathbb{N}_{\alpha-2,T+\alpha}, \mathbb{R}^{+} \cap U)$, $f \in C(\mathbb{N}_{\alpha-2,T+\alpha} \times U \times U, U)$ are given functions, $y : C(\mathbb{N}_{\alpha-2,T+\alpha}, U) \to U$ is a given functional, and for $\varphi : \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha} \to [0,\infty)$,

$$(\Psi^{\beta}u)(t) := [\Delta^{-\beta}\varphi\, u](t+\beta) = \frac{1}{\Gamma(\beta)} \sum_{s=\alpha-\beta-2}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}} \varphi(t,s+\beta)\, u(s+\beta).$$

The plan of this paper is as follows. In Section 2, we recall some definitions and basic lemmas. Also, we derive a representation of the solution to (1.1) by converting the problem to an equivalent fractional sum equation. In Section 3, the existence and uniqueness results of the boundary value problem (1.1) are established by Banach contraction mapping principle and Schaefer's fixed point theorem. An illustrative example is presented in Section 4.

2 Preliminaries

In this section, we introduce notations, definitions, and lemmas that are used in the main results.

Definition 2.1. *We define the generalized falling function by* $t^{\underline{\alpha}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, *for any* t *and* α *for which the right-hand side is defined. If* $t+1-\alpha$ *is a pole of the Gamma function and* $t+1$ *is not a pole, then* $t^{\underline{\alpha}} = 0$.

Lemma 2.1. *[10] If $t \le r$, then $t^{\underline{\alpha}} \le r^{\underline{\alpha}}$ for any $\alpha > 0$.*

Definition 2.2. *For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order fractional sum of f is defined by*

$$\Delta^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} f(s),$$

for $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s+1$.

Definition 2.3. *For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order Riemann-Liouville fractional difference of f is defined by*

$$\Delta^{\alpha} f(t) := \Delta^N \Delta^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t-\sigma(s))^{\underline{-\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \alpha \le N$.

Lemma 2.2. *[10] Let $0 \le N-1 < \alpha \le N$. Then*

$$\Delta^{-\alpha}\Delta^{\alpha} y(t) = y(t) + C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \ldots + C_N t^{\underline{\alpha-N}},$$

for some $C_i \in \mathbb{R}$, with $1 \le i \le N$.

To define the solution of the boundary value problem (1.1) we need the following lemma that deals with a linear variant of the boundary value problem (1.1) and gives a representation of the solution.

Lemma 2.3. *Let $\Lambda \neq 0$, $p, q > 0$, $2 < \alpha \le 3$, $0 < \beta, \theta \le 1$, $\eta \in \mathbb{N}_{\alpha-1,\alpha+T-1}$, functions $h : \mathbb{N}_{\alpha-1,\alpha+T-1} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$ be given. Then the problem*

$$\begin{cases} \Delta^{\alpha} u(t) = h(t+\alpha-1), & t \in \mathbb{N}_{0,T}, \\ u(\alpha-2) = \Delta^{\theta} u(\alpha-\theta-2) = p\, y(u), \\ u(T+\alpha) = q\, \Delta^{-\beta} u(\eta+\beta), \end{cases} \tag{2.1}$$

has the unique solution

$$\begin{aligned} u(t) \;=\; & -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)} \Bigg[\frac{q}{\Gamma(\beta)} \sum_{s=\alpha}^{\eta} \sum_{\xi=0}^{s-\alpha} (\eta+\beta-\sigma(s))^{\underline{\beta-1}} (s-\sigma(\xi))^{\underline{\alpha-1}} h(\xi+\alpha-1) \\ & - \sum_{s=0}^{T} (T+\alpha-\sigma(s))^{\underline{\alpha-1}} h(s+\alpha-1) \Bigg] + \frac{p\, y(u)}{\Gamma(\alpha-1)} \left[t^{\underline{\alpha-2}} - \frac{t^{\underline{\alpha-1}}\Theta}{\Lambda} \right] \\ & + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} h(s+\alpha-1), \end{aligned} \tag{2.2}$$

where

$$\Lambda = \frac{q}{\Gamma(\beta)} \sum_{s=0}^{\eta-\alpha+1} (\eta+\beta-s-\alpha)^{\underline{\beta-1}} (s+\alpha-1)^{\underline{\alpha-1}} - \frac{\Gamma(T+\alpha+1)}{\Gamma(T+2)}, \tag{2.3}$$

$$\Theta = \frac{q}{\Gamma(\beta)} \sum_{s=0}^{\eta-\alpha+2} (\eta+\beta-\alpha-s+1)^{\underline{\beta-1}} (s+\alpha-2)^{\underline{\alpha-2}} - \frac{\Gamma(T+\alpha+1)}{\Gamma(T+3)}. \tag{2.4}$$

Proof. From Lemma 2.2, we find that a general solution for (2.1) can be written as

$$u(t) = C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + C_3 t^{\underline{\alpha-3}} + \Delta^{-\alpha} h(t+\alpha-1), \tag{2.5}$$

for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$.

Using the fractional difference of order $0 < \theta \le 1$ for (2.5), we obtain

$$\begin{aligned} \Delta^{\theta} u(t) \quad = \quad & \frac{C_1}{\Gamma(-\theta)} \sum_{s=\alpha-1}^{t+\theta} (t-\sigma(s))^{\underline{-\theta-1}} s^{\underline{\alpha-1}} + \frac{C_2}{\Gamma(-\theta)} \sum_{s=\alpha-2}^{t+\theta} (t-\sigma(s))^{\underline{-\theta-1}} s^{\underline{\alpha-2}} \\ & + \frac{C_3}{\Gamma(-\theta)} \sum_{s=\alpha-3}^{t+\theta} (t-\sigma(s))^{\underline{-\theta-1}} s^{\underline{\alpha-3}} \\ & + \frac{1}{\Gamma(-\theta)\Gamma(\alpha)} \sum_{s=\alpha}^{t+\theta} \sum_{\xi=0}^{s-\alpha} (t-\sigma(s))^{\underline{-\theta}} (s-\sigma(\xi))^{\underline{\alpha-1}} h(\xi+\alpha-1), \end{aligned}$$

for $t \in \mathbb{N}_{\alpha-\theta-2,T+\alpha-\theta+1}$.

Applying the condition of (2.1): $u(\alpha-2) = \Delta^{\theta} u(\alpha-\theta-2)$, we have $C_3 = 0$.
So,

$$u(t) = C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \Delta^{-\alpha} h(t+\alpha-1). \tag{2.6}$$

From (2.6) and the second condition of (2.1): $u(\alpha-2) = p\,y(u)$, we have

$$C_2 = \frac{p\,y(u)}{\Gamma(\alpha-1)}. \tag{2.7}$$

Hence,

$$u(t) = C_1 t^{\underline{\alpha-1}} + \frac{py(u)}{\Gamma(\alpha-1)} t^{\underline{\alpha-2}} + \Delta^{-\alpha} h(t+\alpha-1), \tag{2.8}$$

for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$.

Using the fractional sum of order $0 < \beta \le 1$ for (2.8), we obtain

$$\Delta^{-\beta} u(t) \quad = \quad \frac{C_1}{\Gamma(\beta)} \sum_{s=\alpha-1}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}} s^{\underline{\alpha-1}} + \frac{py(u)}{\Gamma(\beta)\Gamma(\alpha-1)} \sum_{s=\alpha-2}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}} s^{\underline{\alpha-2}}$$

$$+\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{t-\beta}\sum_{\xi=0}^{s-\alpha}(t-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha-1), \tag{2.9}$$

for $t\in\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta}$.

The third condition of (2.1) implies

$$\begin{aligned}
& q\Delta^{-\beta}u(\eta+\beta)\\
= & \frac{qC_1}{\Gamma(\beta)}\sum_{s=\alpha-1}^{\eta}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}s^{\underline{\alpha-1}}+\frac{p\,q\,y(u)}{\Gamma(\beta)\Gamma(\alpha-1)}\sum_{s=\alpha-2}^{\eta}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}s^{\underline{\alpha-2}}\\
& +\frac{q}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha-1)\\
= & C_1(T+\alpha)^{\underline{\alpha-1}}+\frac{p\,y(u)}{\Gamma(\alpha-1)}(T+\alpha)^{\underline{\alpha-2}}+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}h(s+\alpha-1).
\end{aligned}$$

Solving the above equation for the constant C_1, we get

$$\begin{aligned}
C_1 = & \frac{-p\,q\,y(u)}{\Lambda\Gamma(\beta)\Gamma(\alpha-1)}\sum_{s=\alpha-2}^{\eta}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}s^{\underline{\alpha-2}}+\frac{p\,y(u)}{\Lambda\Gamma(\alpha-1)}(T+\alpha)^{\underline{\alpha-2}}\\
& +\frac{1}{\Lambda\Gamma(\alpha)}\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}h(s+\alpha-1) \qquad (2.10)\\
& -\frac{q}{\Lambda\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha-1),
\end{aligned}$$

where Λ is defined as (2.3). Substituting C_1 into (2.8), we obtain (2.2). $\square$

3 Main Results

In this section, we wish to establish the existence results for problem (1.1). To accomplish this, let $\mathcal{C}=C(\mathbb{N}_{\alpha-3,\alpha+T},\mathbb{R})$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}}=\max\{\|u\|,\|\Delta^{\mu}\Delta^{\nu}u\|\},$$

where $\|u\|=\max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}|u(t)|$ and $\|\Delta^{\mu}\Delta^{\nu}u\|=\max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}|\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)|$. Also define an operator $F:\mathcal{C}\to\mathcal{C}$ by

$$Fu(t)$$

$$
\begin{aligned}
= & -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\left[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}f(\xi+\alpha-1,u(\xi+\alpha-1),\right.\\
& \Delta^{\mu}\Delta^{\nu}u(\xi+\alpha-\mu-\nu+1))-\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}f(s+\alpha-1,u(s+\alpha-1),\\
& \left.\Delta^{\mu}\Delta^{\nu}u(s+\alpha-\mu-\nu+1))\right]+\frac{p\,y(u)}{\Gamma(\alpha-1)}\left[t^{\underline{\alpha-2}}-\frac{t^{\underline{\alpha-1}}\Theta}{\Lambda}\right] \qquad (3.1)\\
& +\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}f(s+\alpha-1,u(s+\alpha-1),\Delta^{\mu}\Delta^{\nu}u(s+\alpha-\mu-\nu+1)),
\end{aligned}
$$

for $t\in\mathbb{N}_{\alpha-3,\alpha+T}$, where $\Lambda\neq 0,\ \Theta$ are defined as (2.3),(2.4), respectively. The problem (1.1) has solutions if and only if the operator F has fixed points.

Our first result is based on Banach contraction mapping principle.

Theorem 3.1. *Assume that*

(H_1) *There exist constants* $\gamma_1,\gamma_2>0$ *such that, for each* $t\in\mathbb{N}_{\alpha-3,\alpha+T}$ *and for all* $u,v\in\mathcal{C}$,

$$
\begin{aligned}
& |f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2))-f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))|\\
\leq\ & \gamma_1\,|u(t)-v(t)|+\gamma_2\,|\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)-\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2)|.
\end{aligned}
$$

(H_2) *There exists a constant* $\omega>0$ *such that, for all* $u,v\in\mathcal{C}$,

$$|y(u)-y(v)|\leq\omega|u-v|.$$

(H_3) $\gamma\Omega+\omega\Phi<\frac{(T+2)(T+1)}{(T+\alpha+2)(T+\alpha+1)}$,

where

$$
\begin{aligned}
\gamma &= \max\{\gamma_1+\gamma_2\} && (3.2)\\
\Omega &= \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(\alpha+\beta+1)\Gamma(T)}-\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|+\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} && (3.3)\\
\Phi &= \frac{p(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)}\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right]. && (3.4)
\end{aligned}
$$

Then the boundary value problem (1.1) *has at least one solution on* $\mathbb{N}_{\alpha-3,\alpha+T}$.

Proof. Denote that,

$$\mathcal{H}|u-v|(t)=\big|f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2))-f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))\big|.$$

For all $u, v \in \mathcal{C}$, by computing directly, we have

$$\begin{aligned}
& \|Fu - Fv\| \\
= \;& \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}} \Bigg| -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\left[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\mathcal{H}|u-v|(\xi)\right. \\
& \left.-\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\right] + \left[t^{\underline{\alpha-2}} - \frac{t^{\underline{\alpha-1}}\Theta}{\Lambda}\right]\frac{p\,|y(u)-y(v)|}{\Gamma(\alpha-1)} \\
& +\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg| \\
\leq \;& (\gamma\|u-v\|_{\mathcal{C}})\left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right] \\
& +\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right]\frac{(\omega\|u-v\|_{\mathcal{C}})\;p\,(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)} \\
= \;& (\gamma\|u-v\|_{\mathcal{C}})\;\Omega + (\omega\|u-v\|_{\mathcal{C}})\;\Phi,
\end{aligned}$$

and

$$\begin{aligned}
& \left\|\Delta^{\mu}\Delta^{\nu}Fu - \Delta^{\mu}\Delta^{\nu}Fv\right\| \\
= \;& \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}} \left|\,(\Delta^{\mu}\Delta^{\nu}Fu)\,(t-\mu-\nu+2) - (\Delta^{\mu}\Delta^{\nu}Fv)\,(t-\mu-\nu+2)\right| \\
< \;& \left(\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{T+\alpha-\nu+2}\sum_{\xi=\alpha-1}^{s+\nu}(T+\alpha-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(\xi))^{\underline{-\nu-1}}\right)\times \\
& (T+\alpha+2)^{\underline{\alpha-1}}\left[\frac{(\gamma\|u-v\|_{\mathcal{C}})}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right. \\
& \left.+\frac{p\;(\omega\|u-v\|_{\mathcal{C}})}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\right] + p\;(\omega\|u-v\|_{\mathcal{C}})\;\frac{(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)}\times \\
& \left(\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{T+\alpha-\nu+2}\sum_{\xi=\alpha-2}^{s+\nu}(T+\alpha-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(\xi))^{\underline{-\nu-1}}\right) \\
& +\left(\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{T+\alpha-\nu+2}\sum_{r=\alpha}^{s+\nu}(T+\alpha-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(r))^{\underline{-\nu-1}}\right)\times \\
& \frac{(\gamma\|u-v\|_{\mathcal{C}})}{\Gamma(\alpha)}\sum_{\xi=0}^{T+2}(T+\alpha+2-\sigma(\xi))^{\underline{\alpha-1}} \\
< \;& \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\left[\gamma\Omega+\omega\Phi\right]\|u-v\|_{\mathcal{C}}.
\end{aligned}$$

Thus, $\|Fu-Fv\|_{\mathcal{C}} \leq \dfrac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\left[\gamma\Omega+\omega\Phi\right]\|u-v\|_{\mathcal{C}}$.

By (H_3), we get that F is a contraction mapping, and then Theorem 3.1 implies that boundary value problem (1.1) has unique solution on $\mathbb{N}_{\alpha-3,\alpha+T}$. This completes the proof. □

The second result is based on Schaefer's fixed point theorem.

Theorem 3.2. *(Arzelá-Ascoli Theorem) [18] A set of function in $C[a,b]$ with the sup norm, is relatively compact if and only it is uniformly bounded and equicontinuous on $[a,b]$.*

Theorem 3.3. *[18] If a set is closed and relatively compact then it is compact.*

Theorem 3.4. ***[Schaefer's fixed point theorem]*** *[19] Let X be a Banach space and $T : X \to X$ be a continuous and compact mapping. If the set*

$$\{x \in X \ : \ x = \lambda T(x), \ \textit{for some } \lambda \in (0,1)\}$$

is bounded, then T has a fixed point.

We shall use Schaefer's fixed point theorem to prove that the operator F defined as (3.1), has a fixed point.

Theorem 3.5. *Suppose that there exist constants $L_1, L_2 > 0$ such that, for each $t \in \mathbb{N}_{\alpha-3,\alpha+T}$ and $u \in \mathcal{C}$,*

$$\begin{aligned} |f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2))| &\leq L_1 \max\{\|u\|, \|\Delta^{\mu}\Delta^{\nu}u\|\}, \\ |y(u)| &\leq L_2. \end{aligned}$$

Then the problem (1.1) has at least one solution on $\mathbb{N}_{\alpha-3,\alpha+T}$.

Proof. We divide the proof into four steps.

Step I. Verify F map bounded sets into bounded sets in $C(\mathbb{N}_{\alpha-3,\alpha+T})$.
Let $u \in B_L = \{u \in C(\mathbb{N}_{\alpha-3,\alpha+T}) : \|u\|_{\mathcal{C}} \leq L\}$, and choosing a constant

$$L \geq \frac{L_2\Phi(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1) - L_1\Omega\,(T+\alpha+2)(T+\alpha+1)}.$$

Denote that

$$\begin{aligned} \mathcal{H}|u-v|(t) &:= |f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)) - f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))| \\ &\leq \|f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)) - f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))\| \\ &=: \mathcal{H}\|u-v\|(t). \end{aligned}$$

For each $u \in B_L$, we obtain

$$\begin{aligned}
&\|Fu\| \\
=\;& \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}} \Bigg| -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\Bigg[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\mathcal{H}|u-v|(\xi) \\
&-\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg] + \left[t^{\underline{\alpha-2}} - \frac{t^{\underline{\alpha-1}}\Theta}{\Lambda}\right]\frac{p\,|y(u)|}{\Gamma(\alpha-1)} \\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg| \\
\leq\;& L_1\|u\|_{\mathcal{C}}\left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right] \\
&+\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right]\frac{pL_2\,(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)} \\
\leq\;& L_1 L\,\Omega + L_2\,\Phi.
\end{aligned}$$

and

$$\begin{aligned}
&\|\Delta^{\mu}\Delta^{\nu}Fu\| = \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}\left|\,(\Delta^{\mu}\Delta^{\nu}Fu)\,(t-\mu-\nu+2)\right| \\
=\;& \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}\Bigg\{\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{t-\nu+2}\sum_{\xi=\alpha-1}^{s+\nu}(t-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(\xi))^{\underline{-\nu-1}}\times \\
&\xi^{\underline{\alpha-1}}\Bigg[\frac{(L_1\|u\|_{\mathcal{C}})}{|\Lambda|\Gamma(\alpha)}\Bigg|\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}} \\
&-\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\Bigg| + \frac{pL_2}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\Bigg] \\
&+\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{t-\nu+2}\sum_{\xi=\alpha-2}^{s+\nu}(t-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(\xi))^{\underline{-\nu-1}}\xi^{\underline{\alpha-2}}\times \\
&\left[\frac{pL_2}{\Gamma(\alpha-1)}\,(T-\alpha+2)^{\underline{\alpha-2}}\right] + \frac{(L_1\|u\|_{\mathcal{C}})}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{t-\nu+2}\sum_{r=\alpha}^{s+\nu}(t-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}\times \\
&(s-\sigma(r))^{\underline{-\nu-1}}\left[\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{r-\alpha}(r-\sigma(\xi))^{\underline{\alpha-1}}\right]\Bigg\} \\
<\;& \left\{\frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\right\}L_1L\left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\times\right. \\
&\left.\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right] + \left\{\frac{(T+\alpha+2)(T+\alpha+1)}{(T+3)(T+2)}\right\}\times
\end{aligned}$$

$$
\begin{aligned}
& \frac{pL_2}{\Gamma(\alpha-1)}(T+\alpha+2)^{\underline{\alpha-2}}\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right] \\
< \; & \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\left[L_1 L\Omega + L_2\Phi\right].
\end{aligned}
$$

Hence, $\|Fu\|_{\mathcal{C}} \leq L$ where Ω and Φ are defined on 3.3 and 3.4, respectively. Thus F is uniformly bounded.

Step II. Show that F is continuous on B_L.
Let $\epsilon > 0$ there exists $\delta = \max\{\delta_1, \delta_2\} > 0$ such that, for each $t \in \mathbb{N}_{\alpha-3,\alpha+T}$ and for all $u, v \in B_L$ with

$$
\max\{|u(t)-v(t)|, |\Delta^\mu\Delta^\nu u(t-\mu-\nu+2) - \Delta^\mu\Delta^\nu v(t-\mu-\nu+2)|\} < \delta_1,
$$

we have

$$
\mathcal{H}|u-v| < \frac{\epsilon\,(T+2)(T+1)}{2\Omega\,(T+\alpha+2)(T+\alpha+1)},
$$

and for all $u, v \in B_L$ with $|u-v| < \delta_2$, we have

$$
|y(u)-y(v)| < \frac{\epsilon\,(T+2)(T+1)}{2\Phi\,(T+\alpha+2)(T+\alpha+1)}.
$$

Then, we have

$$
\begin{aligned}
& \|Fu(t)-Fv(t)\| \\
= \; & \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}\Bigg| -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\left[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\mathcal{H}|u-v|(\xi)\right. \\
& \left. -\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\right] + \left[t^{\underline{\alpha-2}} - \frac{t^{\underline{\alpha-1}}\Theta}{\Lambda}\right]\frac{p\,|y(u)-y(v)|}{\Gamma(\alpha-1)} \\
& +\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg| \\
\leq \; & \mathcal{H}\|u-v\|\left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\cdot\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right] \\
& +\|y(u)-y(v)\|\,\frac{p\,(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)}\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right] \\
= \; & \Omega\,\mathcal{H}\|u-v\| + \Phi\,\|y(u)-y(v)\|.
\end{aligned}
$$

Similarly to the proof above and Theorem 3.1, we obtain

$$
\big\|\left(\Delta^\mu\Delta^\nu Fu\right)(t-\mu-\nu+2) - \left(\Delta^\mu\Delta^\nu Fv\right)(t-\mu-\nu+2)\big\|
$$

$$< \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\Big[\Omega\,\mathcal{H}\|u-v\| + \Phi\,\|y(u)-y(v)\|\Big] \;<\; \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

Hence, $\|Fu - Fv\|_{\mathcal{C}} \le \epsilon$. This means that F is continuous on B_L.

Step III. Examine $F(B_L)$ is equicontinuous with B_L. For any $\epsilon > 0$, there exists $\delta = \max\{\delta_1, \delta_2, \delta_3\} > 0$ such that, for $t_1, t_2 \in \mathbb{N}_{\alpha-3,\alpha+T}$

$$\left|t_2^{\underline{\alpha}} - t_1^{\underline{\alpha}}\right| \;<\; \frac{\epsilon\,\Gamma(\alpha+1)\,(T+2)(T+1)}{3L_1(T+\alpha+2)(T+\alpha+1)} \qquad \text{whenever} \quad |t_2-t_1| < \delta_1,$$

$$\left|t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right| \;<\; \frac{\epsilon\,|\Lambda|\,(T+2)(T+1)}{3(T+\alpha+2)(T+\alpha+1)\,\left[L_1\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha-1)}\right| + \frac{pL_2\,|\Theta|}{\Gamma(\alpha-1)}\right]} \quad \text{whenever} \quad |t_2-t_1| < \delta_2,$$

$$\left|t_2^{\underline{\alpha-2}} - t_1^{\underline{\alpha-2}}\right| \;<\; \frac{\epsilon\Gamma(\alpha-1)(T+2)(T+1)}{3pL_2(T+\alpha+2)(T+\alpha+1)} \qquad \text{whenever} \quad |t_2-t_1| < \delta_3.$$

Then, we have

$$\begin{aligned}
&|Fu(t_2) - Fu(t_1)| \\
= &\;\Bigg| -\frac{t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\Bigg[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}} \times \\
&f(\xi+\alpha-1, u(\xi+\alpha-1), \Delta^{\mu}\Delta^{n}u(\xi+\alpha-\mu-\nu+1)) - \sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}} \times \\
&f(s+\alpha-1, u(s+\alpha-1), \Delta^{\mu}\Delta^{n}u(s+\alpha-\mu-\nu+1))\Bigg] \\
&+\frac{p\,y(u)}{\Gamma(\alpha-1)}\left[\left(t_2^{\underline{\alpha-2}} - t_1^{\underline{\alpha-2}}\right) - \left(t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right)\frac{\Theta}{\Lambda}\right] \\
&+\frac{1}{\Gamma(\alpha)}\Bigg[\sum_{s=0}^{t_2-\alpha}(t_2-\sigma(s))^{\underline{\alpha-1}} f(s+\alpha-1, u(s+\alpha-1), \Delta^{\mu}\Delta^{n}u(s+\alpha-\mu-\nu+1)) \\
&-\sum_{s=0}^{t_1-\alpha}(t_1-\sigma(s))^{\underline{\alpha-1}} f(s+\alpha-1, u(s+\alpha-1), \Delta^{\mu}\Delta^{n}u(s+\alpha-\mu-\nu+1))\Bigg]\Bigg| \\
\le &\;\left|t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right|\left[\frac{L_1}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right| + \frac{pL_2}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\right] \\
&+\frac{L_1}{\Gamma(\alpha)}\left[\sum_{s=0}^{t_2-\alpha}(t_2-\sigma(s))^{\underline{\alpha-1}} + \sum_{s=0}^{t_1-\alpha}(t_1-\sigma(s))^{\underline{\alpha-1}}\right] + \left|t_2^{\underline{\alpha-2}} - t_1^{\underline{\alpha-2}}\right|\frac{pL_2}{\Gamma(\alpha-1)} \\
= &\;\left|t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right|\left[\frac{L_1}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right| + \frac{pL_2}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\right]
\end{aligned}$$

$$+\frac{L_1}{\Gamma(\alpha+1)}\big|t_2^{\underline{\alpha}}-t_1^{\underline{\alpha}}\big|+\frac{pL_2}{\Gamma(\alpha-1)}\big|t_2^{\underline{\alpha-2}}-t_1^{\underline{\alpha-2}}\big|.$$

So $\big\|Fu-Fv\big\|<\epsilon$.

Similarly to the proof above and Theorem 3.1, we obtain

$$\begin{aligned}
&\big\|\Delta^{\mu}\Delta^{\nu}Fu-\Delta^{\mu}\Delta^{\nu}Fu\big\| \\
<\;& \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\Bigg\{\big|t_2^{\underline{\alpha-1}}-t_1^{\underline{\alpha-1}}\big|\left[\frac{L_1}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)}-\frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right.\\
&\left.+\frac{pL_2}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\right]+\frac{L_1}{\Gamma(\alpha+1)}\big|t_2^{\underline{\alpha}}-t_1^{\underline{\alpha}}\big|+\frac{pL_2}{\Gamma(\alpha-1)}\big|t_2^{\underline{\alpha-2}}-t_1^{\underline{\alpha-2}}\big|\Bigg\}\\
<\;& \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.
\end{aligned}$$

Thus, $\|Fu(t_2)-Fu(t_1)\|_{\mathcal{C}}\le\epsilon$. This means that $F(B_L)$ is an equicontinuous set.

As a consequence of Steps I to III together with the Arzelá-Ascoli theorem, its imply that $F:C(\mathbb{N}_{\alpha-3,\alpha+T})\to C(\mathbb{N}_{\alpha-3,\alpha+T})$ is completely continuous.

Step IV. *A priori bounds.* We show that the set

$$E=\big\{u\in C(\mathbb{N}_{\alpha-3,\alpha+T}):u=\lambda Fu\quad\text{for some}\quad 0<\lambda<1\big\}\;\;\text{is bounded.}$$

Let $u\in E$. Then $u(t)=\lambda(Fu)(t)$ for some $0<\lambda<1$. Thus, for each $t\in\mathbb{N}_{\alpha-3,\alpha+T}$, we have

$$|\lambda Fu(t)|<|Fu(t)|<L_1L\,\Omega+L_2\,\Phi:=\Im.$$

So, we have $\big\|\lambda Fu\big\|<\Im$. Similarly to the proof above and Theorem 3.1, we obtain

$$\big\|\lambda\,\Delta^{\mu}\Delta^{\nu}Fu\big\|\;<\;\frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\Im\;=:\;\widetilde{\Im}.$$

Hence, $\|\lambda Fu\|_{\mathcal{C}}\le\widetilde{\Im}$. This shows that E is bounded.

By of the Schaefer's fixed point theorem, we conclude that F has a fixed point which is a solution of the problem (1.1). □

4 Some examples

In this section, in order to illustrate our results, we consider some examples.

Example 4.1. *Consider the following boundary value problem*

$$\Delta^{\frac{5}{2}}u(t) = \frac{e^{-\sin^2(t+\frac{3}{2})}}{\left(t+\frac{15}{2}\right)^2} \cdot \frac{\left|u\left(t+\frac{3}{2}\right)\right| + |\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{25}{12}\right)|}{\left|u\left(t+\frac{3}{2}\right)\right|+1}, \quad t \in \mathbb{N}_{0,4}, \tag{4.1}$$

$$u\left(\frac{1}{2}\right) = \Delta^{\frac{1}{4}}u\left(\frac{1}{4}\right) = \frac{1}{2}\sum_{i=0}^{7} C_i u(t_i), \quad t_i = i - \frac{1}{2}, \tag{4.2}$$

$$u\left(\frac{13}{2}\right) = \frac{1}{3}\Delta^{-\frac{1}{3}}u\left(\frac{29}{6}\right). \tag{4.3}$$

where C_i *are given positive constants with* $\sum_{i=0}^{7} C_i < \frac{1}{10\,e^{20}}$.

Here $p = \frac{1}{2},\ q = \frac{1}{3},\ \theta = \frac{1}{4},\ \alpha = \frac{5}{2},\ \beta = \frac{1}{3},\ \mu = \frac{2}{3},\ \nu = \frac{3}{4},\ \eta = \frac{9}{2},\ T = 4,$ $f\left(t, u(t), \Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right)\right) = \frac{e^{-\sin^2 t}}{(t+6)^2} \cdot \frac{|u(t)|+|\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{7}{12}\right)|}{|u(t)|+1}$ and $y(u) = \sum_{i=0}^{7} C_i u(t_i),\ t_i = i - \frac{1}{2}$.

Let $t \in \mathbb{N}_{-\frac{1}{2},\frac{13}{2}}$ and $u, v \in \mathbb{R}$, then

$$|\Lambda| = 7.781 \neq 0, \quad \Theta = 1.278, \quad \Omega \approx 106.039, \quad \Phi \approx 3.119.$$

Since $\quad |f\left(t, u(t), \Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right)\right) - f\left(t, v(t), \Delta^{\mu}\Delta^{\nu}v\left(t-\mu-\nu+2\right)\right)|$ $\leq \frac{4}{1849}\left|u(t)-v(t)\right| + \frac{4}{1849}\left|\Delta^{\mu}\Delta^{\nu}u\left(t+\frac{7}{12}\right) - \Delta^{\mu}\Delta^{\nu}v\left(t+\frac{7}{12}\right)\right|$ is satisfied with $\gamma = \max\{\gamma_1+\gamma_2\} = \frac{8}{1849}$.

Also, we get $|y(u)-y(v)| = \left|\sum_{i=0}^{7} C_i u(t_i) - \sum_{i=0}^{7} C_i v(t_i)\right| \leq \sum_{i=0}^{7} C_i |u(t_i)-v(t_i)|$, so (H_2) holds with $\omega = \sum_{i=0}^{7} C_i < \frac{1}{10\,e^{20}}$.

We can show that

$$\frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\left[\gamma\Omega + \omega\Phi\right] \approx 0.975 < 1.$$

Hence, by Theorem 3.1, the problem (4.1)-(4.3) has unique solution. □

Example 4.2. *Consider the following boundary value problem*

$$\Delta^{\frac{5}{2}}u(t) = \frac{t+\frac{3}{2}}{10\pi}\left[2\sin\left|u\left(t+\frac{3}{2}\right)\right| + \cos\left|\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{25}{12}\right)\right|\right], \ t \in \mathbb{N}_{0,4}, \tag{4.4}$$

$$u\left(\frac{1}{2}\right) = \Delta^{\frac{1}{4}}u\left(\frac{1}{4}\right) = \frac{1}{4}\sum_{i=0}^{7} C_i \frac{|u(t_i)|}{1+|u(t_i)|}, \quad t_i = i - \frac{1}{2}, \tag{4.5}$$

$$u\left(\frac{13}{2}\right) = \frac{1}{5}\Delta^{-\frac{1}{3}}u\left(\frac{29}{6}\right), \tag{4.6}$$

where C_i *are given positive constants with* $\sum_{i=0}^{7} C_i < \frac{1}{e}$.

Here $p = \frac{1}{4},\ q = \frac{1}{5},\ \alpha = \frac{5}{2},\ \beta = \frac{1}{3},\ \theta = \frac{1}{4},\ \mu = \frac{2}{3},\ \nu = \frac{3}{4},\ \eta = \frac{9}{2},\ T = 4,$ $f\left(t, u(t), \Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right)\right) = \frac{t}{10\pi}\left[2\sin|u(t)| + \cos\left|\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{7}{12}\right)\right|\right]$ and $y(u) = \sum_{i=0}^{7} C_i \frac{|u(t_i)|}{1+|u(t_i)|},\quad t_i = i - \frac{1}{2}$. Clearly for $t \in \mathbb{N}_{-\frac{1}{2},\frac{13}{2}}$, we have

$$\begin{aligned} \left|f(t,u(t),\Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right))\right| &\leq \frac{13}{20\,\pi}\max\{2,1\} \approx 0.414 \qquad \left(L_1 = \frac{13}{20\,\pi}\right) \\ |y(u)| &\leq \sum_{i=0}^{7} C_i \frac{|u(t_i)|}{1+|u(t_i)|} < \frac{1}{e} = L_2. \end{aligned}$$

Hence, by Theorem 3.5, the problem (4.4)-(4.6) has at least one solution. □

Acknowledgments. This research was funded by King Mongkut's University of Technology North Bangkok. Contract no.KMUTNB-GRAD-59-04

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.6, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Hesitant fuzzy mighty filters of BE-algebras

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Abstract. The notion of hesitant fuzzy mighty filter of a BE-algebra is introduced and related properties are investigated. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy mighty filter. We construct a new quotient structure of a transitive BE-algebra using a hesitant fuzzy filter and study some properties of it.

1. Introduction

In 2007, Kim and Kim [5] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Song et al. [8] considered the fuzzification of ideals in BE-algebras. They introduced the notion of fuzzy ideals in BE-algebras, and investigated related properties. They gave characterizations of a fuzzy ideal in BE-algebras.

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [9] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [3, 7, 11, 12, 13, 14, 15]). In [4], Y. B. Jun and S. S. Ahn introduced the notion of a hesitant fuzzy filter and investigated some properties of it. The authors [2] defined a hesitant fuzzy implicative filter in a BE-algebra and discussed some properties of it.

In this paper, we introduce the notion of hesitant fuzzy mighty filter of a BE-algebra, and investigate some properties of it. We consider characterizations of a hesitant fuzzy mighty filter of a BE-algebra. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy mighty filter. We construct a new quotient structure of a transitive BE-algebra using a hesitant fuzzy filter and study some properties of it.

2. Preliminaries

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[0]**2010 Mathematics Subject Classification**: 06F35; 03G25; 06D72.

[0]**Keywords**: BE-algebra; (mighty) filter; hesitant (mighty) filter.

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By a *BE-algebra* ([5]) we mean a system $(X; *, 1)$ of type $(2, 0)$ which the following axioms hold:

(2.1) $(\forall x \in X)\,(x * x = 1)$,
(2.2) $(\forall x \in X)\,(x * 1 = 1)$,
(2.3) $(\forall x \in X)\,(1 * x = x)$,
(2.4) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)$ (exchange).

We introduce a relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 1$.

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies: for any $x, y, z \in X$, $y * z \leq (x * y) * (x * z)$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if it satisfies: for any $x, y, z \in X$, $x * (y * z) = (x * y) * (x * z)$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see [5]).

Every self distributive BE-algebra $(X; *, 1)$ satisfies the following properties:

(2.5) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z)$,
(2.6) $(\forall x, y \in X)\,(x * (x * y) = x * y)$,
(2.7) $(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))$,

Definition 2.1. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* of X ([5]) if

(F1) $1 \in F$;
(F2) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

F is a *mighty filter* ([6]) of X if it satisfies (F1) and

(F3) $(\forall x, y, z \in X)(z * (y * x), z \in F \Rightarrow ((x * y) * y) * x \in F)$.

Theorem 2.2. ([6]) *A filter F of a BE-algebra X is mighty if and only if*

(2.8) $(\forall x, y \in X)(y * x \in F \Rightarrow ((x * y) * y) * x \in F)$.

Definition 2.3. ([9]) Let E be a reference set. A *hesitant fuzzy set* on E is defined in terms of a function that when applied to E returns a subset of $[0, 1]$, which can be viewed as the following mathematical representation:

$$H_E := \{(e, h_E(e)) | e \in E\}$$

where $h_E : E \to \mathscr{P}([0, 1])$.

Definition 2.4. Given a non-empty subset A of a BE-algebra X, a hesitant fuzzy set

$$H_X := \{(x, h_X(x)) | x \in X\}$$

on satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A$$

is called a *hesitant fuzzy set related to* A (briefly, A-*hesitant fuzzy set*) on X, and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a mapping from X to $\mathscr{P}([0,1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

For a hesitant set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of a BE-algebra X and a subset γ of $[0,1]$, the hesitant fuzzy γ-inclusive set of H_X, denoted by $H_X(\gamma)$, is defined to be the set

$$H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}.$$

For any hesitant fuzzy set $H_X = \{(x, h_X(x)|x \in X\}$ and $G_X = \{(x, g_X(x))|x \in X\}$, we call H_X a *hesitant fuzzy subset* of G_X, denoted by $H_X \widetilde{\subseteq} G_X$, if $h_X(x) \subseteq g_X(x)$ for all $x \in X$.

3. Hesitant fuzzy mighty filters

Definition 3.1. Given a non-empty subset (subalgebra as much as possible) A of a BE-algebra X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy subalgebra of* X *related to* A (briefly, A-*hesitant fuzzy subalgebra* of X) ([4]) if it satisfies the following condition: $h_A(x) \cap h_A(y) \subseteq h_A(x * y)$ for any $x, y \in A$. An A-hesitant fuzzy subalgebra of X with $A = X$ is called a *hesitant fuzzy subalgebra* of X. An A-hesitant fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a *hesitant fuzzy filter of* X *related to* A (briefly, A-*hesitant fuzzy filter* of X) ([4]) if it satisfies the following condition:

(3.1) $(\forall x \in A)(h_A(x) \subseteq h_A(1))$,
(3.2) $(\forall x, y \in A)(h_A(x * y) \cap h_A(x) \subseteq h_A(y))$.

An A-hesitant fuzzy filter of X with $A = X$ is called a *hesitant fuzzy filter* of X.

Proposition 3.2. ([4]) *Let* $H_A := \{(x, h_A(x))|x \in X\}$ *be an* A*-hesitant fuzzy filter of a* BE*-algebra* X *where* A *is a subalgebra of* X*. Then the following assertions are valid.*

(i) $(\forall x, y \in A)(x \le y \Rightarrow h_A(x) \subseteq h_A(y))$,
(ii) $(\forall x, y, z \in A)(z \le x * y \Rightarrow h_A(y) \supseteq h_A(x) \cap h_A(z))$,
(iii) $(\forall x, y, z \in A)(h_A(x * (y * z)) \cap h_A(y) \subseteq h_A(x * z))$,
(iv) $(\forall a, x \in A)(h_A(a) \subseteq h_A((a * x) * x)$.

Proposition 3.3. *Every hesitant fuzzy filter of a* BE*-algebra* X *is a hesitant fuzzy subalgebra of* X*.*

Proof. Let $H_X = \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy filter of X. For any $x, y \in X$, we have $h_X(x) \cap h_X(y) \subseteq h_X(1) \cap h_X(y) = h_X(y * (x * y)) \cap h_X(y) \subseteq h_X(x * y)$. Hence H_X is a hesitant fuzzy subalgebra of X. □

The converse of Proposition 3.3 may not be true in general (see Example 3.4).

Example 3.4. Let $X = \{0, 1, a, b, c\}$ be a BE-algebra ([4]) with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \big\{(1, [0, 1]), (a, (0, \tfrac{1}{8})), (b, (\tfrac{1}{4}, \tfrac{3}{4}])), (c, (0, \tfrac{1}{2}))\big\}$$

Then H_X is a hesitant fuzzy subalgebra of X, but not a hesitant fuzzy filter of X since $h_X(b * a) \cap h_X(b) = h_X(1) \cap h_X(b) = [0, 1] \cap (\frac{1}{4}, \frac{3}{4}] \nsubseteq h_X(a) = (0, \frac{1}{8})$.

Definition 3.5. Given a non-empty subset (subalgebra as much as possible) A of a BE-algebra X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy mighty filter of X related to A* (briefly, *A-hesitant fuzzy mighty filter* of X) if it satisfies (3.1) and

(3.3) $(\forall x, y, z \in A)(h_A(z * (y * x)) \cap h_A(z) \subseteq h_A(((x * y) * y) * x)$.

An A-hesitant fuzzy mighty filter of X with $A = X$ is called a *hesitant fuzzy mighty filter* of X.

Example 3.6. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra ([6]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	d	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(1, [0, 1]), (a, [\tfrac{3}{4}, 1]), (b, [\tfrac{1}{2}, 1]), (c, [\tfrac{1}{2}, 1]), (d, \{\tfrac{3}{4}, 1\}), (0, \{\tfrac{1}{2}, 1\})\}$$

It is easy to check that H_X is a hesitant fuzzy fuzzy mighty filter of X.

Proposition 3.7. *Every hesitant fuzzy mighty filter of a BE-algebra X is a hesitant fuzzy filter of X.*

Proof. Let $H_X = \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy mighty filter of X. Putting $y := 1$ in (3.3), we have $h_X(z * (1 * x)) \cap h_X(z) = h_X(z * x) \cap h_X(z) \subseteq h_X(((x * 1) * 1) * x) = h_X(x)$. Hence H_X is a hesitant fuzzy filter of X. □

The converse of Proposition 3.7 may not be true in general (see Example 3.8).

Example 3.8. Let $X = \{1, a, b, c, d\}$ be a BE-algebra ([5]) with the following Cayley table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(1, [0, 1]), (a, [\tfrac{1}{2}, 1]), (b, [\tfrac{1}{4}, 1]), (c, [\tfrac{1}{5}, 1]), (d, \{\tfrac{3}{4}, 1\})\}.$$

It is easy to check that H_X is a hesitant fuzzy fuzzy filter of X, but not a hesitant fuzzy mighty filter of X since $h_X(1 * (c * a)) \cap h_X(1) = h_X(1) = [0, 1] \nsubseteq h_X(((a * c) * c) * a) = h_X(a) = [\frac{1}{2}, 1]$.

Theorem 3.9. *Any hesitant fuzzy filter $H_X = \{(x, h_X(x))|x \in X\}$ of a BE-algebra X is mighty if and only if it satisfies*

(3.4) $(\forall x, y \in X)(h_X(y * x) \subseteq h_X(((x * y) * y) * x))$.

Proof. Assume that a hesitant fuzzy filter H_X is mighty. Setting $z := 1$ in (3.3), we have $h_X(1 * (y * x)) \cap h_X(1) = h_X(y * x) \subseteq h_X(((x * y) * y) * x)$. Hence (3.4) holds.

Conversely, suppose that the hesitant fuzzy filter $H_X = \{(x, h_X(x))|x \in X\}$ satisfies the condition (3.4). Using (3.2) and (3.4), we have $h_X(z * (y * x)) \cap h_X(z) \subseteq h_X(y * x) \subseteq h_X(((x * y) * y) * x)$, for any $x, y \in X$. Hence H_X is a hesitant fuzzy mighty filter of X. □

Proposition 3.10. *Let $H_X = \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy mighty filter of a BE-algebra X. Then $X_{H_X} := \{x \in X|h_X(x) = h_X(1)\}$ is a mighty filter of X.*

Proof. Clearly, $1 \in X_{H_X}$. Let $z*(y*x), z \in X_{H_X}$. Then $h_X(z*(y*x)) = h_X(1)$ and $h_X(z) = h_X(1)$. It follows from (3.3) that $h_X(z * (y * x)) \cap h_X(z) = h_X(1) \subseteq h_X(((x * y) * y) * x)$. By (3.1), we get $h_X(((x * y) * y) * x) = h_X(1)$. Hence $((x * y) * y) * x \in X_{H_X}$. Therefore X_{H_X} is a mighty filter of X. □

Theorem 3.11. *Let $H_X = \{(x, h_X(x))|x \in X\}$ and $G_X = \{(x, g_X(x))|x \in X\}$ be hesitant fuzzy filters of a transitive BE-algebra such that $H_X \widetilde{\subseteq} G_X$ and $h_X(1) = g_X(1)$. If H_X is mighty, then so is G_X.*

Proof. Let $x, y \in X$. Note that $y * ((y * x) * x) = (y * x) * (y * x) = 1$. Since H_X is a hesitant fuzzy mighty filter of a BE-algebra X, by (3.4) and $H_X \widetilde{\subseteq} G_X$ we have $h_X(1) = h_X(y * ((y * x) * x)) \subseteq h_X(((((y*x)*x)*y)*y)*((y*x)*x)) \subseteq g_X(((((y*x)*x)*y)*y)*((y*x)*x))$. Since $h_X(1) = g_X(1)$, we get $g_X((y * x) * ((((y * x) * x) * y) * y) * x)) = g_X(((((y * x) * x) * y) * y) * ((y * x) * x)) = g_X(1)$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} g_X(y*x) =& g(1)\cap g_X(y*x) \\ =& g_X((y*x)*(((((y*x)*x)*y)*y)*x))\cap g_X(y*x) \\ \subseteq& g_X(((((y*x)*x)*y)*y)*x). \end{aligned} \tag{3.5}$$

Since X is transitive, we get

$$\begin{aligned} [((((y*x)*x)*y)*y)*x]*[((x*y)*y)*x] & \\ &\geq ((x*y)*y)*((((y*x)*x)*y)*y) \\ &\geq (((y*x)*x)*y)*(x*y) \\ &\geq x*((y*x)*x) \\ &= (y*x)*(x*x) \\ &= (y*x)*1 = 1. \end{aligned}$$

It follows from Proposition 3.2 that $g_X(((((y*x)*x)*y)*y)*x)\cap g_X(1) = g_X(((((y*x)*x)*y)*y)*x) \subseteq g_X(((x*y)*y)*x)$. Using (3.5), we have $g_X(y*x) \subseteq g_X(((((y*x)*x)*y)*y)*x) \subseteq g_X(((x*y)*y)*x)$. Therefore $g_X(y*x) \subseteq g_X(((x*y)*y)*x)$. By Theorem 3.9, G_X is a hesitant fuzzy mighty filter of X. □

Corollary 3.12. *Every hesitant fuzzy filter H_X of a transitive BE-algebra X is mighty if and only if the hesitant fuzzy filter $H_{\{1\}}$ is mighty.*

Proof. Straightforward. □

Let $H_X := \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy filter of a transitive BE-algebra X. Define a binary relation " $\sim_{h_X}$ " on X by putting $x \sim_{h_X} y$ if and only if $h_X(x*y) = h_X(y*x) = h_X(1)$ for any $x, y \in X$.

Lemma 3.13. *The relation " $\sim_{h_X}$ " is an equivalence relation on a transitive BE-algebra X.*

Proof. For any $x \in X$, $x*x = 1$ by (2.1). So $h_X(x*x) = h_X(1)$, hence $x \sim_{h_X} x$, which $\sim_{h_X}$ is reflexive. Suppose that $x \sim_{h_X} y$ for any $x, y \in X$. Then $h_X(x*y) = h_X(y*x) = h_X(1)$. Hence $\sim_{h_X}$ is symmetric. Assume that $x \sim_{h_X} y$ and $y \sim_{h_X} z$ for any $x, y, z \in X$. Then $h_X(x*y) = h_X(y*x) = h_X(1)$ and $h_X(y*z) = h_X(z*y) = h_X(1)$. By transitivity, $(x*y)*[(y*z)*(x*z)] = 1$ and $(z*y)*[(y*x)*(z*x)] = 1$. By Proposition 3.2, we have $h_X(x*y)\cap h_X(y*z) = h_X(1) \subseteq h_X(x*z)$ and $h_X(z*y)\cap h_X(y*x) = h_X(1) \subseteq h_X(z*x)$. Hence $h_X(z*x) = h_X(z*x) = h_X(1)$, i.e., $x \sim_{h_X} z$. Thus $\sim_{h_X}$ is an equivalence relation on X. □

Lemma 3.14. *The relation " $\sim_{h_X}$ " is a congruence relation on a transitive BE-algebra X.*

Proof. If $x \sim_{h_X} y$ and $u \sim_{h_X} v$ for any $x, y, u, v \in X$, then $h_X(x*y) = h_X(y*x) = h_X(1)$ and $h_X(u*v) = h_X(v*u) = h_X(1)$. By transitivity, $(u*v)*[(x*u)*(x*v)] = 1$ and

$(v*u)*[(x*v)*(x*u)] = 1$, it follows from Proposition 3.2 that $h_X(1) = h_X(u*v) \subseteq h_X((x*u)*((x*v))$ and $h_X(1) = h_X(v*u) \subseteq h_X((x*v)*(x*u))$. Hence $h_X((x*u)*(x*v)) = h_X(1)$ and $h_X((x*v)*(x*u)) = h_X(1)$. Therefore $x*u \sim_{h_X} x*v$. By a similar way, we can prove that $x*v \sim_{h_X} y*v$. Therefore $\sim_{h_X}$ is a congruence relation on X. □

X is decomposed by the congruence relation $\sim_{h_X}$. The class containing x is denoted by $[x]_{h_X}$. Denote $X/h_X := \{[x]_{h_X} | x \in X\}$. We define a binary relation $*'$ on X/h_X by $[x]_{h_X} *' [y]_{h_X} := [x*y]_{h_X}$. This definition is well defined since $\sim_{h_X}$ is a congruence relation on X.

Lemma 3.15. $[1]_{h_X} = X_{H_X}$.

Proof. $[1]_{h_X} = \{x \in X | 1 \sim_{h_X} x\} = \{x \in X | h_X(1*x) = h_X(x*1) = h_X(1)\} = \{x \in X | h_X(x) = h_X(1)\} = X_{H_X}$. □

Theorem 3.16. *Let X be a transitive BE-algebra X. Then $(X/h_X; *', [1]_{h_X})$ is a transitive BE-algebra.*

Proof. Straightforward. □

Theorem 3.17. *A hesitant fuzzy filter of a transitive BE-algebra X is mighty if and only if every filter of the quotient algebra X/h_X is mighty.*

Proof. Assume that a hesitant fuzzy filter H_X is mighty and let $x, y \in X$ be such that $[y]_{h_X} *' [x]_{h_X} \in [1]_{h_X}$. Then $h_X(y*x) = h_X(1)$. It follows from (2.3) and (3.3) that $h_X(1*(y*x)) \cap h_X(1) = h_X(y*x) = h_X(1) \subseteq h_X(((x*y)*y)*x)$. Hence $h_X(((x*y)*y)*x) = h_X(1)$. So $((([x]_{h_X} *' [y]_{h_X}) *' [y]_{h_X})) *' [x]_{h_X} = [((x*y)*y)*x]_{h_X} \in [1]_{h_X}$ which proves that $\{[1]_{h_X}\}$ is a mighty filter of X/h_X. By Corollary 3.13, every filter of X/h_X is mighty.

Conversely, suppose that every filter of the quotient algebra X/h_X is mighty and let $x, y \in X$ be such that $y*x \in [1]_{h_X}$. Then $h_X(y*x) = h_X(1)$ and so $[y]_{h_X} *' [x]_{h_X} \in [1]_{h_X}$. Since $\{[1]_{h_X}\}$ is a mighty filter of X/h_X, it follows from Theorem 2.2 that $[((x*y)*y)*x]_{h_X} = (([x]_{h_X} *' [y]_{h_X}) *' [y]_{h_X}) *' [x]_{h_X} \in [1]_{h_X}$. Hence $h_X((((x*y)*y)*)*x) = h_X(1)$. Therefore $h_X(y*x) = h_X(((x*y)*y))*x)$. Thus H_X is a hesitant fuzzy filter of Theorem 3.9. □

Theorem 3.18. *A hesitant fuzzy set $H_X := \{(x, h_X(x) | x \in X\}$ of a BE-algebra X is a hesitant fuzzy mighty filter of X if and only if the set $H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}$ is a mighty filter of X for all $\gamma \in \mathscr{P}([0,1])$ whenever it is nonempty.*

Proof. Suppose that H_X is a hesitant fuzzy mighty filter of X. Let $x, y, z \in X$ and $\gamma \in \mathscr{P}([0,1])$ be such that $z*(y*x) \in H_X(\gamma)$ and $z \in H_X(\gamma)$. Then $h_X(z*(y*x)) \supseteq \gamma$ and $h_X(z) \supseteq \gamma$. It follows from (3.1) and (3.3) that $h_X(1) \supseteq h_X(((x*y)*y)*x) \supseteq h_X(z*(y*x)) \cap h_X(z) \supseteq \gamma$. Hence $1 \in H_X(\gamma)$ and $((x*y)*y)*x \in H_X(\gamma)$, and therefore $H_X(\gamma)$ is a mighty filter of X.

Conversely, assume that $H_X(\gamma)$ is a mighty filter of X for all $\gamma \in \mathscr{P}([0,1])$ with $H_X(\gamma) \neq \emptyset$. For any $x \in X$, let $h_X(x) = \gamma$. Then $x \in H_X(\gamma)$. Since $H_X(\gamma)$ is a mighty filter of X, we have

$1 \in h_X(\gamma)$ and so $h_X(x) = \gamma \subseteq h_X(1)$. For any $x, y, z \in X$, let $h_X(z * (y * x)) = \gamma_{z*(y*x)}$ and $h_X(z) = \gamma_z$. Let $\gamma := \gamma_{z*(y*x)} \cap \gamma_z$. Then $z * (y * x) \in H_X(\gamma)$ and $z \in H_X(\gamma)$ which imply that $((x * y) * y) * x \in H_X(\gamma)$. Hence $h_X(((x * y) * y) * x) \supseteq \gamma = \gamma_{z*(y*x)} \cap \gamma_z = h_X(z * (y * x)) \cap h_X(z)$. Thus H_X is a hesitant fuzzy mighty filter of X. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.6, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

A Class of New General Iteration Approximation of Common Fixed Points for Total Asymptotically Nonexpansive Mappings in Hyperbolic Spaces

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Abstract. In this paper, we introduce and study a class of new general iteration processes for two finite families of total asymptotically nonexpansive mappings in hyperbolic spaces, which includes asymptotically nonexpansive mapping, (generalized) nonexpansive mapping of all normed linear spaces, Hadamard manifolds and CAT(0) spaces as special cases. Some important related properties to the new general iterative processes are also given and analyzed, and Δ-convergence and strong convergence of the iteration in hyperbolic spaces are proved. Furthermore, some meaningful illustrations for clarifying our results and two open questions are proposed. The results presented in this paper extend and improve the corresponding results announced in the current literature.

Key Words and Phrases: common fixed point, new general iterative approximation, Δ-convergence and strong convergence, total asymptotically nonexpansive mapping, hyperbolic space.

AMS Subject Classification: 47H09, 47H10, 54E70.

1 Introduction and preliminaries

Let $(\mathcal{H}, d)$ be a metric space, $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$ be two finite families of nonlinear mappings on nonempty set $K \subset \mathcal{H}$. Suppose that $\{\alpha_{in}\}$ and $\{\beta_{in}\}$ are two real sequences in $[a, b]$ for some $a, b \in (0, 1)$ and $\theta_{in} := \frac{\beta_{in}}{1-\alpha_{in}}$. For $r \geq 2$ and $n \geq 1$, in this paper, we consider the following general iterative sequence $\{x_n\}$:

$$\begin{aligned}
x_{n+1} &= W(T_1^n y_{n+r-2}, W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), \alpha_{1n}),\\
y_{n+r-2} &= W(T_2^n y_{n+r-3}, W(y_{n+r-3}, S_2^n y_{n+r-3}, \theta_{2n}), \alpha_{2n}),\\
y_{n+r-3} &= W(T_3^n y_{n+r-4}, W(y_{n+r-4}, S_3^n y_{n+r-4}, \theta_{3n}), \alpha_{3n}),\\
&\ \ \vdots\\
y_{n+1} &= W(T_{r-1}^n y_n, W(y_n, S_{r-1}^n y_n, \theta_{(r-1)n}), \alpha_{(r-1)n}),\\
y_n &= W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}).
\end{aligned} \tag{1.1}$$

Remark 1.1 For appropriate and suitable choices of the nonlinear mappings $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$, the positive integer r and the underlying spaces, the iteration (1.1) includes a number of known iterative processes, which were studied previously by many authors. For more details, see [1–20] and the references therein, and the following examples:

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Example 1.1 If $\beta_{in} = 0$ for $i = 1, 2, 3, \cdots, r$ and all $n \geq 1$, and $\{\alpha_{in}\}$ is a real sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$, then the sequence $\{x_n\}$ in (1.1) reduces to

$$\begin{aligned}
x_{n+1} &= \alpha_{1n} y_{n+r-2} + (1-\alpha_{1n}) T_1^n y_{n+r-2},\\
y_{n+r-2} &= \alpha_{2n} y_{n+r-3} + (1-\alpha_{2n}) T_2^n y_{n+r-3},\\
y_{n+r-3} &= \alpha_{3n} y_{n+r-4} + (1-\alpha_{3n}) T_3^n y_{n+r-4},\\
&\vdots\\
y_{n+1} &= \alpha_{(r-1)n} y_n + (1-\alpha_{(r-1)n}) T_{r-1}^n y_n,\\
y_n &= \alpha_{rn} x_n + (1-\alpha_{rn}) T_r^n x_n,
\end{aligned} \tag{1.2}$$

which was considered by Yildirim and Ozdemir [1] when $\{T_i\}_{i=1}^r$ is a family of asymptotically quasi-nonexpansive self-mappings on $K \subset \mathcal{H}$ and $\mathcal{H}$ is a Banach space. Further, the iteration process (1.2) was introduced and studied by Basarir and Sahin [2] for a generalized nonexpansive mapping of the CAT(0) spaces.

Example 1.2 For $r = 3$ and $\alpha_{in} = 0$, then (1.1) changes into the iterative process introduced by Noor [3], which was dealt for variational inequalities of the Hilbert spaces. Moreover, a unified treatment regarding the iterative process for nonexpansive mapping in hyperbolic spaces was considered by Akbulut and Gündüz [4]. For many more, see, for example, the research works of Sahin and Basarir [5], Suantai [6] and many others in the literature.

Example 1.3 Let $r = 2$, and $\alpha_{1n} = 1$, and $\alpha_{2n} = 0$, and $T_2 = S_2$, then (1.1) becomes to the following iteration:

$$x_{n+1} = T_1^n y_n, y_n = W(x_n, T_2^n x_n, \theta_{2n}). \tag{1.3}$$

The iteration (1.3) is called a modified hybrid Picard-Mann iteration process, which was introduced and studied by Thakur et al. [7] in CAT(0) space. This process (1.3) is independent of Picard and Mann iterative process and the convergence process is faster than Picard and Mann iteration process. For more on (hybrid) Picard-Mann iteration process and a comparison between different process of modified hybrid Picard-Mann iteration process, see, for example, [7, 8] and the references therein.

Example 1.4 Let $r = 2$, and $\alpha_{1n} = 0$, and $\beta_{1n} = 1$, $\alpha_{2n} = 1$, then (1.1) is equivalent to

$$x_{n+1} = W(x_n, S^n x_n, \theta_n),$$

which is well-known modified Mann iteration process, and was studied by Schu [9] in Banach spaces.

In 2013, Fukhar-ud-din and Khan [21] pointed out "structural properties of the space under consideration are very important in establishing the fixed point property of the space, for example, strict convexity, uniform convexity and uniform smoothness etc". In fact, in recent decades, motivated and governed by questions in most of science problems about hyperbolic groups, the study on hyperbolic spaces has been developed unremittingly in geometric group theory and metric fixed point theory in normed linear spaces or Banach spaces. Especially, the concept of hyperbolic spaces introduced by Kohlenbach [22] and defined below, is more restrictive and more general than that of being considered in [23] and in [24], respectively (see also [25]). Furthermore, all normed linear spaces, convex subsets wherein Hadamard manifolds and CAT(0) spaces are the special cases of the class of hyperbolic spaces due to Kohlenbach [22].

Definition 1.1 A hyperbolic spaces is a metric space $(\mathcal{H}, d)$ together with a mapping $W : \mathcal{H}^2 \times [0,1] \to \mathcal{H}$ satisfying

(i) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1-\alpha) d(u, y)$,
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,
(iii) $W(x, y, \alpha) = W(y, x, (1-\alpha))$,
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1-\alpha) d(z, w)$ for all $u, x, y, z, w \in \mathcal{H}$ and $\alpha, \beta \in [0,1]$.

Remark 1.1 (1) The class of hyperbolic spaces is general in nature and its important example is the open unit ball B in a complex domain C with respect to the Poincare metric (also called

"Poincare distance")

$$d_B(x,y) := \arg\tanh\left|\frac{x-y}{1-x\overline{y}}\right| = \arg\tanh(1-\sigma(x,y))^{\frac{1}{2}},$$

where $\sigma(x,y) := \frac{(1-|x|^2)(1-|y|^2)}{|1-x\overline{y}|^2}$ for all $x,y \in B$. Further, the above example can be extended from C to general complex Hilbert spaces $(H, \langle\cdot\rangle)$ (see [21, 22]).

(2) A metric space $(\mathcal{H}, d)$ satisfying only (i) in Definition 1.1 is a convex metric space introduced by Takahashi [26]. A nonempty subset K of a hyperbolic space $\mathcal{H}$ is convex if $W(x,y,\alpha) \in K$ for all $x,y \in K$ and $\alpha \in [0,1]$. For more on hyperbolic spaces and a comparison between different notions of hyperbolic space, see, for example, [27] and the references therein.

(3) A hyperbolic space is uniformly convex if for any $r > 0$ and $\epsilon \in (0,2]$, and all $u,x,y \in \mathcal{H}$, there exists $\delta \in (0,1]$ such that

$$d(W(x,y,\frac{1}{2}),u) \le (1-\delta)r,$$

provided $\max\{d(x,u), d(y,u)\} \le r$ and $d(x,y) \ge r\epsilon$ (see [28, 29]). A map $\eta : (0,+\infty)\times(0,2] \to (0,1]$, which provides such $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\epsilon \in (0,2]$, is known as a modulus of uniform convexity of $\mathcal{H}$. We call η monotone if it decreases with r (for fixed ϵ), i.e., for all $\epsilon > 0$, $r_2 \ge r_1 > 0(\eta(r_2,\epsilon) \le \eta(r_1,\epsilon))$. CAT(0) spaces are uniformly convex hyperbolic spaces with modulus of uniform convexity $\eta(r,\varepsilon) = \frac{\varepsilon^2}{8}$ (see [28, 30]). Thus, the class of uniformly convex hyperbolic spaces includes both uniformly convex normed spaces and CAT(0) spaces as special cases.

In the sequel, let $(\mathcal{H}, d)$ be a metric space, and let K be a nonempty subset of $\mathcal{H}$. We shall denote the fixed point set of a self-mapping on K of T by $F(T) = \{x \in K : Tx = x\}$.

Definition 1.2 A mapping $T : K \to K$ is said to be

(i) semi-compact if every bounded sequence $\{x_n\} \subset K$, satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence;

(ii) nonexpansive if $d(Tx,Ty) \le d(x,y)$ for any $x,y \in K$;

(iii) quasi-nonexpansive if $d(Tx,p) \le d(x,p)$ for all $x \in K$ and $p \in F(T) \neq \emptyset$;

(iv) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0,+\infty)$ and $\lim_{n\to\infty} k_n = 0$ such that

$$d(T^n x, T^n y) \le (1+k_n)d(x,y), \quad \forall x,y \in K, n \ge 1;$$

(v) asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\} \subset [0,+\infty)$ and $\lim_{n\to\infty} k_n = 0$ such that

$$d(T^n x, p) \le (1+k_n)d(x,p), \quad \forall x \in K, p \in F(T), n \ge 1;$$

(vi) $(\{\mu_n\},\{\xi_n\},\rho)$-total asymptotically nonexpansive, if there exist nonnegative sequences $\{\mu_n\}$, $\{\xi_n\}$ with $\mu_n \to 0$, $\xi_n \to 0$ and a strictly increasing continuous function $\rho : [0,+\infty) \to [0,+\infty)$ with $\rho(0) = 0$ such that

$$d(T^n x, T^n y) \le d(x,y) + \mu_n \rho\big(d(x,y)\big) + \xi_n, \quad \forall x,y \in K, n \ge 1;$$

(vii)$(\{\mu_n\},\{\xi_n\},\rho)$-total asymptotically quasi-nonexpansive, if there exist nonnegative sequences $\{\mu_n\}$, $\{\xi_n\}$ with $\mu_n \to 0$, $\xi_n \to 0$ and a strictly increasing continuous function $\rho : [0,+\infty) \to [0,+\infty)$ with $\rho(0) = 0$ such that

$$d(T^n x, p) \le d(x,p) + \mu_n \rho\big(d(x,p)\big) + \xi_n, \quad \forall x \in K, p \in F, n \ge 1;$$

(viii) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \le L d(x,y), \quad \forall x,y \in K, n \ge 1.$$

Remark 1.2 From Definition 1.2, it follows that a (quasi-)nonexpansive mapping is an asymptotically (quasi-)nonexpansive mapping with $k_n \equiv 0$ for $n \ge 1$, and each asymptotically (quasi-)nonexpansive mapping is a $(\{\mu_n\},\{\xi_n\},\rho)$-total asymptotically (quasi-)nonexpansive mapping with $\xi_n = 0$, and $\rho(t) = t \ge 0$. However, in general, the converse of these statement is not true.

As all we know, the study of such types of problems on the iterative approximation of (common) fixed points for generalizations of nonexpansive mappings in hyperbolic spaces, is motivated by an increasing interest in the problems of finding a common fixed point of some nonlinear mappings, which is the only main tool for analysis of generalized nonexpansive mappings and provides us a general and unified framework for studying the existence of fixed points of various nonlinear mappings arising in many branches of nonlinear analysis, topology and applied mathematics, etc.

Inspired and motivated and by the above recent works, in this paper, we shall study some important related properties to the new general iterative process (1.1) for two finite families of total asymptotically nonexpansive mappings as well as two finite families of total asymptotically quasi-nonexpansive mappings in hyperbolic spaces. Results concerning Δ-convergence as well as strong convergence of this iteration are proved. The results presented in the paper extend and improve some recent results given in [1, 2, 4–7, 9, 21].

In order to define the concept of Δ-convergence in the general setup of hyperbolic spaces, in the next moment, we first give some basic concepts.

In 1976, Lim [31] introduced the notion of asymptotic center and, consequently, coined the concept of Δ-convergence in a general setting of a metric space. Kirk and Panyanak [32] proposed an analogous version of convergence in geodesic spaces, namely Δ-convergence, which was originally introduced by Lim [31]. Further, Kirk and Panyanak [32] showed that Δ-convergence coincides with the usual weak convergence in Banach spaces and both concepts share many useful properties.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space $\mathcal{H}$. For $x \in \mathcal{H}$, we define a continuous functional $r(\cdot, \{x_n\}) : \mathcal{H} \to [0, +\infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $\hat{r}(\{x_n\})$ of $\{x_n\}$ is given by

$$\hat{r}(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in \mathcal{H}\}.$$

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to $K \subset \mathcal{H}$ is defined as follows:

$$A_K(\{x_n\}) = \{x \in \mathcal{H} : r(x, \{x_n\}) \le r(y, \{x_n\}), \forall y \in K\},$$

which is the set of minimizers for $r(\cdot, \{x_n\})$. Further, it is simply denoted by $A(\{x_n\})$ when the asymptotic center is taken with respect to $\mathcal{H}$, and a sequence $\{x_n\}$ in $\mathcal{H}$ is said to Δ-converge to $x \in \mathcal{H}$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

It is well known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that "bounded sequences have unique asymptotic centers with respect to closed convex subsets". The following lemma ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 1.1 ([30]) Let $(\mathcal{H}, d, W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in $\mathcal{H}$ has a unique asymptotic center with respect to any nonempty closed convex subset K of $\mathcal{H}$.

In the sequel, we need the following lemmas.

Lemma 1.2 ([10]) Let $(\mathcal{H}, d, W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in \mathcal{H}$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathcal{H}$ such that for some $c \ge 0$,

$$\limsup_{n\to\infty} d(x_n, x) \le c, \quad \limsup_{n\to\infty} d(y_n, x) \le c, \quad \lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = c,$$

Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 1.3 ([10]) Let K be a nonempty closed convex subset of uniformly convex hyperbolic space, and let $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \zeta$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \zeta$, then $\lim_{m\to\infty} y_m = y$.

Lemma 1.4 ([33]) Let $\{a_n\}$, $\{b_n\}$ and $\{\omega_n\}$ be nonnegative real sequences satisfying

$$a_{n+1} \leq (1+\omega_n)a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \omega_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exist. If there exists a subseqence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $\lim_{n\to\infty} a_n = 0$.

2 Some important related properties

Throughout in this paper, we assume that $I = \{1, 2, \cdots, r\}$, $\{T_i\}_{i=1}^{r}$ and $\{S_i\}_{i=1}^{r}$ are two finite families of total asymptotically nonexpansive mappings on a nonempty subset K of the hyperbolic space $\mathcal{H}$ defined by Definition 1.2, for each $i \in I$ and all $n \geq 1$, $\{\alpha_{in}\}$, $\{\beta_{in}\}$ and $\{\theta_{in}\}$ are the same as in (1.1). We start with the following important related property of the general iterative process (1.1) for two finite families of total asymptotically nonexpansive mappings in a hyperbolic space.

Theorem 2.1 Let K be a nonempty closed and convex subset of a hyperbolic space $\mathcal{H}$. For $i \in I$, let $T_i : K \to K$ be a $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \mu_n^i = 0$ and $\lim_{n\to\infty} \xi_n^i = 0$, and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ satisfying $\rho^i(0) = 0$, and let $S_i : K \to K$, be a $(\{\hat{\mu}_n^i\}, \{\hat{\xi}_n^i\}, \hat{\rho}^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \hat{\mu}_n^i = 0$ and $\lim_{n\to\infty} \hat{\xi}_n^i = 0$, and a strictly increasing continuous function $\hat{\rho}^i : [0, +\infty) \to [0, +\infty)$ satisfying $\hat{\rho}^i(0) = 0$. Assume that $F = \bigcap_{i=1}^{r}(F(T_i) \cap F(S_i)) \neq \emptyset$, and for each $i \in I$, the following conditions hold:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\mu}_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\xi}_n^i < +\infty$;

(ii) there exists a constant $M^* > 0$ such that

$$\rho^i(r) \leq M^* r, \quad \hat{\rho}^i(r) \leq M^* r, \quad \forall r > 0.$$

Then, for the sequence $\{x_n\}$ in (1.1), $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$.

Proof. Set $\mu_n = \max_{i\in I}\{\mu_n^i, \hat{\mu}_n^i\}$, and $\xi_n = \max_{i\in I}\{\xi_n^i, \hat{\xi}_n^i\}$, $\rho = \max_{i\in I}\{\rho^i, \hat{\rho}^i\}$. By condition (i), we know that $\sum_{n=1}^{\infty} \mu_n < +\infty$, $\sum_{n=1}^{\infty} \xi_n < +\infty$. For any $p \in F$ and all $n \geq 1$, it follows from (1.1) that

$$\begin{aligned}
d(y_n, p) &\leq \alpha_{rn} d(T_r^n x_n, p) + (1-\alpha_{rn}) d(W(x_n, S_r^n x_n, \theta_{rn}), p) \\
&\leq \alpha_{rn} d(T_r^n x_n, p) + \beta_{rn} d(x_n, p) + (1-\alpha_{rn}-\beta_{rn}) d(S_r^n x_n, p) \\
&\leq \alpha_{rn}[d(x_n, p) + \mu_n^r \rho^r\big(d(x_n, p)\big) + \xi_n^r] + \beta_{rn} d(x_n, p) \\
&\quad +(1-\alpha_{rn}-\beta_{rn})[d(x_n, p) + \hat{\mu}_n^r \hat{\rho}^r\big(d(x_n, p)\big) + \hat{\xi}_n^r] \\
&\leq \alpha_{rn}[d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n] + \beta_{rn} d(x_n, p) \\
&\quad +(1-\alpha_{rn}-\beta_{rn})[d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n] \\
&\leq \alpha_{rn}[(1+\mu_n M^*) d(x_n, p) + \xi_n] + \beta_{rn} d(x_n, p) \\
&\quad +(1-\alpha_{rn}-\beta_{rn})[(1+\mu_n M^*) d(x_n, p) + \xi_n] \\
&\leq (1+\mu_n M^*) d(x_n, p) + \xi_n
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
d(y_{n+1}, p) &\leq \alpha_{(r-1)n} d(T_{r-1}^n y_n, p) + (1-\alpha_{(r-1)n}) d(W(y_n, S_{r-1}^n y_n, \theta_{(r-1)n}), p) \\
&\leq \alpha_{(r-1)n} d(T_{r-1}^n y_n, p) + \beta_{(r-1)n} d(y_n, p) \\
&\quad +(1-\alpha_{(r-1)n}-\beta_{(r-1)n}) d(S_{r-1}^n y_n, p) \\
&\leq \alpha_{(r-1)n}[d(y_n, p) + \mu_n \rho\big(d(y_n, p)\big) + \xi_n] + \beta_{(r-1)n} d(y_n, p) \\
&\quad +(1-\alpha_{(r-1)n}-\beta_{(r-1)n})[d(y_n, p) + \mu_n \rho\big(d(y_n, p)\big) + \xi_n] \\
&\leq \alpha_{(r-1)n}[(1+\mu_n M^*) d(y_n, p) + \xi_n] + \beta_{(r-1)n} d(y_n, p) \\
&\quad +(1-\alpha_{(r-1)n}-\beta_{(r-1)n})[(1+\mu_n M^*) d(y_n, p) + \xi_n] \\
&\leq (1+\mu_n M^*) d(y_n, p) + \xi_n.
\end{aligned} \tag{2.2}$$

Similarly, we have

$$d(y_{n+r-2},p) \le (1+\mu_n M^*)d(y_{n+r-3},p)+\xi_n,$$
$$d(x_{n+1},p) \le (1+\mu_n M^*)d(y_{n+r-2},p)+\xi_n.$$

Thus,

$$\begin{aligned}
d(x_{n+1},p) \quad &\le (1+\mu_n M^*)^r d(x_n,p)+\sum_{j=1}^{r-1}(1+\mu_n M^*)^j\xi_n\\
&\le d(x_n,p)\Big[1+\tbinom{r}{1}\mu_n M^*+\tbinom{r}{2}(\mu_n M^*)^2+\tbinom{r}{3}(\mu_n M^*)^3\\
&\quad +\cdots+\tbinom{r}{r}(\mu_n M^*)^r\Big]+\sum_{j=1}^{r-1}(1+\mu_n M^*)^j\xi_n\\
&\le (1+a_n^r\mu_n)d(x_n,p)+\sum_{j=1}^{r-1}(1+\mu_n M^*)^j\xi_n\\
&\le (1+M_1\mu_n)d(x_n,p)+M_2\xi_n,
\end{aligned}$$

where $a_n^r=\binom{r}{1}M^*+\binom{r}{2}(M^*)^2\mu_n+\binom{r}{3}(M^*)^3(\mu_n)^2+\cdots+\binom{r}{r}(M^*)^r(\mu_n)^{r-1}$, and by virtue of condition(i), there exist positive constants M_1 and M_2 such that $a_n^r \le M_1$, $\sum_{j=1}^{r-1}(1+\mu_n M^*)^j \le M_2$ for each $n \ge 1$. Applying Lemma 1.4 to the above inequality, we have $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F$. □

In 1993, Bruck et al. [34] introduced a notion of asymptotically nonexpansive mapping in the intermediate sense. More accurately, a mapping $T : K \to K$ is said to be asymptotically nonexpansive mapping in the intermediate sense, provided that T is uniformly continuous and $\limsup_{n\to\infty}\sup_{x,y\in K}\{d(T^n x,T^n y)-d(x,y)\} \le 0$. Put $\xi_n=\max\{0,\sup_{x,y\in K}\{d(T^n x,T^n y)-d(x,y)\}\}$ and $\sum_{n=1}^{\infty}\xi_n^i<+\infty$, then $d(T^n x,T^n y)\le d(x,y)+\xi_n$ for any $n\ge 1$ and $x,y\in K$. In more detail, see, for example, [20] and the references therein.

The following result can be obtained from Theorem 2.1 immediately.

Corollary 2.1 Let K be a nonempty closed and convex subset of a hyperbolic space $\mathcal{H}$. For $i \in I$, let $T_i : K \to K$ be a $\{\xi_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense and let $S_i : K \to K$ be a $\{\hat{\xi}_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense. If $\sum_{n=1}^{\infty}\xi_n^i<+\infty$, $\sum_{n=1}^{\infty}\hat{\xi}_n^i<+\infty$ for $i\in I$ and $F=\bigcap_{i=1}^{r}(F(T_i)\cap F(S_i))\neq\emptyset$, then, for the sequence $\{x_n\}$ in (1.1), $\lim_{n\to\infty} d(x_n,p)$ exists for all $p\in F$.

Proof. Let $\xi_n=\max_{i\in I}\{\xi_n^i,\hat{\xi}_n^i\}$, then $\sum_{n=1}^{\infty}\xi_n<+\infty$. The rest of the proof is trivial. □

Corollary 2.2 Let K be a nonempty closed and convex subset of a hyperbolic space $\mathcal{H}$. Let $T_i : K \to K$ be a $\{k_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty}k_n^i<+\infty$ and $S_i : K \to K$ be a $\{\hat{k}_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty}\hat{k}_n^i<+\infty$ for $i\in I$. Assume that $F=\bigcap_{i=1}^{r}(F(T_i)\cap F(S_i))\neq\emptyset$. Then, for the sequence $\{x_n\}$ in (1.1), $\lim_{n\to\infty} d(x_n,p)$ exists for all $p\in F$.

Proof. Taking $k_n=\max_{i\in I}\{k_n^i,\hat{k}_n^i\}$, then $\sum_{n=1}^{\infty}k_n<+\infty$. Let $\rho^i(t)=\hat{\rho}^i(t)=t$, $\xi_n^i=\hat{\xi}_n^i=0$, $\mu_n^i=k_n^i$ in Theorem 2.1 for $i\in I$. Then all the conditions in Theorem 2.1 are satisfied and so the result holds. □

Theorem 2.2 Let K be a nonempty closed and convex subset of a uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i\in I$, let $T_i : K\to K$ be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\},\{\xi_n^i\},\rho^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty}\mu_n^i=0$ and $\lim_{n\to\infty}\xi_n^i=0$, and a strictly increasing continuous function $\rho^i:[0,+\infty)\to[0,+\infty)$ satisfying $\rho^i(0)=0$, and let $S_i : K\to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $(\{\hat{\mu}_n^i\},\{\hat{\xi}_n^i\},\hat{\rho}^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty}\hat{\mu}_n^i=0$ and $\lim_{n\to\infty}\hat{\xi}_n^i=0$, and a strictly increasing continuous function $\hat{\rho}^i:[0,+\infty)\to[0,+\infty)$ satisfying $\hat{\rho}^i(0)=0$. Suppose that $F=\bigcap_{i=1}^{r}(F(T_i)\cap F(S_i))\neq\emptyset$ and the conditions (i) and (ii) in Theorem 2.1 hold. Then, for $i\in I$ and the sequence $\{x_n\}$ generated by (1.1), we have

$$\lim_{n\to\infty} d(x_n,T_i x_n)=\lim_{n\to\infty} d(x_n,S_i x_n)=0.$$

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Proof. It follows from Theorem 2.1 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Assume that $\lim_{n\to\infty} d(x_n, p) = c > 0$. Otherwise the proof is trivial.

Take lim sup on both sides of inequalities (2.1) and (2.2). Since $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, we have $\limsup_{n\to\infty} d(y_n, p) \le c$ and $\limsup_{n\to\infty} d(y_{n+1}, p) \le c$. Similarly, we get $\limsup_{n\to\infty} d(y_{n+r-2}, p) \le c$, and so in total

$$\limsup_{n\to\infty} d(y_{n+k-1}, p) \le c, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.3}$$

Carry lim inf on both side of (2.4). Since

$$d(x_{n+1}, p) \le (1 + \mu_n M^*)^{r-1} d(y_n, p) + \sum_{j=1}^{r-2} (1 + \mu_n M^*)^j \xi_n \tag{2.4}$$

we have

$$\liminf_{n\to\infty} d(y_n, p) \ge c,$$

$$d(x_{n+1}, p) \le (1 + \mu_n M^*)^{r-k} d(y_{n+k-1}, p) + \sum_{j=1}^{r-k-1} (1 + \mu_n M^*)^j \xi_n, \quad \forall k = 2, 3, \cdots, r-1.$$

Also taking lim inf on both side of the above estimate, then we get

$$\liminf_{n\to\infty} d(y_{n+k-1}, p) \ge c, \quad \forall k = 2, 3, \cdots, r-1.$$

Thus, in total,

$$\liminf_{n\to\infty} d(y_{n+k-1}, p) \ge c, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.5}$$

Combining (2.3) and (2.5), we have

$$\lim_{n\to\infty} d(y_{n+k-1}, p) = c, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.6}$$

For $k = 1$ in (2.6), we get

$$\lim_{n\to\infty} d(W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}), p) = c. \tag{2.7}$$

Moreover,

$$\begin{aligned} d(W(x_n, S_r^n x_n, \theta_{rn}), p) &\le \theta_{rn} d(x_n, p) + (1 - \theta_{rn}) d(S_r^n x_n, p) \\ &\le \theta_{rn} d(x_n, p) + (1 - \theta_{rn})[(1 + \mu_n M^*) d(x_n, p) + \xi_n] \\ &\le (1 + \mu_n M^*) d(x_n, p) + \xi_n \end{aligned}$$

implies that

$$\limsup_{n\to\infty} d(W(x_n, S_r^n x_n, \theta_{rn}), p) \le c. \tag{2.8}$$

Obviously,

$$\limsup_{n\to\infty} d(T_r^n x_n, p) \le c. \tag{2.9}$$

It follows from (2.7)-(2.9) and Lemma 1.2 that

$$\lim_{n\to\infty} d(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn})) = 0. \tag{2.10}$$

Again, for $k = 2, 3, \cdots, r-1$, (2.6) can be expressed as

$$\lim_{n\to\infty} d(W(T^n_{r-(k-1)}y_{n+k-2}, W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}), \alpha_{(r-k+1)n}), p) = c. \tag{2.11}$$

By (2.3) and the inequality

$$\begin{aligned}
&d(W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}), p)\\
&\le \theta_{(r-k+1)n}d(y_{n+k-2}, p) + (1-\theta_{(r-k+1)n})d(S^n_{r-(k-1)}y_{n+k-2}, p)\\
&\le \theta_{(r-k+1)n}d(y_{n+k-2}, p) + (1-\theta_{(r-k+1)n})[(1+\mu_n M^*)d(y_{n+k-2}, p) + \xi_n]\\
&\le (1+\mu_n M^*)d(y_{n+k-2}, p) + \xi_n,
\end{aligned}$$

now we know that

$$\limsup_{n\to\infty} d(W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}), p) \le c. \tag{2.12}$$

Further,

$$\limsup_{n\to\infty} d(T^n_{r-(k-1)}y_{n+k-2}, p) \le c, \quad \forall k = 2, 3, \cdots, r-1. \tag{2.13}$$

From (2.11)-(2.13) and Lemma 1.2, it follows that

$$\lim_{n\to\infty} d(T^n_{r-(k-1)}y_{n+k-2}, W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n})) = 0 \tag{2.14}$$

for $k = 2, 3, \cdots, r-1$ and for $k = r$, we have

$$\lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d(W(T^n_1 y_{n+r-2}, W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), \alpha_{1n}), p) = c. \tag{2.15}$$

Applying (2.3), the following estimate

$$\begin{aligned}
&d(W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), p)\\
&\le \theta_{1n}d(y_{n+r-2}, p) + (1-\theta_{1n})d(S^n_1 y_{n+r-2}, p)\\
&\le \theta_{1n}d(y_{n+r-2}, p) + (1-\theta_{1n})[(1+\mu_n M^*)d(y_{n+r-2}, p) + \xi_n]\\
&\le (1+\mu_n M^*)d(y_{n+r-2}, p) + \xi_n
\end{aligned}$$

implies that

$$\limsup_{n\to\infty} d(W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), p) \le c. \tag{2.16}$$

Also,

$$\limsup_{n\to\infty} d(T^n_1 y_{n+r-2}, p) \le c. \tag{2.17}$$

Hence, (2.15)-(2.17) and Lemma 1.2 imply that

$$\lim_{n\to\infty} d(T^n_1 y_{n+r-2}, W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n})) = 0. \tag{2.18}$$

Observe that

$$\begin{aligned}
d(x_{n+1}, T^n_1 y_{n+r-2}) &= d(W(T^n_1 y_{n+r-2}, W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), \alpha_{1n}), T^n_1 y_{n+r-2})\\
&\le (1-\alpha_{1n})d(W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), T^n_1 y_{n+r-2})\\
&\quad +\alpha_{1n}d(T^n_1 y_{n+r-2}, T^n_1 y_{n+r-2}).
\end{aligned}$$

Based on (2.18), this implies

$$\lim_{n\to\infty} d(x_{n+1}, T^n_1 y_{n+r-2}) = 0. \tag{2.19}$$

Similarly, since $a \le \alpha_{in}, \beta_{in} \le b$ for all $i \in I$, we have

$$\begin{aligned} d(x_{n+1}, p) &\le \alpha_{1n} d(T_1^n y_{n+r-2}, p) + (1-\alpha_{1n}) d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \\ &\le \alpha_{1n} d(x_{n+1}, p) + \alpha_{1n} d(x_{n+1}, T_1^n y_{n+r-2}) \\ &\quad +(1-\alpha_{1n}) d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \\ &\le \frac{\alpha_{1n}}{1-\alpha_{1n}} d(x_{n+1}, T_1^n y_{n+r-2}) + d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \\ &\le \frac{b}{1-b} d(x_{n+1}, T_1^n y_{n+r-2}) + d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p). \end{aligned} \tag{2.20}$$

Taking lim inf on both side of the estimate (2.20) and using (2.19), we have

$$\liminf_{n\to\infty} d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \ge c. \tag{2.21}$$

Combining (2.16) and (2.21), we get

$$\lim_{n\to\infty} d(W(y_{n+r-2},, S_1^n y_{n+r-2}, \theta_{1n}), p) = c. \tag{2.22}$$

By Lemma 1.2 and (2.22), we have

$$\lim_{n\to\infty} d(y_{n+r-2}, S_1^n y_{n+r-2}) = 0.$$

In a similar way, for $k = 2, 3, \cdots, r-1$, we compute

$$\begin{aligned} & d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) \\ &= d(W(T_{r-(k-1)}^n y_{n+k-2}, W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), \alpha_{(r-k+1)n}), \\ & T_{r-(k-1)}^n y_{n+k-2}) \\ &\le (1-\alpha_{(r-k+1)n}) d(W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), T_{r-(k-1)}^n y_{n+k-2}) \\ &\quad +\alpha_{(r-k+1)n} d(T_{r-(k-1)}^n y_{n+k-2}, T_{r-(k-1)}^n y_{n+k-2}). \end{aligned}$$

Utilizing (2.14), we have

$$\lim_{n\to\infty} d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) = 0, \quad \forall k = 2, 3, \cdots, r-1. \tag{2.23}$$

For $k = 1$, we calculate

$$\begin{aligned} d(y_n, T_r^n x_n) &= d(W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}), T_r^n x_n) \\ &\le \alpha_{rn} d(T_r^n x_n, T_r^n x_n) + (1-\alpha_{rn}) d(W(x_n, S_r^n x_n, \theta_{rn}), T_r^n x_n). \end{aligned}$$

Now, using (2.10), we have

$$\lim_{n\to\infty} d(y_n, T_r^n x_n) = 0. \tag{2.24}$$

Reasoning as above, we get that

$$d(y_n, p) \le \frac{b}{1-b} d(T_r^n x_n, y_n) + d(W(x_n, S_r^n x_n, \theta_{rn}), p). \tag{2.25}$$

Setting lim inf on both sides of the estimate (2.25) and utilizing (2.6) and (2.24), we know

$$\liminf_{n\to\infty} d(W(x_n, S_r^n x_n, \theta_{rn}), p) \ge c. \tag{2.26}$$

Inequalities (2.8) and (2.26) collectively imply that

$$\lim_{n\to\infty} d(W(x_n, S_r^n x_n, \theta_{rn}), p) = c. \tag{2.27}$$

Consequently, Lemma 1.2 and (2.27) imply that

$$\lim_{n\to\infty} d(x_n, S_r^n x_n) = 0. \tag{2.28}$$

Note that

$$\begin{aligned} d(x_n, T_r^n x_n) &\le d(x_n, y_n) + d(y_n, T_r^n x_n) \\ &\le \alpha_{rn} d(x_n, T_r^n x_n) + (1-\alpha_{rn}) d(W(x_n, S_r^n x_n, \theta_{rn}), x_n) + d(y_n, T_r^n x_n) \\ &\le (1-\theta_{rn}) d(x_n, S_r^n x_n) + \frac{1}{1-\alpha_{rn}} d(y_n, T_r^n x_n) \\ &\le \frac{1-2a}{1-b} d(x_n, S_r^n x_n) + \frac{1}{1-b} d(y_n, T_r^n x_n). \end{aligned}$$

From (2.24) and (2.28), we have

$$\lim_{n\to\infty} d(x_n, T_r^n x_n) = 0. \tag{2.29}$$

Moreover

$$\begin{aligned} d(x_n, y_n) &\le \alpha_{rn} d(x_n, T_r^n x_n) + (1-\alpha_{rn}) d(x_n, W(x_n, S_r^n x_n, \theta_{rn})) \\ &\le \alpha_{rn} d(x_n, T_r^n x_n) + (1-\alpha_{rn}-\beta_{rn}) d(x_n, S_r^n x_n) \\ &\le b d(x_n, T_r^n x_n) + (1-2a) d(x_n, S_r^n x_n). \end{aligned}$$

By (2.28) and (2.29), we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{2.30}$$

Again, reasoning as above, we have

$$\begin{aligned} d(y_{n+k-1}, p) &\le d(W(y_{n+k-2}, S^n_{r-(k-1)} y_{n+k-2}, \theta_{(r-k+1)n}), p) \\ &\quad + \frac{b}{1-b} d(T^n_{r-(k-1)} y_{n+k-2}, y_{n+k-1}). \end{aligned}$$

Now, Utilizing (2.6) and (2.23), we get

$$\liminf_{n\to\infty} d(W(y_{n+k-2}, S^n_{r-(k-1)} y_{n+k-2}, \theta_{(r-k+1)n}), p) \ge c. \tag{2.31}$$

Thus, (2.12) and (2.31) imply in total

$$\lim_{n\to\infty} d(W(y_{n+k-2}, S^n_{r-(k-1)} y_{n+k-2}, \theta_{(r-k+1)n}), p) = c,$$

and by Lemma 1.2, we conclude that

$$\lim_{n\to\infty} d(y_{n+k-2}, S^n_{r-(k-1)} y_{n+k-2}) = 0, \quad \forall k = 2, 3, \cdots, r-1. \tag{2.32}$$

Also,

$$\begin{aligned} & d(y_{n+k-2}, T^n_{r-(k-1)} y_{n+k-2}) \\ & \le d(y_{n+k-2}, y_{n+k-1}) + d(y_{n+k-1}, T^n_{r-(k-1)} y_{n+k-2}), \\ & S^n_{r-(k-1)} y_{n+k-2}, \theta_{(r-k+1)n}), \alpha_{(r-k+1)n})) + d(y_{n+k-1}, T^n_{r-(k-1)} y_{n+k-2}) \\ & \le d(y_{n+k-1}, T^n_{r-(k-1)} y_{n+k-2}) + \alpha_{(r-k+1)n} d(y_{n+k-2}, T^n_{r-(k-1)} y_{n+k-2}) \\ & \quad + (1-\alpha_{(r-k+1)n}) d(y_{n+k-2}, W(y_{n+k-2}, S^n_{r-(k-1)} y_{n+k-2}, \theta_{(r-k+1)n})) \\ & \le d(y_{n+k-1}, T^n_{r-(k-1)} y_{n+k-2}) + \alpha_{(r-k+1)n} d(y_{n+k-2}, T^n_{r-(k-1)} y_{n+k-2}) \\ & \quad + (1-\alpha_{(r-k+1)n} - \beta_{(r-k+1)n}) d(y_{n+k-2}, S^n_{r-(k-1)} y_{n+k-2}) \\ & \le \frac{1}{1-b} d(y_{n+k-1}, T^n_{r-(k-1)} y_{n+k-2}) + \frac{1-2a}{1-b} d(y_{n+k-2}, S^n_{r-(k-1)} y_{n+k-2}). \end{aligned}$$

Now, it follows from (2.23) and (2.32) that

$$\lim_{n\to\infty} d(y_{n+k-2}, T^n_{r-(k-1)}y_{n+k-2}) = 0, \quad \forall k = 2, 3, \cdots, r-1. \tag{2.33}$$

For $k = 2, 3, \cdots, r-1$, we have

$$d(y_{n+k-2}, y_{n+k-1}) \le d(y_{n+k-2}, T^n_{r-(k-1)}y_{n+k-2}) + d(T^n_{r-(k-1)}y_{n+k-2}, y_{n+k-1}).$$

Hence, (2.23) and (2.33) imply that

$$\lim_{n\to\infty} d(y_{n+k-2}, y_{n+k-1}) = 0. \tag{2.34}$$

Additionally,

$$d(x_n, y_{n+k-1}) \le d(x_n, y_n) + d(y_n, y_{n+1}) + \cdots + d(y_{n+k-2}, y_{n+k-1}).$$

By (2.30) and (2.34), we have

$$\lim_{n\to\infty} d(x_n, y_{n+k-1}) = 0, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.35}$$

Let $L = \max_{i\in I}\{L_i, \hat{L}_i\}$, where L_i and $\hat{L}_i$ are Lipschitz constants for T_i and S_i for $i \in I$, respectively. Since each T_i is uniformly L-Lipschitzian for $i \in I$, we have

$$\begin{aligned} d(x_n, T^n_i x_n) &\le d(x_n, y_{n+r-i-1}) + d(y_{n+r-i-1}, T^n_i x_n) \\ &\le d(x_n, y_{n+r-i-1}) + d(y_{n+r-i-1}, T^n_i y_{n+r-i-1}) + d(T^n_i y_{n+r-i-1}, T^n_i x_n) \\ &\le (1+L)d(x_n, y_{n+r-i-1}) + d(y_{n+r-i-1}, T^n_i y_{n+r-i-1}) \end{aligned}$$

for $1 \le i \le r-1$.

It follows from (2.33) and (2.35) that

$$\lim_{n\to\infty} d(x_n, T^n_i x_n) = 0, \quad \forall 1 \le i \le r-1. \tag{2.36}$$

Moreover,

$$\begin{aligned} d(x_n, T_i x_n) &\le d(x_n, T^n_i x_n) + d(T^n_i x_n, T^n_i y_{n+r-i-1}) + d(T^n_i y_{n+r-i-1}, T_i x_n) \\ &\le d(x_n, T^n_i x_n) + Ld(x_n, y_{n+r-i-1}) + Ld(T^{n-1}_i y_{n+r-i-1}, x_n) \\ &\le d(x_n, T^n_i x_n) + 2Ld(x_n, y_{n+r-i-1}) + Ld(T^{n-1}_i y_{n+r-i-1}, y_{n+r-i-1}). \end{aligned}$$

Thus, (2.33), (2.35) and (2.36) (or (2.29)) imply that $d(x_n, T_i x_n) \to 0$ as $n \to \infty$ and so

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad \forall 1 \le i \le r.$$

Similarly, we have

$$\lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad \forall 1 \le i \le r.$$

This completes the proof. □

The following results can be obtained from Theorem 2.2 immediately. The proof is similar to Corollaries 2.1 and 2.2, respectively, and so they are omitted.

Corollary 2.3 Assume that K and F are the same as in Theorem 2.2. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{\xi^i_n\}$-asymptotically nonexpansive mapping in the intermediate sense and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{\xi}^i_n\}$-asymptotically nonexpansive mapping in the intermediate sense. If $\sum_{n=1}^{\infty} \xi^i_n < +\infty$ and $\sum_{n=1}^{\infty} \hat{\xi}^i_n < +\infty$ for $i \in I$, then, for the sequence $\{x_n\}$ in (1.1),

$$\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad \forall i \in I.$$

Corollary 2.4 Suppose that K and F are the same as in Theorem 2.2. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{k_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} k_n^i < +\infty$, and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{k}_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \hat{k}_n^i < +\infty$. Then,

$$\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad i \in I,$$

where $\{x_n\}$ is the sequence defined by (1.1).

Remark 2.1 (1) It is worth mentioning that Theorems 2.1-2.2 can easily be extended to a more general class of total asymptotically quasi-nonexpansive mappings for the iteration process (1.1). And the proofs of Theorems 2.1-2.2 are greatly differ from those of Lemmas 2.1 and 2.2 in [21]. Further, Corollaries 2.1 and 2.3 (Corollaries 2.2 and 2.4, respectively) are so.

(2) Moreover, conclusion of the Theorem 2.2 (Corollaries 2.3 and 2.4, respectively) can be extended to a more general class of weakly total-asymptotically quasi-nonexpansive mappings (weakly asymptotically quasi-nonexpansive mappings asymptoticallyin in the intermediate sense and weakly quasi-nonexpansive mappings). For concepts of the weakly properly, see, for example, Fukhar-ud-din and Khan [21].

3 Approximation of common fixed points

In this section, we approximate common fixed points of two finite families of total asymptotically nonexpansive mappings in a hyperbolic space. More briefly, we establish Δ-convergence and strong convergence of the iteration process (1.1) for two finite families of total asymptotically nonexpansive mappings in a hyperbolic space.

Theorem 3.1 Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i \in I$, let $T_i : K \to K$, $i \in I = \{1, 2, 3, \cdots, r\}$ be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \mu_n^i = 0$ and $\lim_{n\to\infty} \xi_n^i = 0$, and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ satisfying $\rho^i(0) = 0$, and let $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $(\{\hat{\mu}_n^i\}, \{\hat{\xi}_n^i\}, \hat{\rho}^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \hat{\mu}_n^i = 0$ and $\lim_{n\to\infty} \hat{\xi}_n^i = 0$, and with a strictly increasing continuous function $\hat{\rho}^i : [0, +\infty) \to [0, +\infty)$ satisfying $\hat{\rho}^i(0) = 0$. Assume that $F = \bigcap_{i=1}^{r} (F(T_i) \cap F(S_i)) \neq \emptyset$ and for $i \in I$, the following conditions hold:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\mu}_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\xi}_n^i < +\infty$.

(ii)There exists a constant $M^* > 0$ such that $\rho^i(r) \leq M^* r$ and $\hat{\rho}^i(r) \leq M^* r$ for all $r > 0$.

Then the sequence $\{x_n\}$ defined in (1.1) Δ-converges to a common fixed point $p \in F$.

Proof. Since the sequence $\{x_n\}$ is bounded (by Theorem 2.1), therefore Lemma 1.1 asserts that $\{x_n\}$ has a unique asymptotic center in K. That is, $A(\{x_n\}) = \{x\}$. Let $\{v_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. Then, by Theorem 2.2, we have

$$\lim_{n\to\infty} d(v_n, T_i v_n) = \lim_{n\to\infty} d(v_n, S_i v_n) = 0, \quad \forall i \in I. \tag{3.1}$$

We claim that v is the common fixed point of $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$.

For each $i \in I$, define a sequence $\{z_m\}$ in K by $z_m = T_i^m v$. Then, we calculate

$$\begin{aligned} d(z_m, v_n) \quad & \leq d(T_i^m v, T_i^m v_n) + d(T_i^m v_n, T_i^{m-1} v_n) + \cdots + d(T_i v_n, v_n) \\ & \leq [d(v, v_n) + \mu_m^i \rho^i(d(v, v_n)) + \xi_m^i] + \sum_{j=0}^{m-1} d(T_i^{j+1} v_n, T_i^j v_n). \end{aligned}$$

Since each T_i is uniformly L_i-Lipschitzian with the Lipschitz constant L_i for $i \in I$, the above estimate yields

$$d(z_m, v_n) \leq [(1 + \mu_m M^*) d(v, v_n) + \xi_m] + mL d(T_i v_n, v_n), \tag{3.2}$$

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where $L = \max_{i\in I}\{L_i, \hat{L}_i\}$.

Taking lim sup on both sides of (3.2) and using (3.1), we have

$$r(z_m, \{v_n\}) = \limsup_{n\to\infty} d(z_m, v_n) \le \limsup_{n\to\infty} d(v, v_n) = r(v, \{v_n\}),$$

which implies that $|r(z_m, \{v_n\}) - r(v, \{v_n\})| \to 0$ as $m \to \infty$. It follows Lemma 1.3 that $\lim_{m\to\infty} T_i^m v = v$. by the uniform continuity of T_i, we know that

$$T_i(v) = T(\lim_{m\to\infty} T_i^m v) = \lim m \to \infty T_i^{m+1} v = v.$$

From the arbitrariness of $i \in I$, we conclude that v is the common fixed point of $\{T_i\}_{i=1}^r$. Similarly, we can show that v is the common fixed point of $\{S_i\}_{i=1}^r$. Hence, $v \in F$.

Next, we claim that the common fixed point v is the unique asymptotic center for each subsequence $\{v_n\}$ of $\{x_n\}$.

Contrarily, $v \neq x$. It follows Theorem 2.1 that $\lim_{n\to\infty} d(x_n, v)$ exists, and by the uniqueness of asymptotic centers, we have

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) \quad &< \limsup_{n\to\infty} d(v_n, x) \le \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(v_n, v),\end{aligned}$$

a contradiction. Therefore $v = x$. Since $\{v_n\}$ is an arbitrary subsequence of $\{x_n\}$, $A(\{v_n\}) = \{x\}$ for all subsequence $\{v_n\}$ of $\{x_n\}$, this proves that $\{x_n\}$ Δ-converges to a common fixed point x of $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$. □

From Theorem 3.1, we have the following result.

Corollary 3.1 Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{\xi_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{\xi}_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense. If for all $i \in I$, $\sum_{n=1}^\infty \xi_n^i < +\infty$ and $\sum_{n=1}^\infty \hat{\xi}_n^i < +\infty$, and $F = \bigcap_{i=1}^r (F(T_i) \cap F(S_i)) \neq \emptyset$, then the sequence $\{x_n\}$ defined in (1.1) Δ-converges to a common fixed point $p \in F$.

Corollary 3.2 Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{k_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^\infty k_n^i < +\infty$, and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{k}_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^\infty \hat{k}_n^i < +\infty$. Assume that $F = \bigcap_{i=1}^r (F(T_i) \cap F(S_i)) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (1.1) Δ-converges to a common fixed point $p \in F$.

Proof. Based on Corollaries 2.2 and 2.4, and the proof of Theorem 3.1 in [21], the result holds. □

In order to prove strong convergence of the iteration (1.1) for two finite families of total asymptotically nonexpansive mappings in a hyperbolic space, we first give the following conditions:

(**H**) There exists a nondecreasing self-mapping on $[0, +\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, +\infty)$ such that $d(x, Tx) \ge f(d(x, F(T)))$ for all $x \in K$, where $T : K \to K$ is a nonlinear mapping with $F(T) \neq \emptyset$ and $d(x, F(T)) = \inf\{d(x, y) : y \in F(T)\}$.

The condition (**H**) was introduced by Senter and Dotson [35]. Further, based on works of [21, 36, 37], for two finite families of total asymptotically nonexpansive mappings $\{T_i, i \in I\}_{i=1}^r$ and $\{S_i, i \in I\}_{i=1}^r$ on $K \subset \mathcal{H}$ with $F = \bigcap_{i=1}^n (F(T_i) \cap F(S_i)) \neq \emptyset$, condition (**H**) becomes as follows:

(**A**) $d(x, Tx) \ge f(d(x, F))$ or $d(x, Sx) \ge f(d(x, F))$ holds for $x \in K$ and for at least one $T \in \{T_i\}_{i=1}^r$ or $S \in \{S_i\}_{i=1}^r$, where $d(x, F) = \inf\{d(x, y) : y \in F\}$.

(**B**) $d(x, T_i x) + d(x, S_i x) \ge f(d(x, F))$ for $x \in K$ and $i \in I$.

(**C₁**) $\frac{1}{2r}\left(\sum_{i=1}^r d(x, T_i x) + \sum_{i=1}^r d(x, S_i x)\right) \ge f(d(x, F))$ for $x \in K$.

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$(\mathbf{C_2})$ $\frac{1}{2}\big(\max_{1\le i\le r} d(x,T_ix)+\max_{1\le i\le r} d(x,S_ix)\big)\ge f(d(x,F))$ for $x\in K$.

$(\mathbf{C_3})$ $\max\big\{\max_{1\le i\le r} d(x,T_ix),\max_{1\le i\le r} d(x,S_ix)\big\}\ge f(d(x,F))$ for $x\in K$.

Note that the conditions $(\mathbf{A})$, $(\mathbf{B})$ and $(\mathbf{C_1})$-$(\mathbf{C_3})$ are equivalent to the condition $(\mathbf{H})$, if $T_i=S_i$ for $i\in I$. We shall use condition $(\mathbf{C_1})$ or $(\mathbf{C_2})$ or $(\mathbf{C_3})$ to study strong convergence of the iteration (1.1).

Now we give the following lemma for proving the strong convergence.

Lemma 3.1 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and $\{x_n\}$ be as in Theorem 3.1. Then $\{x_n\}$ converges strongly to some $p\in F$ if and only if

$$\liminf_{n\to\infty} d(x_n,F)=0.$$

Proof. If $\{x_n\}$ converges strongly to $p\in F$, then $\lim_{n\to\infty} d(x_n,p)=0$. Since $0\le d(x_n,F)\le d(x_n,p)$, we have $\liminf_{n\to\infty} d(x_n,F)=0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F)=0$. It follows from Theorem 2.1 that $\lim_{n\to\infty} d(x_n,F)$ exists. Now $\liminf_{n\to\infty} d(x_n,F)=0$ reveals that $\lim_{n\to\infty} d(x_n,F)=0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. By last inequalities in the proof of Theorem 2.1

$$d(x_{n+1},p)\le(1+M_1\mu_n)d(x_n,p)+M_2\xi_n,$$

taking infimum on $p\in F$ on both sides in the above inequality, we have

$$d(x_{n+1},F)\le(1+M_1\mu_n)d(x_n,F)+M_2\xi_n.$$

On account of $\sum_{n=1}^\infty\mu_n<\infty$, $\sum_{n=1}^\infty\xi_n<\infty$, set $e^{M_1\sum_{n=1}^\infty\mu_n}=M$. Let $\forall\varepsilon>0$. Since $\lim_{n\to\infty} d(x_n,F)=0$, for any given $\varepsilon>0$, there exists a positive integer n_0 such that

$$d(x_{n_0},F)<\frac{\varepsilon}{4(M+1)}\quad\text{and}\quad\sum_{n=n_0}^\infty\xi_n<\frac{\varepsilon}{2MM_2}\tag{3.3}$$

The first inequality in (3.3) implies that there exists $p_0\in F$ such that $d(x_{n_0},p_0)<\frac{\varepsilon}{2(M+1)}$. Hence, for any $n\ge n_0$ and $m\ge1$, we have

$$\begin{aligned}
d(x_{n_0+m},x_{n_0}) &\le d(x_{n_0+m},p_0)+d(x_{n_0},p_0)\\
&\le[e^{M_1\sum_{k=n_0}^{n_0+m-1}\mu_k}+1]d(x_{n_0},p_0)+M_2[\xi_{n_0+m-1}\\
&\quad+\xi_{n_0+m-2}e^{M_1\mu_{n_0+m-1}}+\xi_{n_0+m-3}e^{M_1\sum_{k=n_0+m-2}^{n_0+m-1}\mu_k}\\
&\quad+\cdots+\xi_{n_0}e^{M_1\sum_{k=n_0+1}^{n_0+m-1}\mu_k}]\\
&\le(M+1)d(x_{n_0},p_0)+MM_2\sum_{n=n_0}^\infty\xi_n\\
&<(M+1)\frac{\varepsilon}{2(M+1)}+MM_2\frac{\varepsilon}{2MM_2}=\varepsilon.
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}$. Sine K is a closed subset of a complete hyperbolic space $\mathcal{H}$, it is complete. We can assume that $\lim_{n\to\infty}x_n=q$, and $q\in K$. It is easy to see that $F(T)$ is a close subset in K, so is $F(T)$. Since $\lim_{n\to\infty} d(x_n,F)=0$, we obtain $q\in F(T)$. This completes the proof. □

We now establish strong convergence of the iteration process (1.1) based on Theorem 2.2.

Theorem 3.2 Suppose that K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Theorem 3.1, and $\{T_i\}_{i=1}^r$, and $\{S_i\}_{i=1}^r$, satisfies condition $(\mathbf{C_1})$ (or $(\mathbf{C_2})$, or $(\mathbf{C_3})$). Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to some $p\in F$.

Proof. It follows from Theorem 2.1 that $\lim_{n\to\infty} d(x_n,F)$ exists. Moreover, Theorem 2.2 implies that $\lim_{n\to\infty} d(x_n,T_ix_n)=\lim_{n\to\infty} d(x_n,S_ix_n)=0$ for each $i\in I$. Thus, the condition $(\mathbf{C_1})$ (or $(\mathbf{C_2})$, or $(\mathbf{C_3})$) guarantees that $\lim_{n\to\infty} f(d(x_n,F))=0$. Since f is nondecreasing with $f(0)=0$,

it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Then, Lemma 3.1 implies that $\{x_n\}$ converges strongly to a common fixed point $p \in F$. $\square$

From Theorem 3.2, we have the following results.

Corollary 3.3 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Corollary 3.1. Suppose that $\{T_i\}_{i=1}^r$, and $\{S_i\}_{i=1}^r$, satisfies condition $(\mathbf{C_1})$ (or $(\mathbf{C_2})$, or $(\mathbf{C_3})$). Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to some $p \in F$.

Corollary 3.4 Assume that K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F are the same as in Corollary 3.2, and $\{T_i\}_{i=1}^r$, and $\{S_i\}_{i=1}^r$, satisfies condition $(\mathbf{C_1})$ (or $(\mathbf{C_2})$, or $(\mathbf{C_3})$). Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to some $p \in F$.

Theorem 3.3 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Theorem 3.1. Suppose that either $T_l \in \{T_i\}_{i=1}^r$ or $S_l \in \{S_i\}_{i=1}^r$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to $p \in F$.

Proof. Let $T_l \in \{T_i\}_{i=1}^r$ is semi-compact. By Theorem 2.2, we know that $\lim_{n\to\infty} d(T_i x_n, x_n) = 0$ for all $i \in I$. By Theorem 2.1, $\{x_n\}$ is bounded and T_l is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q$ as $j \to \infty$. By continuity of T_i and Theorem 2.2, we obtain

$$d(q, T_i q) = \lim_{j\to\infty} d(x_{n_j}, T_i x_{n_j}) = 0, \quad i \in I.$$

This implies that q is the common fixed point of $\{T_i\}_{i=1}^r$. Similarly, we can show that q is the common fixed point of $\{S_i\}_{i=1}^r$. Hence, $q \in F$. Again, by Theorem 2.1, $\lim_{n\to\infty} d(x_n, q)$ exists. Therefore, q is the strong limit of the sequence $\{x_n\}$. As a result, $\{x_n\}$ converges strongly to a point q. $\square$

From Theorem 3.3, we have the following results.

Corollary 3.5 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Corollary 3.1. Suppose that either $T_l \in \{T_i\}_{i=1}^r$ or $S_l \in \{S_i\}_{i=1}^r$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to $p \in F$.

Corollary 3.6 Suppose that K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and $\{x_n\}$ be the same as in Corollary 3.2, and either $T_l \in \{T_i\}_{i=1}^r$ or $S_l \in \{S_i\}_{i=1}^r$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to $p \in F$.

Remark 3.1 (1) If the uniformly convex hyperbolic spaces with modulus of uniform convexity reduce to CAT(0) spaces, and iterative process (1.1) reduce to iterative process (1.3), Theorem 3.1, Lemma 3.1, Theorem 3.2 reduce to Theorems 3.1-3.3 proved by Thakur et al. [7], respectively.

(2) If $r = 3$ and $\alpha_{in} = 0$ and $S_1 = S_2 = \cdots = S_r = T$, Theorem 3.1, Lemma 3.1, Theorem 3.2 and Theorem 3.3 become to Theorems 1-4 in [5], respectively.

(3) If the uniformly convex hyperbolic spaces with modulus of uniform convexity reduce to CAT(0) spaces, and $r = 3$ and $\alpha_{in} = 0$ and $S_1^n = S_2^n = \cdots = S_r^n = T$, where T is a nonexpansive mappings on $K \subset \mathcal{H}$, Theorem 3.1, Lemma 3.1, Theorem 3.2 are equivalent to Theorems 1-3 of [6], respectively.

4 Concluding remarks

In this paper, we introduced and studied the following new general iteration for two finite families of total asymptotically nonexpansive mappings in hyperbolic spaces $\mathcal{H}$:

$$\begin{aligned}
x_{n+1} &= W(T_1^n y_{n+r-2}, W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), \alpha_{1n}),\\
y_{n+r-2} &= W(T_2^n y_{n+r-3}, W(y_{n+r-3}, S_2^n y_{n+r-3}, \theta_{2n}), \alpha_{2n}),\\
y_{n+r-3} &= W(T_3^n y_{n+r-4}, W(y_{n+r-4}, S_3^n y_{n+r-4}, \theta_{3n}), \alpha_{3n}),\\
&\ \vdots\\
y_{n+1} &= W(T_{r-1}^n y_n, W(y_n, S_{r-1}^n y_n, \theta_{(r-1)n}), \alpha_{(r-1)n}),\\
y_n &= W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}),
\end{aligned} \tag{4.1}$$

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where $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$ be two finite families of total asymptotically nonexpansive mappings on nonempty closed and convex subset $K \subset \mathcal{H}$, $\{\alpha_{in}\}$ and $\{\beta_{in}\}$ are two double real sequences in $[0,1]$, and for each $i \in I = \{1, 2, \cdots, r\}$, $r \geq 2$ and $n \geq 1$, $\theta_{in} := \frac{\beta_{in}}{1-\alpha_{in}}$.

In order to prove Δ-convergence and strong convergence of the iteration (4.1) in hyperbolic spaces, we gave and analyzed some important related properties to the new general iterative processes (4.1), and proposed some meaningful illustrations for clarifying the results presented in this paper, which show that our results extend and improve the corresponding results of iterative approximation for asymptotically (quasi-)nonexpansive mapping, (generalized) (quasi-)nonexpansive mapping of all normed linear spaces, Hadamard manifolds and CAT(0) spaces as special cases. Our results extended and improved the corresponding results of [1, 2, 4–7, 9, 21].

It is well known that iterative processes as ubiquitous in the area of abstract nonlinear analysis and still remain as a main tool for approximation of fixed points of generalizations of nonexpansive maps. Furthermore, the analysis of general iterative processes, in a more general setup, is a problem of interest in theoretical numerical analysis. Therefore, on two finite families of total asymptotically nonexpansive mappings in the setting of the general iteration (4.1), the following two **open questions** will be worth further studying:.

(1) If some errors are added in the iteration (4.1), such as the iterative approximating scheme (3.1) in [11], can the Δ-convergence and strong convergence presented in this paper be proved?

(2) When T_i and S_i $(i \in I)$ in (4.1) become total asymptotically quasi-nonexpansive mappings, whether do the results of Theorems 3.1-3.3 hold?

Acknowledgements

This work was partially supported by the Scientific Research Project of Sichuan University of Science & Engineering (2015RC07) and co-financed by the Scientific Research Fund of Sichuan Provincial Education Department (16ZA0256).

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On Simpson's type inequalities utilizing fractional integrals

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October 10, 2016

Abstract

In the present article, we establish an integral identity for Riemann-Liouville fractional integrals. Some Simpson type integral inequalities utilizing this integral identity are obtained. It is worth mentioning that the presented results have close connection with those in [M. Z Sarikaya, E. Set, M. E Ozdemir, On new inequalities of Simpson's type for s-convex functions, Computers and Mathematics with Applications, 60 (2010), 2191–2199)].

Subject class: [2000] 26A15, 26A51, 26D10.
keywords: Simpson's Inequality, Convex Functions, Power-mean Inequality, Riemann-Liouville Fractional Integral.

1. Introduction

The following definition for convex functions is well known in the mathematical literature:

A function $f : \Phi \neq I \subseteq R \to R.$ is said to be convex on I, if inequality

$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right)$, for all $x, y \in I, t \in [0,1]$

Many inequalities have been established for convex functions but the most famous is the Simpson's inequality, due to its rich geometrical significance and applications, which is stated as [9]:

Theorem 1 *Let $f : [a,b] \to R$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$, then we have the following inequality:*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4 . \tag{1}$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see [[9]-[11]].

In [10], Dragomir et. al proved the following recent developments on Simpson's inequality for which the remainder is expressed in terms of derivatives lower than the fourth.

Theorem 2 *Let $f : [a,b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f' \in L[a,b]$. Then we have the following inequality:*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{3}\left\|f'\right\|_1 , \tag{2}$$

where $\left\|f'\right\|_1 = \int_a^b |f'(x)\,dx|$.

The bound of (2) for L-Lipschitzian mapping was given in [8] by $\frac{5}{36}(b-a)$.

In [8], Sarikaya et. al presented inequalities for differentiable convex functions which are linked with Simpson's inequality, and the main inequality in [8], pointed out, is as follows.

Theorem 3 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 (interior of I) such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|$ is $s-$convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\quad\le (b-a)\frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+1}(s+1)(s+2)}\left(\,|f'(a)|+|f'(b)|\,\right).\end{aligned} \tag{3}$$

Proposition 1 *Under the assumptions of Theorem 3 with $s = 1$, we have the following inequality,*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{5(b-a)}{72}\left(\,|f'(a)|+|f'(b)|\,\right). \tag{4}$$

Proposition 2 *Under the assumptions of Theorem 3 with $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, we have the following inequality,*

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$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{5\,(b-a)}{72}(\ |f'(a)|+|f'(b)|\ ). \tag{5}$$

Theorem 4 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|^q$ is $s-$convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q}\times$$
$$\left\{\left(\left[\frac{2\times 5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(b)|^q+\left[\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(a)|^q\right)^{1/q}\right.$$
$$\left.+\left(\left[\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(b)|^q+\left[\frac{2\times 5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(a)|^q\right)^{1/q}\right\}.$$

Proposition 3 *Under the assumptions of Theorem 4 with $s = 1$, we have the following inequality,*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q}\times$$
$$\left\{\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}+\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}\right\}.$$

Proposition 4 *Under the assumptions of Theorem 4 with $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, we have the following inequality,*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{72}(5)^{1-1/q}\times$$
$$\left\{\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}+\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}\right\}$$

Definition 1 *Let $f \in L^1[a,b]$. The left$-$sided and right$-$sided Riemann$-$Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad a < x$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x < b$$

respectively, where $\Gamma(.)$ is Gamma function and its definition is $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$. It is to be noted that $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Properties relating to this operator can be found in [5] and for useful details on Simpson's type inequalities connected with fractional integral inequalities, the interested readers are directed to [1]

The main aim of this paper is to establish new Simpson's type inequalities for Riemann−Liouville fractional integral using the convexity as well as concavity, for the class of functions whose derivatives in absolute value at certain powers are convex functions. we will derive a general integral identity for convex functions.

2. Main Results

In order to prove our main results we need the following integral identity:

Lemma 1 *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following identity for Riemann−Liouville fractional integrals holds:*

$$\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right]$$
$$= \frac{b-a}{2^{\alpha+1}}\left[I_1 + I_2 + (2^{\alpha}-1)(I_3+I_4)\right],$$

where

$I_1 = \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) f'(tb + (1-t)\frac{a+b}{2})dt,$
$I_2 = \int_0^1 \left(\frac{1}{2}(1-t)^{\alpha} - \frac{1}{6}\right) f'(ta + (1-t)\frac{a+b}{2})dt,$
$I_3 = \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right) f'(tb + (1-t)\frac{a+b}{2})dt,$
$I_4 = \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)} - \frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} + \frac{1}{3}\right) f'(ta + (1-t)\frac{a+b}{2})dt.$

Proof. Integrating by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) f'(tb + (1-t)\frac{a+b}{2})dt \\
&= \left.\frac{2\left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) f'(tb + (1-t)\frac{a+b}{2})dt}{b-a}\right|_0^1 \\
&\qquad - \frac{2\alpha}{b-a}\int_0^1 (1-t)^{\alpha+1} f(tb + (1-t)\frac{a+b}{2})dt \\
&= \frac{2}{b-a}\left[\frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2\alpha}{b-a}\int_0^1 (1-t)^{\alpha+1} f(tb + (1-t)\frac{a+b}{2})dt \\
&= \frac{2}{b-a}\left[\frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^{\alpha}\alpha}{(b-a)^{\alpha}}J_3.
\end{aligned}$$

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$$
\begin{aligned}
I_3 &= \int_0^1 \left(\frac{1}{2\left(2^\alpha-1\right)}\left(1+t\right)^\alpha - \frac{1}{2\left(2^\alpha-1\right)} - \frac{1}{3}\right) f'(tb+(1-t)\frac{a+b}{2})dt \\
&= \left.\frac{2\left[\frac{1}{2(2^\alpha-1)}\left(1+t\right)^\alpha - \frac{1}{2(2^\alpha-1)} - \frac{1}{3}\right] f(tb+(1-t)\frac{a+b}{2})dt)}{b-a}\right|_0^1 \\
&\qquad - \frac{2\alpha}{(b-a)\left(2^\alpha-1\right)}\int_0^1 (1+t)^{\alpha+1} f(tb+(1-t)\frac{a+b}{2})dt) \\
\left(2^\alpha-1\right) I_3 &= \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right] + \frac{2\left(\alpha+1\right)}{b-a}\int_0^1 (1+t)^{\alpha+1} f(tb+(1-t)\frac{a+b}{2})dt) \\
&= \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^\alpha\alpha}{(b-a)^{\alpha+1}}J_2.
\end{aligned}
$$

Analogously:
$I_2 = \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^\alpha\alpha}{(b-a)^\alpha}J_1.$
$\left(2^\alpha-1\right) I_4 = \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^\alpha\alpha}{(b-a)^{\alpha+1}}J_4.$
Adding above equalities, we get

$$
\begin{aligned}
\frac{2}{b-a}\left[\frac{1}{6}f\left(a\right)+\frac{1}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f\left(b\right)\right] - \frac{\alpha}{2\left(b-a\right)^\alpha}\left[J_1+J_2+J_3+J_4\right] \\
= I_1+I_2+\left(2^\alpha-1\right)\left(I_3+I_4\right).
\end{aligned}
$$

Now making suitable substitutions, we have
$J_1 = \int_0^1 (1-t)^{\alpha+1} f'(ta+(1-t)\frac{a+b}{2})dt = \frac{2^\alpha}{(b-a)^\alpha}\int_a^{a+b/2} (u-a)^{\alpha-1} f(u)du$
$J_2 = \int_0^1 (1+t)^{\alpha+1} f'(tb+(1-t)\frac{a+b}{2})dt = \frac{2^\alpha}{(b-a)^\alpha}\int_{a+b/2}^{b} (u-a)^{\alpha-1} f(u)du$
$J_1+J_2 = \frac{2^\alpha}{(b-a)^\alpha}\int_a^b (u-a)^{\alpha-1} f(u)du = \frac{2^\alpha\Gamma(\alpha)}{(b-a)^\alpha}J_{b^-}^\alpha f(a),$
likewise
$J_3 = \int_0^1 (1-t)^{\alpha+1} f'(tb+(1-t)\frac{a+b}{2})dt = \frac{2^\alpha}{(b-a)^\alpha}\int_{a+b/2}^{b} (b-u)^{\alpha-1} f(u)du$
$J_4 = \int_0^1 (1+t)^{\alpha+1} f'(ta+(1-t)\frac{a+b}{2})dt = \frac{2^\alpha}{(b-a)^\alpha}\int_a^{a+b/2} (b-u)^{\alpha-1} f(u)du$
$J_3+J_4 = \frac{2^\alpha}{(b-a)^\alpha}\int_a^b (b-u)^{\alpha-1} f(u)du = \frac{2^\alpha\Gamma(\alpha)}{(b-a)^\alpha}J_{a^+}^\alpha f(b),$
which completes our proof. ◻

Theorem 5 *Let f and f' be defined as in Theorem 4 and if $|f'|$ is convex on $[a,b]$, then the following identity for Riemann−Liouville fractional integrals holds:*

$$
\begin{aligned}
\left[\frac{1}{6}f\left(a\right)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f\left(b\right)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right] \\
\leq \frac{(b-a)}{2^\alpha}(\psi_1+\psi_2)(\ |f'(a)|+|f'(b)|\ ). \qquad (6)
\end{aligned}
$$

where $\psi_1 = K_1+K_2, \quad \psi_2 = K_3+K_4$

Proof. By using the properties of modulus on Lemma 1, we have

$$
\begin{aligned}
&\left|\left[\frac{1}{6}f\left(a\right)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f\left(b\right)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\right| \leq \frac{b-a}{2^{\alpha+1}}\times \\
&\left[\frac{2c-\alpha+2}{6\left(\alpha+1\right)}+\left\{\left(\frac{2^\alpha-1}{3}+\frac{1}{2}\right)(2d-3)-\frac{1}{\alpha+1}\left(\frac{5d}{3}-\frac{2^{\alpha+1}+1}{2}\right)\right\}\right](\ |f'(a)|+|f'(b)|,
\end{aligned}
$$

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where $c=\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}$ and $d^{\alpha}=\frac{2(2^{\alpha}-1)}{3}+1$.

Using convexity of $|f'|$, we have

$$\begin{aligned}
|I_1| &\leq \int_0^1\left(\frac{1}{6}-\frac{1}{2}(1-t)^{\alpha}\right)|f'(tb+(1-t)\frac{a+b}{2})|dt \\
&= \int_0^1\left(\frac{1}{6}-\frac{1}{2}(1-t)^{\alpha}\right)|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)|dt \\
&\leq \int_0^1\left(\frac{1}{6}-\frac{1}{2}(1-t)^{\alpha}\right)\left\{\left(\frac{1+t}{2}\right)\left|f'(b)\right|+\left(\frac{1-t}{2}\right)\left|f'(a)\right|\right\}dt \\
&= \frac{K_1}{2}\left|f'(b)\right|+\frac{K_2}{2}\left|f'(a)\right|.
\end{aligned}$$

Analogously:

$$|I_2|\leq \frac{K_1}{2}\left|f'(a)\right|+\frac{K_2}{2}\left|f'(b)\right|.$$

Using the convexity on $|f'|$ and the fact that for $\alpha\in(0,1]$ and $\forall\ t\in[0,1]$,

$$\begin{aligned}
|I_3| &\leq \int_0^1\left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha}-\frac{1}{2(2^{\alpha}-1)}-\frac{1}{3}\right)|f'(ta+(1-t)\frac{a+b}{2})|dt \\
&= \int_0^1\left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha}-\frac{1}{2(2^{\alpha}-1)}-\frac{1}{3}\right)|f'(\frac{1+t}{2}a+\frac{1-t}{2}b)|dt \\
&\leq \int_0^1\left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha}-\frac{1}{2(2^{\alpha}-1)}-\frac{1}{3}\right)\left\{\left(\frac{1+t}{2}\right)\left|f'(a)\right|+\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right\}dt \\
&= \frac{K_3}{2}\left|f'(a)\right|+\frac{K_4}{2}\left|f'(b)\right|.
\end{aligned}$$

Similarly

$$|I_4|\leq \frac{K_3}{2}\left|f'(b)\right|+\frac{K_4}{2}\left|f'(a)\right|.$$

To get desired result, adding above inequalities and it is very easy to check

$$\begin{aligned}
K_1 &= \int_0^{1-c}\left(\frac{1}{2}(1-t)^{\alpha}-\frac{1}{6}\right)dt=-\frac{1}{6}(1-c)-\frac{1}{2(\alpha+1)}c^{\alpha+1}+\frac{1}{2(\alpha+1)}, \\
K_2 &= \int_{1-c}^1\left(\frac{1}{6}-\frac{1}{2}(1-t)^{\alpha}\right)dt=\frac{1}{6}-\frac{1}{6}(1-c)-\frac{1}{2(\alpha+1)}c^{\alpha+1},
\end{aligned}$$

$$\begin{aligned}
K_3 &= \int_0^{d-1}\left(\frac{1}{2(2^{\alpha}-1)}-\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha}+\frac{1}{3}\right)dt \\
&= \left[\frac{1}{3}+\frac{1}{2(2^{\alpha}-1)}\right](d-1)-\frac{d^{\alpha+1}}{2(2^{\alpha}-1)(\alpha+1)}+\frac{1}{2(2^{\alpha}-1)(\alpha+1)}, \\
K_4 &= \int_{d-1}^1\left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha}-\frac{1}{2(2^{\alpha}-1)}-\frac{1}{3}\right)dt \\
&= \frac{2^{\alpha+1}}{2(2^{\alpha}-1)(\alpha+1)}-\left[\frac{1}{3}+\frac{1}{2(2^{\alpha}-1)}\right]-\frac{d^{\alpha+1}}{2(2^{\alpha}-1)(\alpha+1)}+\left[\frac{1}{3}+\frac{1}{2(2^{\alpha}-1)}\right](d-1).
\end{aligned}$$

This completes the proof. ◻

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Remark 1 *If we take* $\alpha = 1$ *in Theorem 5 then inequality* (6) *reduces to inequality* (4).

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

Theorem 6 *Let f and f' be defined as in Theorem 4 and if $|f'|^q$ is a convex on $[a,b]$, with $q \geq 1$, then the following inequality holds*:

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right]\right| \leq \frac{(b-a)}{2^{\alpha}}$$

$$\left[\psi_1{}^{1-1/q}\left\{\left(\frac{K_5|f'(a)|^q + K_6|f'(b)|^q}{2}\right)^{1/q} + \left(\frac{K_5|f'(a)|^q + K_6|f'(b)|^q}{2}\right)^{1/q}\right\} + \right.$$

$$\left.\psi_2{}^{1-1/q}\left\{\left(\frac{K_7|f'(a)|^q + K_8|f'(b)|^q}{2}\right)^{1/q} + \left(\frac{K_7|f'(a)|^q + K_8|f'(b)|^q}{2}\right)^{1/q}\right\}\right]. \quad (7)$$

Proof. Using the well-known power-mean integral inequality for $q > 1$, we have

$$|I_1| \leq \left(\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right)\right| dt\right)^{1-1/q} \left(\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right)\right| \left|f'\left(ta + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}$$

Using the convexity of $|f'|^q$, we have

$$|I_1| \leq \psi_1{}^{1-1/q}\left(K_5\frac{|f'(a)|^q}{2} + K_6\frac{|f'(b)|^q}{2}\right)^{1/q}.$$

Analogously:

$$|I_2| \leq \psi_1{}^{1-1/q}\left(K_5\frac{|f'(b)|^q}{2} + K_6\frac{|f'(a)|^q}{2}\right)^{1/q}.$$

$|I_2| \leq \psi_2{}^{1-1/q}\left(\int_0^1((1+t)^{\alpha+1} - 2^{\alpha}(1+t) + \alpha 2^{\alpha}(1-t))\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}$.

By the convexity of $|f'|^q$, we have

$$|I_3| \leq \psi_2{}^{1-1/q}\left(K_7\frac{|f'(a)|^q}{2} + K_8\frac{|f'(b)|^q}{2}\right)^{1/q}.$$

Analogously:

$$|I_4| \leq \psi_2{}^{1-1/q}\left(K_7\frac{|f'(b)|^q}{2} + K_8\frac{|f'(a)|^q}{2}\right)^{1/q}.$$

It is very easy to check that

$K_5 = \int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right)\right|(1+t)\,dt = \frac{3(\alpha+1)+4\alpha(\alpha+2)c-\alpha(\alpha+1)c^2}{12(\alpha+1)(\alpha+2)} - \frac{1}{8}$,

$K_6 = \int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right)\right|(1-t)\,dt = \frac{2\alpha c^2-\alpha+4}{24(\alpha+2)}$,

$K_7 = \int_0^1 \left|\frac{1}{2(2^{\alpha}-1)} - \frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} + \frac{1}{3}\right|(1+t)\,dt,$

$= \frac{1}{2(2^{\alpha}-1)}\left[\left(d^2 - \frac{5}{2}\right)\left(\frac{2^{\alpha}-1}{3} + \frac{1}{2}\right) - \frac{1}{(\alpha+2)}\left(\frac{5}{3}d^2 - \frac{2^{\alpha+1}+1}{2}\right)\frac{1}{3} + \frac{1}{2(2^{\alpha}-1)}\right]$

$K_8 = \int_0^1 \left|\frac{1}{2(2^{\alpha}-1)} - \frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} + \frac{1}{3}\right|(1-t))dt$

$= \frac{1}{2(2^{\alpha}-1)}\left[\left(\frac{1}{2} - (2-d)^2\right)\left(\frac{2^{\alpha}-1}{3} + \frac{1}{2}\right) + \frac{1}{(\alpha+1)}\left(\frac{1}{2} - \frac{5d}{3}(2-d)\right) + \right.$
$\left.\frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{2^{\alpha+2}+1}{2} - \frac{5}{3}d^2\right)\right].$

This completes the proof. ⊔⊓

Remark 2 *If we take $\alpha = 1$ in Theorem 6, then inequality (7) reduces to inequality as obtained in Proposition 3.*

In the following theorem, we obtain estimate of Simpson's inequality (1) for concave functions.

Theorem 7 *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) such that $f' \in L^1[a, b]$. If $|f'|^q$ is concave on $[a, b]$, for some fixed $p > 1$ with $q = \frac{p}{p-1}$, then the following inequality for fractional integrals holds for $\alpha > 0$:*

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right]\right| \leq \frac{(b-a)}{2^{\alpha+1}} \times$$
$$\left[\psi_1\left\{\left|f'\left(\frac{K_5 b + K_6 a}{\psi_1}\right)\right| + \left|f'\left(\frac{K_5 a + K_6 b}{\psi_1}\right)\right|\right\}\right.$$
$$\left. + \psi_2\left(2^{\alpha} - 1\right)\left|f'\left(\frac{K_7 b + K_8 a}{\psi_2}\right)\right| + \left|f'\left(\frac{K_7 a + K_8 b}{\psi_2}\right)\right|\right]. \quad (8)$$

Proof. Using the concavity of $|f'|^q$ and the power-mean inequality, we obtain

$$\begin{aligned}|f'|^q &> t|f'|^q + (1-t)|f'|^q \\ &\geq t|f'|^q + (1-t)|f'|^q.\end{aligned}$$

Hence

$$|f'(tx + (1-t)y)| \geq t|f'(x)| + (1-t)|f'(y)|.$$

So $|f'|$ is also concave. By the Jensen integral inequality, we have

$$\begin{aligned}|I_1| &\leq \left(\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right)\right| dt\right)\left|f''\left(\frac{\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right)\right|\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt}{\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right)\right| dt}\right)\right| \\ &= \psi_1\left|f'\left(\frac{K_5 b + K_6 a}{\psi_1}\right)\right|.\end{aligned}$$

Analogously:

$$\begin{aligned}|I_2| &\leq \psi_1\left|f'\left(\frac{K_5 a + K_6 b}{\psi_1}\right)\right|, \\ |I_3| &\leq \psi_2\left|f'\left(\frac{K_7 b + K_8 a}{\psi_2}\right)\right|, \\ |I_4| &\leq \psi_2\left|f'\left(\frac{K_7 a + K_8 b}{\psi_2}\right)\right|.\end{aligned}$$

This completes the proof. ∎

Corollary 1 *If we take $\alpha = 1$ in Theorem 7, then inequality (8) becomes as:*

$$\begin{aligned}&\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\qquad \leq \frac{5(b-a)}{72}\left[\left|f'\left(\frac{29a + 61b}{90}\right)\right| + \left|f'\left(\frac{61a + 29b}{90}\right)\right|\right]. \quad (9)\end{aligned}$$

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Remark 3 *Inequality* (9) *is an generalization of obtained inequality as in [9, Theorem 8]*

3. Acknowledgments

The author S. Qaisar is grateful to Dr S.M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Pakistan for providing excellent research facilities. S. Qaisar was partially supported by the Higher Education Commission of Pakistan. (Grant No. 21-52/ SRGP / RD / HEC / 2014.)

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The permanence and global attractivity in a nonautonomous Gilpin-Ayala competition system with several delayed negative feedbacks

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Abstract: In this paper, a nonautonomous delayed Gilpin-Ayala competition system without instantaneous negative feedbacks (i.e., pure-delay-type system) is investigated. By the techniques of comparison arguments and constructing Lyapunov functionals something different to usual case, several results to guarantee the permanence of the system are derived by means of Ahmad and Lazer's definitions of lower and upper averages of a function. Moreover, the sufficient conditions for the global attractivity of the positive solution are also obtained, in which it is not necessarily to require the exponent of nonlinear intraspecific interference to exceed that of nonlinear interspecific interactions. These results are more general and practical, and possess a wide range of applications. Obviously, they are basically an extension of many existing conclusions for nonlinear competitive systems.

Keywords: Permanence; Global attractivity; Nonlinear competition; Lyapunov functionals; Pure-delays

1 Introduction

The permanence and global stability of ecological systems are always the most important and ubiquitous problems in mathematical biology. As pointed out by Li and Kuang [1], more realistic and interesting models of single or multiple species growth should take into account both the seasonality of the changing environment and the effects of time delays. Moreover, in view of the fact that in real-life species interactions, instantaneous responses are rare or weak relatively to delayed responses, more realistic models should consist of delay differential systems instead of the ones with instantaneous feedbacks. Recently, some model with discrete delay and distributed delay was studied [2–5]. In the meantime, some scholars [6,7] argue that continuously distributed delays as ecologically and biologically are more realistic than discrete delays to species interactions, which is proved true by Caperon [8]. Therefore, a reasonable alternative way is to study the pure-delay-type systems with both discrete delays and continuously distributed delays.

One the other hand, it is well know that for Lotka-Volterra model with delays, the stability is ordinarily delineated in two ways: the one that contain delay independent terms which dominate other intra-specific and inter-specific interaction effects with and without delays, called a "no-pure-delay-type", and the other with only delay feedbacks, is named as "pure-delay-type". For no-pure-delay-type system, one can use the no-delay terms to control the delay terms. Various results have been obtained recently under so-called diagonally dominant conditions and the conditions are often independent of delays (see [9–13]). However, for the pure-delay-type

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systems, the analysis of the permanence and the global asymptotic stability of the system is very difficult, let along the nonlinear type system.

Motivated by the works on Gilpin-Ayala competition systems with delays (see [12, 14–16]), in particular, strongly stimulated by the works [1, 17–19], which all contain several time delay, we consider the following Gilpin-Ayala competitive system with several discrete arguments and continuous time delays

$$\begin{aligned}\dot{x}_i(t) = x_i(t)\Big[r_i(t) - &\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)x_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) \\ &-\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)x_j^{\beta_{ijl}}(t+s)ds\Big].\end{aligned} \tag{1.1}$$

The aim of this paper is, by developing the analytic technique the analytic technique of the literatures [10, 11, 14–16, 21, 22], to obtain conditions which guarantee the permanence of the system (1.1); after that, by constructing a suitable Lyapunov functional, sufficient conditions about the global attractivity of the positive solution of system (1.1) are gained.

For convenience, we will use following notations in the rest of this paper, let $\tau_{ijk} = \sup\{\tau_{ijk}(t) \mid t \in R\}$ and $\tau = \max\{\tau_{ijk},\ \sigma_{ijl}\}$, then we have $0 < \tau_{ijk},\ \sigma_{ijl} \le \tau$. Denote by $\Psi_{ijk}(t) = t - \tau_{ijk}(t)$, and the functions $\Psi_{ijk}^{-1}(t)$ is the inverse functions of $\Psi_{ijk}(t)$, respectively. In this paper, for system (1.1) we always assume that

(H_1) $\alpha_{ijk} > 0,\ \beta_{ijl} > 0$.

(H_2) $r_i(t),\ a_{ijk}(t),\ \tau_{ijk}(t)$, are positively continuous and bounded functions on $[c, +\infty)$.

(H_3) Functions $b_{ijl}(t, s)$ are defined on $[c, +\infty) \times [-\tau, 0]$ such that they are integrable with respect to s, and $\int_{-\sigma_{ijl}}^{0} b_{ijl}(t, s)ds$ are positive, continuous and bounded above with respect to t on $[c, +\infty)$.

(H_4) $\tau_{ijk}(t)$ are nonnegative, continuous and bounded, $\Psi_{ijk}(t) = t - \tau_{ijk}(t)$ are all invertible. Furthermore, it is differentiable and satisfy $1 - \tau'_{ijk}(t) > 0$ $(t \ge c)$.

Stimulated by the application of system (1.1) to population dynamics, we assume that solutions of system (1.1) satisfy the following initial condition

$$x_i(\theta) = \phi_i(\theta) \ge 0,\ \theta \in [-\tau, 0],\ \ \phi_i(0) > 0,\ \sup_{\theta\in[-\tau,0]} \phi_i(\theta) < +\infty. \tag{1.2}$$

2 Basic results

Let $g(t)$ be a continuous function define on $[c, +\infty)$. Denote

$$g^u = \sup\{g(t) \mid c \le t < +\infty\},\ \ g^l = \sup\{g(t) \mid c \le t < +\infty\}.$$

According to Ahmad and Lazer [10], we define the lower and upper averages of a function $g(t)$. If $c \le t_1 < t_2$, set

$$A[g,\ t_1,\ t_2] = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} g(s)ds.$$

The lower and upper averages of $g(t)$ denoted by $m[g]$ and $M[f]$ are follows

$$m[g] = \lim_{s\to+\infty} \inf\{A[g,\ t_1,\ t_2] \mid t_2 - t_1 \ge s\},$$

and

$$M[g] = \lim_{s\to+\infty} \sup\{A[g,\ t_1,\ t_2] \mid t_2 - t_1 \ge s\}.$$

Since the set $\{A[g,\ t_1,\ t_2] \mid t_2 - t_1 \ge s\}$ decreases as s increases, the limits exist; and since $g^l \le A[g,\ t_1,\ t_2] \le g^u$, it follows that $g^l \le m[g] \le A[g,\ t_1,\ t_2] \le M[g] \le g^u$.

Definition 2.1. The system of differential equation

$$\dot{x}(t) = F(t,\ x(t)),\qquad x \in R^n$$

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is said to be permanent if there exits a compact set D in $R_+^n = \{(x_1,\ x_2,\ ...,\ x_n) \in R^n \mid x_i > 0\ (i = 1, 2, ..., n)\}$, such that all solutions starting in the interior of R_+^n ultimately enter D.

Now we consider following single species Logistic type equation

$$\dot{x}(t) = x(t)\Big[r(t) - \sum_{k=1}^{n} a_k(t)x^{\alpha_k}(t)\Big]. \tag{2.1}$$

Where $r(t)$ and $a_k(t)$ $(k = 1, 2, ..., n)$ are all continuous functions on $[0, +\infty)$, $r(t)$ may be negative, $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in 1,\ 2,\ ...,\ n$ such that $m[a_k] > 0$, and α_k $(k = 1, 2, ..., n)$ are positive constants.

From the Lemma of [11], we have

Lemma 2.1. Suppose that $m[r] > 0$, $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in \{1,\ 2,\ ...,\ n\}$ such that $m[a_k] > 0$, then any solution $x(t)$ of (2.1) with initial value $x(t_0) > 0$ is bounded above and below on $[t_0, +\infty)$ and globally attractive. Specially, if $r(t),\ a_k(t)$ $(k = 1, 2, ..., n)$ are continuous T-periodic functions, then (2.1) has a unique positive, global attractive T-periodic solution $x^*(t)$.

As a matter of fact, according to Lemma 2.2 of [11], if $r(t)$ may be negative but $M[r] > 0$, $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in \{1,\ 2,\ ...,\ n\}$ such that $m[a_k] > 0$, then we have Lemma 2.2 below corresponding to Lemma 2.1:

Lemma 2.2. Assume that $M[r] > 0$ and $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in \{1,\ 2,\ ...,\ n\}$ such that $m[a_k] > 0$, then any solution $x(t)$ of (2.1) with initial value $x(t_0) > 0$ is bounded above and below by strictly positive real numbers on $[t_0, +\infty)$ and globally attractive. Specially, if $r(t),\ a_k(t)$ $(k = 1, 2, ..., n)$ are all continuous T-periodic functions, then system (2.1) has a unique positive, globally asymptotically stable T-periodic solution $x^*(t)$.

By developing the analytic technique of [11, 16], it is not difficult to verify the following results

Lemma 2.3. If $(H_2) - (H_4)$ are hold, then we have

$$M\Big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t - \tau_{ijk}(t))\Big] = M\Big[\frac{a_{ijk}\big(\Psi_{ijk}^{-1}(t)\big)}{1 - \tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big].$$

$$m\Big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t - \tau_{ijk}(t))\Big] = m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1 - \tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big].$$

where $X_i(t)$ is the unique solution of the Logistic system corresponding to Eqs. (1.1) with initial condition $X_i(t_0) > 0$.

Proof. From $(H_2) - (H_4)$ and Lemma 2.1, 2.2, we infer that $\tau_{ijk}(t)$, $\frac{a_{ijk}\left(\Phi_{ijk}^{-1}(t)\right)}{1-\tau'_{ijk}\left(\Phi_{ijk}^{-1}(t)\right)}$ and $X_j^{\alpha_{ijk}}(t)$ are all bounded, we claim that

$$\int_{t_1-\tau_{ijk}(t_1)}^{t_1} \frac{a_{ijk}\big(\Phi_{ijk}^{-1}(s)\big)}{1 - \tau'_{ijk}\big(\Phi_{ijk}^{-1}(s)\big)}X_j^{\alpha_{ijk}}(s)ds,\ \int_{t_2}^{t_2-\tau_{ijk}(t_2)} \frac{a_{ijk}\big(\Phi_{ijk}^{-1}(s)\big)}{1 - \tau'_{ijk}\big(\Phi_{ijk}^{-1}(s)\big)}X_j^{\alpha_{ijk}}(s)ds$$

are all bounded above and below. Then from the definition of lower and upper averages of a function, we obtain that for $t_2 > t_1 \geq t_0$

$$M\Big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t - \tau_{ijk}(t))\Big] = \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} a_{ijk}(t)X_j^{\alpha_{ijk}}(t - \tau_{ijk}(t))ds \mid t_2 - t_1 \geq s\Big\}$$

$$= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2 - t_1}\int_{t_1-\tau_{ijk}(t_1)}^{t_2-\tau_{ijk}(t_2)} \frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1 - \tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)ds \mid t_2 - t_1 \geq s\Big\}$$

$$= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2 - t_1}\Big(\int_{t_1-\tau_{ijk}(t_1)}^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^{t_2-\tau_{ijk}(t_2)}\Big)\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1 - \tau'_{ijk}\big(\Psi_{ijk}^{-1}(t)\big)}dt \mid t_2 - t_1 \geq s\Big\}$$

$$= \lim_{s\to+\infty} \sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2} \frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}dt \mid t_2-t_1\geq s\Big\} = M\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}\Big].$$

Similarly, we can testify that the equality for the case of $m\big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\big]$ is also true.

Lemma 2.4. If $(H_2)-(H_4)$ hold, then

$$M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big] = M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big],$$
$$m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big] = m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big].$$

where $X_i(t)$ is the unique solution of the Logistic system corresponding to Eqs. (1.1) with initial condition $X_i(t_0)>0$.

Proof. From $(H_2)-(H_4)$ and Lemma 2.1, 2.2, it follows that $b_{ijl}(t,.)$ and $\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)ds,\ X_j^{\beta_{ijl}}(t)$ are all bounded functions, we conclude that

$$\int_{-\sigma_{ijl}}^{0}\int_{t_1+s}^{t_1} b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(s)ds,\ \int_{-\sigma_{ijl}}^{0}\int_{t_2}^{t_2+s} b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(s)ds$$

are all bounded. Therefore, according to the definition of lower and upper averages of a function, we find that for $t_2>t_1\geq t_0$

$$\begin{aligned}
&M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big]\\
&= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big(\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ij}}(t+s)ds\Big)dt \mid t_2-t_1\geq s\Big\}\\
&= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{-\sigma_{ijl}}^{0}\int_{t_1+s}^{t_2+s} b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(t)dt \mid t_2-t_1\geq s\Big\}ds\\
&= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{-\sigma_{ijl}}^{0}\Big(\int_{t_1+s}^{t_1}+\int_{t_1}^{t_2}+\int_{t_2}^{t_2+s}\Big)b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(t)dt \mid t_2-t_1\geq s\Big\}ds\\
&= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big(\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big)dt \mid t_2-t_1\geq s\Big\}\\
&= M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big].
\end{aligned}$$

In a similar way, we can show that the equality for the case of $m\big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big]$ is also hold.

3 Permanence

In this section, we are mainly concerned with the permanence of the system (1.1)-(1.2). Firstly, for the sake of the permanence with regarding to the system (1.1), we introduce the following notations

$$a^*_{ijk}(t) = a_{ijk}(t)\exp\Big\{\alpha_{ijk}\int_t^{t-\tau_{ijk}(t)} r_i(s)ds\Big\},$$
$$b^*_{ijl}(t) = \int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)\exp\Big\{\beta_{ijl}\int_t^{t+s} r_i(u)du\Big\}ds.$$

Then, let us consider the following logistic type equation corresponding to Eqs. (1.1)

$$\dot{x}_i(t) = x_i(t)\Big[r_i(t)-\sum_{k=1}^{k_{ii}} a^*_{iik}(t)x_i^{\alpha_{iik}}(t)-\sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b^*_{iil}(t,s)dsx_i^{\beta_{iil}}(t)\Big]. \tag{3.1}$$

Theorem 3.1. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)\ M\Big[r_i(t)-\sum_{j=1,\ j\neq i}^{n}\Big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))+\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big)\Big]>0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0) > 0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. Firstly, we show that any positive solution of system (1.1) is ultimately bounded above by some positive constant. Let $x(t) = \big(x_1(t),\ ...,\ x_n(t)\big)$ be any positive solution of system (1.1), then it follows from (1.1) that for all $t \geq 0$

$$\dot{x}_i(t) \leq r_i(t)x_i(t). \tag{3.2}$$

Thus for any $t \geq 0,\ s \leq 0$ and $t+s \geq 0$, by integrating (2.11) over interval $[t+s, t]$ we derive

$$x_i(t+s) \geq x_i(t)\exp\Big\{\int_t^{t+s} r_i(s)ds\Big\} \quad \text{for, } t \geq \tau. \tag{3.3}$$

Integrate with (3.3), we obtain directly from the system (1.3) that

$$\begin{aligned}\dot{x}_i(t) &= x_i(t)\Big[r_i(t) - \sum_{j=1}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)x_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) - \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)x_j^{\beta_{ijl}}(t+s)ds\Big]\\ &\leq x_i(t)\Big[r_i(t) - \sum_{k=1}^{k_{ii}} a_{iik}(t)x_i^{\alpha_{iik}}(t-\tau_{iik}(t)) - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b_{iil}(t,s)x_i^{\beta_{iil}}(t+s)ds\Big]\\ &\leq x_i(t)\Big[r_i(t) - \sum_{k=1}^{k_{ii}} a_{iik}^*(t)x_i^{\alpha_{iik}}(t) - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b_{iil}^*(t,s)dsx_i^{\beta_{iil}}(t)\Big].\end{aligned} \tag{3.4}$$

By using the comparison theorem, we find

$$x_i(t) \leq X_i(t), \ \text{ for all } \ t \geq t_0. \tag{3.5}$$

Where $X_i(t)$ is the positive solution of system (3.1) with initial condition $X_i(0)$ which satisfies $x_i(0) \leq X_i(0)$. From Lemma 2.1, Lemma 2.2 and (3.5), it is not difficult to obtain that

$$\limsup_{t\to+\infty} x_i(t) \leq X_i(t), \ \text{ for all } \ t \geq t_0.$$

Hence, for a sufficiently small $\varepsilon > 0$, there exists a $T_{i1}(\varepsilon) > 0$ such that for $t \geq T_{i1}(\varepsilon)$

$$x_i(t) \leq X_i(t) \leq X_i(t) + \varepsilon. \tag{3.6}$$

Now choose $M_0 = \sup\{X_i(t) + \varepsilon \mid t \geq 0,\ i = 1,\ 2,\ ...,\ n\}$, then M_0 does not depend on any solution of system (3.1), also $x_i(t) \leq M_0$, for all $t \geq T_1$, where $T_1 = \max_{1\leq i\leq n}\{T_{i1}\}$.

Secondly, we shall show that any positive solution of system (1.1) is ultimately bounded below by some positive constant. To this end, we proceed with following two steps.

Step 1: We show that there exists $\epsilon_0 > 0$ such that $\limsup_{t\to+\infty} x_i(t) \geq \epsilon_0$, for all $i = 1,\ 2,\ ...,\ n$. For the convenience of the following discuss, for any constant $\varepsilon > 0$, we denote by

$$\begin{aligned}R_i(t,\varepsilon) = r_i(t) &- \sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)\Big(X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) + \varepsilon\Big)\\ &- \sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)\Big(X_j^{\beta_{ijl}}(t+s) + \varepsilon\Big)ds\Big]\end{aligned}$$

On the one hand, according to (H_5) in Theorem 3.1, one finds that for any given small number $\varepsilon > 0$, there is $M[R_i(t,\varepsilon)] > 0$ $(i = 1, 2, ..., n)$. Therefore, we can choose a sufficiently small number $\epsilon_0 > 0,\ \delta > 0$ such that

$$M\Big[R_i(t,\varepsilon) - \sum_{k=1}^{k_{ii}} a_{iik}(t)\epsilon_0^{\alpha_{iik}} - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b_{iil}(t,s)ds\epsilon_0^{\beta_{iil}}\Big] \geq \delta,$$

for all $i = 1,\ 2,\ ...,\ n$, i.e.,

$$\lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big[R_i(t,\varepsilon) - \sum_{k=1}^{k_{ii}} a_{iik}(t)\epsilon_0^{\alpha_{iik}} - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b_{iil}(t,s)ds\epsilon_0^{\beta_{iil}}\Big]dt \mid t_2 - t_1 \geq s\Big\} \geq \delta.$$

Which implies that

$$\lim_{s\to+\infty}\sup\Big\{\int_{t_1}^{t_2}\Big[R_i(t,\varepsilon)-\sum_{k=1}^{k_{ii}}a_{iik}(t)\epsilon_0^{\alpha_{iik}}\\ -\sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0}b_{iil}(t,s)\epsilon_0^{\beta_{iil}}ds\Big]dt \mid t_2-t_1\ge s\Big\}=+\infty.$$

Therefore, there must exist $\lambda>0$ and a positive number $\gamma_0>0$ such that

$$\int_t^{t+\lambda}\Big[R_i(t,\varepsilon)-\sum_{k=1}^{k_{ii}}a_{iik}(t)\epsilon_0^{\alpha_{iik}}-\sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0}b_{iil}(t,s)ds\epsilon_0^{\beta_{iil}}\Big]dt\ge\gamma_0,\ \text{ for all } t\ge T_2. \tag{3.7}$$

Now we claim that the following inequality holds

$$\limsup_{t\to+\infty}x_i(t)\ge\epsilon_0,\ \text{ for all } i=1,\ 2,\ ...,\ n. \tag{3.8}$$

By way of contradiction, suppose that $\limsup_{t\to+\infty}x_i(t)<\epsilon_0$ for a certain $p\in\{1,\ 2,\ ...,\ n\}$, then there exists $T_2>T_1$ such that $x_p(t)<\delta$, for all $t\ge T_2$. This, together with the (3.6), gives out that for all $t\ge T_2$

$$\begin{aligned}\dot{x}_p(t)&=x_p(t)\Big[r_p(t)-\sum_{j=1}^{n}\Big(\sum_{k=1}^{k_{pj}}a_{pjk}(t)x_j^{\alpha_{pjk}}(t-\tau_{pjk}(t))+\sum_{l=1}^{l_{pj}}\int_{-\sigma_{pjl}}^{0}b_{pjl}(t,s)x_j^{\beta_{pjl}}(t+s)ds\Big)\Big]\\ &\ge x_p(t)\Big[r_p(t)-\sum_{j=1,\ j\ne p}^{n}\sum_{k=1}^{k_{pj}}a_{pjk}(t)\Big(X_j^{\alpha_{pjk}}(t-\tau_{pjk}(t))+\varepsilon\Big)\\ &\quad-\sum_{j=1,\ j\ne p}^{n}\sum_{l=1}^{l_{pj}}\int_{-\sigma_{pjl}}^{0}b_{pjl}(t,s)\Big(X_j^{\beta_{pjl}}(t+s)+\varepsilon\Big)ds\Big]\\ &\quad-\sum_{k=1}^{k_{pp}}a_{ppk}(t)\epsilon_0^{\alpha_{ppk}}-\sum_{l=1}^{l_{pp}}\int_{-\sigma_{ppl}}^{0}b_{ppl}(t,s)ds\epsilon_0^{\beta_{ppl}}\Big]\\ &\ge x_p(t)\Big[R_p(t,\varepsilon)-\sum_{k=1}^{k_{pp}}a_{ppk}(t)\epsilon_0^{\alpha_{ppk}}-\sum_{l=1}^{l_{pp}}\int_{-\sigma_{ppl}}^{0}b_{ppl}(t,s)ds\epsilon_0^{\beta_{ppl}}\Big].\end{aligned} \tag{3.9}$$

An integration of (3.9) over time interval $[T_2,t]$ leads to

$$x_p(t)\ge x_p(T_2)\exp\Big\{\int_{T_2}^{t}\Big[R_p(t,\varepsilon)-\sum_{k=1}^{k_{pp}}a_{ppk}(t)\epsilon_0^{\alpha_{ppk}}-\sum_{l=1}^{l_{pp}}\int_{-\sigma_{ppl}}^{0}b_{ppl}(t,s)ds\epsilon_0^{\beta_{ppl}}\Big]\Big\}. \tag{3.10}$$

Obviously, which, together with (3.7) result into the conclusion that $x_p(t)\to+\infty$ as $t\to+\infty$, which contradicts to the boundedness of $x_i(t)$, for all $t\ge T_{i1}$ in (3.6). Hence, the inequality (3.8) is true.

Step 2: We shall prove that there exists a constant $m_0>0$, m_0 is independent of any solution of system (1.1), i.e., there is a positive constant $m_0>0$ such that for any solution $x(t)=(x_1(t),\ ...,\ x_n(t))$, one has

$$\liminf_{t\to+\infty}x_i(t)\ge m_0,\ \text{ for all } i=1,\ 2,\ ...,\ n. \tag{3.11}$$

Assume that it is not true, then there exist a certain integer $q\in\{1,\ 2,\ ...,\ n\}$ and a sequence of initial functions $\{\phi_q^{(k)}(t)\}_{k=1}^{+\infty}$ for system (1.1) such that $x_q^{(k)}(t)=x_q(t,\phi_q^{(k)}),\ k=1,\ 2,\ ...$ satisfy

$$\liminf_{t\to+\infty}x_q^{(k)}(t)\le\frac{\epsilon_0}{(k+1)^2},\ \text{ for all } k=1,\ 2,\ ... \tag{3.12}$$

For each $k=1,\ 2,\ ...$, from (3.8) we claim that $\limsup_{t\to+\infty}x_q^{(k)}(t)\ge\frac{1}{(k+1)}\epsilon_0$. Hence, by (3.12) one can infer that there exists two time sequences $\{s_n^{(k)}\}$ and $\{t_n^{(k)}\}$ such that for each $k=1,\ 2,\ ...$

$$0<s_1^{(k)}<t_1^{(k)}<s_2^{(k)}<t_2^{(k)}<\ ...\ <s_n^{(k)}<t_n^{(k)}<\ ...\ ,\ \text{ for all } n=1,\ 2,\ ...,$$

$$s_n^{(k)} \to +\infty, \quad t_n^{(k)} \to +\infty, \text{ as } n \to +\infty, \; x_q^{(k)}(t_n^{(k)}) = \frac{\epsilon_0}{(k+1)^2}, \; x_q^{(k)}(s_n^{(k)}) = \frac{\epsilon_0}{(k+1)}. \tag{3.13}$$

$$\frac{\epsilon_0}{(k+1)^2} < x_q^{(k)}(t) < \frac{\epsilon_0}{(k+1)}, \text{ for all } t \in (s_n^{(k)}, \; t_n^{(k)}). \tag{3.14}$$

It follows from (3.6) that for a given small number ϵ_0, there exists $T_2^{(k)} > T_1$ such that $x_i^{(k)}(t) \leq X_i(t) + \epsilon_0, \; t \geq T_2^{(k)}$.

Obviously, by (3.13) there exists a large enough integer $N_1^{(k)} > 0$ such that $s_n^{(k)} > T_2^{(k)} + \tau$ for all $n \geq N_1^{(k)}$ for each $k = 1, 2, \ldots$. Hence, for any $t \in [s_n^{(k)}, \; t_n^{(k)}]$ and $n \geq N_1^{(k)}$, we have

$$\begin{aligned}
\dot{x}_q^{(k)}(t) &= x_q^{(k)}(t)\Big[r_q(t) - \sum_{j=1}^{n}\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\Big(x_j^{(k)}(t-\tau_{qj\nu}(t))\Big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1}^{n}\sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\Big(x_j^{(k)}(t+s)\Big)^{\beta_{qjl}} ds\Big] \\
&\geq x_q^{(k)}(t)\Big[r_q(t) - \sum_{j=1}^{n}\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\Big(X_j^{(k)}(t-\tau_{qj\nu}(t)) + \varepsilon\Big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1}^{n}\sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\Big(X_j^{(k)}(t+s) + \varepsilon\Big)^{\beta_{qjl}} ds\Big] \geq -\gamma x_q^{(k)}(t).
\end{aligned} \tag{3.15}$$

Where

$$\gamma = \sup_{t \in R}\Big\{\sum_{j=1}^{n}\Big[\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\Big(X_j^{(k)}(t-\tau_{qj\nu}(t))+\varepsilon\Big)^{\alpha_{qj\nu}} + \sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\Big(X_j^{(k)}(t+s)+\varepsilon\Big)^{\beta_{qjl}} ds\Big]\Big\}.$$

Therefore, for any $n \geq N_1^{(k)}$ and $k = 1, 2, \ldots$, an integration of (3.15) over $[s_n^{(k)}, \; t_n^{(k)}]$ makes one lead to

$$\begin{aligned}
\frac{\epsilon_0}{(k+1)^2} &= x_q^{(k)}(t_n^{(k)}) \geq x_q^{(k)}(s_n^{(k)}) \exp\big\{-\gamma(t_n^{(k)} - s_n^{(k)})\big\} \\
&= \frac{\epsilon_0}{(k+1)} \exp\big\{-\gamma(t_n^{(k)} - s_n^{(k)})\big\}.
\end{aligned}$$

Which means

$$t_n^{(k)} - s_n^{(k)} \geq \frac{\ln(k+1)}{\gamma}, \text{ for all } n \geq N_1^{(k)}, \; k = 1, 2, \ldots \tag{3.16}$$

It follows from (3.16) that there exists a sufficient large integer K_0 such that

$$t_n^{(k)} - s_n^{(k)} \geq \lambda, \text{ for all } k \geq K_0, \quad n \geq N_1^{(k)}. \tag{3.17}$$

Hence, for any $k \geq K_0, \; n \geq N_1^{(k)}$ and $t \in [s_n^{(k)}, \; t_n^{(k)}]$, it follows from (3.13) and (3.14) that

$$\begin{aligned}
\dot{x}_q^{(k)}(t) &= x_q^{(k)}(t)\big[r_q(t) - \sum_{j=1}^{n}\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\big(x_j^{(k)}(t-\tau_{qj\nu}(t))\big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1}^{n}\sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\big(x_j^{(k)}(t+s)\big)^{\beta_{qjl}} ds\big] \\
&\geq x_q^{(k)}(t)\big[r_q(t) - \sum_{\nu=1}^{\nu_{qq}} a_{qq\nu}(t)\big(\frac{\epsilon_0}{k+1}\big)^{\alpha_{qq\nu}} - \sum_{l=1}^{l_{qq}} \int_{-\sigma_{qql}}^{0} b_{qql}(t,s)ds\big(\frac{\epsilon_0}{k+1}\big)^{\beta_{qql}} \\
&\qquad - \sum_{j=1,\, j\neq q}^{n} \sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\big(X_j^{(k)}(t-\tau_{qj\nu}(t)) + \varepsilon\big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1,\, j\neq p}^{n} \sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\big(X_j^{(k)}(t+s) + \varepsilon\big)^{\beta_{qjl}} ds\big] \\
&\geq x_q^{(k)}(t)\big[r_q(t) - \sum_{\nu=1}^{\nu_{qq}} a_{qq\nu}(t)\epsilon_0^{\alpha_{qq\nu}} - \sum_{l=1}^{l_{qq}} \int_{-\sigma_{qql}}^{0} b_{qql}(t,s)ds\epsilon_0^{\beta_{qql}}
\end{aligned}$$

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$$
\begin{aligned}
&- \sum_{j=1,\ j\neq q}^{n} \sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\big(X_j^{(k)}(t-\tau_{qj\nu}(t))+\varepsilon\big)^{\alpha_{qj\nu}} \\
&- \sum_{j=1,\ j\neq p}^{n} \sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\big(X_j^{(k)}(t+s)+\varepsilon\big)^{\beta_{qjl}} ds\big]. \qquad (3.18)
\end{aligned}
$$

According to (3.7), (3.13) and (3.14), an integration of (3.18) over time interval $[t_n^{(k)}-\lambda,\ t_n^{(k)}]$ makes it reach

$$
\begin{aligned}
\frac{\epsilon_0}{(k+1)^2} &= x_q^{(k)}(t_n^{(k)}) \geq x_q^{(k)}(t_n^{(k)}-\lambda)\exp\Big\{\int_{t_n^{(k)}-\lambda}^{t_n^{(k)}} \big[B_q(t,\epsilon_0) - \sum_{j=1,\ j\neq q}^{n}\big(\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t) \\
&\quad \times \big(X_j^{(k)}(t-\tau_{qj\nu}(t))+\varepsilon\big)^{\alpha_{qj\nu}} + \sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\big(X_j^{(k)}(t+s)+\varepsilon\big)^{\beta_{qjl}} ds\big)\big]dt\Big\} \\
&\quad > \frac{\epsilon_0}{(k+1)^2}\exp\epsilon_0 > \frac{\epsilon_0}{(k+1)^2}. \qquad (3.19)
\end{aligned}
$$

Where

$$
B_q(t,\epsilon_0) = r_q(t) - \sum_{\nu=1}^{\nu_{qq}} a_{qq\nu}(t)\epsilon_0^{\alpha_{qq\nu}} - \sum_{l=1}^{l_{qq}} \int_{-\sigma_{qql}}^{0} b_{qql}(t,s)ds\epsilon_0^{\beta_{qql}}.
$$

Which is contradiction. This shows that there exists a constant $m_0>0$ ($m_0>0$ is independent of any initial function) such that the inequality (2.15) is correct. That is to say, any positive solution $x(t)$ of the initial value problem (1.1)-(1.2) is ultimately bounded below by a positive constant $m_0>0$. From Definition 2.1, the proof of Theorem 3.1 is complete.

Theorem 3.2. In addition to $(H_1)-(H_4)$, assume further that

$$
\begin{aligned}
(H_5)'\ M\big[r_i(t)\big] &- \sum_{j=1,\ j\neq i}^{n} \sum_{k=1}^{k_{ij}} m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)} X_j^{\alpha_{ijk}}(t)\Big] \\
&- \sum_{j=1,\ j\neq i}^{n} \sum_{l=1}^{l_{ij}} m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big] > 0.
\end{aligned}
$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0)>0$. Then the system (1.1)-(1.2) is permanent.

Proof. In order to prove the correct of Theorem 3.2, We only need to show that $(H_5)'$ implies the assumption (H_5). Actually, if take into account the fact that

$$
M[X_{i0}(t)+c] = M[X_{i0}(t)]+c, \qquad m[f_i(t)] \leq A[f_i(t),\ t_1,\ t_2].
$$

Then we may obtain that

$$
\begin{aligned}
&M\big[r_i(t)\big] - \sum_{j=1,\ j\neq i}^{n}\Big(\sum_{k=1}^{k_{ij}} m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}\Big] + \sum_{l=1}^{l_{ij}} m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big]\Big) \\
&-M\big[r_i(t) - \sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) + \sum_{l=1}^{l_{ij}} \int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big] \\
&= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big[r_i(t) - \sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}\Big] \\
&\quad + \sum_{l=1}^{l_{ij}} m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big]\big)\Big]dt \mid t_2-t_1\geq s\Big\} - \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[r_i(t) \\
&\quad - \sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) + \sum_{l=1}^{l_{ij}} \int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big]dt \mid t_2-t_1\geq s\Big\}
\end{aligned}
$$

$$= \lim_{s\to+\infty} \sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big[r_i(t) - \sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} m\big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\big]$$
$$+\sum_{l=1}^{l_{ij}} m\big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big]\big)\Big]dt \mid t_2-t_1\geq s\Big\} - \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[r_i(t)$$
$$-\sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) + \sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big]dt \mid t_2-t_1\geq s\Big\}$$
$$\geq \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2} r_i(t)dt - \sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\big]dt$$
$$+\sum_{l=1}^{l_{ij}}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big]\big)dt \mid t_2-t_1\geq s\Big\} - \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[r_i(t)$$
$$-\sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) + \sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big]dt \mid t_2-t_1\geq s\Big\} = 0.$$

Therefore, we claim from Theorem 3.1 that Theorem 3.2 is correct. The proof is complete.

Theorem 3.3. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)''\ M\big[r_i(t)\big] - \sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} M\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big]$$
$$-\sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}} M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big] > 0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0)>0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. Noticing the following facts that

$$M[X_{i0}(t)+c] = M[X_{i0}(t)]+c,\quad m[f_i(t)]\leq M[f_i(t)]\quad\text{and}\quad \sum_{j=1}^{n} m[f_i(t)]\leq\sum_{i=1}^{n} M[f_i(t)].$$

We find that the condition $(H_5)''$ means the hypothesis $(H_5)'$, and so it does the assumption (H_5). Hence, one can confirm that the result of Theorem 3.3 is also true.

Theorem 3.4. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)'''\ m\big[r_i(t)\big] - \sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} M\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big]$$
$$-\sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}} M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big] > 0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0)>0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. Taking into account the facts that

$$M[X_{i0}(t)+c] = M[X_{i0}(t)]+c,\quad m[f_i(t)]\leq M[f_i(t)].$$

We declare that the assumption $(H_5)''$ can be deduced from the hypothesis $(H_5)'''$, so it is evident that Theorem 3.3 implies the Theorem 3.4.

Theorem 3.5. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)''''\ m\big[r_i(t)\big] - \sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big]$$

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$$-\sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}} m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big] > 0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0) > 0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. As a matter of fact, $m[f_i(t)] \le M[f_i(t)]$ and assumption $(H_5)''''$ means that the hypothesis (H_5) is true, so it follows from Theorem 3.1 that the conclusion of Theorem 3.5 is right.

Remark. 3.1 It is easy to verify that $M[g] = m[g] = \frac{1}{T}\int_0^T g(t)dt$ for a T-periodic function $g(t)$. So if system (1.1) is a periodic system, i.e., $r_i(t)$, $a_{ijk}(t)$, $b_{ijl}(t,\cdot)$ are the continuous T-periodic functions, then $X_i(t)$ in above mentioned Theorems can be replaced by the unique positive T-periodic solution $X_i^*(t)$ of (3.1), and the assumptions of Theorem 3.1-Theorem 3.5 are equivalent to each other.

Remark. 3.2 Theorems 3.1-3.5 generalize the main results of Zhao et al. [11], Chen et al. [14,15] and Xia et al. [16]. We mention here that for general nonautonomous Lotka-Volterra system (1.1), Teng et al. [21, 22] also obtained some similar results as that of Zhao [11]. It is in this sense, our results can also be seen as the generalization of Theorems of [21, 22].

4 Global attractivity

A very basic and important problem accompanying with the ecological dynamics systems is the global stability of the positive solution for the system. In this section, we will devote ourselves to give some new criteria to guarantee global attractivity of the positive solution.

Definition 4.1. The bounded solution $X^*(t) = (x_1^*(t), x_2^*(t), ..., x_n^*(t))$ of system (1.1) with $X^*(t_0) > 0$ is said to be globally attractive, if for any other solution $X(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ with $X(0) > 0$, there is

$$\lim_{t\to+\infty} \mid x_i(t) - x_i^*(t) \mid = 0, \quad i = 1, 2, ..., n.$$

Before we state the main result of this section, we first introduce some notations which will be used in the following discussion. Let $\Phi_{ijk}^{-1}(t)$ be the inverse function of $\Phi_{ijk}(t) = t - \tau_{ijk}(t)$, and

$$A_{ijk}^{(1)}(t) = \frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)},\ A_{ijk}^{(2)}(t) = \frac{a_{ijk}\Big(\Phi_{ijk}^{-1}\big(\Phi_{ijk}^{-1}(t)\big)\Big)}{\Big(1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}\big(\Phi_{ijk}^{-1}(t)\big)\big)\Big)\Big(1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)\Big)},$$

$$B_{ijl}^{(1)}(t) = \int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)ds, \quad B_{ijl}^{(2)}(t) = \int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t} b_{ijl}(\theta-s,s)d\theta ds,$$

$$\big(B_{ijl}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t) = \int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t} A_{i\tilde{j}\tilde{k}}^{(1)}(\theta-s)b_{ijl}(t-s,s)d\theta ds,$$

$$\big(B_{ijl}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t) = \int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t} B_{i\tilde{j}\tilde{l}}^{(1)}(\theta-s)b_{ijl}(t-s,s)d\theta ds.$$

Let $u_i(t) = \ln x_i(t)$, then Eqs. (1.1) can be reformulated as

$$\begin{aligned}\dot{u}_i(t) = r_i(t) &- \sum_{j=1}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)\exp\Big\{\alpha_{ijk}u_j\big(t-\tau_{ijk}(t)\big)\Big\} \\ &-\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)\exp\big\{\beta_{ijl}u_j(t+s)\big\}ds. \end{aligned} \tag{4.1}$$

Now we are in the position of stating the sufficient conditions which guarantee the global attractivity of system (1.1).

Theorem 4.1. In addition to $(H_1)-(H_5)$, we assume further that

(H_6) There exist positive constants $\lambda_i > 0$ $(i = 1, 2, ..., n)$, $\zeta > 0$ such that

$$\liminf_{t\to+\infty}\{\Lambda_i(t)\}>\zeta,\ \ \liminf_{t\to+\infty}\{\Delta_i(t)\}>\zeta.$$

Where
$$\Lambda_i(t)=2\sum_{k=1}^{k_{ii}}\lambda_i A_{iik}^{(1)}(t)-\sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2 M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)+\sum_{k=1}^{k_{ij}}\frac{\lambda_i}{\alpha_{iik}m_{i0}^{\alpha_{iik}}}A_{ijk}^{(1)}(t)\Big]$$
$$-\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2 M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}\tilde{k}}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(1)}(s)ds+\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)ds\Big)$$
$$+\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2 M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big)d\theta+\sum_{l=1}^{l_{i\tilde{j}}}B_{i\tilde{j}l}^{(2)}(t)\Big)\Big],$$
$$\Delta_i(t)=2\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)-\sum_{j=1,j\neq i}^{n}\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2 M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B_{jil}^{(1)}(t)+\sum_{l=1}^{l_{ij}}\frac{\lambda_i}{\beta_{iil}m_{i0}^{\beta_{iil}}}B_{ijl}^{(1)}(t)\Big]$$
$$-\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2 M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B_{jil}^{(1)}(t)\Big(\sum_{k=1}^{k_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}k}(t)}^{t}A_{i\tilde{j}k}^{(1)}(s)ds+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{jil}^{(1)}(t)B_{i\tilde{j}\tilde{l}}^{(2)}(t)\Big)$$
$$+\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2 M_{i0}^{2\beta_{jil}}}{\beta_{iil}}\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\big(B_{jil}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t)+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\big(B_{jil}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t)\Big)\Big].$$

Then the solution $X^*(t)=(x_1^*(t),x_2^*(t),...,x_n^*(t))$ of $(1.1)-(1.2)$ is globally attractive.

Proof. Let $X^*(t)=\big(x_1^*(t),...,x_n^*(t)\big)$ with $x_i^*(t_0)>0$ be a positive solution of (1.1), and $X(t)=\big(x_1(t),...,x_n(t)\big)$ with $x_i(t_0)>0$ be an any given solution of system (1.1). In order to show the global attractivity of the bounded solution $X^*(t)$ of system (1.1), we shall show that the solution $U^*(t)=(u_1^*(t),\ ...,\ u_n^*(t)$ of system (4.1) is globally attractive. Let $U(t)=(u_1(t)\ ,...,\ u_n(t))$ be any other positive solution of system (4.1). According to Theorem 3.1, there exist positive constants $m_{i0},\ M_{i0}\ (i=1,2,...,n)$ and enough large $T>0$ such that for all $t\geq T$, there are

$$m_{i0}\leq u_i(t),\ \ u_i^*(t)\leq M_{i0}\ (i=1,2,...,n).\tag{4.2}$$

Obviously, So to prove the global attractivty of the system (1.1), it is suffices to verify that system (4.1) is globally attractive. Firstly, construct a Lyapunov functional as follows

$$V_1(t)=\sum_{i=1}^{n}\lambda_i\Big[\big(u_i(t)-u_i^*(t)\big)-\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(t)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)ds$$
$$-\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(\theta)\big\}-\exp\big\{\beta_{ijl}u_j^*(\theta)\big\}\Big)d\theta ds\Big]^2.$$

By calculating the right upper derivative of $V_1(t)$, we find

$$\dot{V}_1(t)=-2\sum_{i=1}^{n}\lambda_i\Big[\big(u_i(t)-u_i^*(t)\big)-\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)ds$$
$$-\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(\theta)\big\}-\exp\big\{\beta_{ijl}u_j^*(\theta)\big\}\Big)d\theta ds\Big]$$
$$\times\Big[\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}A_{ijk}^{(1)}(t)\Big(\exp\big\{\alpha_{ijk}u_j(t)\big\}-\exp\big\{\alpha_{ijk}u_j^*(t)\big\}\Big)$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}B_{ijl}^{(1)}(t)\Big(\exp\big\{\beta_{ijl}u_j(t)\big\}-\exp\big\{\beta_{ijl}u_j^*(t)\big\}\Big)\Big]$$
$$\leq-2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}}\lambda_i A_{iik}^{(1)}(t)\Big(\exp\big\{\alpha_{iik}u_i(t)\big\}-\exp\big\{\alpha_{iik}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big)$$

$$
\begin{aligned}
&-2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)\Big(\exp\big\{\beta_{iil}u_i(t)\big\}-\exp\big\{\beta_{iil}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}}\lambda_j A_{jik}^{(1)}(t)\Big(\exp\big\{\alpha_{jik}u_i(t)\big\}-\exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}}\lambda_j B_{jil}^{(1)}(t)\Big(\exp\big\{\beta_{jil}u_i(t)\big\}-\exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_j(t)\big\}-\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_j^*(t)\big\}\Big)\Big]\\
&\qquad\times\Big[\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)ds\Big]\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_j(t)\big\}-\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_j^*(t)\big\}\Big)\Big]\\
&\qquad\times\Big[\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^*(s)\big\}\Big)d\theta ds\Big]\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_j(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_j^*(t)\big\}\Big)\Big]\\
&\qquad\times\Big[\sum_{j=1,j\neq\tilde{j}}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)ds\Big]\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_j(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_j^*(t)\big\}\Big)\Big]\\
&\qquad\times\Big[\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^*(s)\big\}\Big)d\theta ds\Big].
\end{aligned}\tag{4.3}
$$

That is

$$
\begin{aligned}
\dot{V}_1(t)\le&-2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}}\lambda_i A_{iik}^{(1)}(t)\Big(\exp\big\{\alpha_{iik}u_i(t)\big\}-\exp\big\{\alpha_{iik}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big)\\
&-2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)\Big(\exp\big\{\beta_{iil}u_i(t)\big\}-\exp\big\{\beta_{iil}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}}\lambda_j A_{jik}^{(1)}(t)\Big(\exp\big\{\alpha_{jik}u_i(t)\big\}-\exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}}\lambda_j B_{jil}^{(1)}(t)\Big(\exp\big\{\beta_{jil}u_i(t)\big\}-\exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\big\}-\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}^*(t)\big\}\Big)\\
&\qquad\times\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)ds\\
&+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\big\}-\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}^*(t)\big\}\Big)\Big]\\
&\qquad\times\Big[\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^*(s)\big\}\Big)d\theta ds
\end{aligned}
$$

$$
\begin{aligned}
&+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^{*}(t)\big\}\Big)\\
&\qquad\times\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^{*}(s)\big\}\Big)ds\\
&+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^{*}(t)\big\}\Big)\\
&\qquad\times\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^{*}(s)\big\}\Big)d\theta ds. \qquad (4.4)
\end{aligned}
$$

By further using the inequality $a^2+b^2\geq 2ab$, it follows from (4.4) that

$$
\begin{aligned}
\dot{V}_1(t)\leq{}&-2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}}\lambda_i A_{iik}^{(1)}(t)\Big(\exp\big\{\alpha_{iik}u_i(t)\big\}-\exp\big\{\alpha_{iik}u_i^{*}(t)\big\}\Big)\big(u_i(t)-u_i^{*}(t)\big)\\
&-2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)\Big(\exp\big\{\beta_{iil}u_i(t)\big\}-\exp\big\{\beta_{iil}u_i^{*}(t)\big\}\Big)\big(u_i(t)-u_i^{*}(t)\big)\\
&+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}}\lambda_j A_{jik}^{(1)}(t)\Big[\Big(\exp\big\{\alpha_{jik}u_i(t)\big\}-\exp\big\{\alpha_{jik}u_i^{*}(t)\big\}\Big)^2+\big(u_j(t)-u_j^{*}(t)\big)^2\Big]\\
&+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}}\lambda_j B_{jil}^{(1)}(t)\Big[\Big(\exp\big\{\beta_{jil}u_i(t)\big\}-\exp\big\{\beta_{jil}u_i^{*}(t)\big\}\Big)^2+\big(u_j(t)-u_j^{*}(t)\big)^2\Big]\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big[\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)ds\Big(\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\big\}-\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}^{*}(t)\big\}\Big)^2\\
&\qquad+\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^{*}(s)\big\}\Big)^2ds\Big]\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big[B_{ijl}^{(2)}(t)\Big(\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\big\}-\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}^{*}(t)\big\}\Big)^2\\
&\qquad+\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^{*}(s)\big\}\Big)^2d\theta ds\Big]\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big[\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)ds\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^{*}(t)\big\}\Big)^2\\
&\qquad+\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^{*}(s)\big\}\Big)^2ds\Big]\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big[B_{ijl}^{(2)}(t)\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^{*}(t)\big\}\Big)^2\\
&\qquad+\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^{*}(s)\big\}\Big)^2d\theta ds\Big]
\end{aligned}
$$

Now let us define the Lyapunov functional $V_2(t)$ as follows

$$
\begin{aligned}
V_2(t)={}&\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)\big(\Phi_{ijk}^{-1}\big)\int_{s}^{t}A_{ijk}^{(1)}(r)\\
&\qquad\times\Big(\exp\big\{\alpha_{ijk}u_j(r)\big\}-\exp\big\{\alpha_{ijk}u_j^{*}(r)\big\}\Big)^2drds
\end{aligned}
$$

$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}A_{i\tilde{j}\tilde{k}}^{(1)}(\theta-s)\int_{\theta}^{t}b_{ijl}(r-s,s)$$
$$\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2 drd\theta ds$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{ijk}^{-1}(\theta)\big)\int_{\theta}^{t}A_{ijk}^{(1)}(r)$$
$$\times\Big(\exp\big\{\alpha_{ijk}u_j(r)\big\}-\exp\big\{\alpha_{ijk}u_j^*(r)\big\}\Big)^2 drd\theta$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}(\theta-s)\int_{\theta}^{t}b_{ijl}(r-s,s)$$
$$\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2 drd\theta ds.$$

Calculating the derivative of $V_2(t)$ along the positive solution of system (1.1), it follows:

$$\dot{V}_2(t)=\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)dsA_{ijk}^{(1)}(t)$$
$$\times\Big(\exp\big\{\alpha_{ijk}u_j(t)\big\}-\exp\big\{\alpha_{ijk}u_j^*(t)\big\}\Big)^2$$
$$-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_iA_{i\tilde{j}\tilde{k}}^{(1)}(t)\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)$$
$$\times\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)^2 ds$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\big(B_{ijl}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t)\Big(\exp\big\{\beta_{ijl}u_j(t)\big\}-\exp\big\{\beta_{ijl}u_j^*(t)\big\}\Big)^2$$
$$-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_iA_{i\tilde{j}\tilde{k}}^{(1)}(t)\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(r-s,s)$$
$$\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2 drds$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{ijk}^{-1}(\theta)\big)d\theta A_{ijk}^{(1)}(t)$$
$$\times\Big(\exp\big\{\alpha_{ijk}u_j(t)\big\}-\exp\big\{\alpha_{ijk}u_j^*(t)\big\}\Big)^2$$
$$-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_iB_{i\tilde{j}\tilde{l}}^{(1)}(t)\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(r)$$
$$\times\Big(\exp\big\{\alpha_{ijk}u_j(r)\big\}-\exp\big\{\alpha_{ijk}u_j^*(r)\big\}\Big)^2 dr$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\big(B_{ijl}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t)\Big(\exp\big\{\beta_{ijl}u_j(t)\big\}-\exp\big\{\beta_{ijl}u_j^*(t)\big\}\Big)^2$$
$$-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_iB_{i\tilde{j}\tilde{l}}^{(1)}(t)\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(r-s,s)$$
$$\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2 drds. \tag{4.5}$$

Finally, we consider the following Lyapunov functional $V(t)$

$$V(t) = V_1(t) + V_2(t). \tag{4.6}$$

Calculating the upper right derivative of $V(t)$ along the solution of system (1.2), and integrating with the above-mentioned analysis, one claims that

$$\begin{aligned}
D^+V(t) \le& -2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}} \lambda_i A^{(1)}_{iik}(t)\Big(\exp\big\{\alpha_{iik}u_i(t)\big\} - \exp\big\{\alpha_{iik}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big) \\
&-2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}} \lambda_i B^{(1)}_{iil}(t)\Big(\exp\big\{\beta_{iil}u_i(t)\big\} - \exp\big\{\beta_{iil}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big) \\
&+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}} \lambda_j A^{(1)}_{jik}(t)\Big(\exp\big\{\alpha_{jik}u_i(t)\big\} - \exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)^2 \\
&+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}} \lambda_j B^{(1)}_{jil}(t)\Big(\exp\big\{\beta_{jil}u_i(t)\big\} - \exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)^2 \\
&+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ij}} \lambda_i A^{(1)}_{ijk}(t)\big(u_i(t)-u_i^*(t)\big)^2 + \sum_{l=1}^{l_{ij}} \lambda_i B^{(1)}_{ijl}(t)\big(u_i(t)-u_i^*(t)\big)^2\Big] \\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{k_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}} \lambda_{\tilde{j}} A^{(1)}_{\tilde{j}i\tilde{k}}(t)\int_{t-\tau_{ijk}(t)}^{t} A^{(1)}_{ijk}(s)ds\Big(\exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i(t)\big\} - \exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i^*(t)\big\}\Big)^2 \\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{k_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} \lambda_{\tilde{j}} A^{(1)}_{\tilde{j}i\tilde{k}}(t)B^{(2)}_{ijl}(t)\Big(\exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i(t)\big\} - \exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i^*(t)\big\}\Big)^2 \\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{l_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}} \lambda_{\tilde{j}} B^{(1)}_{\tilde{j}i\tilde{l}}(t)\int_{t-\tau_{ijk}(t)}^{t} A^{(1)}_{ijk}(s)ds\Big(\exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i(t)\big\} - \exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i^*(t)\big\}\Big)^2 \\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{l_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} \lambda_{\tilde{j}} B^{(1)}_{\tilde{j}i\tilde{l}}(t)B^{(2)}_{ijl}(t)\Big(\exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i(t)\big\} - \exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i^*(t)\big\}\Big)^2 \\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{k_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ji}} \lambda_j \int_{t-\tau_{jik}(t)}^{t} A^{(2)}_{i\tilde{j}\tilde{k}}(s)dsA^{(1)}_{jik}(t)\Big(\exp\big\{\alpha_{jik}u_i(t)\big\} - \exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)^2 \\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{k_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ji}} \lambda_j \big(B^{(2)}_{jil}\cdot A^{(1)}_{i\tilde{j}\tilde{k}}\big)(t)\Big(\exp\big\{\beta_{jil}u_i(t)\big\} - \exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)^2 \\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{l_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ji}} \lambda_j \int_{t-\tau_{jik}(t)}^{t} B^{(1)}_{i\tilde{j}\tilde{l}}\big(\Phi^{-1}_{jik}(\theta)\big)d\theta A^{(1)}_{jik}(t)\Big(\exp\big\{\alpha_{jik}u_i(t)\big\} - \exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)^2 \\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{l_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ji}} \lambda_j \big(B^{(2)}_{jil}\cdot B^{(1)}_{i\tilde{j}\tilde{l}}\big)(t)\Big(\exp\big\{\beta_{jil}u_i(t)\big\} - \exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)^2.
\end{aligned} \tag{4.7}$$

Meanwhile, by making use of mean value theorem, we can obtain that for any given positive number $\epsilon > 0$, there are

$$\begin{aligned}
&\exp\big\{\epsilon u_i(t)\big\} - \exp\big\{\epsilon u_i^*(t)(t)\big\} = \epsilon\exp\{\epsilon\vartheta_i^{(1)}(t)\}\big(u_i(t)-u_i^*(t)\big), \\
&\exp\big\{\epsilon u_i(t)\big\} - \exp\big\{\epsilon u_i^*(t)\big\} = \frac{\epsilon}{\alpha_{iik}}\exp\{\epsilon\vartheta_i^{(2)}(t)\} \\
&\qquad\qquad \times\big(\exp\big\{\alpha_{iik}u_i(t)\big\} - \exp\big\{\alpha_{iik}u_i^*(t)\big\}\big), \\
&\exp\big\{\epsilon u_i(t)\big\} - \exp\big\{\epsilon u_i^*(t)\big\} = \frac{\epsilon}{\beta_{iil}}\exp\{\epsilon\vartheta_i^{(3)}(t)\}
\end{aligned}$$

$$\times\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big). \tag{4.8}$$

Where $\vartheta_i^{(1)}(t),\ \vartheta_i^{(2)}(t),\ \vartheta_i^{(3)}(t)$ are all lie between $u_i(t)$ and $u_i^*(t)$. Thus, it follows from (4.2) and (4.8) that for any given positive number $\epsilon>0$, we have

$$\begin{aligned}&\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)(t)\}\ge \epsilon m_{i0}^{\epsilon}\big(u_i(t)-u_i^*(t)\big),\\&\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)(t)\}\le \epsilon M_{i0}^{\epsilon}\big(u_i(t)-u_i^*(t)\big).\end{aligned} \tag{4.9}$$

$$\begin{aligned}\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\ge{}&\frac{\epsilon}{\alpha_{iik}}m_{i0}^{\epsilon}\\&\times\big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\big),\\\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\le{}&\frac{\epsilon}{\alpha_{iik}}M_{i0}^{\epsilon}\\&\times\Big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\Big).\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\ge{}&\frac{\epsilon}{\beta_{iil}}m_{i0}^{\epsilon}\\&\times\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big),\\\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\le{}&\frac{\epsilon}{\beta_{iil}}M_{i0}^{\epsilon}\\&\times\Big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\Big).\end{aligned} \tag{4.11}$$

Inequality (4.7), (4.9), (4.10) and (4.11) implies that for $t\ge T_1$

$$\begin{aligned}D^+V(t)\le{}&\sum_{i=1}^n\Big\{\sum_{k=1}^{k_{ii}}-2\lambda_iA_{iik}^{(1)}(t)+\sum_{j=1,j\ne i}^n\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)+\sum_{k=1}^{k_{ij}}\frac{\lambda_i}{\alpha_{iik}m_{i0}^{\alpha_{iik}}}A_{ijk}^{(1)}(t)\Big]\\&+\sum_{j=1}^n\sum_{\tilde{j}=1}^n\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}\tilde{k}}(t)}^tA_{i\tilde{j}\tilde{k}}^{(1)}(s)ds+\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^tA_{i\tilde{j}\tilde{k}}^{(2)}(s)ds\Big)\\&+\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^tB_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big)d\theta+\sum_{l=1}^{l_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(2)}(t)\Big)\Big]\Big\}\\&\times\big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big)\\&+\sum_{i=1}^n\Big\{-2\sum_{l=1}^{l_{ii}}\lambda_iB_{iil}^{(1)}(t)+\sum_{j=1,j\ne i}^n\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B_{jil}^{(1)}(t)+\sum_{l=1}^{l_{ij}}\frac{\lambda_i}{\beta_{iil}m_{i0}^{\beta_{iil}}}B_{ijl}^{(1)}(t)\Big]\\&+\sum_{j=1}^n\sum_{\tilde{j}=1}^n\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B_{jil}^{(1)}(t)\Big(\sum_{k=1}^{k_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}k}(t)}^tA_{i\tilde{j}k}^{(1)}(s)ds+\sum_{\tilde{l}=1}B_{jil}^{(1)}(t)B_{i\tilde{j}\tilde{l}}^{(2)}(t)\Big)\\&+\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\big(B_{jil}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t)+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\big(B_{jil}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t)\Big)\Big]\Big\}\\&\times\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big).\\=:{}&-\sum_{i=1}^n\Lambda_i(t)\mid\big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big)\mid\\&-\sum_{i=1}^n\Delta_i(t)\mid\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big)\mid.\end{aligned} \tag{4.12}$$

At the same time, according to hypotheses (H_6) of Theorem 4.1, we declare that there exists a constant $\zeta>0$ such that $\Lambda_i(t),\ \Delta_i(t)>\zeta$, so it follows from (4.12) that $V(t)$ is nonincreasing, and it not difficult to see that $\dot{u}_i(t)$ are bounded for $t\ge T_1$. Hence, one can further infer that $\mid u_i(t)-u_i^*(t)\mid,\ \mid\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\mid,\ \mid\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\mid$ are

uniformly continuous on $[T_1,+\infty)$. An integration on both sides of (4.10) over time interval $[T_1,t)$ leads to

$$V(t)+\zeta\sum_{i=1}^{n}\int_{T_1}^{t}\Big[\mid\big(\exp\{\alpha_{iik}u_i(s)\}-\exp\{\alpha_{iik}u_i^*(s)\}\big)\big(u_i(s)-u_i^*(s)\big)\mid$$
$$+\mid\big(\exp\{\beta_{iil}u_i(s)\}-\exp\{\beta_{iil}u_i^*(s)\}\big)\big(u_i(s)-u_i^*(s)\big)\mid\Big]ds\le V(T_1)<+\infty.$$

Thus

$$\limsup_{t\to+\infty}\sum_{i=1}^{n}\int_{T_1}^{t}\Big[\mid\big(\exp\{\alpha_{iik}u_i(s)\}-\exp\{\alpha_{iik}u_i^*(s)\}\big)\big(u_i(s)-u_i^*(s)\big)\mid$$
$$+\mid\big(\exp\{\beta_{iil}u_i(s)\}-\exp\{\beta_{iil}u_i^*(s)\}\big)\big(u_i(s)-u_i^*(s)\big)\mid\Big]ds\le\frac{V(T_1)}{\zeta}<+\infty.\tag{4.13}$$

It follows from (4.13) that

$$\mid\big(\exp\{\alpha_{iik}u_i(s)\}-\exp\{\alpha_{iik}u_i^*(s)\}\big)\big(u_i(s)-u_i^*(s)\big)\mid\in L[T_1,+\infty),$$

$$\mid\big(\exp\{\beta_{iil}u_i(s)\}-\exp\{\beta_{iil}u_i^*(s)\}\big)\big(u_i(s)-u_i^*(s)\big)\mid\in L[T_1,+\infty).$$

According to Barbalat's lemma, we conclude that

$$\lim_{t\to+\infty}\mid\big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big)\mid=0.\tag{4.14}$$

$$\lim_{t\to+\infty}\mid\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big)\mid=0.\tag{4.15}$$

By way of contradiction, it easy to obtain from (4.14) and (4.15) that

$$\lim_{t\to+\infty}\mid u_i(t)-u_i^*(t)\mid=0.\tag{4.16}$$

Therefore, the positive solution $X^*(t)$ of the system (1.1) is also globally attractive. This completes the proof.

Theorem 4.2. In addition to $(H_1)-(H_5)$, we assume further that
$(H_6)'$ There exist positive constants $\lambda_i>0$ $(i=1,2,...,n)$, $\zeta>0$ such that

$$\liminf_{t\to+\infty}\{\Lambda_i(t)\}>\zeta.$$

Where
$$\Lambda_i(t)=2\sum_{k=1}^{k_{ii}}\lambda_i\alpha_{iik}m_{i0}^{\alpha_{iik}}A_{iik}^{(1)}(t)-\sum_{j=1,j\ne i}^{n}\Big[\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)+\sum_{k=1}^{k_{ij}}\lambda_iA_{ijk}^{(1)}(t)\Big]$$
$$+2\sum_{l=1}^{l_{ii}}\lambda_i\beta_{iil}m_{i0}^{\beta_{iil}}B_{iil}^{(1)}(t)-\sum_{j=1,j\ne i}^{n}\Big[\sum_{l=1}^{l_{ji}}\lambda_jB_{jil}^{(1)}(t)\beta_{jil}^2M_{i0}^{2\beta_{jil}}+\sum_{l=1}^{l_{ij}}\lambda_iB_{ijl}^{(1)}(t)\Big]$$
$$-\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}\tilde{k}}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(1)}(s)ds+\sum_{l=1}^{l_{i\tilde{j}}}B_{i\tilde{j}l}^{(2)}(t)\Big)$$
$$+\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{\tilde{j}i}}\int_{t-\tau_{jik}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)ds+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big)d\theta\Big)$$
$$+\sum_{l=1}^{l_{ji}}\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}B_{jil}^{(1)}(t)\Big(\sum_{k=1}^{k_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}k}(t)}^{t}A_{i\tilde{j}k}^{(1)}(s)ds+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(2)}(t)\Big)$$
$$+\sum_{l=1}^{l_{ji}}\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}(B_{jil}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)})(t)+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}(B_{jil}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)})(t)\Big)\Big].$$

Then the solution $X^*(t)=(x_1^*(t),x_2^*(t),...,x_n^*(t))$ of $(1.1)-(1.2)$ is globally attractive.

Proof. Let $U^*(t)=(u_1^*(t),\ ...,\ u_n^*(t)$ be the solution of system (4.1), and $U(t)=(u_1(t)\ ,...,\ u_n(t))$

be any other positive solution of system (4.1). Then for the Lyapunov functional $V(t)$ as defined in (4.6), similarly to the discuss of Theorem 4.1, one can obtain that the inequality (4.7) is true. By further making use of (4.9), (4.10) and (4.11), it follows that (4.7) implies

$$\begin{aligned}
D^+V(t) \le \sum_{i=1}^{n}\Big\{ & -2\sum_{k=1}^{k_{ii}}\lambda_i\alpha_{iik}m_{i0}^{\alpha_{iik}}A_{iik}^{(1)}(t) + \sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t) + \sum_{k=1}^{k_{ij}}\lambda_iA_{ijk}^{(1)}(t)\Big] \\
& -2\sum_{l=1}^{l_{ii}}\lambda_i\beta_{iil}m_{i0}^{\beta_{iil}}B_{iil}^{(1)}(t) + \sum_{j=1,j\neq i}^{n}\Big[\sum_{l=1}^{l_{ji}}\lambda_jB_{jil}^{(1)}(t)\beta_{jil}^2M_{i0}^{2\beta_{jil}} + \sum_{l=1}^{l_{ij}}\lambda_iB_{ijl}^{(1)}(t)\Big] \\
& +\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}\tilde{k}}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(1)}(s)ds + \sum_{l=1}^{l_{i\tilde{j}}}B_{i\tilde{j}l}^{(2)}(t)\Big) \\
& +\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{\tilde{j}i}}\int_{t-\tau_{jik}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)ds + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big)d\theta\Big) \\
& +\sum_{l=1}^{l_{ji}}\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}B_{jil}^{(1)}(t)\Big(\sum_{k=1}^{k_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}k}(t)}^{t}A_{i\tilde{j}k}^{(1)}(s)ds + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(2)}(t)\Big) \\
& +\sum_{l=1}^{l_{ji}}\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}(B_{jil}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)})(t) + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}(B_{jil}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)})(t)\Big)\Big]\Big\}\big(u_i(t)-u_i^*(t)\big)^2. \\
& =: -\sum_{i=1}^{n}\Lambda_i(t)\big(u_i(t)-u_i^*(t)\big)^2
\end{aligned} \tag{4.17}$$

An integration on both sides of (4.17) over time interval $[T_1, t)$ leads to

$$V(t) + \zeta\sum_{i=1}^{n}\int_{T_1}^{t}\big(u_i(s)-u_i^*(s)\big)^{|}2ds \le V(T_1) < +\infty.$$

Thus

$$\limsup_{t\to+\infty}\sum_{i=1}^{n}\int_{T_1}^{t}\big(u_i(s)-u_i^*(s)\big)^{|}2ds \le \frac{V(T_1)}{\zeta} < +\infty. \tag{4.18}$$

It follows from (4.18) that

$$\big(u_i(s)-u_i^*(s)\big)^2 \in L[T_1, +\infty),$$

According to Barbalat's lemma, we conclude that

$$\lim_{t\to+\infty}\big(u_i(t)-u_i^*(t)\big)^2 = 0. \tag{4.19}$$

Taking into account the fact that for $t \ge T_1$

$$\big(x_i(t)-x_i^*(t)\big) = \exp\big\{u_i(t)\big\} - \exp\big\{u_i^*(t)\big\}$$

One infers that

$$(m_{i0}) \mid u_i(t)-u_i^*(t) \mid \le \mid x_i(t)-x_i^*(t) \mid \le (M_{i0}) \mid u_i(t)-u_i^*(t) \mid$$

So it follows that

$$\lim_{t\to+\infty} \mid x_i(t)-x_i^*(t) \mid = 0. \tag{4.20}$$

Thus, we have verified that the positive solution $X^*(t)$ of the system (1.1) is globally attractive.

Acknowledgments

The research was supported by the National Natural Science Foundation of China [11501498, 11526183] and the Natural Science Foundation of Shanxi Province [2015021015].

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.6, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Some approximations of the Bateman's $G-$function

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Abstract

In the paper, we presented a family $M(\mu, x)$ of approximations of the Bateman function $G(x)$. The family $M(\mu, x) = G(x)$ for a certain μ whenever x is fixed and it presented asymptotical approximation of the Bateman's $G-$function as $x \to \infty$. We studied the order of convergence of the approximations $M(\mu, x)$ of the function $G(x)$. Some properties and bounds of the error are deduced. We presented new sharp double inequality of $G(x)$ with the upper and lower bounds $M(1, x)$ and $M(\frac{4}{e^2-4}, x)$ (resp.). Also, we show that the approximations $M(\mu, x)$ are better than the approximation $\frac{1}{x} + \frac{1}{2x^2}$ for any μ in an open subinterval of $\left[1, \frac{4}{e^2-4}\right]$.

2010 Mathematics Subject Classification: 33B15, 26D15.

Key Words: Bateman function, digamma function, monotonicity, sharp inequality, approximation, error.

1 Introduction.

In 1953, Erdélyi [6] defined the Bateman's $G-$function as

$$G(x) = \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right), \qquad x \neq 0, -1, -2, ... \tag{1}$$

where the digamma function $\psi(x)$ is given by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x)$$

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and $\Gamma(x)$ is the ordinary gamma function defined by [3]

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad x > 0.$$

The function $G(x)$ is very useful in estimating and summing certain numerical and algebraic series [18]. For more details on bounding the function $\Gamma(x)$ and its logarithmic derivatives $\psi^{(n)}(x)$, please refer to the papers [2]-[5], [7]-[23] and plenty of references therein.

The function $G(x)$ can be also defined by

$$G(x) = \frac{2}{x}\, {}_2F_1(1, x; 1+x; -1),$$

where

$${}_rF_s(a_1, ..., a_r; b_1, ..., b_s; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k ...(a_r)_k}{(b_1)_k ...(b_s)_k} \frac{x^k}{k!}$$

is the generalized hypergeometric series [1] defined for $r, s \in \mathbb{N}$, $a_j \in \mathbb{C}$, $b_j \in \mathbb{C} - \{0, -1, -2, ...\}$ and the Pochhammer symbol $(a)_n$ is defined by

$$(a)_0 = 1 \quad \text{and} \quad (a)_n = \prod_{i=0}^{n-1} (a+i) = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad n \geq 1.$$

The function $G(x)$ satisfies the functional equation [6]:

$$G(1+x) = -G(x) + \frac{2}{x} \tag{2}$$

and it has the integral representation

$$G(x) = 2\int_0^\infty \frac{e^{-xt}}{1+e^{-t}} dt, \qquad x > 0 \tag{3}$$

which can be deduced from the following known integral representation of the digamma [3]

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) dt, \qquad x > 0.$$

Qiu and Vuorinen [24] deduced the inequality

$$\frac{1}{x} + \frac{4(1.5 - \log 4)}{x^2} < G(x) < \frac{1}{x} + \frac{1}{2x^2}, \qquad x > 1/2. \tag{4}$$

Mahmoud and Agarwal [9] presented the following asymptotic formula for Bateman's G-function

$$G(x) \sim \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(2^{2k}-1)B_{2k}}{kx^{2k}}, \qquad x \to \infty \tag{5}$$

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and they deduced the double inequality

$$\frac{1}{2x^2+1.5} < G(x) - \frac{1}{x} < \frac{1}{2x^2}, \qquad x > 0 \tag{6}$$

which improve the lower bound of the inequality (4). Also, Mahmoud and Almuashi [11] proved that the Bateman's $G-$function satisfies the double inequality

$$\sum_{n=1}^{2m} \frac{(2^n-1)B_{2n}}{nx^{2n}} < G(x) - \frac{1}{x} < \sum_{n=1}^{2m-1} \frac{(2^n-1)B_{2n}}{nx^{2n}}, \qquad m \in \mathbb{N} \tag{7}$$

with best bounds, where $B_r{}'s$ are the Bernoulli numbers and they presented some estimates for the error term of a class of the alternating series, which improve and generalize some recent results. Mortici [13] established the inequality

$$0 < \psi(x+v) - \psi(x) \le \psi(v) + \gamma + \frac{1}{v} - v \qquad x \ge 1;\ 0 < v < 1, \tag{8}$$

where γ is the Euler constant, which also improves the inequality (4) of Qiu and Vuorinen. Also, Alzer presented the double inequality [2]

$$\frac{1}{x} - T_n(v;x) - \rho_n(v;x) < \psi(x+v) - \psi(x) < \frac{1}{x} - T_n(v;x),$$

where $n \ge 0$ be an integer, $x > 0$, $0 < v < 1$,

$$T_n(v;x) = (1-v)\left[\frac{1}{v+n+1} + \sum_{i=0}^{n-1} \frac{1}{(x+i+1)(x+i+v)}\right]$$

and

$$\rho_n(v;x) = \frac{1}{x+n+v} \log \frac{(x+n)^{(x+n)(1-v)}(x+n+1)^{(x+n+1)v}}{(x+n+v)^{x+n+v}}.$$

In 2006, Muqattash and Yahdi [17] presented an infinite family of functions $I_a(x) = \psi(x)$ for a certain a when x is fixed. Local and global bounding error functions are found and new inequalities for the Digamma function are introduced. These functions are shown to approximate ψ locally and asymptotically. The approximations are compared to another approximations of the Digamma function. The technique of construct of Muqattash and Yahdi is very useful and can be updated to another functions as we will see in this paper.

In 2014, Guo and Qi improved the results of [8] and presented the two sharp inequalities

$$\ln\left(x+\frac{1}{2}\right) < \psi(x) + \frac{1}{x} < \ln\left(x+e^{-\gamma}\right), \qquad x > 0$$

where the constants $\frac{1}{2}$ and $e^{-\gamma}$ are the best possible, and

$$\ln\left(n+\frac{1}{2}\right) + \gamma < H_n(n) < \ln\left(n+e^{1-\gamma}-1\right) + \gamma, \qquad n \in \mathbb{N}$$

where the n-th harmonic numbers are defined by

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \qquad n \in \mathbb{N}$$

and is related to the Psi function by the relation

$$H_n = \gamma + \psi(n+1).$$

In this paper, we presented a family of functions $M(\mu, x)$ satisfies that for all $x > 0$ there exists $\mu \in [1, 2]$ such that $M(\mu, x) = G(x)$ and is asymptotically equivalent to $G(x)$ as $x \to \infty$. We proved that the approximations $M(\mu, x)$ of the function $G(x)$ are of an order of convergence of $O\left(\ln \frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]}\right)$ for $x > 2$ and $\mu \in \left(1, \frac{4}{e^2-4}\right)$. Some properties and bounds of the error are deduced. Also, we presented a new sharp double inequality of the function $G(x)$ between the lower bound $M(\frac{4}{e^2-4}, x)$ and the upper bound $M(1, x)$. We proved that the approximations $M(\mu, x)$ are better than the approximation $\frac{1}{x} + \frac{1}{2x^2}$ for any μ in an open subinterval of $\left[1, \frac{4}{e^2-4}\right]$.

2 Main Results

Lemma 2.1. *For $x > 0$, we have*

$$\ln\left(1 + \frac{1}{x+2}\right) + \frac{2}{x(x+1)} \le G(x) \le \ln\left(1 + \frac{1}{x+1}\right) + \frac{2}{x(x+1)}. \tag{9}$$

Proof. Consider the function

$$H_\mu(x) = \ln\left(1 + \frac{1}{x+\mu}\right) + \frac{2}{x(x+1)} - G(x), \qquad x > 0;\ \mu > 0$$

which can be represented using (3) by the integral formula

$$H_\mu(x) = \int_0^\infty \frac{e^{-(\mu+1)t}[e^{2t} - 1 - 2te^{\mu t}]}{t(1+e^t)} e^{-xt} dt.$$

The function $m_1(t) = e^{2t} - 1 - 2te^t$ is strictly increasing pass through the origin, then $H_1(x) > 0$, that is

$$\ln\left(1 + \frac{1}{x+1}\right) + \frac{2}{x(x+1)} > G(x).$$

Also, $m_2(t) = e^{2t} - 1 - 2te^{2t}$ is strictly decreasing function pass through the origin, then $H_2(x) < 0$, that is

$$\ln\left(1 + \frac{1}{x+2}\right) + \frac{2}{x(x+1)} < G(x).$$

□

The double inequality (9) show that the function $G(x)$ lies between two functions of the following family of functions

$$M(\mu, x) = \ln\left(1 + \frac{1}{x+\mu}\right) + \frac{2}{x(x+1)} \qquad x > 0;\ \mu > 0. \tag{10}$$

and hence we can conclude the following result:

Theorem 1. *For every $x > 0$, there exists $\mu \in [1, 2]$ such that*

$$M(\mu, x) = G(x).$$

Proof. For a positive fixed x, consider the function $M_2(\mu) = M(\mu, x)$ with $1 \le \mu \le 2$ and $G(x) = \lambda$. $M_2(\mu)$ is a continuous on $[1, 2]$ and using the inequality (9), we obtain

$$M_2(2) \le \lambda \le M_2(1).$$

Then by the Intermediate Value Theorem, there exists $\mu \in [1, 2]$ such that $M_2(\mu) = \lambda$. □

Also, by using the relations

$$\frac{\partial M(\mu, x)}{\partial x} = -\frac{2\mu + 2\mu^2 + 2x + 8\mu x + 4\mu^2 x + 7x^2 + 8\mu x^2 + 6x^3 + x^4}{x^2(1+x)^2(\mu + \mu^2 + x + 2\mu x + x^2)} < 0$$

and

$$\frac{\partial M(\mu, x)}{\partial \mu} = \frac{-1}{(x+\mu+1)(x+\mu)} < 0,$$

we obtain the following properties of the family $M(\mu, x)$.

Lemma 2.2.

1. *$M_1(x) = M(\mu, x)$ is a positive and strictly decreasing as a function of x, $x > 0$.*
2. *$M_2(\mu) = M(\mu, x)$ is strictly decreasing as a function of μ, $1 \le \mu \le 2$*

and hence

$$0 < M(2, x) \le M(\mu, x) \le M(1, x), \qquad x > 0;\ \mu \in [1, 2]. \tag{11}$$

Now, we will show that the family $M(\mu, x)$ presented asymptotical approximation of the Bateman's $G-$function for all $\mu \in [1, 2]$.

Theorem 2. *For all $\mu \in [1, 2]$, the Bateman's $G-$function and the family $M(\mu, x)$ are asymptotically equivalent as $x \to \infty$, that is*

$$\lim_{x\to\infty} \frac{G(x)}{M(\mu, x)} = 1$$

and this is written symbolically as $G(x) \sim M(\mu, x)$.

Proof. Using the inequality (9), we get

$$M(2,x) \leq G(x) \leq M(1,x) \tag{12}$$

and hence

$$\frac{M(2,x)}{M(1,x)} \leq \frac{G(x)}{M(1,x)} \leq 1.$$

But

$$\lim_{x\to\infty} \frac{M(2,x)}{M(1,x)} = \frac{12+34x+23x^2+6x^3+x^4}{(3+x)(4+10x+5x^2+x^3)} = 1$$

and then

$$\lim_{x\to\infty} \frac{G(x)}{M(1,x)} = 1. \tag{13}$$

Similarly, we have

$$\lim_{x\to\infty} \frac{G(x)}{M(2,x)} = 1. \tag{14}$$

Using the inequality (11), we obtain

$$\frac{G(x)}{M(1,x)} \leq \frac{G(x)}{M(\mu,x)} \leq \frac{G(x)}{M(2,x)}. \tag{15}$$

From (13), (14) and (15), we get

$$1 \leq \lim_{x\to\infty} \frac{G(x)}{M(\mu,x)} \leq 1.$$

□

Now, we will study the error of the approximation $M(\mu,x)$ of the function $G(x)$.

Theorem 3. *For any $\mu \in [1,2]$, the error*

$$e_\mu(x) = G(x) - M(\mu,x)$$

approaches zero as $x \to \infty$ and

$$G(x) = \ln\left(1+\frac{1}{x+\mu}\right) + \frac{2}{x(x+1)} + \mathrm{O}\left(\ln\left(1+\frac{1}{(x+1)(x+3)}\right)\right). \tag{16}$$

Proof. From inequality (12), we have

$$M(2,x) - M(\mu,x) \leq G(x) - M(\mu,x) \leq M(1,x) - M(\mu,x)$$

and using (11), we get

$$M(2,x) - M(1,x) \leq M(2,x) - M(\mu,x).$$

Hence

$$0 \leq |G(x) - M(\mu,x)| \leq M(1,x) - M(2,x) \tag{17}$$

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or

$$0 \leq |e_\mu(x)| \leq \ln\left(1 + \frac{1}{(x+1)(x+3)}\right). \tag{18}$$

Then

$$G(x) = M(\mu, x) + \mathrm{O}\left(\ln\left(1 + \frac{1}{(x+1)(x+3)}\right)\right)$$

and

$$\lim_{x\to\infty} e_\mu(x) = 0.$$

□

As a consequence of the above result, we obtain some bounds of the error $e_\mu(x)$.

Corollary 2.3. *The error $e_\mu(x)$ is uniformly bounded by $\pm \ln\left(1 + \frac{1}{(\varepsilon+1)(\varepsilon+3)}\right)$ $\forall\ x > \varepsilon > 0$ and $\forall\ \mu \in [1, 2]$.*

Proof. Using the inequality (18), we obtain

$$\sup_{0<x<\infty} |e_\mu(x)| \leq \ln\left(1 + \frac{1}{(x+1)(x+3)}\right).$$

Also, the function $g(x) = \ln\left(1 + \frac{1}{(x+1)(x+3)}\right)$ for $x > 0$ is decreasing. Then the errors $e_\mu(x)$ are uniformly bounded between $-\ln\left(1 + \frac{1}{(\varepsilon+1)(\varepsilon+3)}\right)$ and $\ln\left(1 + \frac{1}{(\varepsilon+1)(\varepsilon+3)}\right)$. □

3 The best bounds of the double inequality (9).

Firstly, we will prove the following auxiliary results:

Lemma 3.1.

$$\lim_{x\to\infty}\left(\frac{1}{e^{G(x+2)} - 1} - x\right) = 1 \tag{19}$$

and

$$\lim_{x\to\infty} \frac{G'(x+2)e^{G(x+2)}}{(e^{G(x+2)} - 1)^2} = -1. \tag{20}$$

Proof. Using the double inequality (6) with

$$\beta(x) = \frac{1}{x} + \frac{1}{2x^2 + 3/2} \quad \text{and} \quad \alpha(x) = \frac{1}{x} + \frac{1}{2x^2},$$

we get

$$\lim_{x\to\infty}\left(\frac{1}{e^{\alpha(x+2)} - 1} - x\right) \leq \lim_{x\to\infty}\left(\frac{1}{e^{G(x+2)} - 1} - x\right) \leq \lim_{x\to\infty}\left(\frac{1}{e^{\beta(x+2)} - 1} - x\right).$$

But

$$\lim_{x\to\infty}\left(\frac{1}{e^{\alpha(x+2)} - 1} - x\right) = \lim_{x\to\infty}\left(\frac{1}{\left[1 + \frac{1}{x} - \frac{1}{x^2} + \frac{2}{3x^3} + \frac{5}{12x^4} - O\left(\frac{1}{x^5}\right)\right] - 1} - x\right) = 1$$

and

$$\lim_{x\to\infty}\left(\frac{1}{e^{\beta(x+2)}-1}-x\right)=\lim_{x\to\infty}\left(\frac{1}{\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{1}{24x^4}-O\left(\frac{1}{x^5}\right)\right]-1}-x\right)=1.$$

Also, using the double inequality (6), we have

$$\lim_{x\to\infty}\frac{G'(x+2)e^{\alpha(x+2)}}{(e^{\beta(x+2)}-1)^2}\le\lim_{x\to\infty}\frac{G'(x+2)e^{G(x+2)}}{(e^{G(x+2)}-1)^2}\le\lim_{x\to\infty}\frac{G'(x+2)e^{\beta(x+2)}}{(e^{\alpha(x+2)}-1)^2}.$$

Now, using the asymptotic formula for Bateman's G-function (5), we obtain

$$G'(x)=\frac{-1}{x^2}-O\left(\frac{1}{x^3}\right).$$

Then

$$\lim_{x\to\infty}\frac{G'(x+2)e^{\alpha(x+2)}}{(e^{\beta(x+2)}-1)^2}=\lim_{x\to\infty}\frac{\left[\frac{-1}{(x+2)^2}-O\left(\frac{1}{x^3}\right)\right]\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{5}{12x^4}-O\left(\frac{1}{x^5}\right)\right]}{\left(\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{1}{24x^4}-O\left(\frac{1}{x^5}\right)\right]-1\right)^2}=-1$$

and

$$\lim_{x\to\infty}\frac{G'(x+2)e^{\beta(x+2)}}{(e^{\alpha(x+2)}-1)^2}=\lim_{x\to\infty}\frac{\left[\frac{-1}{(x+2)^2}-O\left(\frac{1}{x^3}\right)\right]\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{1}{24x^4}-O\left(\frac{1}{x^5}\right)\right]}{\left(\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{5}{12x^4}-O\left(\frac{1}{x^5}\right)\right]-1\right)^2}=-1$$

□

Now, we will present the sharp bounds of the double inequality (9).

Theorem 4. *For all* $x\in(0,\infty)$

$$\ln\left(1+\frac{1}{x+\frac{4}{e^2-4}}\right)+\frac{2}{x(x+1)}<G(x)<\ln\left(1+\frac{1}{x+1}\right)+\frac{2}{x(x+1)},\tag{21}$$

where the constants 1 *and* $\frac{4}{e^2-4}$ *are the best possible.*

Proof. Using the inequality (9) and functional equation (2), we get

$$0<\frac{1}{e^{G(x+2)}-1}-x<2.$$

Now consider the two functions

$$f(x)=e^{G(x+2)}-1,\qquad x>0$$

and

$$q(x)=\frac{1}{f(x)}-x,\qquad x>0.$$

Then $f'(x) = G'(x+2)e^{G(x+2)} < 0$ and $f(x)$ is strictly decreasing function. Hence $\frac{1}{f(x)}$ is strictly increasing function. Since $\frac{d}{dx}\frac{1}{f(x)}|_{x=0} \simeq 0.91$, and $\frac{d}{dx}\frac{1}{f(x)}|_{x=1} \simeq 0.96$. Then the function $\frac{1}{f(x)}$ is convex and $\frac{d}{dx}\frac{1}{f(x)}$ is increasing function. Thus we get

$$\frac{d}{dx}\frac{1}{f(x)} < \lim_{x\to\infty}\frac{d}{dx}\frac{1}{f(x)} = -\lim_{x\to\infty}\frac{G'(x+2)e^{G(x+2)}}{(e^{G(x+2)}-1)^2}.$$

Using the limit (20), we obtain

$$\frac{d}{dx}\frac{1}{f(x)} < 1, \qquad x > 0.$$

Then $q(x)$ is strictly decreasing function for all $x > 0$, where $\frac{dq(x)}{dx} = \frac{d}{dx}\frac{1}{f(x)} - 1 < 0$. Hence

$$\lim_{x\to\infty} q(x) < q(x) < \lim_{x\to 0^+} q(x)$$

and using the limit (19) and $G(2) = 2 - \ln 4$, we have

$$1 < q(x) < \frac{4}{e^2-4}. \tag{22}$$

with best bounds. □

In the proof of theorem (4), we proved that the function $\frac{1}{f(x)}$ is convex. Also, the second derivatives of the functions $q(x)$ and $\frac{1}{f(x)}$ have the same sign, then we get the following resuts:

Corollary 3.2. *The function $q(x)$ is strictly decreasing and convex for all $x > 0$.*

Corollary 3.3. *For every $x > 0$ there exists a unique number $\mu \in \left(1, \frac{4}{e^2-4}\right)$ such that $G(x) = M(\mu, x)$. Conversely for every $\mu \in \left(1, \frac{4}{e^2-4}\right)$ there exists a unique number $x > 0$ such that $M(\mu, x) = G(x)$.*

Proof. The function $q(x)$ is strictly decreasing from $(0,\infty)$ onto $\left(1, \frac{4}{e^2-4}\right)$ then the mapping $q(x) : (0,\infty) \to \left(1, \frac{4}{e^2-4}\right)$ is bijective and the proof is easy consequence of this result. □

Corollary 3.4. *For $x > 2$ and $\mu \in \left(1, \frac{4}{e^2-4}\right)$ we have*

1) the errors $e_\mu(x)$ are uniformly bounded by $\pm \ln\left(\frac{4(2e^2-4)}{3(3e^2-8)}\right)$.

2) $G(x) = M(\mu, x) + O\left(\ln\frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]}\right)$.

Proof. Analogues to inequality (17), we can deduce for all $x > 2$ and $\mu \in \left(1, \frac{4}{e^2-4}\right)$ that

$$0 \le |G(x) - M(\mu, x)| \le \left|M(1,x) - M\left(\frac{4}{e^2-4}, x\right)\right|$$

which is equivalent to

$$0 \le |e_\mu(x)| \le \left|\ln\frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]}\right| \le \left|\ln\left(\frac{4(2e^2-4)}{3(3e^2-8)}\right)\right|.$$

□

4 Comparing approximations

Firstly, we will prove the following one side inequality the function $G(x)$ which proves a special case of a conjecture posed in [9] and proved in [11] about the best bounds of the Bateman's function but with different proof.

Lemma 4.1. *For all $x > 0$, we have*

$$G(x) - \frac{1}{x} > \frac{1}{2x^2} - \frac{1}{4x^4}. \tag{23}$$

Proof. Consider the function

$$K(x) = G(x) - \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{4x^4}, \qquad x > 0.$$

Using the integral representation (3) of $G(x)$ and the formula

$$\frac{1}{x^r} = \frac{1}{(r-1)!}\int_0^\infty t^{r-1}e^{-xt}dt, \qquad r \in N$$

we get

$$K(x) = \int_0^\infty \varphi(t)\frac{e^{-xt}}{1+e^t}dt,$$

where

$$\varphi(t) = e^t - 1 - \frac{1}{2}t(1+e^t) + \frac{1}{24}t^3(1+e^t).$$

But

$$\begin{aligned}\varphi(t) &= \sum_{k=4}^\infty \frac{t^k}{k!} - \frac{1}{2}\sum_{k=3}^\infty \frac{t^{k+1}}{k!} + \frac{1}{24}\sum_{k=1}^\infty \frac{t^{k+3}}{k!} \\ &= \sum_{k=0}^\infty \frac{t^{(k+4)}}{(k+4)!}(1 + \frac{1}{24}(k+4)[(k+3)(k+2) - 12]) \\ &= \sum_{k=0}^\infty \frac{t^{(k+5)}}{(k+5)!}(1 + \frac{1}{24}k(k+5)(k+7)) > 0.\end{aligned}$$

Hence $\varphi(x) > 0$ and then $K(x) > 0$. □

As by-product of the the inequalities (6) and (23), we obtain the following double inequality.

Corollary 4.2. *For all $x > 1$, we have*

$$0 < \frac{(2x+1)(x-1)(x^2+1)}{2x^4(x+1)} < 2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} < \frac{2x^2 - x + 1}{2x^2(x+1)}. \tag{24}$$

Now, we will prove the following auxiliary results:

Lemma 4.3. *For all $x > x_0 \approx 2.5315129$, we have*

$$\frac{1}{e^{2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2}} - 1} - x > \frac{1}{e^{\frac{2x^2-x+1}{2x^2(x+1)}} - 1} - x > 1. \tag{25}$$

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Proof. Using the inequality (24), we have

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} - \ln\left(\frac{x+2}{x+1}\right) < u(x)$$

where

$$u(x) = \frac{2x^2 - x + 1}{2x^2(x+1)} - \ln\left(\frac{x+2}{x+1}\right), \qquad x > 0.$$

Then

$$u'(x) = \frac{(x - \frac{3+\sqrt{17}}{2})(x - \frac{3-\sqrt{17}}{2})}{x^3(x+1)^2},$$

and the function $u(x)$ has only one positive critical point at $x_m = \frac{3+\sqrt{17}}{2}$. Now,

$$u(x_m) = \frac{10}{(3+\sqrt{17})^2} - \ln\frac{7+\sqrt{17}}{5+\sqrt{17}} \approx -0.00113 < 0,$$

$$\lim_{x\to\infty} u(x) = 0$$

and

$$\lim_{x\to 0^-} u(x) = \infty.$$

Hence $u(x)$ has only one positive root $x_0 \approx 2.5315129$ and

$$u(x) < 0, \qquad \forall x > x_0.$$

Then

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} < \ln\left(\frac{x+2}{x+1}\right), \qquad \forall x > x_0.$$

□

Lemma 4.4. *For all $x > x_1 \approx 2.6925094$, we have*

$$\frac{1}{e^{2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2}} - 1} - x < \frac{4}{e^2 - 4}. \tag{26}$$

Proof. Using the inequality (24), we have

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} - \ln\left(\frac{e^2 + (e^2-4)x)}{4 + (e^2-4)x}\right) > v(x),$$

where

$$v(x) = \frac{(2x+1)(x-1)(x^2+1)}{2x^4(x+1)} - \ln\left(\frac{e^2 + (e^2-4)x)}{4 + (e^2-4)x}\right), \qquad x > 1.$$

Hence

$$v'(x) = \frac{L(x)}{S(x)},$$

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where

$$L(x) = 8e^2 + (-32 + 16e^2 + 2e^4)x + (-32 - 12e^2 + 6e^4)x^2 + (48 - 36e^2 + 5e^4)x^3 + (32 - 4e^2)x^4$$
$$+(-16 - 4e^2 + e^4)x^5 + (64 - 24e^2 + 2e^4)x^6$$

and

$$S(x) = x^5(x+1)^2(4e^2 + (e^4 - 16)x + (16 - 8e^2 + e^4)x^2 > 0, \qquad x > 0.$$

The function $L''(x)$ is a polynomial of fourth degree has one positive root at $x_I \approx 2.31866$ with $L''(3) < 0$, then $L(x)$ is concave function on (x_I, ∞). Also, $L(x_I) > 0$ and $\lim_{x\to\infty} L(x) = -\infty$. Hence, the function $L(x)$ has only one root on (x_I, ∞) at $x_3 \approx 4.0635204$, where $L(4.063) > 0$ and $L(4.064) < 0$. Then $L(x) > 0$ on $[x_I, x_3)$ and $L(x) < 0$ for all $x > x_3$. Hence $v(x)$ is increasing on (x_I, x_3) and decreasing function on (x_3, ∞) and it has a maximum point at x_3. But $v(2.69) < 0$ and $v(2.7) > 0$ and then $v(x)$ has a root $x_1 \approx 2.6925094 \in (x_I, x_3)$. Also, $\lim_{x\to\infty} v(x) = 0$, then we have

$$v(x) > 0, \qquad x > x_1$$

and hence

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} - \ln\left(\frac{e^2 + (e^2-4)x)}{4 + (e^2-4)x}\right) > 0, \qquad x > x_1.$$

□

Theorem 5. *For a fixed $x > x_1$, consider I_x be the nonempty open interval of $\left[1, \frac{4}{e^2-4}\right]$ defined by*

$$I_x = \left(\frac{1}{e^{-\frac{2}{x(x+1)}+\frac{1}{x}+\frac{1}{2x^2}} - 1} - x, \frac{1}{e^{2G(x)-\frac{2}{x(x+1)}-\frac{1}{x}-\frac{1}{2x^2}} - 1} - x\right).$$

For any $\mu \in I_x$, we have

$$|e_\mu(x)| < \left|G(x) - \left(\frac{1}{x} + \frac{1}{2x^2}\right)\right|.$$

Proof. Using the inequalities (25) and (26), we obtain

$$I_x \subset \left[1, \frac{4}{e^2-4}\right].$$

For any positive real number μ,

$$\frac{1}{e^{-\frac{2}{x(x+1)}+\frac{1}{x}+\frac{1}{2x^2}} - 1} - x < \mu \text{ iff } -M(\mu, x) > -\frac{1}{x} - \frac{1}{2x^2}$$

and hence

$$\frac{1}{e^{-\frac{2}{x(x+1)}+\frac{1}{x}+\frac{1}{2x^2}} - 1} - x < \mu \text{ iff } G(x) - M(\mu, x) > G(x) - \frac{1}{x} - \frac{1}{2x^2}. \tag{27}$$

Also,

$$\frac{1}{e^{2G(x)-\frac{2}{x(x+1)}-\frac{1}{x}-\frac{1}{2x^2}} - 1} - x > \mu \text{ iff } 2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} < \ln\left(1 + \frac{1}{x+\mu}\right)$$

and hence

$$\frac{1}{e^{2G(x)-\frac{2}{x(x+1)}-\frac{1}{x}-\frac{1}{2x^2}}-1}-x>\mu \text{ iff } G(x)-M(\mu,x)<-G(x)+\frac{1}{x}+\frac{1}{2x^2}. \tag{28}$$

From the inequalities (27) and (28) we have

$$G(x)-\frac{1}{x}-\frac{1}{2x^2}<G(x)-M(\mu,x)<-G(x)+\frac{1}{x}+\frac{1}{2x^2}, \qquad \forall\mu\in I_x.$$

Thus

$$|G(x)-M(\mu,x)|<\left|G(x)-\left(\frac{1}{x}+\frac{1}{2x^2}\right)\right|, \qquad \forall\mu\in I_x. \tag{29}$$

□

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Volume 23, Number 7 November 30, 2017
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

Editor in Chief: George Anastassiou
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DIFFERENTIAL EQUATIONS ASSOCIATED WITH MODIFIED DEGENERATE BERNOULLI AND EULER NUMBERS

TAEKYUN KIM, DAE SAN KIM, HYUCK IN KWON, AND JONG JIN SEO

Abstract. In this paper, we consider some ordinary differential equations associated with modified degenerate Euler and Bernoulli numbers and give some new identities for these numbers arising from our differential equations.

1. Introduction

As is well known, Bernoulli numbers are defined by the generating function

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\quad (\text{see [1–12]})\,, \tag{1.1}$$

and the Euler numbers are given by generating function

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\quad (\text{see [7, 8]})\,. \tag{1.2}$$

In [2], L. Carlitz considered the degenerate Bernoulli and Euler numbers which are defined by the generating functions

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}=\sum_{n=0}^{\infty}\beta_n(\lambda)\frac{t^n}{n!}, \tag{1.3}$$

and

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(\lambda)\frac{t^n}{n!}. \tag{1.4}$$

Note that $\lim_{\lambda\to 0}\beta_n(\lambda)=B_n$ and $\lim_{\lambda\to 0}\mathcal{E}_n(\lambda)=E_n$, $(n\geq 0)$.

Now, we define the modified degenerate Bernoulli and Euler numbers which are slightly different from the Carlitz degenerate Bernoulli and Euler numbers as follows:

$$\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}=\sum_{n=0}^{\infty}\tilde{\beta}_n(\lambda)\frac{t^n}{n!},\quad (\text{see [3]})\,, \tag{1.5}$$

and

$$\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}=\sum_{n=0}^{\infty}\tilde{\mathcal{E}}_n(\lambda)\frac{t^n}{n!},\quad (\text{see [9]})\,. \tag{1.6}$$

2010 Mathematics Subject Classification : 05A19, 11B37, 11B68, 11B83, 34A30.

Key words and phrases: modified degenerate Bernoulli and Euler numbers, differential equations.

From (1.5) and (1.4), we easily note that

$$\lim_{\lambda\to 0}\tilde{\beta}_n(\lambda)=B_n \quad\text{and}\quad \lim_{\lambda\to 0}\tilde{\mathcal{E}}_n(\lambda)=E_n,\quad (n\geq 0). \tag{1.7}$$

For $r\in\mathbb{N}$, the higher-order modified Bernoulli and Euler numbers are also defined by the generating functions

$$\left(\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^r=\sum_{n=0}^{\infty}\tilde{\beta}_n^{(r)}(\lambda)\frac{t^n}{n!}, \tag{1.8}$$

and

$$\left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}\right)^r=\sum_{n=0}^{\infty}\tilde{\mathcal{E}}_n^{(r)}(\lambda)\frac{t^n}{n!}. \tag{1.9}$$

Recall that the higher order Bernoulli and Euler numbers are given by the generating functions

$$\left(\frac{t}{e^t-1}\right)^r=\sum_{n=0}^{\infty}B_n^{(r)}\frac{t^n}{n!}, \tag{1.10}$$

and

$$\left(\frac{2}{e^t+1}\right)^r=\sum_{n=0}^{\infty}E_n^{(r)}\frac{t^n}{n!},\quad \text{(see [6, 11])}. \tag{1.11}$$

From (1.8), (1.9), (1.10) and (1.11), we note that

$$\lim_{\lambda\to 0}\tilde{\beta}_n^{(r)}(\lambda)=B_n^{(r)} \quad\text{and}\quad \lim_{\lambda\to 0}\tilde{\mathcal{E}}_n^{(r)}(\lambda)=E_n^{(r)}.$$

In [1], Bayad-Kim studied the following nonlinear differential equations:

$$F_q^N=\frac{1}{(N-1)!}\sum_{k=1}^{N}a_k(N)F_q^{(k-1)},\quad (N\in\mathbb{N}), \tag{1.12}$$

where $F^{(k)}=F^{(k)}(t)=\left(\frac{d}{dt}\right)^k F$.

For $F_q(t)=\frac{1}{qe^t\pm 1}$, Bayad-Kim gave explicit formulae for Apostol-Bernoulli and Apostol-Euler numbers and polynomials which are derived from (1.12).

In [4], Guo-Qi obtained the following results

$$\left(\frac{d}{dt}\right)^k\left(\frac{1}{\lambda e^{\alpha t}-1}\right)=(-1)^k\alpha^k\sum_{m=1}^{k+1}(m-1)!S_2(k+1,m)\left(\frac{1}{\lambda e^{\alpha t}-1}\right)^m, \tag{1.13}$$

and

$$\left(\frac{1}{\lambda e^{\alpha t}-1}\right)^k=\frac{1}{(k-1)!}\sum_{m=1}^{k}\frac{(-1)^{m-1}}{\alpha^{m-1}}S_1(k,m)\left(\frac{d}{dt}\right)^{m-1}\left(\frac{1}{\lambda e^{\alpha t}-1}\right), \tag{1.14}$$

where $k\in\mathbb{N}$, and $S_1(k,m)$ and $S_2(k,m)$ are respectively the Stirling numbers of the first kind and of the second kind (see [4, 10]). However, the results of Guo-Qi are immediately obtained from the paper of Bayad-Kim in [1] by replacing q by λ and t by αt (α =constnat).

Recently, Kim-Kim studied the nonlinear differential equations given by

$$\left(\frac{d}{dt}\right)^N\left(\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}\pm 1}\right)=\frac{(-1)^N}{(1+\lambda t)^N}\sum_{i=1}^{N+1}a_i(N,\lambda)F^i, \tag{1.15}$$

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where

$$F = F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}} \pm 1} \quad (\text{see [7]}).$$

From (1.15), we derived some new identities involving degenerate Euler and Bernoulli polynomials.

In this paper, along the same line as [7] we study some ordinary differential equations arising from the generating functions of the modified degenerate Bernoulli and Euler numbers. From those equations, we derive some new identities for the modified degenerate Bernoulli and Euler numbers.

2. Differential equations associated with modified degenerate Bernoulli and Euler numbers

Let

$$F = F(t) = \left((1+\lambda)^{\frac{t}{\lambda}} \pm 1\right)^{-1}. \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned} F^{(1)} = \frac{dF}{dt} &= -\left((1+\lambda)^{\frac{t}{\lambda}} \pm 1\right)^{-2}(1+\lambda)^{\frac{t}{\lambda}}\frac{1}{\lambda}\log(1+\lambda) \\ &= -\frac{1}{\lambda}\log(1+\lambda)\left((1+\lambda)^{\frac{t}{\lambda}} \pm 1\right)^{-2}\left((1+\lambda)^{\frac{t}{\lambda}} \pm 1 \mp 1\right) \\ &= -\frac{1}{\lambda}\log(1+\lambda)\left(F \mp F^2\right), \end{aligned} \tag{2.2}$$

$$\begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} \\ &= -\frac{1}{\lambda}\log(1+\lambda)\left(F^{(1)} \mp 2FF^{(1)}\right) \\ &= -\frac{1}{\lambda}\log(1+\lambda)(1 \mp 2F)F^{(1)} \\ &= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^2(1 \mp 2F)\left(F \mp F^2\right) \\ &= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^2\left(F \mp 3F^2 + 2F^3\right). \end{aligned} \tag{2.3}$$

Thus we are led to put

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t) \\ &= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^N \sum_{i=1}^{N+1} a_{i-1}^{\pm}(N) F^i, \quad (N = 0, 1, 2, \dots), \end{aligned} \tag{2.4}$$

where $a_{i-1}^{+}(N)$ corresponds to $\left((1+\lambda)^{\frac{t}{\lambda}} + 1\right)^{-1}$ and $a_{i-1}^{+}(N)$ does to $\left((1+\lambda)^{\frac{t}{\lambda}} - 1\right)^{-1}$.

Now, from (2.4), we have

$$\begin{aligned} & F^{(N+1)} \\ = & \frac{d}{dt}F^{(N)} \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1} a_{i-1}^{\pm}(N)\, iF^{i-1}F^{(1)} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N+1}\left\{\sum_{i=1}^{N+1} ia_{i-1}^{\pm}(N)\,F^{i} \mp \sum_{i=2}^{N+2}(i-1)\,a_{i-2}^{\pm}(N)\,F^{i}\right\} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N+1}\Big\{a_0^{\pm}(N)\,F \mp (N+1)\,a_N^{\pm}(N)\,F^{N+2} \\
&\quad + \sum_{i=2}^{N+1}\left(ia_{i-1}^{\pm}(N) \mp (i-1)\,a_{i-2}^{\pm}(N)\right)F^{i}\Big\}.
\end{aligned}
$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$
F^{(N+1)} = \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N+1}\sum_{i=1}^{N+2} a_{i-1}^{\pm}(N+1)\,F^{i}. \tag{2.6}
$$

Comparing the coefficients on both sides of (2.5) and (2.6), we obtain

$$
a_0^{\pm}(N+1) = a_0^{\pm}(N), \tag{2.7}
$$

$$
a_{N+1}^{\pm}(N+1) = \mp(N+1)\,a_N^{\pm}(N), \tag{2.8}
$$

and

$$
a_{i-1}^{\pm}(N+1) = ia_{i-1}^{\pm}(N) \mp (i-1)\,a_{i-2}^{\pm}(N), \tag{2.9}
$$

for $2 \le i \le N+1$.

Also, by (1.12), we get

$$
F = F^{(0)} = a_0^{\pm}(0)\,F. \tag{2.10}
$$

Thus, by (2.10), we see that

$$
a_0^{\pm}(0) = 1. \tag{2.11}
$$

It is easy to show that

$$
\begin{aligned}
F^{(1)} &= -\frac{1}{\lambda}\log(1+\lambda)\sum_{i=1}^{2} a_{i-1}^{\pm}(1)\,F^{i} \\
&= -\frac{1}{\lambda}\log(1+\lambda)\left(a_0^{\pm}(1)\,F + a_1^{\pm}(1)\,F^{2}\right) \\
&= -\frac{1}{\lambda}\log(1+\lambda)\left(F \mp F^{2}\right).
\end{aligned} \tag{2.12}
$$

Thus, by comparing the coefficients on both sides of (2.12), we have

$$
a_0^{\pm}(1) = 1, \quad a_1^{\pm}(1) = \mp 1. \tag{2.13}
$$

From (2.7) and (2.8), we note that

$$
a_0^{\pm}(N+1) = a_0^{\pm}(N) = \cdots = a_0^{\pm}(0) = 1, \tag{2.14}
$$

and

$$
\begin{aligned}
a_{N+1}^{+}(N+1) &= -(N+1)\,a_N^{+}(N) \\
&= (-1)^{2}(N+1)\,Na_{N-1}^{+}(N-1) \\
&\;\;\vdots
\end{aligned} \tag{2.15}
$$

$$\begin{aligned}&= (-1)^{N+1}(N+1)!a_0^{+}(0)\\ &= (-1)^{N+1}(N+1)!,\end{aligned}$$

$$\begin{aligned} a_{N+1}^{-}(N+1) &= (N+1)\,a_N^{-}(N) \\ &= (N+1)\,Na_{N-1}^{-}(N-1) \\ &\;\vdots \\ &= (N+1)!a_0^{-}(0) \\ &= (N+1)!. \end{aligned} \tag{2.16}$$

By (2.15) and (2.16), we easily get

$$a_{N+1}^{\pm}(N+1) = (\mp 1)^{N+1}(N+1)!. \tag{2.17}$$

Observe also that the matrix $\left(a_i^{+}(j)\right)_{0\le i,j\le N}$ and $\left(a_i^{-}(j)\right)_{0\le i,j\le N}$ are as follows:

$$\begin{array}{c} \\ 0\\1\\2\\3\\ \\ N \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & & N \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ & (-1)1! & & & & \\ & & (-1)^2 2! & & & \\ & & & (-1)^3 3! & & \\ & & \mathbf{0} & & \ddots & \\ & & & & & (-1)^N N! \end{bmatrix} \end{array} = \left(a_i^{+}(j)\right)_{0\le i,j\le N}$$

and

$$\begin{array}{c} \\ 0\\1\\2\\3\\ \\ N \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & & N \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ & 1! & & & & \\ & & 2! & & & \\ & & & 3! & & \\ & & \mathbf{0} & & \ddots & \\ & & & & & N! \end{bmatrix} \end{array} = \left(a_i^{-}(j)\right)_{0\le i,j\le N}$$

For $i=2$ in (2.9), we have

$$\begin{aligned} &a_1^{\pm}(N+1) \\ &= \mp a_0^{\pm}(N) + 2a_1^{\pm}(N) \\ &= \mp a_0^{\pm}(N) + 2\left(\mp a_0^{\pm}(N-1) + 2a_1^{\pm}(N-1)\right) \\ &= \mp\left(a_0^{\pm}(N) + 2a_0^{\pm}(N-1)\right) + 2^2 a_1^{\pm}(N-1) \\ &= \mp\left(a_0^{\pm}(N) + 2a_0^{\pm}(N-1)\right) + 2^2\left(\mp a_0^{\pm}(N-2) + 2a_1^{\pm}(N-2)\right) \\ &= \mp\left(a_0^{\pm}(N) + 2a_0^{\pm}(N-1) + 2^2 a_0^{\pm}(N-2)\right) + 2^3 a_1^{\pm}(N-2) \\ &\;\vdots \end{aligned} \tag{2.18}$$

$$= \mp \sum_{i=0}^{N-1} 2^i a_0^{\pm}(N-i) + 2^N a_1^{\pm}(1) = \mp \sum_{i=0}^{N} 2^i a_0^{\pm}(N-i).$$

Let us take $i = 3$ in (2.9). Then, we note that

$$\begin{aligned} (2.19) \qquad & a_2^{\pm}(N+1) \\ &= \mp 2a_1^{\pm}(N) + 3a_2^{\pm}(N) \\ &= \mp 2a_1^{\pm}(N) + 3\left(\mp 2a_1^{\pm}(N-1) + 3a_2^{\pm}(N-1)\right) \\ &= \mp 2\left(a_1^{\pm}(N) + 3a_1^{\pm}(N-1)\right) + 3^2 a_2^{\pm}(N-1) \\ &= \mp 2\left(a_1^{\pm}(N) + 3a_1^{\pm}(N-1)\right) + 3^2\left(\mp 2a_1^{\pm}(N-2) + 3a_2^{\pm}(N-2)\right) \\ &= \mp 2\left(a_1^{\pm}(N) + 3a_1^{\pm}(N-1) + 3^2 a_1^{\pm}(N-2)\right) + 3^3 a_2^{\pm}(N-2) \\ &\vdots \\ &= \mp 2\sum_{i=0}^{N-2} 3^i a_1^{\pm}(N-i) + 3^{N-1} a_2^{\pm}(2) \\ &= \mp 2\sum_{i=0}^{N-1} 3^i a_1^{\pm}(N-i). \end{aligned}$$

For $i = 4$ in (2.9), we have

$$\begin{aligned} (2.20) \qquad & a_3^{\pm}(N+1) \\ &= \mp 3a_2^{\pm}(N) + 4a_3^{\pm}(N) \\ &= \mp 3a_2^{\pm}(N) + 4\left(\mp 3a_2^{\pm}(N-1) + 4a_3^{\pm}(N-1)\right) \\ &= \mp 3\left(a_2^{\pm}(N) + 4a_2^{\pm}(N-1)\right) + 4^2 a_3^{\pm}(N-1) \\ &= \mp 3\left(a_2^{\pm}(N) + 4a_2^{\pm}(N-1)\right) + 4^2\left(\mp 3a_2^{\pm}(N-2) + 4a_3^{\pm}(N-2)\right) \\ &= \mp 3\left(a_2^{\pm}(N) + 4a_2^{\pm}(N-1) + 4^2 a_2^{\pm}(N-2)\right) + 4^3 a_3^{\pm}(N-2) \\ &\vdots \\ &= \mp 3\sum_{i=0}^{N-3} 4^i a_2^{\pm}(N-i) + 4^{N-2} a_3^{\pm}(3) \\ &= \mp \sum_{i=0}^{N-2} 4^i a_2^{\pm}(N-i). \end{aligned}$$

Continuing this process, we can deduce that

$$(2.21) \qquad a_j^{\pm}(N+1) = \mp j \sum_{i=0}^{N-j+1} (j+1)^i a_{j-1}^{\pm}(N-i),$$

for $1 \le j \le N$.

Now, we give explicit expression for $a_j^{\pm}(N+1)$, $(1 \le j \le N)$.

$$(2.22) \qquad a_1^{\pm}(N+1) = \mp \sum_{i_1=0}^{N} 2^{i_1},$$

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$$a_2^{\pm}(N+1) = \mp 2\sum_{i_2=0}^{N-1} 3^{i_2} a_1^{\pm}(N-i_2) \tag{2.23}$$
$$= \mp 2\sum_{i_2=0}^{N-1} 3^{i_2}(\mp 1)\sum_{i_1=0}^{N-i_2-1} 2^{i_1}$$
$$= (\mp 1)^2\, 2! \sum_{i_2=0}^{N-1}\sum_{i_1=0}^{N-1-i_2} 3^{i_2}2^{i_1},$$

and, by (2.23), we get

$$a_3^{\pm}(N+1) \tag{2.24}$$
$$= \mp 3\sum_{i_3=0}^{N-2} 4^{i_3} a_2^{\pm}(N-i_3)$$
$$= \mp 3\sum_{i_3=0}^{N-2} 4^{i_3}(\mp 1)^2\, 2! \sum_{i_2=0}^{N-i_3-2}\sum_{i_1=0}^{N-i_3-i_2-2} 3^{i_2}2^{i_1}$$
$$= (\mp 1)^3\, 3! \sum_{i_3=0}^{N-2}\sum_{i_2=0}^{N-2-i_3}\sum_{i_1=0}^{N-2-i_3-i_2} 4^{i_3}3^{i_2}2^{i_1}.$$

So, we can deduce that

$$a_j^{\pm}(N+1) = (\mp 1)^j\, j! \sum_{i_j=0}^{N-j+1}\sum_{i_{j-1}=0}^{N-j+1-i_j}\cdots\sum_{i_1=0}^{N-j+1-i_j-\cdots-i_2} (j+1)^{i_j}\, j^{i_{j-1}}\cdots 2^{i_1}, \tag{2.25}$$

where $1 \le j \le N$.

Remark. Observe that $a_{N+1}^{\pm}(N+1) = (\mp 1)^{N+1}(N+1)!$ is the same as the above expression with $j = N+1$. Therefore, by (2.4) and (2.25), we obtain the following theorem.

Theorem 1. *The ordinary differential equations*

$$F^{(N)} = \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^N \sum_{i=1}^{N+1} a_{i-1}^{-}(N)\, F^i, \quad (N = 0, 1, 2, \ldots),$$

have a solution $F = F(t) = \frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}$, *where* $a_0^{-}(N) = 1$,

$$a_j^{-}(N) = j! \sum_{i_j=0}^{N-j}\sum_{i_{j-1}=0}^{N-j-i_j}\cdots\sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (j+1)^{i_j}\, j^{i_{j-1}}\cdots 2^{i_1},$$

for $1 \le j \le N$.

Theorem 2. *The ordinary differential equations*

$$F^{(N)} = \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^N \sum_{i=1}^{N+1} a_{i-1}^{+}(N)\, F^i, \quad (N = 0, 1, 2, \ldots),$$

have a solution $F = F(t) = \frac{1}{(1+\lambda)^{\frac{t}{\lambda}}+1}$, *where* $a_0^+(N) = 1$,

$$a_j^+(N) = (-1)^j j! \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}} \cdots 2^{i_1},$$

for $1 \leq j \leq N$.

Now, we observe that

$$\begin{aligned}
&\sum_{k=0}^{\infty} \tilde{\mathcal{E}}_{k+N}(\lambda) \frac{t^k}{k!} \\
&= \left(\sum_{k=0}^{\infty} \tilde{\mathcal{E}}_k(\lambda) \frac{t^k}{k!} \right)^{(N)} \\
&= 2 \left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^{(N)} \\
&= 2 \left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} a_{i-1}^+(N) \left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^i \\
&= \left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} a_{i-1}^+(N)\, 2^{1-i} \left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^i \\
&= \sum_{k=0}^{\infty} \left(\left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N)\, \mathcal{E}_k^{(i)}(\lambda) \right) \frac{t^k}{k!}.
\end{aligned} \tag{2.26}$$

Thus, by comparing the coefficients on both sides of (2.26), we get

$$\tilde{\mathcal{E}}_{k+N}(\lambda) = \left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N)\, \tilde{\mathcal{E}}_k^{(i)}(\lambda), \tag{2.27}$$

for $k, N = 0, 1, 2, \ldots$.

Therefore, by (2.27), we obtain the following theorem.

Theorem 3. *For* $k, N = 0, 1, 2, \ldots,$ *we have*

$$\tilde{\mathcal{E}}_{k+N}(\lambda) = \left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N)\, \tilde{\mathcal{E}}_k^{(i)}(\lambda),$$

where $a_0^+(N) = 1$,

$$a_j^+(N) = (-1)^j j! \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}} \cdots 2^{i_1}, \tag{2.28}$$

where $1 \leq j \leq N$.

Corollary 4. $\tilde{\mathcal{E}}_N(x) = \left(-\frac{1}{\lambda} \log(1+\lambda) \right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N)$.

Replacing t by $\frac{t}{\lambda} \log(1+\lambda)$ in (1.11), we obtain

$$\sum_{n=0}^{\infty} \tilde{\mathcal{E}}_n^{(r)}(\lambda) \frac{t^n}{n!} = \left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^r \tag{2.29}$$

$$= \sum_{n=0}^{\infty} E_n^{(r)} \frac{\left(\frac{1}{\lambda}\log(1+\lambda)t\right)^n}{n!}.$$

Thus, by (2.29), we get

$$\tilde{\mathcal{E}}_n^{(r)}(\lambda) = \left(\frac{1}{\lambda}\log(1+\lambda)\right)^n E_n^{(r)}, \quad (n \geq 0). \tag{2.30}$$

From (2.30), we obtain the following corollary.

Corollary 5. *For $k, N = 0, 1, 2, \ldots,$ we have*

$$E_{k+N} = (-1)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^{+}(N) E_k^{(i)},$$

where $a_j^{+}(N)(0 \leq j \leq N)$ are as in (2.28).

From (1.3), we note that

$$\begin{aligned} &\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1} \\ &= \sum_{k=0}^{\infty} \tilde{\beta}_k(\lambda) \frac{t^{k-1}}{k!} \\ &= \sum_{k=1}^{\infty} \tilde{\beta}_k(\lambda) \frac{t^{k-1}}{k!} + \tilde{\beta}_0(\lambda)\frac{1}{t} \\ &= \sum_{k=0}^{\infty} \tilde{\beta}_{k+1}(\lambda) \frac{t^k}{(k+1)!} + \frac{\lambda}{\log(1+\lambda)} t^{-1}. \end{aligned} \tag{2.31}$$

Thus, by (2.31), we get

$$\begin{aligned} &\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{(N)} \\ &= \sum_{k=N}^{\infty} \tilde{\beta}_{k+1}(\lambda) \frac{(k)_N}{(k+1)!} t^{k-N} \\ &\quad + (-1)^N N! \frac{\lambda}{\log(1+\lambda)} t^{-N-1}. \end{aligned} \tag{2.32}$$

From (2.32), we note that

$$\begin{aligned} &t^{N+1}\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{(N)} \\ &= \sum_{k=N}^{\infty} \tilde{\beta}_{k+1}(\lambda) \frac{(k)_N}{(k+1)!} t^{k+1} + (-1)^N N! \frac{\lambda}{\log(1+\lambda)} \\ &= \sum_{k=N+1}^{\infty} \tilde{\beta}_k(\lambda)(k-1)_N \frac{t^k}{k!} + (-1)^N N! \frac{\lambda}{\log(1+\lambda)}. \end{aligned} \tag{2.33}$$

On the other hand, by Theorem 1, we get

$$
\begin{aligned}
(2.34)\qquad & t^{N+1}\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{(N)} \\
&= t^{N+1}\left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1}a_{i-1}^{-}(N)\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{i} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1}a_{i-1}^{-}(N)\,t^{N+1-i}\left(\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{i} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1}a_{i-1}^{-}(N)\,t^{N+1-i}\sum_{l=0}^{\infty}\tilde{\beta}_{l}^{(i)}(\lambda)\frac{t^{l}}{l!} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=0}^{N}a_{N-i}^{-}(N)\sum_{l=0}^{\infty}\tilde{\beta}_{l}^{(N+1-i)}(\lambda)\frac{t^{l+i}}{l!} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=0}^{N}\sum_{l=0}^{\infty}a_{N-i}^{-}(N)\,\tilde{\beta}_{l}^{(N+1-i)}(\lambda)\frac{t^{l+i}}{l!} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=0}^{N}\sum_{k=i}^{\infty}a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)\frac{t^{k}}{(k-i)!}
\end{aligned}
$$

From (2.34), we have

$$
\begin{aligned}
(2.35)\qquad & t^{N+1}\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{(N)} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N} \\
&\quad\times\left\{\sum_{k=0}^{N}\sum_{i=0}^{k}a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)\,(k)_{i}\frac{t^{k}}{k!}\right. \\
&\quad\left.+\sum_{k=N+1}^{\infty}\sum_{i=0}^{N}a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)\,(k)_{i}\frac{t^{k}}{k!}\right\}.
\end{aligned}
$$

Comparing (2.33) and (2.35), we obtain the following theorem.

Theorem 6. *Let N be a positive integer. Then*

(i) $\tilde{\beta}_{k}(\lambda)=\frac{1}{(k-1)_{N}}\left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=0}^{N}a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)\,(k)_{i}$, *where* $k\geq N+1$, $(k)_{N}=k(k-1)\cdots(k-N+1)$ *for* $N\geq 1$, *and* $(k)_{0}=1$.

(ii) *For* $1\leq k\leq N$, *we have*

$$\sum_{i=0}^{k}a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)\,(k)_{i}=0,$$

where $a_{0}^{-}(N)=1$,

$$(2.36)\qquad a_{j}^{-}(N)=j!\sum_{i_{j}=0}^{N-j}\sum_{i_{j-1}=0}^{N-j-i_{j}}\cdots\sum_{i_{1}=0}^{N-j-i_{j}-\cdots i_{2}}(j+1)^{i_{j}}\,j^{i_{j-1}}\cdots 2^{i_{1}},$$

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$(1 \le j \le N)$.

Replacing t by $\frac{t}{\lambda}\log(1+\lambda)$ in (1.10), we get

$$\left(\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \left(\frac{1}{\lambda}\log(1+\lambda)\right)^{n-r} \frac{t^n}{n!}. \tag{2.37}$$

Thus, from (2.37), we have

$$\tilde{\beta}_n^{(r)}(\lambda) = \left(\frac{1}{\lambda}\log(1+\lambda)\right)^{n-r} B_n^{(r)}, \quad \text{for } n \ge 0. \tag{2.38}$$

From (2.38), we obtain the following corollary.

Corollary 7. *Let N be any positive integer. Then*

(i) $B_k = \frac{(-1)^N}{(k-1)_N} \sum_{i=0}^{N} a_{N-i}^{-}(N) B_{k-i}^{(N+1-i)}(k)_i$, *for* $k \ge N+1$,

(ii) $\sum_{i=0}^{k} a_{N-i}^{-}(N) B_{k-i}^{(N+1-i)}(k)_i = 0$, *for* $1 \le k \le N$, *where* $a_j^{-}(N)$ $(0 \le j \le N)$ *are as in (2.36).*

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ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN BANACH SPACES

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Abstract. Let

$$\begin{aligned} M_1 f(x,y): &= \frac{3}{4}f(x+y) - \frac{1}{4}f(-x-y) \\ &+\frac{1}{4}f(x-y) + \frac{1}{4}f(y-x) - f(x) - f(y), \end{aligned}$$

$$M_2 f(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y).$$

We solve the additive-quadratic ρ-functional inequalities

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\|, \tag{0.1}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$ and

$$\|M_2 f(x,y)\| \le \|\rho M_1 f(x,y)\|, \tag{0.2}$$

where ρ is a fixed complex number with $|\rho| < 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [23] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [22] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 6, 7, 10, 13, 14, 16, 17, 18, 19, 20, 21, 24, 25]).

In Section 2, we solve the additive-quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in Banach spaces.

2010 *Mathematics Subject Classification.* Primary 39B62, 39B52.

Key words and phrases. Hyers-Ulam stability; additive-quadratic ρ-functional inequality; Banach space.

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In Section 3, we solve the additive-quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in Banach spaces.

In this paper, assume that X is a complex normed space and that Y is a complex Banach space.

2. Additive-quadratic ρ-functional inequality (0.1) in Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < \frac{1}{2}$.

We solve and investigate the additive-quadratic ρ-functional inequality (0.1) in normed spaces.

Lemma 2.1.
(i) *If a mapping $f : X \to Y$ satisfies $M_1 f(x,y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*
(ii) *If a mapping $f : X \to Y$ satisfies $M_2 f(x,y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

Proof. (i)

$$M_1 f_o(x,y) = f_o(x+y) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. So f_o is the Cauchy additive mapping.

$$M_1 f_e(x,y) = \frac{1}{2} f_e(x+y) + \frac{1}{2} f_e(x-y) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. So f_o is the quadratic mapping.

(ii)

$$M_2 f_o(x,y) = 2 f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_o is the Cauchy additive mapping.

$$M_2 f_e(x,y) = 2 f_e\left(\frac{x+y}{2}\right) + 2 f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_e is the quadratic mapping.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping and the quadratic mapping. □

Lemma 2.2.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\| \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies (2.1), then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (2.1).

Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\|f(2x) - 2f(x)\| \le 0$$

and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y)-f(x)-f(y)\| &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \\ &= |\rho|\|f(x+y)-f(x)-f(y)\| \end{aligned}$$

and so

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get

$$\|f(0)\| \leq \|2\rho f(0)\|.$$

So $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\|\frac{1}{2}f(2x)-2f(x)\| \leq 0$$

and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{4}f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned} &\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| \\ &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &= |\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| \end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in complex Banach spaces for an odd mapping case.

Theorem 2.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an odd mapping such that*

$$\Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j},\frac{y}{2^j}\right) < \infty, \tag{2.4}$$

$$\|M_1 f(x,y)\| \leq \|\rho M_2 f(x,y)\| + \varphi(x,y) \tag{2.5}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leq \frac{1}{2}\Psi(x,x) \tag{2.6}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x, x) \tag{2.7}$$

for all $x \in X$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned} \tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an odd mapping, A is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.6).

It follows from (2.4) and (2.5) that

$$\begin{aligned}\|A(x+y) - A(x) - A(y)\| &= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty}\left\|2^n\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &+ \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\ &= \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \end{aligned}$$

for all $x, y \in X$. So

$$\|A(x+y) - A(x) - A(y)\| \le \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.2, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.6). Then we have

$$\begin{aligned}\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A, as desired. $\square$

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Corollary 2.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\| + \theta(\|x\|^r + \|y\|^r) \tag{2.9}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Theorem 2.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an odd mapping satisfying (2.5) and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x,x) \tag{2.11}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\varphi(2^j x, 2^j x) \end{aligned}\tag{2.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.12) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.12), we get (2.11).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying (2.9). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^r}\|x\|^r \tag{2.13}$$

for all $x \in X$.

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in complex Banach spaces for an even mapping case.

Theorem 2.7. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (2.5) and*

$$\Psi(x,y) : \quad = \quad \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.14}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{2}\Psi(x,x) \tag{2.15}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5), we get

$$\left\|\frac{1}{2}f(2x) - 2f(x)\right\| \le \varphi(x,x) \tag{2.16}$$

for all $x \in X$. So

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} \frac{4^{j+1}}{2}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned}\tag{2.17}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.17) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an even mapping, Q is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.17), we get (2.15).

It follows from (2.5) and (2.14) that

$$\begin{aligned}&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right\| \\ &= \lim_{n\to\infty}\left\|4^n\left(\frac{1}{2}f\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty}\left\|4^n\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| + \lim_{n\to\infty} 4^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\ &= \left\|\rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\|\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right\| \\ &\qquad \le \left\|\rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\|\end{aligned}$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.15). Then we have

$$\begin{aligned}\|Q(x) - T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| \\ &\leq 4^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q, as desired. □

Corollary 2.8. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{4\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Theorem 2.9. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (2.5) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty \tag{2.18}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2}\Psi(x, x) \tag{2.19}$$

for all $x \in X$.

Proof. It follows from (2.16) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{2}\varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\leq \sum_{j=l}^{m-1}\frac{1}{2\cdot 4^j}\varphi(2^j x, 2^j x)\end{aligned} \tag{2.20}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.20) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.20), we get (2.19).

The rest of the proof is similar to the proof of Theorem 2.7. □

Corollary 2.10. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{4\theta}{4 - 2^r}\|x\|^r \tag{2.21}$$

for all $x \in X$.

Remark 2.11. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive-quadratic ρ-functional inequality (0.2) in complex Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < 1$.

We solve and investigate the additive-quadratic ρ-functional inequality (0.2) in complex normed spaces.

Lemma 3.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_2 f(x, y)\| \leq \|\rho M_1 f(x, y)\| \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and (3.1), then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &= \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \\ &\leq |\rho|\|f(x+y) - f(x) - f(y)\| \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0 \tag{3.3}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned}\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| \\ = \left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| \\ \le |\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (3.1) in complex Banach spaces for an odd mapping case.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an odd mapping satisfying*

$$\Psi(x,y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j},\frac{y}{2^j}\right) < \infty,$$

$$\|M_2 f(x,y)\| \le \|\rho M_1 f(x,y)\| + \varphi(x,y) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \Psi(x,0) \tag{3.5}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4), we get

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right)-f(x)\right\| \le \varphi(x,0) \tag{3.6}$$

for all $x \in X$. So

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^j},0\right)\end{aligned} \tag{3.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an odd mapping, A is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying*

$$\|M_2 f(x, y)\| \le \|\rho M_1 f(x, y)\| + \theta(\|x\|^r + \|y\|^r) \tag{3.8}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2^r\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be an odd mapping satisfying* (3.4) *and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \Psi(x, 0) \tag{3.9}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l+1}^{m}\frac{1}{2^j}\varphi(2^j x, 0) \end{aligned} \tag{3.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.10) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.10), we get (3.9).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (3.8). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^r\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (3.1) in complex Banach spaces for an even mapping case.

Theorem 3.6. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (3.4) and*

$$\Psi(x,y) : \quad = \quad \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \Psi(x,0) \tag{3.11}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| = \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi(x,0) \tag{3.12}$$

for all $x \in X$. So

$$\begin{aligned}
\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \quad &\le \quad \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\
&\le \quad \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j}, 0\right) \qquad (3.13)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.13) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an even mapping, Q is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.13), we get (3.11).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.7. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.8). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2^r\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Theorem 3.8. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (3.4) and*

$$\Psi(x,y) := \sum_{j=1}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \Psi(x,0) \tag{3.14}$$

for all $x \in X$.

Proof. It follows from (3.12) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{4} \varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f\left(2^j x\right) - \frac{1}{4^{j+1}} f\left(2^{j+1} x\right) \right\| \\ &\leq \sum_{j=l+1}^{m} \frac{1}{4^j} \varphi(2^j x, 0) \end{aligned} \tag{3.15}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.15) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.15), we get (3.14).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.9. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (3.8). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^r \theta}{4 - 2^r} \|x\|^r$$

for all $x \in X$.

Remark 3.10. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

Acknowledgments

This research was supported by Hanshin University Research Grant.

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STABILITY OF ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN BANACH SPACES

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ABSTRACT. Let

$$\begin{aligned} M_1 f(x,y) : &= \frac{3}{4}f(x+y) - \frac{1}{4}f(-x-y) \\ &\quad + \frac{1}{4}f(x-y) + \frac{1}{4}f(y-x) - f(x) - f(y), \end{aligned}$$

$$M_2 f(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y).$$

We solve the additive-quadratic ρ-functional inequalities

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\|, \tag{0.1}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$ and

$$\|M_2 f(x,y)\| \le \|\rho M_1 f(x,y)\|, \tag{0.2}$$

where ρ is a fixed complex number with $|\rho| < 1$.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [31] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation.* In particular, every solution of the Cauchy equation is said to be an *additive mapping.* Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping.* The stability of quadratic functional equation was proved by Skof [30] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have

2010 *Mathematics Subject Classification.* Primary 39B62, 47H10, 39B52.

Key words and phrases. Hyers-Ulam stability; additive-quadratic ρ-functional inequality; fixed point; Banach space.

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been extensively investigated by a number of authors (see [1, 3, 7, 10, 17, 18, 19, 20, 21, 24, 25, 26, 27, 28, 29, 32, 33]).

We recall a fundamental result in fixed point theory.

Theorem 1.1. [4, 9] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 15, 16, 22]).

In Section 2, we solve the additive-quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in Banach spaces by using the fixed point method.

In Section 3, we solve the additive-quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in Banach spaces by using the fixed point method.

In this paper, assume that X is a complex normed space and that Y is a complex Banach space.

2. Additive-quadratic ρ-functional inequality (0.1) in Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < \frac{1}{2}$.

We solve and investigate the additive-quadratic ρ-functional inequality (0.1) in complex normed spaces.

Lemma 2.1.
(i) *If a mapping $f : X \to Y$ satisfies $M_1 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*
(ii) *If a mapping $f : X \to Y$ satisfies $M_2 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

Proof. (i)

$$M_1 f_o(x, y) = f_o(x + y) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. So f_o is the Cauchy additive mapping.

$$M_1 f_e(x, y) = \frac{1}{2} f_e(x + y) + \frac{1}{2} f_e(x - y) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. So f_o is the quadratic mapping.

(ii)

$$M_2 f_o(x,y) = 2f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_o is the Cauchy additive mapping.

$$M_2 f_e(x,y) = 2f_e\left(\frac{x+y}{2}\right) + 2f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_e is the quadratic mapping.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping and the quadratic mapping. □

Lemma 2.2.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\| \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies* (2.1), *then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (2.1).

Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 2f(x)\| \le 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \\ &= |\rho|\|f(x+y) - f(x) - f(y)\| \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|f(0)\| \le \|2\rho f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|\frac{1}{2}f(2x) - 2f(x)\| \le 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned}\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &= |\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in complex Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leq \frac{L}{2}\varphi(x,y) \tag{2.4}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|M_1f(x,y)-\rho M_2f(x,y)\| \leq \varphi(x,y) \tag{2.5}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leq \frac{L}{2(1-L)}\varphi(x,x) \tag{2.6}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5), we get

$$\|f(2x)-2f(x)\| \leq \varphi(x,x) \tag{2.7}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h)=\inf\left\{\mu\in\mathbb{R}_+ : \|g(x)-h(x)\|\leq\mu\varphi(x,x),\ \forall x\in X\right\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S, d) is complete (see [14]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x)-h(x)\| \leq \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \le 2\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\ &\le 2\varepsilon\frac{L}{2}\varphi(x,x) = L\varepsilon\varphi(x,x) \end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.7) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{2}\varphi(x,x)$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \le \mu\varphi(x,x)$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L}d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)}\varphi(x,x)$$

for all $x \in X$.

It follows from (2.4) and (2.5) that

$$\begin{aligned} &\left\|A(x+y) - A(x) - A(y) - \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \\ &= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right. \\ &\qquad \left. -2^n\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $A : X \to Y$ is additive. □

Corollary 2.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying*

$$\|M_1 f(x,y) - \rho M_2 f(x,y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{2.9}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f_o(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Theorem 2.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x,y) \tag{2.10}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_e(x) - Q(x)\| \le \frac{L}{2(1-L)}\varphi(x,x) \tag{2.11}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5) for f_e, we get

$$\left\|\frac{1}{2}f(2x) - 2f(x)\right\| \le \varphi(x,x) \tag{2.12}$$

for all $x \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.3.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \le 4\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right)\\ &\le 4\varepsilon\frac{L}{4}\varphi(x,x) = L\varepsilon\varphi(x,x)\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

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for all $g, h \in S$.

It follows from (2.12) that

$$\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \leq 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{2}\varphi(x,x)$$

for all $x \in X$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(x\right) = 4Q\left(\frac{x}{2}\right) \tag{2.13}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.13) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \quad \leq \quad \mu\varphi\left(x, x\right)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - Q(x)\| \leq \frac{L}{2(1-L)}\varphi\left(x, x\right)$$

for all $x \in X$.

It follows from (2.4) and (2.5) that

$$\begin{aligned}
&\left\| \frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y) \right. \\
&\left. - \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right) \right\| \\
&= \lim_{n\to\infty} \left\| 4^n \left(\frac{1}{2}f\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right) \right. \\
&\left. - 4^n\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right) \right\| \\
&\leq \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
&\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y) \\
&\qquad = \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic. $\square$

Corollary 2.6. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{4\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking $\varphi(x, y) = \theta(\|x\|^r+\|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{2-r}$ and we get the desired result. □

Theorem 2.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.5). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(1 - L)}\varphi(x, x)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{1}{2}\varphi(x, x)$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.8. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying (2.9). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.7 by taking $\varphi(x, y) = \theta(\|x\|^r+\|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. □

Theorem 2.9. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.5). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2(1 - L)}\varphi(x, x)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

It follows from (2.12) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{2} \varphi(x, x)$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 2.10. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \quad \leq \quad \frac{4\theta}{4 - 2^r} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.9 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-2}$ and we get desired result. □

Remark 2.11. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive-quadratic ρ-functional inequality (0.2) in complex Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < 1$.

We solve and investigate the additive-quadratic ρ-functional inequality (0.2) in complex normed spaces.

Lemma 3.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_2 f(x, y)\| \leq \|\rho M_1 f(x, y)\| \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and (3.1), then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\| \leq 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2} f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|f(x+y)-f(x)-f(y)\| &= \left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \\ &\leq |\rho|\|f(x+y)-f(x)-f(y)\|\end{aligned}$$

and so

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|4f\left(\frac{x}{2}\right)-f(x)\right\| \leq 0 \tag{3.3}$$

and so $f\left(\frac{x}{2}\right)=\frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned}&\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| \\ &= \left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| \\ &\leq |\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (3.1) in complex Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leq \frac{L}{2}\varphi(x,y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|M_2f(x,y)-\rho M_1f(x,y)\| \leq \varphi(x,y) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ a such that

$$\|f(x)-A(x)\| \leq \frac{1}{1-L}\varphi(x,0)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4), we get

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right)-f(x)\right\| \leq \varphi(x,0) \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0)=0\}$$

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and introduce the generalized metric on S:

$$d(g,h) = \inf \{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu \varphi(x,0), \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [14]).

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying*

$$\|M_2 f(x,y) - \rho M_1 f(x,y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{3.6}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2^r \theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ and we get desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x,y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{1-L}\varphi(x,0)$$

for all $x \in X$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 3.2.

Letting $y = 0$ in (3.4), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| = \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi(x,0) \tag{3.7}$$

for all $x \in X$.

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 3.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.6). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2^r\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{2-r}$ and we get desired result. □

Theorem 3.6. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (3.4). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{L}{1 - L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.5) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(2x, 0) \le L\varphi(x, 0)$$

for all $x \in X$.

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.7. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying (3.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^r\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.6 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. □

Theorem 3.8. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{L}{1 - L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.7) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{4} \varphi(2x, 0) \leq L\varphi(x, 0)$$

for all $x \in X$.

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 3.9. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.6). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^r \theta}{4 - 2^r} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.8 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-2}$ and we get desired result. □

Remark 3.10. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

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Global Attractivity and the Periodic Nature of Third Order Rational Difference Equation

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ABSTRACT

The main target of our study to cover the solutions behavior of the following difference equation

$$x_{n+1} = ax_n + bx_{n-1} + \frac{c + dx_{n-2}}{e + fx_{n-2}}, \quad n = 0, 1, ...,$$

where the parameters $a,\ b,\ c,\ d,\ e$ and f are positive real numbers and the initial conditions $x_{-2},\ x_{-1}$ and x_0 are positive real numbers.

Keywords: stability, boundedness, periodicity, global attractor, difference equations.

Mathematics Subject Classification: 39A10

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1. INTRODUCTION

Our objective in this research is to study character of global stability and the periodicity of the solutions of the recursive sequence

$$x_{n+1} = ax_n + bx_{n-1} + \frac{c + dx_{n-2}}{e + fx_{n-2}}, \tag{1}$$

where the following parameters $a,\ b,\ c,\ d,\ e$ and f are defined as positive real numbers and the initial conditions $x_{-2},\ x_{-1}$ and x_0 are also defined as positive real numbers.

The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of discrete dynamical systems and difference equations have appeared recently in many areas. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering, economics, probability theory, genetics, psychology and resource management [12]. It is very interesting to investigate the behavior of solutions of a higher-order rational difference equation and to discuss the local asymptotic stability of its equilibrium points. Rational difference equations have been studied by several authors. Especially there has been a great interest in the study of the attractivity of the solutions of such equations. For more results for the rational difference equations, we refer the interested reader to [1–30].

The study of the nonlinear rational difference equations of a higher order is quite challenging and rewarding, and the results about these equations offer prototypes towards the development of the basic theory of the global behavior of nonlinear difference equations of a big order, recently, many researchers have investigated the behavior

of the solution of difference equations for example: Abo-Zeid and Al-Shabi [1] investigated the global stability, and periodic nature of the positive solutions of the difference equation

$$x_{n+1} = \frac{A+Bx_n}{C+Dx_n x_{n-2}}.$$

Belhannache et al. [5] studied the global behavior of positive solutions of the following third order difference equation

$$x_{n+1} = \frac{A+Bx_{n-1}}{C+Dx_n^p x_{n-2}^q}.$$

Dehghan and Rastegar [11], deal with the qualitative behavior of solutions of the higher-order non-linear difference equation

$$x_{n+1} = \frac{p+qx_n+rx_{n-k}}{1+x_{n-k}}.$$

Din [14] investigated the local asymptotic stability, global stability, the periodic character, semicycle analysis and the boundedness nature of the following rational difference equation

$$x_{n+1} = \frac{A+Bx_n+Cx_{n-k}}{1+x_n+x_{n-k}}.$$

In [16] Elabbasy et al. investigated the global stability character, boundedness and the periodicity of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}}{Ax_n+Bx_{n-1}+Cx_{n-2}}.$$

Elsayed [22] investigated the local and global stability, boundedness character and obtained the solution of some special cases of the following recursive sequence

$$x_{n+1} = ax_{n-1} + \frac{bx_n x_{n-1}}{cx_n+dx_{n-2}}.$$

A. El-Moneam, and Zayed [20]-[21] studied the periodicity, the boundedness and the global stability of the positive solutions of the following nonlinear difference equations

$$\begin{aligned} x_{n+1} &= Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{bx_{n-k}}{dx_{n-k}-ex_{n-l}}. \\ x_{n+1} &= Ax_n + Bx_{n-k} + Cx_{n-l} + +Dx_{n-\sigma} + \frac{bx_{n-k}+hx_{n-l}}{dx_{n-k}+ex_{n-l}}. \end{aligned}$$

Su and Li [52] studied the global asymptotic stability of the nonlinear difference equation

$$x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-1}}.$$

Yalçınkaya et al. [54] considered the dynamics of the difference equation

$$x_{n+1} = \frac{ax_{n-k}}{b+cx_n^p}.$$

For some related work see [31–57].

2. SOME BASIC PROPERTIES AND DEFINITIONS

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let $F : I^{k+1} \to I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k},\ x_{-k+1},\ ...,\ x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. (Equilibrium Point) A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 2. (Periodicity) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 3. (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{ for all } \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have $\lim_{n\to\infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \tfrac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A. [47] Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then $|p| + |q| < 1$, is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

Remark: Theorem A can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ... \tag{4}$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1, 2, ...\}$. Then Eq. (4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Theorem B. [48] Let $g : [a, b]^{k+1} \to [a, b]$, be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, ... \tag{5}$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq k+1$; the function $g(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, ..., m_{k+1}), \qquad M = g(M_1, M_2, ..., M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, ..., k+1$, we set

$$\begin{aligned} m_i &= \left\{ \begin{matrix} m, & \text{if } g \text{ is non-decreasing in } z_i, \\ M, & \text{if } g \text{ is non-increasing in } z_i, \end{matrix} \right\}, \\ Mi &= \left\{ \begin{matrix} M, & \text{if } g \text{ is non-decreasing in } z_i, \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{matrix} \right\}. \end{aligned}$$

Then there exists exactly one equilibrium point $\bar{x}$ of Equation (5), and every solution of Equation (5) converges to $\bar{x}$.

3. LOCAL STABILITY OF THE EQUILIBRIUM POINT OF EQ.(1)

This section deals with study the local stability character of the equilibrium point of Eq.(1)

Eq.(1) has equilibrium point and is given by

$$\overline{x} = a\overline{x} + b\overline{x} + \frac{c + d\overline{x}}{e + f\overline{x}} \qquad \Rightarrow \qquad \overline{x}(1 - a - b) = \frac{c + d\overline{x}}{e + f\overline{x}},$$

$$f(1 - a - b)\overline{x}^2 + [e(1 - a - b) - d]\,\overline{x} - c = 0$$

If $d > e(1 - a - b) > 0$, then the only positive equilibrium point of Eq.(1) is given by

$$\overline{x} = \tfrac{[d - e(1-a-b)] + \sqrt{[d - e(1-a-b)]^2 + 4fc(1-a-b)}}{2f(1-a-b)}.$$

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a continuous function defined by

$$f(u, v, w) = au + bv + \frac{c + dw}{e + fw}. \tag{6}$$

Therefore it follows that

$$\frac{\partial f(u, v, w)}{\partial u} = a, \quad \frac{\partial f(u, v, w)}{\partial v} = b, \quad \frac{\partial f(u, v, w)}{\partial w} = \frac{(de - fc)}{(e + fw)^2}.$$

Then we see that

$$\frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial u} = a = -a_2, \quad \frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial v} = b = -a_1, \quad \frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial w} = \frac{de - fc}{(e + f\overline{x})^2} = -a_0.$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} + a_2 y_n + a_1 y_{n-1} + a_0 y_{n-2} = 0, \tag{7}$$

whose characteristic equation is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{8}$$

Theorem 1. Assume that
$$\frac{|de-fc|}{(e+f\overline{x})^2}<1-a-b.$$
Then the positive equilibrium point of Eq.(1) is locally asymptotically stable.

Proof: It follows by Theorem A that, Eq.(7) is asymptotically stable if all roots of Eq.(8) lie in the open disc $|\lambda|<1$ that is if
$$|a_2|+|a_1|+|a_0|<1 \qquad \Rightarrow \qquad |a|+|b|+\left|\frac{de-fc}{(e+f\overline{x})^2}\right|<1,$$
and so
$$a+b+\frac{|de-fc|}{(e-f\overline{x})^2}<1,$$
or
$$\frac{|de-fc|}{(e+f\overline{x})^2}<1-a-b.$$
The proof is complete.

4. BOUNDEDNESS OF SOLUTIONS OF EQ.(1)

Here we study the boundedness nature of solutions of Eq.(1).

Theorem 2. Every solution of Eq.(1) is bounded if $a+b+\dfrac{d}{e}<1$.

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that
$$x_{n+1}=ax_n+bx_{n-1}+\frac{c+dx_{n-2}}{e+fx_{n-2}}\le ax_n+bx_{n-1}+\frac{c+dx_{n-2}}{e}.$$
Then
$$x_{n+1}\le ax_n+bx_{n-1}+\frac{d}{e}x_{n-2}+\frac{c}{e} \quad \text{for all} \quad n\ge 1.$$
By using a comparison, we can write the right hand side as follows
$$y_{n+1}=ay_n+by_{n-1}+\frac{d}{e}y_{n-2}+\frac{c}{e},$$
and this equation is locally asymptotically stable if $a+b+\frac{d}{e}<1$, and converges to the equilibrium point $\overline{y}=\frac{c}{e\left(1-a-b-\frac{d}{e}\right)}$. Therefore
$$\limsup_{n\to\infty} x_n\le\frac{c}{e\left(1-a-b-\frac{d}{e}\right)}.$$
Thus the solution is bounded.

Theorem 3. Every solution of Eq.(1) is unbounded if $a>1$ $(or\ b>1)$.

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). Then from Eq.(1) we see that
$$x_{n+1}=ax_n+bx_{n-1}+\tfrac{c+dx_{n-2}}{e+fx_{n-2}}>ax_n \quad \text{for all} \quad n\ge 1.$$
We see that the right hand side can write as follows
$$y_{n+1}=ay_n \quad \Rightarrow \quad y_n=a^n y_0,$$
and this equation is unstable because $a>1$, and $\lim\limits_{n\to\infty} y_n=\infty$. Then by using ratio test $\{x_n\}_{n=-2}^{\infty}$ is unbounded from above (when $b>1$ is similar).

5. EXISTENCE OF PERIOD TWO SOLUTIONS

In this section we study the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

Theorem 4. Eq.(1) has positive prime period two solutions if and only if

$$(i)\ (eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)>0,\quad B=b-a-1.$$

Proof: First suppose that there exists a prime period two solution $...,p,q,p,q,...,$ of Eq.(1). We will prove that Condition (i) holds. We see from Eq.(1) that

$$\begin{aligned} p &= aq+bp+\frac{c+dq}{e+fq},\qquad q=ap+bq+\frac{c+dp}{e+fp}. \\ p(1-b)-aq &= \frac{c+dq}{e+fq},\qquad q(1-b)-ap=\frac{c+dp}{e+fp}. \end{aligned}$$

Then

$$ep(1-b)+pqf(1-b)-aeq-afq^2=c+dq,$$

and

$$eq(1-b)+pqf(1-b)-aep-afp^2=c+dp.$$

Then

$$ep(1-b)+pqf(1-b)-afq^2=c+(d+ae)q, \tag{9}$$

and

$$eq(1-b)+pqf(1-b)-afp^2=c+(d+ae)p. \tag{10}$$

Subtracting (9) from (10) gives

$$e(1-b)(p-q)+af(p-q)(p+q)=-(d+ae)(p-q).$$

Since $p\neq q$, it follows that

$$\begin{aligned} e(1-b)+af(p+q) &= -(d+ae), \\ p+q &= \frac{e(b-1-a)-d}{af}. \end{aligned}$$

or

$$p+q=\frac{eB-d}{af},\qquad B=b-a-1. \tag{11}$$

Again, adding (9) and (10) yields

$$\begin{aligned} &e(1-b)(p+q)+2pqf(1-b)-af(p^2+q^2)=2c+(d+ae)(p+q), \\ &2pqf(1-b)-af((p+q)^2-2pq)=2c+(p+q)(d+ae-e(1-b)). \end{aligned} \tag{12}$$

It follows by (11), (12) and the relation

$$p^2+q^2=(p+q)^2-2pq\quad \text{for all }\ p,q\in R,$$

that

$$2pqf(1-b)+2afpq=af(p+q)^2+2c+(p+q)(d+e(a-1+b)).$$

and

$$2pqf((1-b)+a)=2c+(p+q)\left\{d+e(a-1+b)+af(p+q)\right\}.$$

From Eq. (11) we have

$$\begin{aligned} 2pqf((1-b)+a) &= 2c+(p+q)\{d+e(a-1+b)+e(b-1-a)-d\}, \\ 2pqf((1-b+a)) &= 2c+(p+q)\{-2e+2eb\}, \end{aligned}$$

$$\begin{aligned} pqf(-B) &= c+(p+q)e(b-1) \\ pqfB &= e(1-b)\left(\frac{eB-d}{af}\right)-c. \end{aligned}$$

Thus

$$pq=\frac{e^2(1-b)B-ed(1-b)-af}{aBf^2}. \tag{13}$$

Now it is clear from Eq.(11) and Eq.(13) that p and q are the two distinct roots of the quadratic equation

$$\begin{aligned} t^2-\left(\tfrac{eB-d}{af}\right)t+\left(\tfrac{e^2(1-b)B-ed(1-b)-acf}{aBf^2}\right) &= 0, \\ aBf^2t^2-(eB-d)Bft+\left(e^2(1-b)B-ed(1-b)-acf\right) &= 0, \end{aligned} \tag{14}$$

and so

$$(eB-d)^2B^2f^2>4aBf^2\left(e^2(1-b)B-ed(1-b)-acf\right),$$

or

$$(eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)>0.$$

Therefore Inequality (i) holds.

Second suppose that Inequality (i) is true. We will show that Eq.(1) has a prime period two solution. Assume that

$$p=\frac{(eB-d)Bf+\sqrt{\zeta}}{2aBf^2}, \quad q=\frac{(eB-d)Bf-\sqrt{\zeta}}{2aBf^2},$$

where $\zeta=(eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)$.

We see from Inequality (i) that

$$(eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)>0,$$

which equivalents to

$$(eB-d)^2B^2f^2>4aBf^2\left(e^2(1-b)B-ed(1-b)-acf\right).$$

Therefore p and q are distinct real numbers. Set $x_{-2}=p$, $x_{-1}=q$ and $x_0=p$. We wish to show that $x_1=x_{-1}=q$ and $x_2=x_0=p$. It follows from Eq.(1) that

$$x_1=ap+bq+\frac{c+dp}{e+fp}=\tfrac{a(eB-d)Bf+a\sqrt{\zeta}}{2aBf^2}+\tfrac{b(eB-d)Bf-b\sqrt{\zeta}}{2aBf^2}+\frac{c+\left(\frac{d(eB-d)Bf+d\sqrt{\zeta}}{2aBf^2}\right)}{e+\left(\frac{(eB-d)Bf^2+f\sqrt{\zeta}}{2aBf^2}\right)}.$$

Multiplying the denominator and numerator by $2aBf^2$ gives

$$x_1=a(eB-d)Bf+a\sqrt{\zeta}+b(eB-d)Bf-b\sqrt{\zeta}+\frac{2acBf^2+\left(d(eB-d)Bf+d\sqrt{\zeta}\right)}{2aeBf^2+\left((eB-d)Bf^2+f\sqrt{\zeta}\right)}.$$

By simple computations we can see that

$$x_1=\frac{(eB-d)Bf+\sqrt{\zeta}}{2aBf^2}=q.$$

Similarly as before one can easily show that $x_2=p$. Then it follows by induction that

$$x_{2n}=p \quad \text{and} \quad x_{2n+1}=q \quad \text{for all} \quad n\geq -2.$$

Thus Eq.(1) has the prime period two solution ...,p,q,p,q,..., where p and q are the distinct roots of the quadratic equation (14) and the proof is complete.

6. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In this section we investigate the global asymptotic stability of Eq.(1).

Theorem 5. The equilibrium point $\overline{x}$ is a global attractor of Eq.(1) if one of the following statements holds

$$de \geq fc \text{ and } (1-a-b)\,e \geq d. \tag{15}$$
$$de < fc \text{ and } (1-a-b) \geq 0. \tag{16}$$

Proof: Let α and β be a real numbers and assume that $g : [\alpha,\beta]^3 \longrightarrow [\alpha,\beta]$ be a function defined by

$$g(u,v,w) = au + bv + \frac{c+dw}{e+fw}.$$

Then

$$\frac{\partial g(u,v,w)}{\partial u} = a, \quad \frac{\partial g(u,v,w)}{\partial v} = b, \quad \frac{\partial g(u,v,w)}{\partial w} = \frac{de-fc}{(e+fw)^2}.$$

We consider the two cases:-

Case (1): Assume that (15) is true, then we can easily see that the function $g(u,v,w)$ increasing in u, v and w.

Suppose that (m, M) is a solution of the system $M = g(M,M,M)$ *and* $m = g(m,m,m)$. Then from Eq.(1), we see that

$$\begin{aligned} M &= aM + bM + \frac{c+dM}{de+fM}, \quad m = am + bm + \frac{c+dm}{e+fm}, \\ M(1-a-b) &= \frac{c+dM}{e+fM}, \quad m(1-a-b) = \frac{c+dm}{e+fm}, \end{aligned}$$

then

$$MAe + AfM^2 = c + dM, \quad mAe + Afm^2 = c + dm, \quad A = 1-a-b.$$

Subtracting this two equations we obtain

$$(M-m)\{Ae + Af(M+m) - d\} = 0,$$

under the conditions $Ae \geq d,\ a < 1$, we see that $M = m$. It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete.

Case (2): Assume that (16) is true, then we can easily see that the function $g(u,v,w)$ increasing in u, v and decreasing in w.

Suppose that (m, M) is a solution of the system $M = g(M,M,m)$ *and* $m = g(m,m,M)$.Then from Eq.(1), we see that

$$\begin{aligned} M &= aM + bM + \frac{c+dm}{e+fm}, \quad m = am + bm + \frac{c+dM}{e+fM}, \\ MA &= \frac{c+dm}{e+fm}, \quad mA = \frac{c+dM}{e+fM}, \end{aligned}$$

then

$$MAe + MAfm = c + dm, \quad mAe + fMmA = c + dM.$$

Subtracting we obtain

$$(M-m)\,(Ae+d) = 0,$$

under the conditions $(1-a-b) > 0$, we see that $M = m$. Also, from Theorem B, we see that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete.

7. NUMERICAL EXAMPLES

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1. We assume $x_{-2} = .5,\ x_{-1} = 3,\ x_0 = 9,\ a = .2,\ b = .7,\ c = .2,\ d = .6,\ e = 1.3,\ f = 5.3$**.** See Fig. 1.

Example 2. See Fig. 2, since $x_{-2} = .5,\ x_{-1} = 3,\ x_0 = 9,\ a = .4,\ b = .6,\ c = .2,\ d = .6,\ e = 1.3,\ f = 5.3$.

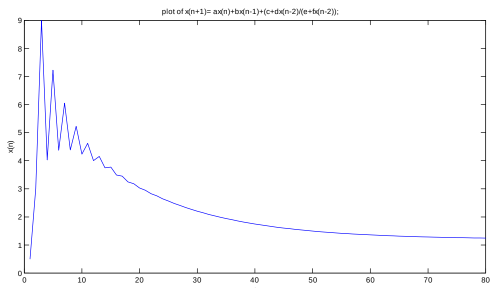

Figure 1.

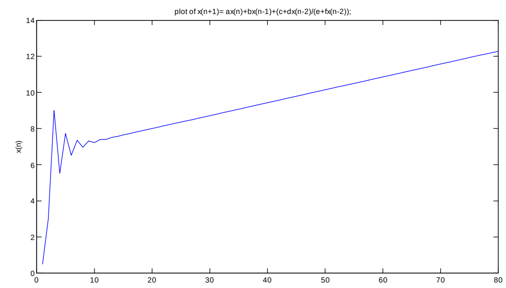

Figure 2.

Example 3. We consider $x_{-2} = 2.5,\ x_{-1} = 3,\ x_0 = 9,\ a = .4,\ b = .5,\ c = 2,\ d = 6,\ e = 3,\ f = 5$. See Fig. 3.

Example 4. See Fig. 4, since $x_{-2} = 2.5,\ x_{-1} = 3,\ x_0 = 9,\ a = 1,\ b = .5,\ c = 2,\ d = 6,\ e = 3,\ f = 5$.

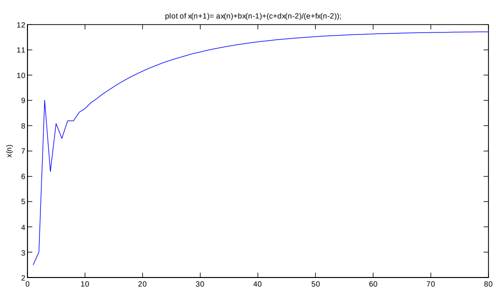

Figure 3.

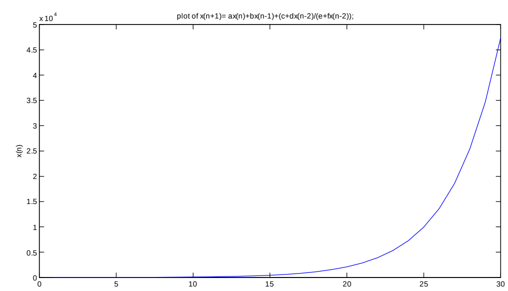

Figure 4.

Example 5. Fig. 5. shows the solutions when $a = .7,\ b = .5,\ c = .2,\ d = .1,\ e = .3,\ f = .5,\ x_{-2} = 2.5,\ x_{-1} = .3,\ x_0 = .9$.

Example 6. Fig. 6. shows the period two solutions when $a = .6,\ b = .5,\ c = .82,\ d = .7,\ e = .3,\ f = .5,\ x_{-2} = p,\ x_{-1} = q,\ x_0 = p$. $\left(\text{Since } p, q = \frac{(eB-d)Bf \pm \sqrt{\zeta}}{2aBf^2}\right)$.

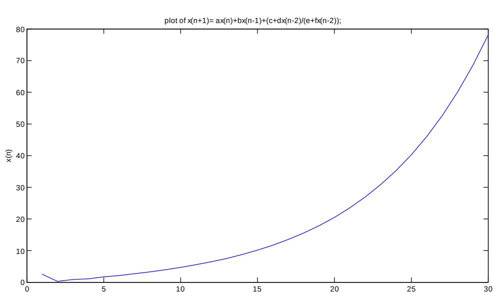

Figure 5.

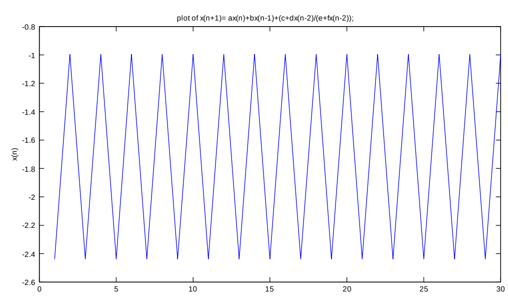

Figure 6.

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Asymptotically stability of solutions of fuzzy differential equations in the quotient space of fuzzy numbers

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Abstract

In this paper, we investigate essentially stability theory for the fuzzy differential equations in the quotient space of fuzzy numbers by Lyapunov-like functions. By using the differential inequalities and the comparison principle for Lyapunov-like functions, we give some sufficient criterias for the asymptotically stability, equi-asymptotically stability and uniformly asymptotically stability of the trivial solution of the fuzzy differential equations.

Keywords: Fuzzy number; Quotient space; Fuzzy differential equation; Asymptotically stability

1 Introduction

Recently, the study of fuzzy differential equations has been gained importance due to its application. Subsequently, the existence and uniqueness of solutions of the initial value ptoblems for fuzzy differential equations under kinds of conditions were studied in [8, 9, 11, 14, 18, 24] and the relationship between a solution and its approximate solutions to fuzzy differential equations were established in [19, 25, 26]. Further, the essentially stability theory for fuzzy differential equations by Lyapunov-like functions were investigated in [2, 12, 28]. In particular, Hien [4] researched the asymptotic stability of solutions of fuzzy differential equations by Lyapunovs second method.

The above these results of fuzzy differential equations based on well known and widely used Hukuhara difference [6] and the H-differentiability of Puri and Ralescu [20]. But in many applications the Hukuhara difference appears to have several limitations and to be very restrictive [1, 8]. In [15, 16], Mareš presented a natural equivalence relation between fuzzy quantities. This equivalence relation can be used to partition of the set of fuzzy quantities into equivalence classes having the desired group properties for the addition operation [7, 17, 27]. Hong and Do [5] defined a more refined equivalence relation than Mareš [15] and improved Mareš's results. In [21], Qiu et al. showed that the method of finding the inverse operation of fuzzy numbers in the sense of Mareš is very intuitive. As an application of the main results, it is shown that if we identify every fuzzy number with the corresponding equivalence class, there wound be more differentiable fuzzy functions than what is found in the literature. After that, the fuzzy differential equations in the quotient space of fuzzy numbers were investigated [23, 22]. In this paper, we shall study the stability of the trivial solution of the fuzzy differential equations in the quotient space of fuzzy numbers by Lyapunov's second method.

2 Preliminaries

A fuzzy set $\widetilde{x}$ of $\mathbb{R}$ is characterized by a membership function $\mu_{\widetilde{x}} : \mathbb{R} \to [0,1]$. For each such fuzzy set $\widetilde{x}$, we denote by $[\widetilde{x}]^{\alpha} = \{x \in \mathbb{R} : \mu_{\widetilde{x}}(x) \geq \alpha\}$ for any $\alpha \in (0,1]$, its α-level set. We define the set

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$[\widetilde{x}]^0$ by $[\widetilde{x}]^0 = \overline{\bigcup_{\alpha \in (0,1]} [\widetilde{x}]^\alpha}$, where $\overline{A}$ denotes the closure of a crisp set A. A fuzzy set $\widetilde{x}$ is said to be a fuzzy number if it satisfies the following conditions [3]:

(1) $\widetilde{x}$ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $\mu_{\widetilde{x}}(x_0) = 1$;
(2) $\widetilde{x}$ is convex, i.e., $\mu_{\widetilde{x}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\widetilde{x}}(x_1), \mu_{\widetilde{x}}(x_2)\}$, for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in (0,1)$;
(3) $\widetilde{x}$ is upper semi-continuous;
(4) $[\widetilde{x}]^0$ is compact.

Equivalently, a fuzzy number $\widetilde{x}$ is a fuzzy set with non-empty bounded closed level sets $[\widetilde{x}]^\alpha = [\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ for all $\alpha \in [0,1]$, where $[\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ denotes a closed interval with the left end point $\widetilde{x}_L(\alpha)$ and the right end point $\widetilde{x}_R(\alpha)$. We denote the class of fuzzy numbers by $\mathscr{F}$. We say that a fuzzy number $\widetilde{s} \in \mathscr{F}$ is symmetric [15], if $\mu_{\widetilde{s}}(x) = \mu_{\widetilde{s}}(-x)$, for all $x \in \mathbb{R}$, i.e., $\widetilde{s} = -\widetilde{s}$. The set of all symmetric fuzzy numbers will be denoted by $\mathscr{S}$.

Definition 2.1 *[5] Let $\widetilde{x}, \widetilde{y} \in \mathscr{F}$. We say that $\widetilde{x}$ is equivalent to $\widetilde{y}$ and write $\widetilde{x} \sim \widetilde{y}$ if and only if there exist symmetric fuzzy numbers $\widetilde{s}_1, \widetilde{s}_2 \in \mathscr{S}$ such that $\widetilde{x} + \widetilde{s}_1 = \widetilde{y} + \widetilde{s}_2$.*

The equivalence relation defined above is reflexive, symmetric and transitive [15]. Let $\langle \widetilde{x} \rangle$ denote the equivalence class containing the element $\widetilde{x}$ and denote the set of equivalence classes by $\mathscr{F}/\mathscr{S}$.

Definition 2.2 *[10] Let $f : [a,b] \to \mathbb{R}$. f is said be of bounded variation if there exists a $C > 0$ such that*

$$\sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)| \leq C$$

for every partition $a = x_0 < x_1 < \cdots < x_n = b$ on $[a,b]$. The total variation of f on $[a,b]$ is defined by

$$V_a^b(f) = \sup_p \sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)|,$$

where p represents all partitions of $[a,b]$. The set of all functions of bounded variation on $[a,b]$ is denoted by $BV[a,b]$.

Definition 2.3 *[7] For a fuzzy number $\widetilde{x}$, we define a function $\widetilde{x}_M : [0,1] \to \mathbb{R}$ by assigning the midpoint of each α-level set to $\widetilde{x}_M(\alpha)$ for all $\alpha \in [0,1]$, i.e.,*

$$\widetilde{x}_M(\alpha) = \frac{\widetilde{x}_L(\alpha) + \widetilde{x}_R(\alpha)}{2}.$$

Then the function $\widetilde{x}_M : [0,1] \to \mathbb{R}$ will be called the midpoint function of the fuzzy number $\widetilde{x}$.

Lemma 2.1 *[21] For any $\tilde{x} \in \mathscr{F}$, the midpoint function $\tilde{x}_M$ is continuous from the right at 0 and continuous from the left on $[0,1]$. Furthermore it is a function of bounded variation on $[0,1]$.*

Definition 2.4 *[16] Let $\widetilde{x} \in \mathscr{F}$ and let $\widehat{x}$ be a fuzzy number such that $\widetilde{x} = \widehat{x} + \widetilde{s}$ for some $\widetilde{s} \in \mathscr{S}$, if $\widehat{x} = \widetilde{y} + \widetilde{s}_1$ for some $\widetilde{y} \in \mathscr{F}$ and $\widetilde{s}_1 \in \mathscr{S}$, then $\widetilde{s}_1 = \widetilde{0}$. Then the fuzzy number $\widehat{x}$ will be called the Mareš core of the fuzzy number $\widetilde{x}$.*

Definition 2.5 *[22] Define $d_{\sup} : \mathscr{F}/\mathscr{S} \times \mathscr{F}/\mathscr{S} \to \mathbb{R}^+ \cup \{0\}$ by*

$$d_{\sup}(\langle \widetilde{x} \rangle, \langle \widetilde{y} \rangle) = \sup_{\alpha \in [0,1]} \left| M_{\langle \widetilde{x} \rangle}(\alpha) - M_{\langle \widetilde{y} \rangle}(\alpha) \right|,$$

for any $\langle \widetilde{x} \rangle, \langle \widetilde{y} \rangle \in \mathscr{F}/\mathscr{S}$.

We know that $(\mathscr{F}/\mathscr{S}, d_{\sup})$ is a metric space [21].

3 Main results

Definition 3.1 *[22] For each $m(t) \in C[J,\mathbb{R}]$, where J is a subinterval of $(0,+\infty)$, we will define $d^+ : C[J,\mathbb{R}] \to \mathbb{R}$ by*

$$d^+m(t) = \overline{\lim_{h\to 0^+}} \frac{1}{h}(m(t+h) - m(t)).$$

Definition 3.2 *[23] A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is differentiable at $t_0 \in J$ if for small $|h| > 0$, there exists an $F'(t_0) \in \mathscr{F}/\mathscr{S}$ such that*

$$\lim_{h\to 0} d_{\sup}\left(\frac{F(t_0+h) - F(t_0)}{h}, F'(t_0)\right) = 0.$$

Definition 3.3 *[23] A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is measurable if F is measurable with respect to $d_{\sup}$.*

A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is called integrably bounded if there exists an integrable function $h : J \to \mathbb{R}^+ \cup \{0\}$ such that $\left|M_{F(t)}(\alpha)\right| \le h(t)$ for all $t \in J$ and $\alpha \in [0,1]$; a mapping $F : J \to \mathscr{F}/\mathscr{S}$ is said to be of uniformly bounded variation with respect to $\alpha \in [0,1]$ (for short, of uniformly bounded variation) if there exists a constant $K > 0$ such that $V_0^1\left(M_{F(t)}\right) \le K$, for each $t \in J$ [23].

Definition 3.4 *[23] Let $F : J \to \mathscr{F}/\mathscr{S}$ be measurable. The integral of F over J, denoted $\int_J F(t)dt$, is a mapping $M_{\int_J F(t)dt} : [0,1] \to \mathbb{R}$, which is defined by the equation*

$$M_{\int_J F(t)dt}(\alpha) = \int_J M_{F(t)}(\alpha)dt$$

for each $\alpha \in [0,1]$. The mapping F is said to be integrable over J if there exists an $\langle \widetilde{x}_0\rangle \in \mathscr{F}/\mathscr{S}$ such that $M_{\int_J F(t)dt} = M_{\langle \widetilde{x}_0\rangle}$. In this case, we denote the integral by

$$\int_J F(t)dt = \langle \widetilde{x}_0\rangle .$$

Assume that $f : \mathbb{R}_+ \times S(\rho) \to \mathscr{F}/\mathscr{S}$ is continuous and of uniformly bounded variation, where $S(\rho) = \{\langle \widetilde{x}\rangle \in \mathscr{F}/\mathscr{S} : d_{\sup}(\langle \widetilde{x}\rangle, \langle \widetilde{0}\rangle) < \rho\}$. We consider the initial value problem for the fuzzy differential equation

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0. \tag{1}$$

We assume that $f(t, \langle \widetilde{0}\rangle) = \langle \widetilde{0}\rangle$ so that we have the trivial solution $x(t) = \langle \widetilde{0}\rangle$ for (1).

We shall discuss some simple asymptotically stability results of solutions of (1) by Lyapunov's second method. First, we give some notions of concerning the stability of the trivial solution of (1). Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) existing on $[t_0, +\infty)$. Denote $\mathcal{K} = \{\omega \in C[\mathbb{R}_+, \mathbb{R}_+], \omega(0) = 0, \omega(\cdot)$ is increasing $\}$.

Definition 3.5 *The trivial solution $x(t) = \langle \widetilde{0}\rangle$ of (1) is said to be*

$(S1)$ stable, if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that if $d_{\sup}(x_0, \langle \widetilde{0}\rangle) < \delta$ then

$$d_{\sup}(x(t), \langle \widetilde{0}\rangle) < \varepsilon, \qquad t \ge t_0;$$

$(S2)$ uniformly stable, if δ in $(S1)$ is independent of t_0;

$(S3)$ asymptotically stable, if it is stable and for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0) > 0$ and $T = T(t_0, x_0, \varepsilon) > 0$ such that if $d_{\sup}(x_0, \langle \widetilde{0}\rangle) < \delta$ then

$$d_{\sup}(x(t), \langle \widetilde{0}\rangle) < \varepsilon, \qquad t \ge t_0 + T;$$

$(S4)$ equi-asymptotically stable, if T in $(S3)$ is independent of x_0;

$(S5)$ uniformly asymptotically stable, if it is uniformly stable and δ and T in $(S4)$ are independent of t_0.

Lemma 3.1 *[13] Suppose that $g(t,\varphi)$ be a continuous function on $\mathbb{R}_+^2$ and $r(t) = r(t,t_0,\varphi_0)$, $\varphi(t_0) = \varphi_0$ be the maximal solution of the scalar differential equation:*

$$\frac{d\varphi}{dt} = g(t,\varphi), \quad \varphi(t_0) = \varphi_0 \geq 0, \tag{2}$$

existing on $[t_0, +\infty)$. Let $m(t)$ be a continuous function on $\mathbb{R}_+$ satisfies

$$d^+m(t) = \overline{\lim_{h\to 0^+}} \frac{m(t+h) - m(t)}{h} \leq g(t, m(t)), \qquad t \geq t_0.$$

Then $m(t) \leq r(t)$, for each $t \geq t_0$ if $m(t_0) \leq \varphi_0$.

Let $V(t,\langle\widetilde{x}\rangle) : \mathbb{R}_+ \times S(\rho) \to \mathbb{R}$ be a given function. Then we define

$$D_f^+ V(t,\langle\widetilde{x}\rangle) = \overline{\lim_{h\to 0^+}} \frac{1}{h}\left(V(t+h, \langle\widetilde{x}\rangle + hf(t,\langle\widetilde{x}\rangle)) - V(t,\langle\widetilde{x}\rangle)\right),$$

where $f(\cdot)$ is the right-hand side of (1). Note that, if $V(t,x)$ is Lipchitzian in x, then we have

$$d^+V(t,x(t)) \leq D_f^+ V(t,x(t)).$$

Lemma 3.2 *[22] Suppose that*
(1) $|V(t,\langle\widetilde{x}\rangle) - V(t,\langle\widetilde{y}\rangle)| \leq L(t) d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;
(2) $D_f^+ V(t,\langle\widetilde{x}\rangle) \leq g(t, V(t,\langle\widetilde{x}\rangle))$, $g(\cdot,\cdot) \in C[\mathbb{R}_+^2, \mathbb{R}]$.
If $x(t) = x(t,t_0,x_0)$ is any solution of (1) through (t_0,x_0) existing on $[t_0,+\infty)$ such that $V(t_0,x_0) \leq \varphi_0$, then we have

$$V(t,x(t)) \leq r(t,t_0,\varphi_0), \qquad t \geq t_0,$$

where $r(t,t_0,\varphi_0)$ is the maximal solution of the scalar differential equation (2) existing on $[t_0,+\infty)$.

Lemma 3.3 *Suppose that*
(1) $|V(t,\langle\widetilde{x}\rangle) - V(t,\langle\widetilde{y}\rangle)| \leq L(t) d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;
(2) $D_f^+ V(t,\langle\widetilde{x}\rangle) \leq -\omega(h(t,\langle\widetilde{x}\rangle)) + g(t, V(t,\langle\widetilde{x}\rangle))$, $h(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $\omega(\cdot) \in \mathcal{K}$ *and* $g(t,\varphi) \in C[\mathbb{R}_+^2, \mathbb{R}]$ *is nondecreasing with respect to φ for each $t \in \mathbb{R}_+$.*

If $x(t) = x(t,t_0,x_0)$ is any solution of (1) through (t_0,x_0) existing on $[t_0,+\infty)$ such that $V(t_0,x_0) \leq \varphi_0$, then we have

$$V(t,x(t)) + \int_{t_0}^{t} \omega(h(s,x(s)))ds \leq r(t,t_0,\varphi_0), \qquad t \geq t_0,$$

where $r(t,t_0,\varphi_0)$ is the maximal solution of the scalar differential equation (2) existing on $[t_0,+\infty)$.

Proof. Let $m(t) = V(t,x(t)) + \int_{t_0}^{t} \omega(h(s,x(s)))ds \geq V(t,x(t))$ for each $t \geq t_0$. Then $m(t_0) = V(t_0,x_0) \leq \varphi_0$ and for small $h > 0$,

$$\begin{aligned}
m(t+h) - m(t) &= V(t+h, x(t+h)) + \int_{t_0}^{t+h} \omega(h(s,x(s)))ds \\
&- V(t,x(t)) - \int_{t_0}^{t} \omega(h(s,x(s)))ds \\
&= V(t+h, x(t+h)) - V(t+h, x(t) + hf(t,x(t))) \\
&+ V(t+h, x(t) + hf(t,x(t))) - V(t,x(t)) + \int_{t}^{t+h} \omega(h(s,x(s)))ds \\
&\leq L(t+h) d_{\sup}(x(t+h), x(t) + hf(t,x(t))) \\
&+ V(t+h, x(t) + hf(t,x(t))) - V(t,x(t)) + \int_{t}^{t+h} \omega(h(s,x(s)))ds.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
d^+m(t) &= \overline{\lim_{h\to 0^+}} \frac{m(t+h)-m(t)}{h} \\
&\leq D_f^+V(t,x(t)) + \overline{\lim_{h\to 0^+}} \frac{1}{h}\int_t^{t+h} \omega(h(s,x(s)))ds \\
&+ L(t)\overline{\lim_{h\to 0^+}} \frac{1}{h} d_{\sup}(x(t+h), x(t)+hf(t,x(t))) \\
&= D_f^+V(t,x(t)) + \omega(h(t,x(t))) \\
&+ L(t)\overline{\lim_{h\to 0^+}} d_{\sup}\left(\frac{x(t+h)-x(t)}{h}, f(t,x(t))\right) \\
&= D_f^+V(t,x(t)) + \omega(h(t,x(t))) + L(t)d_{\sup}\left(x'(t), f(t,x(t))\right) \\
&= D_f^+V(t,x(t)) + \omega(h(t,x(t))) \leq g(t,V(t,x(t))),
\end{aligned}$$

for each $t \geq t_0$. By the monotonicity of $g(t,\varphi)$ with respect to φ for each $t \geq t_0$, we have

$$d^+m(t) \leq g(t,V(t,x(t))) \leq g(t,m(t)),$$

for each $t \geq t_0$. By Lemma 3.1, we obtain

$$V(t,x(t)) + \int_{t_0}^t \omega(h(s,x(s)))ds = m(t) \leq r(t,t_0,\varphi_0), \qquad t \geq t_0.$$

□

Theorem 3.1 *Suppose that there exists a function $V(t,\langle\widetilde{x}\rangle)$ satisfies the following conditions:*

(1) $|V(t,\langle\widetilde{x}\rangle) - V(t,\langle\widetilde{y}\rangle)| \leq L(t)d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;

(2) $\omega(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq V(t,\langle\widetilde{x}\rangle)$, $V(t,\langle\widetilde{0}\rangle) = 0$, $\omega(\cdot) \in \mathcal{K}$;

(3) $D_f^+V(t,\langle\widetilde{x}\rangle) \leq g(t,V(t,\langle\widetilde{x}\rangle))$, $g(\cdot,\cdot) \in C[\mathbb{R}_+^2, \mathbb{R}], g(t,0) = 0$.

If the solution $\varphi(t) = 0$ of (2) is asymptotically stable, then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is asymptotically stable.

Proof. If the solution $\varphi(t) = 0$ of (2) is asymptotically stable, then by (S3) of Definition 3.5, we have it is stable. Thus, by Theorem 3.1 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is stable.

Since for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, x_0, \varepsilon)$ such that if $0 \leq \varphi_0 < \delta_0$ then

$$|\varphi(t,t_0,\varphi_0)| < \omega(\varepsilon), \qquad t \geq t_0 + T.$$

Since $V(t,\langle\widetilde{0}\rangle) = 0$, we have

$$V(t_0,\langle\widetilde{x}\rangle) = |V(t_0,\langle\widetilde{x}\rangle) - V(t_0,\langle\widetilde{0}\rangle)| \leq L(t_0)d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle),$$

for each $\langle\widetilde{x}\rangle \in S(\rho)$. Thus, there exists $\delta = \delta(t_0)$ such that if $d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle) < \delta$, then $V(t_0,\langle\widetilde{x}\rangle) < \delta_0$.

Let $x(t) = x(t,t_0,x_0)$ be any solution of (1) through (t_0,x_0) existing on $[t_0,+\infty)$. Next, we shall show that if $d_{\sup}(x_0,\langle\widetilde{0}\rangle) < \delta$ then $d_{\sup}(x(t),\langle\widetilde{0}\rangle) < \varepsilon$ for each $t \geq t_0 + T$. By the conditions (1), (3) and Lemma 3.2, we get

$$V(t,x(t)) \leq r(t,t_0,V(t_0,x_0)), \qquad t \geq t_0 + T,$$

where $r(t,t_0,V(t_0,x_0))$ is the maximal solution of the scalar differential equation (2) existing on $[t_0,+\infty)$. Since $V(t_0,x_0) < \delta_0$, we have $r(t,t_0,V(t_0,x_0)) < \omega(\varepsilon)$ for each $t \geq t_0 + T$ and therefore

$$V(t,x(t)) \leq r(t,t_0,V(t_0,x_0)) < \omega(\varepsilon), \qquad t \geq t_0 + T.$$

By the condition (2), we get

$$\omega(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) \leq V(t, x(t)) < \omega(\varepsilon), \qquad t \geq t_0 + T.$$

By the monotonicity of ω, we have

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Hence, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is asymptotically stable. □

Theorem 3.2 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the conditions (1), (2) and (3) of Theorem 3.1. If the solution $\varphi(t) = 0$ of (2) is equi-asymptotically stable, then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is equi-asymptotically stable.*

Proof. In fact, we can show Theorem 3.2 by a similar method of Theorem 3.1. □

Theorem 3.3 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the following conditions:*
(1) $|V(t, \langle\widetilde{x}\rangle) - V(t, \langle\widetilde{y}\rangle)| \leq L(t) d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot, \cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;
(2) $\omega_1(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq V(t, \langle\widetilde{x}\rangle) \leq \omega_2(t, d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_1(\cdot), \omega_2(t, \cdot) \in \mathcal{K}$;
(3) $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq -\beta V(t, \langle\widetilde{x}\rangle), \beta > 0$.
Then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is equi-asymptotically stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) through (t_0, x_0) existing on $[t_0, +\infty)$. By Theorem 3.2 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is stable. Thus, taking $\varepsilon = \rho$, there exists a $\delta = \delta(t_0, \rho)$ such that if $d_{\sup}(x_0, \langle\widetilde{0}\rangle) < \delta$, then

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \rho, \qquad t \geq t_0.$$

Let the function $g(t, \varphi) = -\beta\varphi$, $(t, \varphi) \in \mathbb{R}_+^2$ and $\varphi_0 = V(t_0, x_0)$ in Lemma 3.2. Then we know that

$$r(t, t_0, \varphi_0) = V(t_0, x_0) e^{-\beta(t-t_0)}, \qquad t \geq t_0,$$

is the unique solution of the scalar differential equation (2). Thus, by Lemma 3.2, we obtain

$$V(t, x(t)) \leq V(t_0, x_0) e^{-\beta(t-t_0)}, \qquad t \geq t_0.$$

For any given $\varepsilon > 0$, we take $T = T(t_0, \varepsilon) = \dfrac{1}{\beta} \ln \dfrac{\omega_2(t_0, \delta)}{\omega_1(\varepsilon)} + 1$. Then, by the condition (2), we get

$$\begin{aligned}
\omega_1(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) &\leq V(t, x(t)) \leq V(t_0, x_0) e^{-\beta(t-t_0)} \\
&\leq e^{-\beta} \omega_2(t_0, d_{\sup}(x_0, \langle\widetilde{0}\rangle)) \frac{\omega_1(\varepsilon)}{\omega_2(t_0, \delta)} \\
&\leq e^{-\beta} \omega_2(t_0, \delta)) \frac{\omega_1(\varepsilon)}{\omega_2(t_0, \delta)} \\
&= e^{-\beta} \omega_1(\varepsilon) < \omega_1(\varepsilon),
\end{aligned}$$

which implies that

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Hence, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is equi-asymptotically stable. □

Theorem 3.4 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the following conditions:*
(1) $|V(t, \langle\widetilde{x}\rangle) - V(t, \langle\widetilde{y}\rangle)| \leq L(t) d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot, \cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;
(2) $\omega_1(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq V(t, \langle\widetilde{x}\rangle) \leq \omega_2(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_1(\cdot), \omega_2(\cdot) \in \mathcal{K}$;
(3) $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq g(t, V(t, \langle\widetilde{x}\rangle))$, $g(\cdot, \cdot) \in C[\mathbb{R}_+^2, \mathbb{R}], g(t, 0) = 0$.
If the solution $\varphi(t) = 0$ of (2) is uniformly asymptotically stable, then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly asymptotically stable.

Proof. If the solution $\varphi(t) = 0$ of (2) is uniformly asymptotically stable, then by (S5) of Definition 3.5, we have it is uniformly stable. Thus, by Theorem 3.3 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly stable.

Since for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_0 > 0$ and $T = T(\varepsilon)$ such that if $0 \leq \varphi_0 < \delta_0$ then

$$|\varphi(t, t_0, \varphi_0)| < \omega_1(\varepsilon), \qquad t \geq t_0 + T.$$

Since $\omega_1(\cdot), \omega_2(\cdot) \in \mathcal{K}$, there exist a $\delta > 0$ such that $\omega_2(\delta) < \omega_1(\delta_0)$.

Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) through (t_0, x_0) existing on $[t_0, +\infty)$. Next, we shall show that if $d_{\sup}(x_0, \langle\widetilde{0}\rangle) < \delta$ then $d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon$ for each $t \geq t_0 + T$. By the conditions (1), (3) and Lemma 3.2, we get

$$V(t, x(t)) \leq r(t, t_0, \omega_1^{-1}(V(t_0, x_0))), \qquad t \geq t_0 + T,$$

where $r(t, t_0, \omega_1^{-1}(V(t_0, x_0)))$ is the maximal solution of the scalar differential equation (2) existing on $[t_0, +\infty)$. By the condition (2), we have

$$V(t_0, x_0) \leq \omega_2(d_{\sup}(x_0, \langle\widetilde{0}\rangle)) \leq \omega_2(\delta) < \omega_1(\delta_0).$$

Thus, by the monotonicity of ω_1, we have $\omega_1^{-1}(V(t_0, x_0)) \leq \delta_0$, which implies that

$$r(t, t_0, \omega_1^{-1}(V(t_0, x_0))) < \omega_1(\varepsilon), \quad t \geq t_0 + T$$

and therefore

$$V(t, x(t)) \leq r(t, t_0, \omega_1^{-1}(V(t_0, x_0))) < \omega_1(\varepsilon), \qquad t \geq t_0 + T.$$

By the condition (2), we get

$$\omega_1(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) \leq V(t, x(t)) < \omega_1(\varepsilon), \qquad t \geq t_0 + T.$$

By the monotonicity of ω_1, we have

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Hence, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly asymptotically stable. □

Theorem 3.5 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the following conditions:*

(1) $|V(t, \langle\widetilde{x}\rangle) - V(t, \langle\widetilde{y}\rangle)| \leq L(t) d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot, \cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;

(2) $\omega_1(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq V(t, \langle\widetilde{x}\rangle) \leq \omega_2(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_1(\cdot), \omega_2(\cdot) \in \mathcal{K}$;

(3) $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq -\omega_3(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_3(\cdot) \in \mathcal{K}$.

Then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly asymptotically stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) through (t_0, x_0) existing on $[t_0, +\infty)$. By Theorem 3.4 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly stable. Thus, taking $\varepsilon = \rho$, there exists a $\delta = \delta(\rho)$ such that if $d_{\sup}(x_0, \langle\widetilde{0}\rangle) < \delta$, then

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \rho, \qquad t \geq t_0.$$

Let the function $g(t, \varphi) \equiv 0$, $(t, \varphi) \in \mathbb{R}_+^2$ and $\varphi_0 = V(t_0, x_0)$ in Lemma 3.3. Then we know that $r(t, t_0, \varphi_0) \equiv V(t_0, x_0)$ is the unique solution of the scalar differential equation (2). Thus, by Lemma 3.3, we obtain

$$V(t, x(t)) + \int_{t_0}^{t} \omega_3(d_{\sup}(x(s), \langle\widetilde{0}\rangle)) ds \leq V(t_0, x_0), \qquad t \geq t_0.$$

For any given $\varepsilon > 0$, we take $T = T(\varepsilon) = \dfrac{\omega_2(\delta)}{\omega_3\omega_2^{-1}\omega_1(\varepsilon)} + 1$. Suppose that $d_{\sup}(x(t), \langle\widetilde{0}\rangle) \geq \omega_2^{-1}\omega_1(\varepsilon)$ for each $t \in [t_0, t_0 + T]$. Then, by the condition (2), we get

$$\begin{aligned} V(t, x(t)) &= V(t_0, x_0) - \int_{t_0}^{t} \omega_3(d_{\sup}(x(s), \langle\widetilde{0}\rangle))ds \\ &\leq \omega_2(d_{\sup}(x_0, \langle\widetilde{0}\rangle)) - \omega_3\omega_2^{-1}\omega_1(\varepsilon)(t - t_0) \\ &< \omega_2(\delta) - \omega_3\omega_2^{-1}\omega_1(\varepsilon)(t - t_0), \end{aligned}$$

for each $t \in [t_0, t_0 + T]$. Thus, we obtain

$$0 \leq V(t_0 + T, x(t_0 + T)) < \omega_2(\delta) - \omega_3\omega_2^{-1}\omega_1(\varepsilon)T = -\omega_3\omega_2^{-1}\omega_1(\varepsilon) < 0.$$

This is a contradiction, thus there exists a $t^* \in [t_0, t_0 + T]$ such that

$$d_{\sup}(x(t^*), \langle\widetilde{0}\rangle) < \omega_2^{-1}\omega_1(\varepsilon).$$

Since $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq -\omega_3(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq 0$, we have

$$V(t, x(t)) \leq V(t^*, x(t^*)), \qquad t \geq t^*.$$

Then, by the condition (2), we get

$$\begin{aligned} \omega_1(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) &\leq V(t, x(t)) \leq V(t^*, x(t^*)) \\ &\leq \omega_2(d_{\sup}(x(t^*), \langle\widetilde{0}\rangle)) \\ &< \omega_2\omega_2^{-1}\omega_1(\varepsilon) = \omega_1(\varepsilon), \end{aligned}$$

which implies that $d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon$ for each $t \geq t^*$. Hence, we obtain

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Consequently, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly asymptotically stable. □

Example 3.1 *Define $F : \mathbb{R}_+ \to \mathscr{F}/\mathscr{S}$ by the α-level sets of the fuzzy mapping*

$$\left[\widehat{F(t)}\right]^\alpha = \left[-\frac{2e^{-\alpha}}{1+t}, 0\right], \qquad \alpha \in [0, 1],$$

where $\widehat{F(t)}$ is the Mareš core of $F(t)$, for each $t \in \mathbb{R}_+$. Thus, we have

$$M_{F(t)}(\alpha) = -\frac{e^{-\alpha}}{1+t}, \qquad \alpha \in [0, 1],$$

for each $t \in \mathbb{R}_+$. It is obvious that $M_{F(t)}(\alpha)$ is continuous from the right at 0 and continuous from the left on $[0, 1]$ with respect to α. Since $M_{F(t)}(\alpha)$ is increasing with respect to α, we get

$$V_0^1(M_{F(t)}) = \frac{1 - e^{-1}}{1+t} \leq 1 - e^{-1}, \qquad t \in \mathbb{R}_+.$$

Thus, we obtain that $F(t)$ is of uniformly bounded variation. Since $M_{F(t)}(\alpha)$ is uniformly continuous with respect to $t \in \mathbb{R}_+$, we get that $F(t)$ is continuous with respect to $d_{\sup}$. Define $f : \mathbb{R}_+ \times \mathscr{F}/\mathscr{S} \to \mathscr{F}/\mathscr{S}$ by

$$f(t, \langle\widetilde{x}\rangle) = F(t)\,\langle\widetilde{x}\rangle\,.$$

It is obvious that f is continuous with respect to $d_{\sup}$ and of uniformly bounded variation.

Consider a Lyapunov function $V(t, \langle\widetilde{x}\rangle) = d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)$. *Then* $V(t, \langle\widetilde{0}\rangle) = d_{\sup}(\langle\widetilde{0}\rangle, \langle\widetilde{0}\rangle) = 0$ *and*

$$|V(t, \langle\widetilde{x}\rangle) - V(t, \langle\widetilde{y}\rangle)| = \left|d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle) - d_{\sup}(\langle\widetilde{y}\rangle, \langle\widetilde{0}\rangle)\right| \leq d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle),$$

for any $(t, \langle\widetilde{x}\rangle), (t, \langle\widetilde{y}\rangle) \in \mathbb{R}_+ \times \mathscr{F}/\mathscr{S}$. *By Definition 2.9, for a small* $h > 0$, *we have*

$$\begin{aligned} V(t+h, \langle\widetilde{x}\rangle + hf(t, \langle\widetilde{x}\rangle)) &= d_{\sup}(\langle\widetilde{x}\rangle + hf(t, \langle\widetilde{x}\rangle), \langle\widetilde{0}\rangle) = d_{\sup}(\langle\widetilde{x}\rangle + hF(t)\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle) \\ &= \sup_{\alpha\in[0,1]} \left|M_{\langle\widetilde{x}\rangle}(\alpha) + hM_{F(t)}(\alpha)M_{\langle\widetilde{x}\rangle}(\alpha)\right| \\ &\leq \sup_{\alpha\in[0,1]} \left|M_{\langle\widetilde{x}\rangle}(\alpha)\right| \left(1 + h\sup_{\alpha\in[0,1]} M_{F(t)}(\alpha)\right) \\ &= \left(1 - \frac{he^{-1}}{1+t}\right) d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle). \end{aligned}$$

Hence, we get

$$D_f^+ V(t, \langle\widetilde{x}\rangle) = \overline{\lim_{h\to 0^+}} \frac{1}{h}\left(V(t+h, \langle\widetilde{x}\rangle + hf(t, \langle\widetilde{x}\rangle)) - V(t, \langle\widetilde{x}\rangle)\right) \leq -\frac{e^{-1}}{1+t} d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle).$$

Let $g(t, \varphi) = -\dfrac{e^{-1}}{1+t}\varphi$. *Then, we have*

$$D_f^+ V(t, \langle\widetilde{x}\rangle) \leq g(t, d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) = g(t, V(t, \langle\widetilde{x}\rangle)).$$

It's easy to show that the solution $\varphi = 0$ *of* (2) *is asymptotically stable. Hence, by Theorem 3.1, the trivial solution* $x(t) = \langle\widetilde{0}\rangle$ *of* (1) *is asymptotically stable.*

Acknowledgements

The authors thank the anonymous reviewers for their valuable comments. This work was supported by The National Natural Science Foundation of China (Grant no. 11671001), The Graduate Teaching Reform Research Program of Chongqing Municipal Education Commission (No.YJG143010).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

ON DIFFERENTIAL EQUATIONS ASSOCIATED WITH SQUARED HERMITE POLYNOMIALS

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Abstract. In this paper, we investigate differential equations associated with squared Hermite polynomials and derive some new and explicit identities for these polynomials arising from the differential equations.

1. Introduction

As a method of obtaining new identities for special polynomials and numbers, in [8] T. Kim initiated a remarkable idea of using ordinary differential equations. Namely, he derived a family of nonlinear differential equations, indexed by positive integers, satisfied by the generating function of the Frobenius-Euler numbers and used them in order to get an interesting identity expressing higher-order Frobenius-Euler numbers in terms of (ordinary) Frobenius-Euler numbers. Here, more precisely, the differential equations are satisfied not by the generating function of the Frobenius-Euler numbers but by a constant multiple of that.

This method turned out to be very fruitful and can be applied to many interesting special polynomials and numbers (see [5, 8–11]). For example, linear differential equations are derived for Bessel polynomials, Changhee polynomials, actuarial polynomials, Meixner polynomials of the first kind, Poisson-Charlier polynomials, Laguerre polynomials, Hermite polynomials, and Stirling polynomials, while nonlinear ones are obtained for Bernoulli numbers of the

2010 Mathematics Subject Classification : 05A19, 11B37, 11B83, 34A30.
Key words and phrases: squared Hermite polynomials, differential equations.

second, Boole numbers, Chebyshev polynomials of the first, second, third, and fourth kind, degenerate Euler numbers, degenerate Eulerian polynomials, Korobov numbers, and Legendre polynomials.

To be specific, we will illustrate the results in the case of Bernoulli numbers of the second kind (see [5]). Firstly, it is shown that the function $F = F(t) = \frac{1}{log(1+t)}$ satisfies the family of nonlinear differential equations

$$F^{(N)}(t) = \frac{(-1)^N}{(1+t)^N} \sum_{j=2}^{N+1} (j-1)!(N-1)! H_{N-1,j-2} F^j \qquad (N = 1, 2, \cdots), \tag{1}$$

where H_N are the generalized harmonic numbers defined by

$$\begin{aligned} &H_{N,0} = 1, \quad \text{for all } N, \\ &H_{N,1} = \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1}, \\ &H_{N,j} = \frac{H_{N-1,j-1}}{N} + \frac{H_{N-1,j-1}}{N-1} + \cdots + \frac{H_{j-1,j-1}}{j} \quad (N \geq j \geq 2). \end{aligned} \tag{2}$$

Recall that the Bernoulli numbers of the second b_n are given by the generating function

$$\frac{t}{log(1+t)} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \quad \text{(see [5])}. \tag{3}$$

More generally, the Bernoulli numbers of the second $b_n^{(r)}$ of order r are defined by the generating function

$$\left(\frac{t}{log(1+t)}\right)^r = \sum_{n=0}^{\infty} b_n^{(r)} \frac{t^n}{n!} \quad \text{(see [5])}. \tag{4}$$

Then, secondly the family of differential equations in (1) are used to derive the following interesting identities: for $N = 1, 2, \cdots$ and $n = 0, 1, \cdots$, we have

$$\begin{aligned} &(-1)^n \sum_{j=0}^{\min\{n,N-1\}} (N-j)!(N-1)! H_{N-1,N-1-j} (n)_j b_{n-j}^{(N+1-j)} \\ &= \begin{cases} (-1)^N N!(N)_n & \text{if} \quad 0 \leq n \leq N, \\ \sum_{l=0}^{n-N-1} \binom{N}{l} \frac{b_{n-l}}{n-l} (n)_{l+N+1} & \text{if} \quad n \geq N+1. \end{cases} \end{aligned} \tag{5}$$

As a generalization of the usual factorial $n!$, the double factorial of a positive integer n is defined by

$$n!! = \begin{cases} n \cdot (n-2) \cdots 5 \cdot 3 \cdot 1 & \text{if} \quad n > 0 \text{ odd}, \\ n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2 & \text{if} \quad n > 0, \text{ even}, \\ 1 & \text{if} \quad n = -1, \ 0. \end{cases} \tag{6}$$

(see [1]).

Throughout this paper, the double factorials will be used.

The Hermite polynomials are classical orthogonal polynomials used such diverse areas as combinatorics, numerical analysis, probability, finite element methods, systems theory and quantum mechanics (see [2–4, 6, 7, 12–14]).

With the Roman's definition of Hermite polynomials $H_n(x)$ as

$$H_n(x) = e^{xt - t^2/2}, \tag{7}$$

we see from ([3], p.250) that

$$(1-t^2)^{-1/2}e^{x[t/(1+t)]} = \sum_{n=0}^{\infty}[H_n(\sqrt{x})]^2\frac{t^n}{n!}. \tag{8}$$

For brevity, we denote $[H_n(\sqrt{x})]^2$ by $SH_n(x)$, and hence

$$(1-t^2)^{-1/2}e^{x[t/(1+t)]} = \sum_{n=0}^{\infty} SH_n(x)\frac{t^n}{n!}. \tag{9}$$

In this paper, we would like to derive a family of linear differential equations satisfied by the generating function of the squared Hermite polynomials in (9) and use them in order to get an interesting identity for those polynomials. As an easy consequence of this result, we will have an expression for the squared Hermite polynomials.

2. Differential equations for the squared Hermite polynomials

In this paper, all differentiations are taken with respect to t, while x being fixed.
Let

$$\begin{aligned} F = F(t;x) &= \left(1-t^2\right)^{-\frac{1}{2}} e^{x\left(\frac{t}{t+1}\right)} \\ &= (1-t)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}} e^{x\left(\frac{t}{t+1}\right)}. \end{aligned} \tag{10}$$

Then

$$\begin{aligned} F^{(1)} &= \frac{1}{2}(1-t)^{-\frac{3}{2}}(1+t)^{-\frac{1}{2}}e^{x\left(\frac{t}{t+1}\right)} - \frac{1}{2}(1-t)^{-\frac{1}{2}}(1+t)^{-\frac{3}{2}}e^{x\left(\frac{t}{t+1}\right)} \\ &\quad +(1-t)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}(1+t)^{-2}xe^{x\left(\frac{t}{t+1}\right)} \\ &= \left\{\frac{1}{2}(1-t)^{-1} - \frac{1}{2}(1+t)^{-1} + x(1+t)^{-2}\right\}F. \end{aligned} \tag{11}$$

$$\begin{aligned} F^{(2)} &= \left\{\frac{1}{2}(1-t)^{-2} + \frac{1}{2}(1+t)^{-2} - 2x(1+t)^{-3}\right\}F \\ &\quad + \left\{\frac{1}{2}(1-t)^{-1} - \frac{1}{2}(1+t)^{-1} + x(1+t)^{-2}\right\}^2 F \\ &= \left\{\frac{1}{2}(1-t)^{-2} + \frac{1}{2}(1+t)^{-2} - 2x(1+t)^{-3}\right\}F \\ &\quad + \left\{\frac{1}{4}(1-t)^{-2} + \frac{1}{4}(1+t)^{-2} + x^2(1+t)^{-4}\right. \\ &\quad \left. -\frac{1}{2}(1-t)^{-1}(1+t)^{-1} - x(1+t)^{-3} + x(1-t)^{-1}(1+t)^{-2}\right\}F \\ &= \left\{\frac{3}{4}(1-t)^{-2} - \frac{1}{2}(1-t)^{-1}(1+t)^{-1} + x(1-t)^{-1}(1+t)^{-2}\right. \\ &\quad \left. +\frac{3}{4}(1+t)^{-2} - 3x(1+t)^{-3} + x^2(1+t)^{-4}\right\}F. \end{aligned} \tag{12}$$

So, we are led to put

$$F^{(N)} = \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F. \tag{13}$$

Here $a_{i,j}(N,x)$ are polynomials in x.

$$\begin{aligned}
F^{(N+1)} &= \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} i a_{i,j}(N,x)(1-t)^{-(i+1)}(1+t)^{-j}\right)F \\
&\quad - \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} j a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-(j+1)}\right)F \\
&\quad + \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right. \\
&\quad \left. \times\left\{\frac{1}{2}(1-t)^{-1}-\frac{1}{2}(1+t)^{-1}+x(1+t)^{-2}\right\}\right)F \\
&= \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} (i+\frac{1}{2}) a_{i,j}(N,x)(1-t)^{-(i+1)}(1+t)^{-j}\right)F \\
&\quad - \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} (j+\frac{1}{2}) a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-(j+1)}\right)F \\
&\quad + \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} x a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-(j+2)}\right)F \\
&= \left(\sum_{i=1}^{N+1}\sum_{j=N+1-i}^{2(N+1-i)} (i-\frac{1}{2}) a_{i-1,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F \\
&\quad - \left(\sum_{i=0}^{N}\sum_{j=N-i+1}^{2(N-i)+1} (j-\frac{1}{2}) a_{i,j-1}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F \\
&\quad + \left(\sum_{i=0}^{N}\sum_{j=N-i+2}^{2(N-i)+2} x a_{i,j-2}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F. \qquad (14)
\end{aligned}$$

On the other hand,

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1}\sum_{j=N+1-i}^{2(N+1-i)} a_{i,j}(N+1,x)(1-t)^{-i}(1+t)^{-j}\right)F. \qquad (15)$$

In order to add the sums in (14), we decompose them as follows:

$$\begin{aligned}
\sum_{i=1}^{N+1}\sum_{j=N+1-i}^{2(N+1-i)} &= \sum_{i=1}^{N}\sum_{j=N+2-i}^{2(N-i)+1} + \sum_{i=1}^{N}\sum_{j=N+1-i} \\
&\quad + \sum_{i=1}^{N}\sum_{j=2(N+1-i)} + \sum_{i=N+1}\sum_{j=0}; \qquad (16)
\end{aligned}$$

$$\begin{aligned}
\sum_{i=0}^{N}\sum_{j=N-i+1}^{2(N-i)+1} &= \sum_{i=1}^{N}\sum_{j=N-i+2}^{2(N-i)+1} + \sum_{i=1}^{N}\sum_{j=N-i+1} \\
&\quad + \sum_{i=0}\sum_{j=N+2}^{2N+1} + \sum_{i=0}\sum_{j=N+1}; \qquad (17)
\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{N}\sum_{j=N-i+2}^{2(N-i)+2} &= \sum_{i=1}^{N}\sum_{j=N-i+2}^{2(N-i)+1}+\sum_{i=1}^{N}\sum_{j=2(N-i)+2} \\ &\quad +\sum_{i=0}\sum_{j=N+2}^{2N+1}+\sum_{i=0}\sum_{j=2N+2}. \end{aligned} \tag{18}$$

Now, the sum in (14) can be rewritten as

$$\begin{aligned} & F^{(N+1)} \\ = & \sum_{i=1}^{N}\sum_{j=N+2-i}^{2(N-i)+1}\left\{(i-\frac{1}{2})a_{i-1,j}(N,x)-(j-\frac{1}{2})a_{i,j-1}(N,x)+xa_{i,j-2}(N,x)\right\} \\ & \qquad \times(1-t)^{-i}(1+t)^{-j}F \\ & +\sum_{i=1}^{N}\left\{(i-\frac{1}{2})a_{i-1,N-i+1}(N,x)-(N-i+\frac{1}{2})a_{i,N-i}(N,x)\right\} \\ & \qquad \times(1-t)^{-i}(1+t)^{-(N-i+1)}F \\ & +\sum_{i=1}^{N}\left\{(i-\frac{1}{2})a_{i-1,2(N+1-i)}(N,x)+xa_{i,2(N-i)}(N,x)\right\}(1-t)^{-i}(1+t)^{-2(N+1-i)}F \\ & +\sum_{j=N+2}^{2N+1}\left\{-(j-\frac{1}{2})a_{0,j-1}(N,x)+xa_{0,j-2}(N,x)\right\}(1+t)^{-j}F \\ & -(N+\frac{1}{2})a_{0,N}(N,x)(1+t)^{-(N+1)}F+xa_{0,2N}(N,x)(1+t)^{-(2N+2)}F \\ & +(N+\frac{1}{2})a_{N,0}(N,x)(1-t)^{-(N+1)}F. \end{aligned} \tag{19}$$

Comparing (15) and (19), we obtain: for $1 \le i \le N$, $N-i+2 \le j \le 2(N-i)+1$,

$$a_{i,j}(N+1,x) = (i-\frac{1}{2})a_{i-1,j}(N,x)-(j-\frac{1}{2})a_{i,j-1}(N,x)+xa_{i,j-2}(N,x); \tag{20}$$

for $1 \le i \le N$,

$$a_{i,N-i+1}(N+1,x) = (i-\frac{1}{2})a_{i-1,N-i+1}(N,x)-(N-i+\frac{1}{2})a_{i,N-i}(N,x); \tag{21}$$

for $1 \le i \le N$,

$$a_{i,2(N+1-i)}(N+1,x) = (i-\frac{1}{2})a_{i-1,2(N+1-i)}(N,x)+xa_{i,2(N-i)}(N,x); \tag{22}$$

for $N+2 \le j \le 2N+1$,

$$a_{0,j}(N+1,x) = -(j-\frac{1}{2})a_{0,j-1}(N,x)+xa_{0,j-2}(N,x); \tag{23}$$

$$a_{0,N+1}(N+1,x) = -(N+\frac{1}{2})a_{0,N}(N,x); \tag{24}$$

$$a_{0,2N+2}(N+1,x) = xa_{0,2N}(N,x); \tag{25}$$

$$a_{N+1,0}(N+1,x) = (N+\frac{1}{2})a_{N,0}(N,x). \tag{26}$$

Note here that all of these recurrence relations can be merged into one relation (20), for $0 \le i \le N+1$, $N-i+1 \le j \le 2(N-i+1)$, with the understanding that

$$a_{i,j}(N,x) = 0, \tag{27}$$

unless $0 \leq i \leq N,\ N-i \leq j \leq 2(N-i)$. In addition to these, we have the following initial conditions:

$$F = F^{(0)} = a_{0,0}(0,x)F \longrightarrow a_{0,0}(0,x) = 1, \tag{28}$$

$$\begin{aligned}
F^{(1)} &= \left(\sum_{i=0}^{1} \sum_{j=1-i}^{2(1-i)} a_{i,j}(1,x)(1-t)^{-i}(1+t)^{-j} \right) F \\
&= \left(a_{0,1}(1,x)(1+t)^{-1} + a_{0,2}(1,x)(1+t)^{-2} + a_{1,0}(1,x)(1-t)^{-1} \right) F \\
&= \left(\frac{1}{2}(1-t)^{-1} - \frac{1}{2}(1+t)^{-1} + x(1+t)^{-2} \right) F \\
&\longrightarrow \quad a_{1,0}(1,x) = \frac{1}{2},\ a_{0,1}(1,x) = -\frac{1}{2},\ a_{0,2}(1,x) = x.
\end{aligned} \tag{29}$$

As easy consequences, from (24)-(26) we get

$$\begin{aligned}
a_{N+1,0}(N+1,x) &= \left(N + \frac{1}{2} \right) a_{N,0}(N,x) \\
&= \left(N + \frac{1}{2} \right) \left(N - \frac{1}{2} \right) a_{N-1,0}(N-1,x) \\
&= \cdots \\
&= \left(N + \frac{1}{2} \right) \left(N - \frac{1}{2} \right) \cdots \frac{3}{2} a_{1,0}(1,x) \\
&= \left(\frac{1}{2} \right)^{N+1} (2N+1)!!
\end{aligned} \tag{30}$$

$$\begin{aligned}
a_{0,N+1}(N+1,x) &= -\left(N + \frac{1}{2} \right) a_{0,N}(N,x) \\
&= (-1)^2 \left(N + \frac{1}{2} \right) \left(N - \frac{1}{2} \right) a_{0,N-1}(N-1,x) \\
&= \cdots \\
&= (-1)^N \left(N + \frac{1}{2} \right) \left(N - \frac{1}{2} \right) \cdots \frac{3}{2} a_{0,1}(1,x) \\
&= \left(-\frac{1}{2} \right)^{N+1} (2N+1)!!
\end{aligned} \tag{31}$$

$$\begin{aligned}
a_{0,2N+2}(N+1,x) &= x a_{0,2N}(N,x) = x^2 a_{0,2(N-1)}(N-1,x) \\
&= x^N a_{0,2}(1,x) = x^{N+1} a_{0,0}(0,x) = x^{N+1}.
\end{aligned} \tag{32}$$

Let $N+2 \leq j \leq 2N+1$. Then, from (23), we have

$$a_{0,j}(N+1,x) = x a_{0,j-2}(N,x) - \left(j - \frac{1}{2} \right) a_{0,j-1}(N,x). \tag{33}$$

For $j = N+2$, we get the following:

$$\begin{aligned}
& a_{0,N+2}(N+1,x) \\
=\ & x a_{0,N}(N,x) - \left(N + \frac{3}{2} \right) a_{0,N+1}(N,x) \\
=\ & x a_{0,N}(N,x) - \left(N + \frac{3}{2} \right) \left(x a_{0,N-1}(N-1,x) - \left(N + \frac{1}{2} \right) a_{0,N}(N-1,x) \right) \\
=\ & x \left(a_{0,N}(N,x) - (N + \frac{3}{2}) a_{0,N-1}(N-1,x) \right) \\
& + (-1)^2 \left(N + \frac{3}{2} \right) \left(N + \frac{1}{2} \right) \left(x a_{0,N-2}(N-2,x) - \left(N - \frac{1}{2} \right) a_{0,N-1}(N-2,x) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= x\sum_{k=0}^{N-1}(-1)^k\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\left(N-k+\frac{5}{2}\right)a_{0,N-k}(N-k,x) \\
&\quad +(-1)^N\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\frac{5}{2}a_{0,2}(1,x) \\
&= x\sum_{k=0}^{N}\left(-\frac{1}{2}\right)^k(2N+3)(2N+1)\cdots(2N-2k+5)a_{0,N-k}(N-k,x) \\
&= x\sum_{k=0}^{N}\left(-\frac{1}{2}\right)^k\frac{(2N+3)!!}{(2N-2k+3)!!}a_{0,N-k}(N-k,x). \qquad (34)
\end{aligned}
$$

For $j=N+3$, we obtain the following:

$$
\begin{aligned}
&a_{0,N+3}(N+1,x) \\
&= xa_{0,N+1}(N,x)-\left(N+\frac{5}{2}\right)a_{0,N+2}(N,x) \\
&= xa_{0,N+1}(N,x)-\left(N+\frac{5}{2}\right)\left(xa_{0,N}(N-1,x)-\left(N+\frac{3}{2}\right)a_{0,N+1}(N-1,x)\right) \\
&= x\left(a_{0,N+1}(N,x)-\left(N+\frac{5}{2}\right)a_{0,N}(N-1,x)\right) \\
&\quad (-1)^2\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\left(xa_{0,N-1}(N-2,x)-\left(N+\frac{1}{2}\right)a_{0,N}(N-2,x)\right) \\
&= \cdots \\
&= x\sum_{k=0}^{N-2}(-1)^k\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\cdots\left(N-k+\frac{7}{2}\right)a_{0,n-k+1}(N-k,x) \\
&\quad +(-1)^{N-1}\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\cdots\frac{9}{2}a_{0,4}(2,x) \\
&= x\sum_{k=0}^{N-1}(-1)^k\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\cdots\left(N-k+\frac{7}{2}\right)a_{0,n-k+1}(N-k,x) \\
&= x\sum_{k=0}^{N-1}\left(-\frac{1}{2}\right)^k\frac{(2N+5)!!}{(2N-2k+5)!!}a_{0,N-k+1}(N-k,x). \qquad (35)
\end{aligned}
$$

Continuing this process, we can deduce that, for $N+2\leq j\leq 2N+1$,

$$a_{0,j}(N+1,x)=x\sum_{k=0}^{2N+2-j}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}a_{0,j-k-2}(N-k,x). \qquad (36)$$

Let $1\leq i\leq N$. Then, from (21), we have

$$a_{i,N-i+1}(N+1,x)=\left(i-\frac{1}{2}\right)a_{i-1,N-i+1}(N,x)-\left(N-i+\frac{1}{2}\right)a_{i,N-i}(N,x). \qquad (37)$$

For $i=1$, we obtain the following:

$$
\begin{aligned}
&a_{1,N}(N+1,x) \\
&= \frac{1}{2}a_{0,N}(N,x)-\left(N-\frac{1}{2}\right)a_{1,N-1}(N,x) \\
&= \frac{1}{2}a_{0,N}(N,x)-\left(N-\frac{1}{2}\right)\left(\frac{1}{2}a_{0,N-1}(N-1,x)-(N-\frac{3}{2})a_{1,N-2}(N-1,x)\right) \\
&= \frac{1}{2}\left(a_{0,N}(N,x)-\left(N-\frac{1}{2}\right)a_{0,N-1}(N-1,x)\right)
\end{aligned}
$$

$$
\begin{aligned}
& +(-1)^2\left(N-\frac{1}{2}\right)\left(N-\frac{3}{2}\right)\left(\frac{1}{2}a_{0,N-2}(N-2,x)-\left(N-\frac{5}{2}\right)a_{1,N-3}(N-2,x)\right)\\
= & \cdots\\
= & \frac{1}{2}\sum_{k=0}^{N-1}(-1)^k\left(N-\frac{1}{2}\right)\left(N-\frac{3}{2}\right)\cdots\left(N-\frac{2k-1}{2}\right)a_{0,N-k}(N-k,x)\\
& +(-1)^N\left(N-\frac{1}{2}\right)\cdots\frac{1}{2}a_{1,0}(1,x)\\
= & \frac{1}{2}\sum_{k=0}^{N}(-1)^k\left(N-\frac{1}{2}\right)\left(N-\frac{3}{2}\right)\cdots\left(N-\frac{2k-1}{2}\right)a_{0,N-k}(N-k,x)\\
= & \frac{1}{2}\sum_{k=0}^{N}\left(-\frac{1}{2}\right)^k\frac{(2N-1)!!}{(2N-2k-1)!!}a_{0,N-k}(N-k,x). \qquad (38)
\end{aligned}
$$

For $i=2$, we get the following:

$$
\begin{aligned}
& a_{2,N-1}(N+1,x)\\
= & \frac{3}{2}a_{1,N-1}(N,x)-\left(N-\frac{3}{2}\right)a_{2,N-2}(N,x)\\
= & \frac{3}{2}a_{1,N-1}(N,x)-\left(N-\frac{3}{2}\right)\left(\frac{3}{2}a_{1,N-2}(N-1,x)-\left(N-\frac{5}{2}\right)a_{2,N-3}(N-1,x)\right)\\
= & \frac{3}{2}\left(a_{1,N-1}(N,x)-\left(N-\frac{3}{2}\right)a_{1,N-2}(N-1,x)\right)\\
& +(-1)^2\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\left(\frac{3}{2}a_{1,N-3}(N-2,x)-\left(N-\frac{7}{2}\right)a_{2,N-4}(N-2,x)\right)\\
= & \cdots\\
= & \frac{3}{2}\sum_{k=0}^{N-2}(-1)^k\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\cdots\left(N-\frac{2k+1}{2}\right)a_{1,N-k-1}(N-k,x)\\
& +(-1)^{N-1}\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\cdots\frac{1}{2}a_{2,0}(2,x)\\
= & \frac{3}{2}\sum_{k=0}^{N-1}(-1)^k\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\cdots\left(N-\frac{2k+1}{2}\right)a_{1,N-k-1}(N-k,x)\\
= & \frac{3}{2}\sum_{k=0}^{N-1}\left(-\frac{1}{2}\right)^k\frac{(2N-3)!!}{(2N-2k-3)!!}a_{1,N-k-1}(N-k,x). \qquad (39)
\end{aligned}
$$

Continuing this process, we can deduce that, for $1\le i\le N$,

$$
\begin{aligned}
& a_{i,N-i+1}(N+1,x)\\
= & \frac{2i-1}{2}\sum_{k=0}^{N-i+1}\left(-\frac{1}{2}\right)^k\frac{(2N-2i+1)!!}{(2N-2k-2i+1)!!}a_{i-1,N-k-i+1}(N-k,x). \qquad (40)
\end{aligned}
$$

Let $1\le i\le N$. Then, from (22), we have

$$
\begin{aligned}
& a_{i,2(N+1-i)}(N+1,x)\\
= & \left(i-\frac{1}{2}\right)a_{i-1,2(N+1-i)}(N,x)+xa_{i,2(N-i)}(N,x). \qquad (41)
\end{aligned}
$$

Then, proceeding analogously to the case of (37), we can deduce that, for $1\le i\le N$,

$$
a_{i,2(N+1-i)}(N+1)=\frac{2i-1}{2}\sum_{k=0}^{N-i+1}x^k a_{i-1,2(N-k-i+1)}(N-k,x), \qquad (42)
$$

For $1 \leq i \leq N$, $N-i+2 \leq j \leq 2(N-i)+1$, from (20) we have

$$\begin{aligned} & a_{i,j}(N+1,x) \\ = & \left(i-\frac{1}{2}\right) a_{i-1,j}(N,x) - \left(j-\frac{1}{2}\right) a_{i,j-1}(N,x) + x a_{i,j-2}(N,x). \end{aligned} \tag{43}$$

Let $i=1$, Then, with $N+1 \leq j \leq 2N-1$, (43) becomes

$$a_{1,j}(N+1,x) = \frac{1}{2} a_{0,j}(N,x) + x a_{1,j-2}(N,x) - \left(j-\frac{1}{2}\right) a_{1,j-1}(N,x). \tag{44}$$

For $j = N+1$, we get the following:

$$\begin{aligned} & a_{1,N+1}(N+1,x) \\ = & \frac{1}{2} a_{0,N+1}(N,x) + x a_{1,N-1}(N,x) - \left(N+\frac{1}{2}\right) a_{1,N}(N,x) \\ = & \frac{1}{2} a_{0,N+1}(N,x) + x a_{1,N-1}(N,x) \\ & -\left(N+\frac{1}{2}\right)\left(\frac{1}{2} a_{0,N}(N-1,x) + x a_{1,N-2}(N-1,x) - \left(N-\frac{1}{2}\right) a_{1,N-1}(N-1,x)\right) \\ = & \frac{1}{2}\left(a_{0,N+1}(N,x) - \left(N+\frac{1}{2}\right) a_{0,N}(N-1,x)\right) \\ & +x\left(a_{1,N-1}(N,x) - \left(N+\frac{1}{2}\right) a_{1,N-2}(N-1,x)\right) + (-1)^2\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right) \\ & \times\left(\frac{1}{2} a_{0,N-1}(N-2,x) + x a_{1,N-3}(N-2,x) - \left(N-\frac{3}{2}\right) a_{1,N-2}(N-2,x)\right) \\ & = \cdots \\ & = \frac{1}{2}\sum_{k=0}^{N-2}(-1)^k\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right)\cdots\left(N-\frac{2k-3}{2}\right) a_{0,N-k+1}(N-k,x) \\ & +x\sum_{k=0}^{N-2}(-1)^k\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right)\cdots\left(N-\frac{2k-3}{2}\right) a_{1,N-k-1}(N-k,x) \\ & +(-1)^{N-1}\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right)\cdots\left(\frac{5}{2}\right) a_{1,2}(2,x) \\ & = \sum_{k=0}^{N-1}\left(-\frac{1}{2}\right)^k \frac{(2N+1)!!}{(2N-2k+1)!!}\left(\frac{1}{2} a_{0,N-k+1}(N-k,x) + x a_{1,N-k-1}(N-k,x)\right). \end{aligned} \tag{45}$$

For $j = N+2$, we obtain the following:

$$\begin{aligned} & a_{1,N+2}(N+1,x) \\ = & \frac{1}{2} a_{0,N+2}(N,x) + x a_{1,N}(N,x) - \left(N+\frac{3}{2}\right) a_{1,N+1}(N,x) \\ & = \frac{1}{2} a_{0,N+2}(N,x) + x a_{1,N}(N,x) \\ & -\left(N+\frac{3}{2}\right)\left(\frac{1}{2} a_{0,N+1}(N-1,x) + x a_{1,N-1}(N-1,x) - \left(N+\frac{1}{2}\right) a_{1,N}(N-1,x)\right) \\ & = \frac{1}{2}\left(a_{0,N+2}(N,x) - \left(N+\frac{3}{2}\right) a_{0,N+1}(N-1,x)\right) \\ & +x\left(a_{1,N}(N,x) - \left(N+\frac{3}{2}\right) a_{1,N-1}(N-1,x)\right) \\ & +(-1)^2\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{2}a_{0,N}(N-2,x) + xa_{1,N-2}(N-2,x) - \left(N-\frac{1}{2}\right)a_{1,N-1}(N-2,x)\right)\\
&= \cdots\\
&= \frac{1}{2}\sum_{k=0}^{N-3}(-1)^k\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\left(N-\frac{2k-5}{2}\right)a_{0,N-k+2}(N-k,x)\\
&\quad +x\sum_{k=0}^{N-3}(-1)^k\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\left(N-\frac{2k-5}{2}\right)a_{1,N-k}(N-k,x)\\
&\quad +(-1)^{N-2}\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\frac{9}{2}a_{1,4}(3,x)\\
&= \sum_{k=0}^{N-2}\left(-\frac{1}{2}\right)^k\frac{(2N+3)!!}{(2N-2k+3)!!}\left(\frac{1}{2}a_{0,N-k+2}(N-k,x)+xa_{1,N-k}(N-k,x)\right). \qquad (46)
\end{aligned}
$$

Continuing this process, we can deduce that, for $N+1 \leq j \leq 2N-1$,

$$
\begin{aligned}
&a_{1,j}(N+1,x)\\
= &\sum_{k=0}^{2N-j}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\left(\frac{1}{2}a_{0,j-k}(N-k)+xa_{1,j-k-2}(N-k,x)\right). \qquad (47)
\end{aligned}
$$

Let $i=2$. Then, with $N \leq j \leq 2N-3$, (43) becomes

$$
\begin{aligned}
&a_{2,j}(N+1,x)\\
= &\frac{3}{2}a_{1,j}(N,x)+xa_{2,j-2}(N,x)-\left(j-\frac{1}{2}\right)a_{2,j-1}(N,x). \qquad (48)
\end{aligned}
$$

Then, proceeding analogously to the case of (44), we can deduce that, for $N \leq j \leq 2N-3$,

$$
\begin{aligned}
&a_{2,j}(N+1,x)\\
= &\sum_{k=0}^{2N-j-2}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\left(\frac{3}{2}a_{1,j-k}(N-k,x)+xa_{2,j-k-2}(N-k,x)\right) \qquad (49)
\end{aligned}
$$

Thus we can deduce that, for $1 \leq i \leq N$, $N-i+2 \leq j \leq 2(N-i)+1$,

$$
\begin{aligned}
&a_{i,j}(N+1,x)\\
= &\sum_{k=0}^{2N-j-2i+2}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\\
&\times\left(\frac{2i-1}{2}a_{i-1,j-k}(N-k,x)+xa_{i,j-k-2}(N-k,x)\right) \qquad (50)
\end{aligned}
$$

Our results can be summarized as:

$$
\begin{aligned}
&a_{0,0}(0,x)=1;\\
&a_{N+1,0}(N+1,x)=\left(-\frac{1}{2}\right)^{N+1}(2N+1)!!;\\
&a_{0,N+1}(N+1,x)=\left(-\frac{1}{2}\right)^{N+1}(2N+1)!!;\\
&a_{0,2N+2}(N+1,x)=x^{N+1};\\
&a_{0,j}(N+1,x)=x\sum_{k=0}^{2N+2-j}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}a_{0,j-k-2}(N-k,x)\\
&\quad \text{for } N+2 \leq j \leq 2N+1;
\end{aligned}
$$

$$\begin{aligned}
&a_{i,N-i+1}(N+1,x)=\frac{2i-1}{2}\sum_{k=0}^{N-i+1}\left(-\frac{1}{2}\right)^k\frac{(2N-2i+1)!!}{(2N-2k-2i+1)!!}a_{i-1,N-k-i+1}(N-k,x)\\
&\quad\text{for } 1\le i\le N;\\
&a_{i,2(N+1-i)}(N+1,x)=\frac{2i-1}{2}\sum_{k=0}^{N-i+1}x^k a_{i-1,2(N-k-i+1)}(N-k,x),\\
&\quad\text{for } 1\le i\le N;\\
&a_{i,j}(N+1,x)\\
&\quad=\sum_{k=0}^{2N-j-2i+2}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\left(\frac{2i-1}{2}a_{i-1,j-k}(N-k,x)+xa_{i,j-k-2}(N-k,x)\right),\\
&\quad\text{for } 1\le i\le N, N-i+2\le j\le 2(N-i)+1. \qquad (51)
\end{aligned}$$

From these, we can conclude that, for $0\le i\le N+1$, $N+1-i\le j\le 2(N+1-i)$,

$$\begin{aligned}
a_{i,j}(N+1,x) \quad &= \sum_{k=0}^{2N-j-2i+2}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\\
&\quad\times\left(\frac{2i-1}{2}a_{i-1,j-k}(N-k,x)+xa_{i,j-k-2}(N-k,x)\right), \qquad (52)
\end{aligned}$$

with $a_{0,0}(0,x)=1$, $a_{1,0}(1,x)=\frac{1}{2}$, $a_{0,1}(1,x)=-\frac{1}{2}$, $a_{0,2}(1,x)=x$, except for $i=0$ and $j=N+1$, in which case

$$a_{0,N+1}(N+1,x)=\left(-\frac{1}{2}\right)^{N+1}(2N+1)!!. \qquad (53)$$

Our results can now be stated as the following theorem.

Theorem 1. The ordinary differential equations

$$F^{(N)}=\left(\frac{d}{dt}\right)^N F=\left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)}a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F, \qquad (54)$$

$(N=0,1,2,\cdots,)$ have a solution $F=F(t,x)=(1-t^2)^{-\frac{1}{2}}e^{x\left(\frac{t}{1+t}\right)}$, where, for $0\le i\le N$, $N-i\le j\le 2(N-i)$,

$$\begin{aligned}
a_{i,j}(N,x) \quad &= \sum_{k=0}^{2N-j-2i}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\\
&\quad\times\left(\frac{2i-1}{2}a_{i-1,j-k}(N-k-1,x)+xa_{i,j-k-2}(N-k-1,x)\right), \qquad (55)
\end{aligned}$$

with $a_{0,0}(0,x)=1$, $a_{1,0}(1,x)=\frac{1}{2}$, $a_{0,1}(1,x)=-\frac{1}{2}$, $a_{0,2}(1,x)=x$, except for $i=0$ and $j=N$, in which case

$$a_{0,N}(N,x)=\left(-\frac{1}{2}\right)^N(2N-1)!!. \qquad (56)$$

3. Applications of differential equations

We recall from (9) that the squared Hermite polynomials $SH_k(x)$ are given by the generating function

$$F = F(t;x) = (1-t^2)^{-\frac{1}{2}} e^{\left(\frac{t}{1+t}\right)} = \sum_{k=0}^{\infty} SH_k(x) \frac{t^k}{k!}. \tag{57}$$

Here we derive some new and explicit identities for the squared Hermite polynomials from the differential equations in Theorem 1. Now, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} SH_{k+N}(x) \frac{t^k}{k!} &= \left(\sum_{k=0}^{\infty} SH_k(x) \frac{t^k}{k!} \right)^{(N)} \\
&= \left(\left(1-t^2\right)^{-\frac{1}{2}} e^{x\left(\frac{t}{1+t}\right)} \right)^{(N)} \\
&= \left(\sum_{i=0}^{N} \sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j} \right) F \\
&= \sum_{i=0}^{N} \sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x) \sum_{l=0}^{\infty} (i+l-1)_l \frac{t^l}{l!} \\
&\qquad \times \sum_{m=0}^{\infty} (-1)^m (j+m-1)_m \frac{t^m}{m!} \sum_{n=0}^{\infty} SH_n(x) \frac{t^n}{n!} \\
&= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} \sum_{j=N-i}^{2(N-i)} \sum_{l+m+n=k} \binom{k}{l,m,n} \right. \\
&\qquad \left. \times (-1)^m (i+l-1)_l (j+m-1)_m a_{i,j}(N,x) SH_n(x) \right) \frac{t^k}{k!}.
\end{aligned} \tag{58}$$

From this, we have, for $k, N = 0, 1, 2, \cdots$

$$\begin{aligned}
SH_{k+N}(x) &= \sum_{i=0}^{N} \sum_{j=N-i}^{2(N-i)} \sum_{l+m+n=k} \binom{k}{l,m,n} \\
&\quad \times (-1)^m (i+l-1)_l (j+m-1)_m a_{i,j}(N,x) SH_n(x).
\end{aligned} \tag{59}$$

Thus we obtain the following theorem.

Theorem 2. For $k, N = 0, 1, 2, \cdots$

$$\begin{aligned}
SH_{k+N}(x) &= \sum_{i=0}^{N} \sum_{j=N-i}^{2(N-i)} \sum_{l+m+n=k} \binom{k}{l,m,n} \\
&\quad \times (-1)^m (i+l-1)_l (j+m-1)_m a_{i,j}(N,x) SH_n(x),
\end{aligned}$$

where $a_{i,j}(N,x)$ are as in Theorem 1.

Letting $k = 0$ in (59), we obtain the following result giving expressions for the squared Hermite polynomials $SH_N(x)$.

Theorem 3. For $N = 0, 1, 2, \cdots$

$$SH_N(x) = \sum_{i=0}^{N} \sum_{j=N-i}^{2(N-i)} a_{i,j}(N, x),$$

where $a_{i,j}(N, x)$ are as in Theorem 1.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Quenching for the discrete heat equation with a singular absorption term on finite graphs

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Abstract

We study the quenching for the discrete semi-linear heat equation with singular absorption $u_t = \Delta_\omega u - \lambda u^{-p}$ on finite graph with Dirichlet boundary condition and the positive initial condition $u_0(x)$. When $\lambda^{-p} \geq \max_{x\in S} u_0(x)$, we prove that the solution will quench in finite time by comparison principal. Meanwhile, we study the quenching rate. Moreover, we also prove that there exists a critical exponent λ^* such that the problem admits a global solution for all $\lambda \leq \lambda^*$. Finally, a numerical experiment on two finite graphs is given to illustrate our results.

Keywords: Discrete heat equation; singular absorption; quenching; graphs.
MSC: 35B05, 35B33, 45G05

1 Introduction

Let G be a graph with vertex set V and edge set E, where the vertex set is divided into the boundary vertices ∂S and the interior vertices S which is connected, and we always assume G is a finite, connected, simple (without multiple edges and loops) graph in the following context. In this paper, we mainly study the quenching phenomena for the following semi-linear discrete heat equation with singular absorption on finite graph

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G

$$\begin{cases} u_t = \Delta_\omega u - \lambda u^{-p}, & x \in S \text{ and } t \in (0,T), \\ u(x,t) = 1, & x \in \partial S \text{ and } t \in (0,T), \\ u(x,0) = u_0(x), & x \in S, \end{cases} \tag{1}$$

here p, λ and T are positive constants, the initial value $u_0(x) \in C(V)$ and satisfies $0 < u_0(x) \le 1$ for any $x \in S$. The function space $C(V)$ denotes the set of all functions which are definite on the vertices V of the graph G, and Δ_ω denotes the discrete Laplacian operator on finite graph, which is defined as follows (see [1]),

$$\Delta_\omega u(x) = \sum_{y \in V} [u(y) - u(x)] \cdot \omega(x,y),$$

where the function $\omega(x,y)$ is called the weighted function, and satisfies

$$\begin{aligned} &(i)\ \omega(x,x) = 0, \text{ for any } x \in V, \\ &(ii)\ \omega(x,y) = \omega(y,x) \ge 0, \text{ for any } x, y \in V, \\ &(iii)\ \omega(x,y) = 0, \text{ if and only if } (x,y) \notin E. \end{aligned}$$

Moreover, $d_\omega(x) = \sum\limits_{x \in V} \omega(x,y)$ denotes the degree of the node $x \in V$ of the weighted graph G, and we assume that $d_\omega(x) \le 1$ for any $x \in S$.

By introducing $v(x,t) = 1 - u(x,t)$, it is not difficult to verify that the function $v(x,t)$ satisfies the following initial boundary value problem

$$\begin{cases} v_t = \Delta_\omega v + \lambda(1-v)^{-p}, & x \in S \text{ and } t \in (0,T), \\ v(x,t) = 0, & x \in \partial S \text{ and } t \in (0,T), \\ v(x,0) = 1 - u_0(x), & x \in S. \end{cases} \tag{2}$$

In the continuous case including the local and nonlocal diffusion equation likes (1) or (2), its quenching phenomena has attracted much attention from the work of H. Kawarada [2] in 1975. This type of the diffusion equation with a singular absorption term (or a reaction term) comes form the polarization phenomena in ionic conductors [2], and can be considered as a limiting case of models in chemical catalyst kinetics or models of in enzyme kinetics [4, 5, 3, 6]. The detailed researches on the quenching phenomena can be found in [9, 6, 7, 8] and the references therein. Especially, for the nonlinear diffusion equation

$$u_t - u_{xx} = -u^{-p}, -l < x < l$$

with non-homogeneous Dirichlet boundary condition and the positive initial value, its quenching occurs in finite time for sufficiently large l in[2, 7]. Moreover, the quenching of the semilinear parabolic equation

$$u_t - \Delta u = g(u)$$

with homogeneous Dirichlet boundary condition and the positive initial value was also studied, the readers can refer to [10, 11]. On the other hand, the authors of [9] considered the quenching behaviour of the following nonlocal diffusion equation

$$u_t = J * u - u - \lambda u^{-p},$$

the critical parameter λ^* and the quenching rate and the quenching set were also given.

Recently, the ω-harmonic function and the $\omega-$heat equation were considered by many authors since the discrete heat equation has been widely applied to the fields of heat and energy transfer, electrical networks, image processing and so on [1, 12, 13]. In [14], Y.S. Chung, Y.S. Lee et.al considered the extinction and positivity of the discrete heat equation with absorption on network

$$u_t = \Delta_\omega u - u^p,$$

where $p > 0$. Furthermore, the extinction and positivity for the p, ω-heat equation with absorption was also studied in [16, 15]. Blow-up for the $\omega-$heat equation with a reaction term on graphs

$$u_t = \Delta_\omega u + \lambda u^p,$$

where $p > 0$ was researched in [17, 18]. The asymptotic behavior of solutions for the ω-heat equation with reaction and absorption term was considered in [19].

Motivated by the above works, the purpose of this paper is to discuss the quenching phenomenons for the discrete heat equation with singular absorption term and the non-homonomous Dirichlet boundary conditions. The local existence and uniqueness of solutions are obtained in the next section. In the third section, we will show the comparison principal for the discrete heat equation (1). The sufficient conditions on quenching and quenching rate are proved in the section 4. In the section 5, we mainly discuss the existence of the global solution. In the last section, we give some numerical experiments to illustrate our results.

2 Local existence and uniqueness of solutions

Lemma 2.1 *Suppose $0 < u_0(x) \leq 1$, then, there exists a unique solution $u \in C[0, T) \times C(V)$ for the problem (1). Moreover, if T is finite, then*

$$\lim_{t \to T^-} u(x, t) = 0 \tag{3}$$

for some $x \in S$.

Proof. Since $0 < u_0(x) \leq 1$, there exists a positive constant ε, such that $2\varepsilon < u_0(x) \leq 1$. Set

$$X_0 = \{u \in C[0,t_0] \times C(V), \varepsilon \leq u \leq K \text{ and } u(x) \equiv 1 \text{ for any } x \in \partial S\},$$

where $K > 1$ and

$$t_0 < \min\left\{\frac{K-1}{K}, \frac{\varepsilon}{K+\lambda\varepsilon^{-p}}, \frac{1}{2+\lambda p\varepsilon^{-p-1}}\right\}. \tag{4}$$

Now, we define the operator as follows:

$$T_{u_0}[u](x,t) = \begin{cases} u_0(x) + \int_0^t \Delta_\omega u(x,s)ds - \lambda\int_0^t u^{-p}(x,s)ds, & x \in S, 0 \leq t \leq t_0, \\ 1, & x \in \partial S, 0 \leq t \leq t_0, \end{cases}$$

and the norm of the Banach space X_0

$$\|u(x,t)\|_{X_0} = \max_{x\in V}\max_{t\in[0,t_0]}|u(x,t)|$$

for any $u(x,t) \in X_0$.

First, we prove that the operator T_{u_0} maps X_0 into X_0. It is easy to verify that $T_{u_0}[u](x,t)$ is continuous about the time t for any fixed node $x \in V$. On the other hand, for any $u(x,t) \in X_0$, we have

$$T_{u_0}[u](x,t) \geq 2\varepsilon - (K+\lambda\varepsilon^{-p})t_0 \geq \varepsilon, \tag{5}$$

moreover, we also have

$$T_{u_0}[u](x,t) \leq 1 + Kt_0 = K(\frac{1}{K} + t_0) \leq K. \tag{6}$$

Next, we show that T_{u_0} is a strict contraction in X_0. That is to say, for any $u, v \in X_0$, we get

$$\begin{aligned} \|u-v\|_{X_0} &\leq \left\|\int_0^t \sum_{y\in V}[u(y,s)-v(y,s)]\,\omega(x,y)ds\right\|_{X_0} \\ &+ \left\|\int_0^t [u(x,s)-v(x,s)]ds\right\|_{X_0} + \lambda\left\|\int_0^t [v^{-p}(x,s)-u^{-p}(x,s)]ds\right\|_{X_0} \\ &\leq 2t_0\|u-v\|_{X_0} + \lambda p\left\|\int_0^t |\xi|^{-p-1}|u(x,s)-v(x,s)|ds\right\|_{X_0} \\ &\leq t_0(2+\lambda p\varepsilon^{-p-1})\|u-v\|_{X_0} < \|u-v\|_{X_0}. \end{aligned}$$

Hence, by Banach fixed point theorem, there exists a unique $u \in X_0$ such that $u = T_{u_0(x)}[u]$, so, for any $x \in S$, we have

$$u(x,t) = \begin{cases} u_0(x) + \int_0^t \Delta_\omega u(x,s)ds - \lambda\int_0^t u^{-p}(x,s)ds, & x \in S \\ 1, & x \in \partial S, \end{cases} \tag{7}$$

thus, we can get $u(x,t)$ is the unique solution to the problem (1) in $t \in [0,t_0]$. Now, if $u(x,t_0) > 0$, we can continue the above procedure, and then, the solution can be extend to the time interval $[t_0,t_1]$. This procedure can be continued again and again until $\lim_{t\to T^-} u(x,t) \to 0$ for some time T which may be infinite.

3 Comparison principle

In this section, we mainly show a comparison principal. To do this, we begin with the definition of the super-solution and sub-solution to the problem (1).

Definition 3.1 *A function $\overline{u} \in C(V) \times C[0,T)$ is a super-solution to the problem (1) if $\overline{u}$ is a positive function and satisfies*

$$\begin{cases} \overline{u}_t \geq \Delta_\omega \overline{u} - \lambda \overline{u}^{-p}, & x \in S \text{ and } t \in (0,T), \\ \overline{u}(x,t) \geq 0, & x \in \partial S \text{ and } t \in (0,T), \\ \overline{u}(x,0) \geq u_0(x), & x \in S, \end{cases} \tag{8}$$

Analogously, we say that $\underline{u} \in C(V) \times C[0,T)$ is a sub-solution if it satisfies the reverses above inequalities.

Now, we have the following comparison principle.

Theorem 3.1 ***(Comparison principle)*** *Suppose $\overline{u}$ and $\underline{u}$ be a super-solution and a sub-solution to the problem (1.1), respectively, then $\overline{u} \geq \underline{u}$ in $(x,t) \in V \times [0,T)$.*

Proof. For any $0 < t_0 < T$, set $m = \min_{S\times[0,t_0]} \{\overline{u}, \underline{u}\}$ and $M = \max_{S\times[0,t_0]} \{\overline{u}, \underline{u}\}$, thus, we know that m, M are the positive constants. And then, suppose $v(x,t) = \underline{u} - \overline{u}$. Notice that $v(x,0) > 0$ for any $x \in S$. By the definitions of the super-solution and the sub-solution, we can get

$$v_t \geq \Delta_\omega v - \lambda(\underline{u}^{-p} - \overline{u}^{-p}), \tag{9}$$

let $v^+(x,t) = \max\{v(x,t), 0\} \geq 0$. Thus, multiplying v^+ both sides of the above inequality, and integrating on S, we obtain

$$\begin{aligned} &\frac{1}{2}\left(\int_{x\in S} (v^+(x,t))^2\right)_t \\ &\leq \int_{x\in S} \Delta_\omega v(x,t) v^+(x,t) + \int_{x\in S} (\underline{u}^{p(x)} - \overline{u}^{p(x)}) v^+(x,t), \end{aligned} \tag{10}$$

For the first term of the right part of the above inequality, we have

$$\int_{x\in S} \Delta_\omega v(x,t) v^+(x,t) \leq 0. \tag{11}$$

In fact, let $J(t) = \{x \in V : v(x,t) > 0\}$, if $J(t)$ is empty set, we have the desired results. Now, assume $J(t)$ is not an empty set. Due to $\underline{u}(x,t) \leq 0$, $\overline{u}(x,t) \geq 0$ for any $x \in \partial S$ and $0 \leq t \leq t_0$, so $v(x,t) = \underline{u}(x,t) - \overline{u}(x,t) \leq 0$ for any $x \in \partial S$ and $0 \leq t \leq t_0$.

Now, we get $J(t) \subset S$. Thus, if $x \in J(t)$ and $y \in V \setminus J(t)$, we have $v(x,t) > 0$ and $v(y,t) - v(x,t) < 0$, hence, we have

$$\sum_{x\in J(t)} \sum_{y\in V\setminus J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y) < 0.$$

Furthermore, we get

$$\begin{aligned}
&\sum_{x\in S}\sum_{y\in V} v^+(x,t)[v(y,t) - v(x,t)]\omega(x,y) \\
&= \sum_{x\in J(t)} \sum_{y\in J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y) \\
&+ \sum_{x\in J(t)} \sum_{y\in V\setminus J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y) \\
&= -\frac{1}{2} \sum_{x\in J(t)} \sum_{y\in J(t)} [v(y,t) - v(x,t)]^2\omega(x,y) \\
&+ \sum_{x\in J(t)} \sum_{y\in V\setminus J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y) < 0.
\end{aligned} \tag{12}$$

On the other hand, for any fixed $x \in S$, by mean value theorem, we have

$$\underline{u}^{-p}(x,t) - \overline{u}^{-p}(x,t) = -p\xi^{-p-1}(x,t)v(x,t),$$

where $\xi(x,t) = \theta(x)\underline{u}(x,t) + (1-\theta(x))\overline{u}(x,t)$ and $0 \le \theta(x) \le 1$. And then, we have $m \le \xi(x,t) \le M$. Thus, for the second term of the right part of the inequality (10), we also have

$$\int_{x\in S} (\underline{u}^{p(x)} - \bar{u}^{p(x)})v^+(x,t) \le -m^{-p-1} \int_{x\in S} (v^+(x,t))^2. \tag{13}$$

Combine the inequalities (10), (12) and (13), we obtain

$$\left(\int_{x\in V} (v^+(x,t))^2\right)_t < 0. \tag{14}$$

There exists a contradiction. Hence $J(t) = \emptyset$. By the arbitrariness of t_0, we obtain $\overline{u}(x,t) \ge \underline{u}(x,t)$, for $(x,t) \in V \times [0,T)$.

4 Quenching phenomena and quenching rate

In this section, similar to the method used in [9], we mainly propose the quenching condition and quenching rate. Before the discussions and proofs, we firstly give some notes about the initial value condition and also the boundary condition. Since the absorption term is singular at points which satisfy $u(x) = 0$, we need the initial value $u_0(x) > 0$. Moreover, if $\max_{x\in S} u_0(x) > 1$, we can set

$$U(t) = (\lambda p)^{\frac{1}{p+1}}(A-t)^{\frac{1}{p+1}},$$

6

where $A = \max_{x\in S} u_0(x)$, and then, it is easy to verify that $U(t)$ is a super-solution to the discrete diffusion equation (1) when $U(t) \geq 1$. Thus, by the comparison principle, there exists t_0 such that $1 \geq U(t_0) \geq u(x,t_0)$. Hence, we can discuss the quenching pheromone to the problem (1) with the large initial value beginning with the initial time time $t = t_0$. The following proof can be similarly done. Finally, if we choose the homogenous Dirichlet boundary condition, i.e. set $u(x,t) = 0$ for any $x \in \partial S$, and then, we can also get $U(t)$ is also a super-solution to the problem (1) for any $t < A$, and then, we have $u(x,t)$ always quenches in finite time, i.e. the solution to the problem (1) is not global.

Next, we give the proof of the quenching phenomena about the problem (1), we mainly have the following two results.

Theorem 4.1 *If the initial value* $u_0(x)$ *satisfies that*

$$\max_{x\in S} u_0(x) \leq \lambda^{\frac{1}{p}} < 1, \tag{15}$$

and then, the solution to the problem (1) *quenches in finite time* T*.*

Proof. It is easy to verify that

$$v(x,t) = \begin{cases} \lambda^{\frac{1}{p}}, & x \in S, \\ 1, & x \in \partial S, \end{cases} \tag{16}$$

is the super-solution to the problem (1), thus, by the comparison principle, we have $u(x,t) \leq \lambda^{\frac{1}{p}}$ for any $x \in S$ and $t \in [0,T)$.

Now, assume $u(x,t)$ attains its minimum value at the nodes x^* for any fix time t. At this point, we have

$$\begin{aligned} u_t(x^*,t) &= \sum_{y\in V} u(y,t)\omega(x^*,y) - d^* u(x^*,t) - \lambda u^{-p}(x^*,t) \\ &\leq d^* - d^* u(x^*,t) - \lambda u^{-p}(x^*,t) \\ &\leq -d^* u(x^*,t), \end{aligned} \tag{17}$$

where $d^* = d_\omega(x^*)$. Integrating both sides of the above inequality in $[0,t]$, we can get

$$u(x^*,t) \leq u_0(x^*)e^{-d^*t} \leq \lambda^{\frac{1}{p}} e^{-d^*t}. \tag{18}$$

Thus, for the equality in (17), note that the function $-s^{-p}$ is increasing, hence, choose $t_0 \geq \frac{ln(2d^*)}{pd^*}$, and then, for any $t \geq t_0$, we can also get

$$\begin{aligned} u_t(x^*,t) &\leq d^* - d^* u(x^*,t) - \lambda u^{-p}(x^*,t) \\ &\leq d^* - \frac{\lambda}{2} u^{-p}(x^*,t) - \frac{\lambda}{2} u^{-p}(x^*,t) \\ &\leq d^* - \frac{1}{2} e^{pd^*t} - \frac{\lambda}{2} u^{-p}(x^*,t) \\ &\leq -\frac{\lambda}{2} u^{-p}(x^*,t), \end{aligned} \tag{19}$$

Integrating both sides of the above inequality in $[t_0, t]$, we can obtain

$$\begin{aligned} & u^{p+1}(u(x^*,t)) \\ & \le u^{p+1}(u(x^*,t_0)) - \frac{(p+1)\lambda}{2}(t-t_0) \\ & \le u_0^{p+1}(x^*)e^{-d^*(p+1)t_0} - \frac{(p+1)\lambda}{2}(t-t_0), \end{aligned}$$

from this inequality, we have $u(x,t)$ quenches at finite time T, moreover, we have

$$T \le t_0 + \frac{(p+1)\lambda}{2}u_0^{p+1}(x^*)e^{-d^*(p+1)t_0}. \tag{20}$$

Theorem 4.2 *If $\lambda \ge 1$, then the solution to the problem* (1) *also quenches in finite time.*

Proof. Since $\lambda \ge 1$ and $0 < u_0(x) \le 1$, we have $\max_{x\in V} u_0(x) \le \lambda^{\frac{1}{p+1}}$. Now, it is easy to verify that $v(x,t) \equiv \lambda^{\frac{1}{p+1}}, x \in V$ is a super-solution to the problem (1). Thus, by the comparison principle, we also have $u(x,t) \le \lambda^{\frac{1}{p+1}}$ for any $x \in V$.

Now, also assume $u(x,t)$ attains its minimum value at the nodes x^* for any fix time t. At this point, we have

$$\begin{aligned} u_t(x^*,t) &= \sum_{y\in V} u(y,t)\omega(x^*,y) - d^*u(x^*,t) - \lambda u^{-p}(x^*,t) \\ &\le \lambda^{\frac{1}{p+1}} - d^*u(x^*,t) - \lambda u^{-p}(x^*,t) \\ &\le -d^*u(x^*,t), \end{aligned} \tag{21}$$

Integrating both sides of the above inequality on $[0,t]$, we can get

$$u(x^*,t) \le u_0(x^*)e^{-d^*t}. \tag{22}$$

Thus, for any $t \ge t_0$, from the inequality in (21) and by choosing $t_0 \ge \frac{\ln 2 - \frac{p\ln\lambda}{p+1}}{pd^*}$, it follows that

$$\begin{aligned} u_t(x^*,t) &\le \lambda^{\frac{1}{p+1}} - d^*u(x^*,t) - \lambda u^{-p}(x^*,t) \\ &= \lambda^{\frac{1}{p+1}} - d^*u(x^*,t) - \frac{\lambda}{2}u^{-p}(x^*,t) - \frac{\lambda}{2}u^{-p}(x^*,t) \\ &\le \lambda^{\frac{1}{p+1}} - \frac{\lambda}{2}e^{pd^*t} - \frac{\lambda}{2}u^{-p}(x^*,t) \\ &\le -\frac{\lambda}{2}u^{-p}(x^*,t), \end{aligned} \tag{23}$$

Integrating both sides of the above inequality on $[t_0, t]$, we can obtain

$$\begin{aligned} & u^{p+1}(u(x^*,t)) \\ & \le u^{p+1}(u(x^*,t_0)) - \frac{(p+1)\lambda}{2}(t-t_0) \\ & \le u_0^{p+1}(x^*)e^{-(p+1)d^*t_0} - \frac{(p+1)\lambda}{2}(t-t_0), \end{aligned}$$

by this inequality, we get $u(x,t)$ quenches at finite time T, moreover, we also have

$$T \leq t_0 + \frac{(p+1)\lambda}{2} u_0^{p+1}(x^*) e^{-(p+1)d^* t_0}. \tag{24}$$

Theorem 4.3 ***(The quenching rate)*** *If the solution $u(x,t)$ to the problem (1) quenches in finite time T at the node x^*, and then, we have*

$$\lim_{t\to T^-} (T-t)^{\frac{-1}{p+1}} u(x^*,t) = [(p+1)\lambda]^{\frac{1}{p+1}}.$$

Proof. Since $0 < u_0(x) \leq 1$, and then, it is easy to verify that $v(x,t) \equiv 1$ is a super-solution to the problem (1), by the comparison principle, we know that $0 < u(x,t) \leq 1$ for any $x \in V$ and $t \in [0,T)$.

Now, multiply u^p on the both sides of the discrete heat equation in the problem (1), and then, we get

$$u^p u_t = u^p \Delta_\omega u - \lambda, x \in S, t \in [0,T). \tag{25}$$

Next, we establish the upper bound of the quenching rate. Due to $0 < u(x,t) \leq 1$, we have

$$\begin{aligned} u^p u_t &= u^p \Delta_\omega u - \lambda \\ &= u^p \sum_{y\in V} u(y,t)\omega(x,y) - d_\omega(x) u^{p+1} - \lambda \\ &\geq -u^{p+1} - \lambda \geq -1 - \lambda \end{aligned} \tag{26}$$

for any $x \in S, t \in [0,T)$. Assume that $u(x,t)$ quenches in finite time T at the node x^*, and then, integrating the inequality $u^p u_t \geq -1-\lambda$ on the time t on $[t,T]$ on the node x^*, due to $u(x,t) \to 0$ when $t \to T^-$, we can get

$$u^{p+1}(x^*,t) \leq (p+1)(\lambda+1)(T-t).$$

Moreover, due to the inequality $u^p u_t \geq -u^{p+1} - \lambda$, thus, at the quenching node x^*, we also have

$$u^p u_t(x^*,t) \geq -(p+1)(\lambda+1)(T-t) - \lambda, \tag{27}$$

Integrating again in the time interval $[t,T]$, we have

$$-\frac{1}{p+1} u^{p+1}(x^*,t) \geq \frac{1}{2}(p+1)(\lambda+1)(T-t)^2 - \lambda(T-t), \tag{28}$$

thus, we get

$$\frac{u^{p+1}(x^*,t)}{T-t} \leq (p+1)\lambda \left(-(p+1)\frac{2(\lambda+1)}{\lambda}(T-t) + 1 \right). \tag{29}$$

Now, we establish the lower bound of the quenching rate. By the equation (31) and $0 < u(x,t) \le 1$, we also have

$$u^p u_t = u^p \sum_{y\in V} u(y,t)\omega(x,y) - d_\omega(x)u^{p+1} - \lambda \le u^p - \lambda.$$

Thus, by the inequality (26), at the quenching node x^*, we can obtain the following inequality

$$u^p u_t \le [(p+1)(\lambda+1)(T-t)]^{\frac{p}{p+1}} - \lambda.$$

Integrating in the time interval $[t,T]$, we have

$$\frac{u^{p+1}(x^*,t)}{T-t} \ge (p+1)\lambda \left(-\frac{(p+1)^{\frac{2p+1}{p+1}}(\lambda+1)^{\frac{p}{p+1}}}{(2p+1)\lambda}(T-t)+1 \right). \tag{30}$$

Combine the inequalities (29) and (30), and let $t \to T^-$, we can obtain the need results.

5 The existence of a global solution

In this section, we investigate the existence of a global solution to the problem (1) with the initial value $u_0(x) \equiv 1$ for any $x \in S$. To do this, we begin with the following lemma.

Lemma 5.1 *There exists a small nonnegative constant λ^*, such that if $\lambda \le \lambda^*$, then the eigenvalue problem*

$$\begin{cases} \Delta_\omega u(x) = \lambda u^{-p}(x), & x \in S, \\ u(x) = 1, & x \in \partial S, \end{cases} \tag{31}$$

exists at least one solution.

Proof. Let $C(V)$ denotes the set of all the functions which are defined on the finite graph G with its nodes V, and then, the norm on $C(V)$ is as follows:

$$\|v\|_{C(V)} = \max_{x\in V} v(x). \tag{32}$$

Furthermore, set $C_0(V) = \{v(x) \in C(V) \text{ and } v(x) \equiv 0 \text{ for any } x \in \partial S\}$ and assume that $A = \{v \in C_0(V) : -\varepsilon < v(x) < 1\}$ is a open subset of $C_0(V)$, the nonlinear function $F(\lambda, v) : (-\varepsilon, \varepsilon) \times A \to C(S)$ is defined as

$$F(\lambda, v) = \Delta_\omega v + \lambda(1-v)^{-p}, \tag{33}$$

where ε is a small enough constant.

It is obviously that $F(\lambda, v)$ is differentiable function and $F(0,0) = 0$. Moreover, the Fréchet derivative of $F(\lambda, v)$ at $(0,0)$ is

$$F_v(0,0)[z(x)] = \Delta_\omega z(x) \tag{34}$$

is a continuous linear operator for any $z(x) \in A$. In fact, for any sequence $z_m(x) \to z(x)$, we have $\|\Delta_\omega[z_m(x) - z(x)]\|_{C(V)} \leq |V|\|z_m - z\|_{C(V)}$, so $F_v(0,0)$ is a continuous operator. Moreover, its kernel is the function $z = 0$ (see [1]), and then, it is injective. On the other hand, $F_v(0,0)$ is a linear transformation on finite dimensional space, and then, it is also a compact linear operator, hence, it is also bijective. By the Open-Mapping Theorem we deduce that $F_v(0,0)$ is a linear homeomorphism of $C_0(V)$ into $C_0(V)$. By the Implicit Function Theorem in the appendix A of [20], there exists a neighborhoods $U \in (-\varepsilon, \varepsilon)$ of $\lambda = 0$ and $W \in A$ of $v(x) \equiv 0$ such that $F(\lambda, v_\lambda) = 0$ for any $\lambda \in U$, and $v_\lambda \in W$ is unique. Thus, for any $\lambda < \lambda^* \in U$, suppose $u_\lambda(x) = 1 - v_\lambda(x)$, it is easy to verify that $u_\lambda(x)$ is a solution to the equation (31).

Based on the above lemma, we have the following theorem on the existence of the global solution to the problem (1) with $u_0(x) = 1$.

Theorem 5.1 *There exists a constat λ^*, such that $\lambda \leq \lambda*$, the problem (1) with the initial value $u_0(x) = 1$ has a global solution, while for $\lambda > \lambda^*$, then no global solution exists.*

Proof. Firstly, from the proofs of Theorem 4.1 and 4.2, we have the solution to the problem (1) with the initial value $u_0(x) = 1$ quenching in infinite time is impossible. Moreover, set $w(x,t) = u_t(x,t)$, and then, we get w satisfies

$$\begin{cases} w_t = \Delta_\omega w + p\lambda u^{-p-1} w, & (x,t) \in S \times (0,T), \\ w(x,t) = 0, & (x,t) \in \partial S \times (0,T), \\ w(x,0) = -\lambda, & x \in S. \end{cases} \tag{35}$$

Then, by comparison principle, we obtain that $w = u_t \leq 0$. On the other hand, by the Lemma 5.1, we have λ is small enough, the equation (31) exists a positive solution $v_\lambda(x)$, in fact, it is also a sub-solution to the problem (1) with the initial value $u_0(x) = 1$. Hence, the solution of (1) with the initial value $u_0(x) = 1$ satisfies that, either it quenches in finite time, or it converges to a stationary solution

Next, we discuss the critical exponent of the quenching and the global existence. In fact, If $u(x,t)$ is a global solution to the problem (1), then, we know that $u(x,t) \to u_\infty$ as $t \to \infty$ and u_∞ is a solution the the problem (31), is a stationary solution to the equation (31). Moreover, for any fix constant λ_1, if there exists a solution $v_{\lambda_1}(x)$ to the problem (31), i.e. $v_{\lambda_1}(x)$ satisfies

$$\Delta_\omega v_{\lambda_1}(x) = \lambda v_{\lambda_1}^{-p}(x), \tag{36}$$

furthermore, it is easy to verify that $v_{\lambda_1}(x)$ is a sub-solution to the problem (1) with the initial value $u_0(x) = 1$ and $\lambda \leq \lambda_1$. Thus, the solution to the problem (1) with the initial value $u_0(x) = 1$ is global when $\lambda \leq \lambda_1$. By this monotonicity property given

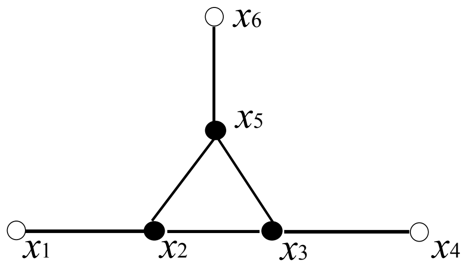

Figure 1: The graph G_1

above discussion, set $\lambda^* = \sup\limits_{\lambda \in B} \lambda$, where the set $B = \{\lambda : v_\lambda(x) \text{ exists to (36)}\}$. This completes the proof.

6 Numerical experiments

In this section, we consider a graph G_1 (as shown in Figure 1), which has six nodes $x_1, x_2, \cdots, x_6$, where x_2, x_3, x_5 are interior and x_1, x_4, x_6 are the boundary. Moreover, we only consider the weight function $\omega \equiv \frac{1}{3}$. Thus, the discrete heat equation in (1) is

$$\begin{cases} u_t(x_2,t) = \frac{1}{3} + \frac{1}{3}u(x_3,t) + \frac{1}{3}u(x_5,t) - u(x_2,t) - \lambda u^{-p}(x_2,t) \\ u_t(x_3,t) = \frac{1}{3} + \frac{1}{3}u(x_2,t) + \frac{1}{3}u(x_5,t) - u(x_3,t) - \lambda u^{-p}(x_3,t) \\ u_t(x_5,t) = \frac{1}{3} + \frac{1}{3}u(x_2,t) + \frac{1}{3}u(x_3,t) - u(x_5,t) - \lambda u^{-p}(x_5,t) \end{cases} \tag{37}$$

Now, we also suppose that the exponent $p = 1.2, \lambda = 0.8$. Moreover, the discrete Laplacian operator Δ_ω on the graph G_1 is as follows:

$$\Delta_\omega = -\frac{1}{3}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \tag{38}$$

Thus, set $U(t) = (u(x_2,t), u(x_3,t), u(x_5,t))^T$, and then, we have the equation (37) can be rewrote as follows:

$$U_t = \frac{\mathbf{1}}{3} + \Delta_\omega * U(t) - 0.8U^{-2}(t), \text{ with } U(0) = (0.3, 0.35, 0.4)^T, \tag{39}$$

where $\mathbf{1} = (1,1,1)^T$.

By Theorem 4.1, we get $U(t)$ quenches in finite time, moreover, U_t blows up in finite time. Since the system (39) is nonlinear, it is difficult to compute its analytic solutions. Hence, we consider its numerical solutions. The explicit difference scheme to the system (39) is as follows:

$$U_{n+1} = U_n + \Delta t(\frac{\mathbf{1}}{3} + \Delta_\omega * U_n - 0.8U_n^{-2}), \text{ with } U_0 = (0.3, 0.35, 0.4)^T, \tag{40}$$

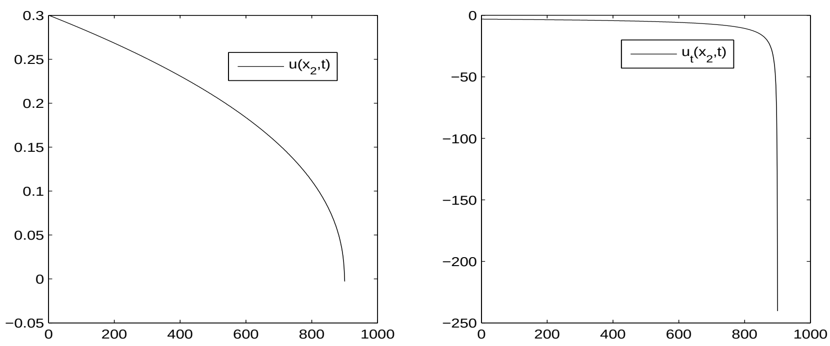

Figure 2: Quenching of $u(x_2, t)$ and Blow-up of $u_t(x_2, t)$ in finite time

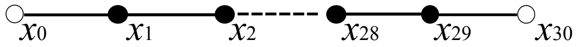

Figure 3: The graph G_2

where U_n denotes $U(n\Delta t)$ for $n = 1, 2, 3, \cdots$ and Δt is the time step which taking as $0.043/n$ in the numerical experiment. The numerical experiment result is shown in Figure 2. From this numerical experiment, we know that the solution $U(t)$ quenches and U_t blows up in finite time.

At the end of this section, we give another example. Now, we consider the discrete heat equation (1) on the following finite graph G_2 (as shown in Figure 3), which has six nodes $x_0, x_2, \cdots, x_{30}$, where $x_1, x_2, \cdots, x_{29}$ are interior and x_0, x_{30} are the boundary. Moreover, we only consider the weight function $\omega(x_i, x_j) \equiv \frac{1}{4}$. Thus, the discrete heat equation in (1) is

$$\begin{cases} u_t(x_1, t) = \frac{1}{4}(1 + u(x_2, t) - 2u(x_1, t)) - \lambda u^{-p}(x_1, t), \\ u_t(x_i, t) = \frac{1}{4}(u(x_{i-1}) + u(x_{i+1}, t) - 2u(x_i, t)) - \lambda u^{-p}(x_i, t), 1 \le i \le 28, \\ u_t(x_{29}, t) = \frac{1}{4}(1 + u(x_{28}, t) - 2u(x_{29}, t)) - \lambda u^{-p}(x_{29}, t), \end{cases} \tag{41}$$

where $\lambda = 1$, $p = 1.2$, and then, let the initial value $u_0(x_i) = 1 - 0.9 \sin\left(\frac{i}{30}\pi\right)$, where $1 \le i \le 29$ and $u(x_0, t) = u(x_{30}, t) = 1$. Thus, by the theorem 4.2, we have the solution $u(x_i, t)$ will quench in finite time. Also since the nonlinear of the system (41), we consider the following difference scheme:

$$V_{n+1} = V_n + \Delta t \left(B + \Delta_\omega V_n - \lambda V_n^{-p}\right), n = 0, 1, 2, \cdots, \tag{42}$$

where $V_n = (u(x_1, n\Delta t), u(x_2, n\Delta t), \cdots, u(x_{29}, n\Delta t))^T$, $B = (1/4, 0, \cdots, 0, 1/4)^T$ is a 29$-$ dimensions vector, $\Delta t = 0.0001/n$ is the time step, and the discrete Laplacian

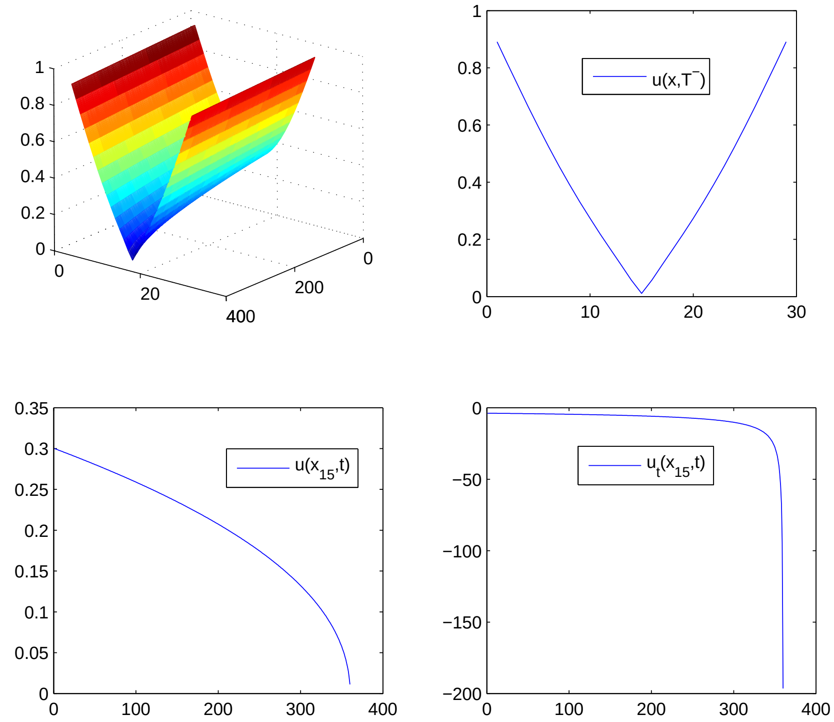

Figure 4: Quenching of $u(x,t)$ and Blow-up of $u_t(x_{15},t)$ in finite time

operator on the graph G_2 is as follows:

$$\Delta_\omega = \frac{1}{4}\begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{pmatrix}_{29\times 29}. \tag{43}$$

Moreover, the initial value $V_0 = (u_0(x_1), u_0(x_2)), \cdots, u_0(x_{29}))$. The numerical experiment results can be found in Figure 4.

7 Conclusion

In this paper, we mainly consider the quenching problem and the global solution of the discrete heat equation with a singular absorption, the quenching time, quenching rate and the critical exponent were also given. We only prove the existence of the critical exponent, its upper and lower bounds may be established by the Kaplan's method in the further work.

Acknowledgments

This work is supported by the Natural Science Foundation of Uygur Autonomous Region of Xinjiang(Grant No.201442137-30) and the National Natural Science Foundation of China (Grant No.11461075).

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Nonlocal fractional-order boundary value problems with generalized Riemann-Liouville integral boundary conditions

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Abstract

In this paper, we study existence and uniqueness of solutions for nonlocal boundary value problems of Caputo fractional differential equations equipped with generalized Riemann-Liouville integral boundary conditions. A variety of fixed point theorems such as Banach's fixed point theorem, nonlinear contractions, Krasnoselskii's fixed point theorem, Schaefer's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder degree theory are applied to obtain the desired results. Several examples are discussed for illustration of the obtained results.

Key words and phrases: Caputo fractional derivative; generalized Riemann-Liouville integral; nonlocal boundary conditions; fixed point theorems.
AMS (MOS) Subject Classifications: 26A33; 34A08

1 Introduction

We investigate the sufficient criteria for existence of solutions for the following Caputo fractional differential equation

$$D^q x(t) = f(t, x(t)), \qquad 0 < t < T, \tag{1}$$

subject to nonlocal generalized Riemann-Liouville fractional integral boundary conditions of the form

$$\begin{aligned} x(0) &= \gamma \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\zeta} \frac{s^{\rho-1} x(s)}{(\zeta^\rho - s^\rho)^{1-\alpha}} ds := \gamma\ {}^\rho I^\alpha x(\zeta), \\ x(T) &= \delta \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_0^{\xi} \frac{s^{\rho-1} x(s)}{(\xi^\rho - s^\rho)^{1-\beta}} ds := \delta\ {}^\rho I^\beta x(\xi), \quad 0 < \zeta, \xi < T, \end{aligned} \tag{2}$$

where D^q denote the Caputo fractional derivative of order q, ${}^\rho I^z$, $z \in \{\alpha, \beta\}$, is the generalized Riemman-Liouville fractional integral of order $z > 0$, $\rho > 0$, ζ, ξ arbitrary, with $\zeta, \xi \in (0, T)$, $\gamma, \delta \in \mathbb{R}$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

As a second problem, we study Caputo fractional differential equation (1) supplemented with a combination of Riemman-Liouville and generalized Riemman-Liouville integral boundary conditions:

$$\begin{aligned} x(0) &= \gamma \frac{1}{\Gamma(\alpha)} \int_0^{\zeta} (\zeta - s)^{\alpha-1} x(s) ds := \gamma \ J^{\alpha} x(\zeta), \\ x(T) &= \delta \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_0^{\xi} \frac{s^{\rho-1} x(s)}{(t^{\rho} - s^{\rho})^{1-\beta}} ds := \delta \ {}^{\rho}I^{\beta} x(\xi), \quad 0 < \zeta, \xi < T, \end{aligned} \tag{3}$$

where J^q is the Riemman-Liouville fractional integral of order $q > 0$ while ${}^{\rho}I^{\beta}$ denote generalized Riemman-Liouville fractional integral of order $\beta > 0$, $\rho > 0$.

The subject of fractional differential equations has evolved into an interesting and popular field of research during the last few decades. The surge in developing several aspects of fractional calculus owes to its extensive applications in several branches of engineering and technical sciences such as physics, chemical technology, population dynamics, biotechnology, biosciences, control theory and economics. The nonlocal nature of fractional derivatives, which takes into account memory and hereditary properties of various materials and processes, has played a key role in improving the mathematical modeling based on integer-order derivatives, for instance, see [1, 2, 3, 4].

Fractional-order boundary value problems supplemented with different kinds of boundary conditions have been studied by several researchers. In particular, integral boundary conditions involving classical, Riemann-Liouville or Hadamard or Erdélyi-Kober type integral operators have received significant attention. In [5], Riemann-Liouville and Hadamard fractional integrals are jointly represented by a single integral, which is called generalized Riemann-Liouville fractional integral (see Definition 2.2). For some recent works on the topic we refer the reader to a series of papers [6]-[20] and the references cited therein.

The purpose of the present study is to develop the existence theory for problems (1)-(2) and (1)-(3) by means of standard tools of fixed point theory. In Section 2 we recall some preliminary facts that we need in the sequel. In Section 3 we present our main results, while Section 4 contains examples illustrating the results obtained in Section 3.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [2, 3] and present preliminary results needed in our proofs later.

Definition 2.1 *The Riemann-Liouville fractional integral of order $q > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.2 *[5] The generalized Riemann-Liouville fractional integral of order $q > 0$ and $\rho > 0$ of a function $f(t)$ for all $0 < t < \infty$, is defined as*

$${}^{\rho}I^q f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} f(s)}{(t^{\rho} - s^{\rho})^{1-q}} ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Remark 2.3 *From the above definition it follows that when $\rho = 1$ we arrive at the standard Riemann-Liouville fractional integral, which is used to define both the Riemann-Liouville and Caputo fractional derivatives, while when $\rho \to 0$ we have*

$$\lim_{\rho \to 0} {}^{\rho}I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\log \frac{t}{s}\right)^{q-1} \frac{f(s)}{s} ds,$$

which is the famous Hadamard fractional integral. See [5].

Definition 2.4 *The Riemann-Liouville fractional derivative of order $q > 0$, $n-1 < q < n$, $n \in \mathbb{N}$, is defined as*

$$D_{0+}^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.5 *The Caputo derivative of order q for a function $f : [0,\infty) \to \mathbb{R}$ can be written as*

$${}^cD^q f(t) = D^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n-1 < q < n.$$

Remark 2.6 *If $f(t) \in C^n[0,\infty)$, then*

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds = I^{n-q} f^{(n)}(t), \ t > 0, \ n-1 < q < n.$$

Lemma 2.7 *Let constants $q > 0$ and $p > 0$. Then:*

$${}^\rho I^q t^p = \frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)} \frac{t^{p+\rho q}}{\rho^q}. \tag{4}$$

Proof. By Definition 2.2, we have

$$\begin{aligned} {}^\rho I^q t^p &= \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} s^p}{(t^\rho - s^\rho)^{1-q}}\, ds = \frac{\rho^{1-q}}{\Gamma(q)} \frac{t^{p+\rho q}}{\rho} \int_0^1 \frac{u^{\frac{p}{\rho}}}{(1-u)^{1-q}}\, du \\ &= \frac{\rho^{1-q}}{\Gamma(q)} \frac{t^{p+\rho q}}{\rho} B\left(\frac{p+\rho}{\rho}, q\right) = \frac{t^{p+\rho q}}{\rho^q} \frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)}. \end{aligned}$$

This completes the proof. □

Lemma 2.8 *For any $y \in AC([0,T],\mathbb{R})$, x is a solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), \quad 1 < q \le 2, \\ x(0) = \gamma\ {}^\rho I^\alpha x(\zeta), \quad x(T) = \delta\ {}^\rho I^\beta x(\xi), \quad 0 < \zeta, \xi < T, \end{cases} \tag{5}$$

if and only if

$$x(t) = J^q y(t) + \frac{\gamma}{\Lambda}(v_4 - t v_3)\ {}^\rho I^\alpha J^q y(\zeta) + \frac{1}{\Lambda}(v_2 + t v_1)\Big(\delta\ {}^\rho I^\beta J^q y(\xi) - J^q y(T)\Big), \tag{6}$$

where

$$\begin{aligned} v_1 &= 1 - \gamma \frac{\zeta^{\rho\alpha}}{\rho^\alpha} \frac{1}{\Gamma(\alpha+1)}, \qquad & v_2 = \gamma \frac{\zeta^{\rho\alpha+1}}{\rho^\alpha} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\alpha+\rho}{\rho})}, \\ v_3 &= 1 - \delta \frac{\xi^{\rho\beta}}{\rho^\beta} \frac{1}{\Gamma(\beta+1)}, \qquad & v_4 = T - \delta \frac{\xi^{\rho\beta+1}}{\rho^\beta} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\beta+\rho}{\rho})}, \end{aligned} \tag{7}$$

and

$$\Lambda = v_1 v_4 + v_2 v_3 \neq 0. \tag{8}$$

Proof. For arbitrary constants $c_0, c_1 \in \mathbb{R}$, the general solution of the fractional differential equation in (5) can be written as [2]

$$x(t) = c_0 + c_1 t + J^q y(t). \tag{9}$$

Applying the generalized fractional integral operator on (9) and using Lemma 2.7, we get

$$^{\rho}I^z x(t) = {}^{\rho}I^z J^q y(t) + c_0 \frac{t^{\rho z}}{\rho^z} \frac{1}{\Gamma(z+1)} + c_1 \frac{t^{\rho z+1}}{\rho^z} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho z+\rho}{\rho})}. \tag{10}$$

Using (9) and (10) in boundary conditions of (5), we get the system

$$\begin{aligned} \left(1 - \gamma \frac{\zeta^{\rho\alpha}}{\rho^\alpha} \frac{1}{\Gamma(\alpha+1)}\right) c_0 - \gamma \frac{\zeta^{\rho\alpha+1}}{\rho^\alpha} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\alpha+\rho}{\rho})} c_1 &= \gamma \ {}^{\rho}I^\alpha J^q y(\zeta), \\ \left(1 - \delta \frac{\xi^{\rho\beta}}{\rho^\beta} \frac{1}{\Gamma(\beta+1)}\right) c_0 + \left(T - \delta \frac{\xi^{\rho\beta+1}}{\rho^\beta} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\beta+\rho}{\rho})}\right) c_1 &= \delta \ {}^{\rho}I^\beta J^q y(\xi) - J^q y(T). \end{aligned} \tag{11}$$

Solving (11) together with the notations (7) and (8), we find that

$$c_0 = \frac{1}{\Lambda}\Big\{\gamma v_4 \ {}^{\rho}I^\alpha J^q y(\zeta) + v_2\Big(\delta \ {}^{\rho}I^\beta J^q y(\xi) - J^q y(T)\Big)\Big\},$$

$$c_1 = \frac{1}{\Lambda}\Big\{v_1\Big(\delta \ {}^{\rho}I^\beta J^q y(\xi) - J^q y(T)\Big) - \gamma v_2 \ {}^{\rho}I^\alpha J^q y(\zeta)\Big\}.$$

Substituting the values of c_0 and c_1 in (9) yields the solution (6). Conversely, it can easily be shown by direct computation that the integral equation (6) satisfies the problem (5). This completes the proof.□

Our next lemma deals with the linear variant of (1)-(3). We do not provide the proof of this result as it is similar to the preceding one.

Lemma 2.9 *For any $y \in AC([0,T], \mathbb{R})$, x is a solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), \quad 1 < q \le 2, \\ x(0) = \gamma \ J^\alpha x(\zeta), \quad x(T) = \delta \ {}^{\rho}I^\beta x(\xi), \quad 0 < \zeta, \xi < T, \end{cases} \tag{12}$$

if and only if

$$x(t) = J^q y(t) + \frac{\gamma}{\Lambda_1}(u_4 - t u_3) \ J^{q+\alpha} y(\zeta) + \frac{1}{\Lambda_1}(u_2 + t u_1)\Big(\delta \ {}^{\rho}I^\beta J^q y(\xi) - J^q y(T)\Big), \tag{13}$$

where

$$u_1 = 1 - \gamma \frac{\zeta^\alpha}{\Gamma(\alpha+1)}, \quad u_2 = \gamma \frac{\zeta^{\alpha+1}}{\Gamma(\alpha+2)}, \ u_3 = 1 - \delta \frac{\xi^{\rho\beta}}{\rho^\beta} \frac{1}{\Gamma(\beta+1)}, \ u_4 = T - \delta \frac{\xi^{\rho\beta+1}}{\rho^\beta} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\beta+\rho}{\rho})}, \tag{14}$$

and

$$\Lambda_1 = u_1 u_4 + u_2 u_3 \neq 0. \tag{15}$$

3 Existence results

Let us denote by $\mathcal{C} = C([0,T], \mathbb{R})$ the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$. By $L^1([0,T], \mathbb{R})$ we mean the Banach space of measurable functions $x : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)| dt$.

In view of Lemma 2.8, we introduce operators $\mathcal{Q}, \widehat{\mathcal{Q}} : \mathcal{C} \to \mathcal{C}$ associated with problems (1)-(2) and (1)-(3) respectively by

$$\begin{aligned}(\mathcal{Q}x)(t) &= J^q f(s,x(s))(t) + \frac{\gamma}{\Lambda}(v_4 - tv_2)\ {}^\rho I^\alpha J^q f(s,x(s))(\zeta) \\ &+\frac{1}{\Lambda}(v_2 + tv_1)\Big(\delta\ {}^\rho I^\beta J^q f(s,x(s))(\xi) - J^q f(s,x(s))(T)\Big),\ t \in [0,T],\end{aligned} \tag{16}$$

$$\begin{aligned}(\widehat{\mathcal{Q}}x)(t) &= J^q f(s,x(s))(t) + \frac{\gamma}{\Lambda_1}(u_4 - tu_3)\ J^{q+\alpha} f(s,x(s))(\zeta) \\ &+\tfrac{1}{\Lambda_1}(u_2 + tu_1)\Big(\delta\ {}^\rho I^\beta J^q(s,x(s))(\xi) - J^q f(s,x(s))(T)\Big),\quad t \in [0,T].\end{aligned} \tag{17}$$

In the sequel, we use the following expression:

$${}^\rho I^h f(s,x(s))(y) = \frac{\rho^{1-h}}{\Gamma(h)}\int_0^y \frac{s^{\rho-1} f(s,x(s))}{(y^\rho - s^\rho)^{1-h}}ds, \quad h \in \{\alpha, \beta\}.$$

Further, we set the constants

$$\begin{aligned}\Omega : &= \frac{T^q}{\Gamma(q+1)} + \frac{|\gamma|(|v_4| + T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\alpha+\rho}{\rho}\Big)} \\ &+\frac{(|v_2| + T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\beta+\rho}{\rho}\Big)} + \frac{T^q}{\Gamma(q+1)}\right).\end{aligned} \tag{18}$$

$$\begin{aligned}\Omega_1 : &= \frac{T^q}{\Gamma(q+1)} + \frac{|\gamma|(|u_4| + T|u_2|)\zeta^{\alpha+q}}{|\Lambda_1|\Gamma(\alpha+q+1)} \\ &+\frac{(|u_2| + T|u_1|)}{|\Lambda_1|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\beta+\rho}{\rho}\Big)} + \frac{T^q}{\Gamma(q+1)}\right).\end{aligned} \tag{19}$$

In the following subsections, we establish several existence and uniqueness results for problems (1)-(2) and (1)-(3) by applying a variety of fixed point theorems. We present in details the proofs for problem (1)-(2), while the proofs for problem (1)-(3) are omitted as they are similar to the ones obtained for problem (1)-(2).

3.1 Existence and uniqueness result via Banach's fixed point theorem

Theorem 3.1 *Assume that:*

(H_1) *there exists a positive constant L such that $|f(t,x) - f(t,y)| \le L|x-y|$, for each $t \in [0,T]$ and $x, y \in \mathbb{R}$.*

If

$$L\Omega < 1, \tag{20}$$

where Ω is defined by (18), then the boundary value problem (1)-(2) has a unique solution on $[0,T]$.

Proof. Observe that a fixed point problem equivalent to problem (1)-(2) is $x = \mathcal{Q}x$, where the operator $\mathcal{Q}$ is defined by (16), and that the existence of a fixed point of the operator $\mathcal{Q}$ implies the existence of a solution for problem (1)-(2). Applying the Banach contraction mapping principle, we shall show that $\mathcal{Q}$ has a unique fixed point. For that we let $\sup_{t\in[0,T]}|f(t,0)| = M < \infty$ and choose $r \ge \frac{M\Omega}{1-L\Omega}$. To show that $\mathcal{Q}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$, we have for any $x \in B_r$ that

$$|(\mathcal{Q}x)(t)| \quad \le \quad \sup_{t\in[0,T]}\left\{J^q|f(s,x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q|f(s,x(s))|(\zeta)\right.$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(\delta\ {}^\rho I^\beta J^q|f(s,x(s))|(\xi)+J^q|f(s,x(s))|(T)\Big)\\
\leq\ & J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(T)\\
&+\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(\zeta)\\
&+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(\xi)\\
&+J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(T)\Big)\\
\leq\ & (L\|x\|+M)J^q(1)(T)+(L\|x\|+M)\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q(1)(\zeta)\\
&+(L\|x\|+M)\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(1)(\xi)+J^q(1)(T)\Big)\\
\leq\ & (Lr+M)\Bigg\{\frac{T^q}{\Gamma(q+1)}+\frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\alpha+\rho}{\rho}\Big)}\\
&+\frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\beta+\rho}{\rho}\Big)}+\frac{T^q}{\Gamma(q+1)}\right)\Bigg\}\\
\leq\ & (Lr+M)\Omega\leq r,
\end{aligned}
$$

which implies that $\mathcal{Q}B_r\subset B_r$.

Next, we let $x,y\in\mathcal{C}$. Then for $t\in[0,T]$, we have

$$
\begin{aligned}
|\mathcal{Q}x(t)-\mathcal{Q}y(t)|\ \leq\ & \sup_{t\in[0,T]}\Bigg\{J^q|f(s,x(s))-f(s,y(s))|(t)\\
&+\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q|f(s,x(s))-f(s,y(s))|(\zeta)\\
&+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(\delta\ {}^\rho I^\beta J^q|f(s,x(s))-f(s,y(s))|(\xi)\\
&+J^q|f(s,x(s))-f(s,y(s))|(T)\Big)\\
\leq\ & L\|x-y\|J^q(1)(T)+L\|x-y\|\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q(1)(\zeta)\\
&+L\|x-y\|\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(1)(\xi)+J^q(1)(T)\Big)\\
=\ & L\Omega\|x-y\|,
\end{aligned}
$$

which leads to $\|\mathcal{Q}x-\mathcal{Q}y\|\leq L\Omega\|x-y\|$. As $L\Omega<1$, $\mathcal{Q}$ is a contraction. Therefore, it follows by the Banach's contraction mapping principle that $\mathcal{Q}$ has a fixed point which in fact is the unique solution of problem (1)-(2). The proof is completed. □

Theorem 3.2 *Assume that (H_1) holds. If*

$$L\Omega_1<1, \tag{21}$$

where Ω_1 is defined by (19), then the boundary value problem (1)-(3) has a unique solution on $[0,T]$.

3.2 Existence result via Krasnoselskii's fixed point theorem

Lemma 3.3 *(Krasnoselskii's fixed point theorem) [21]. Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A,B be the operators such that (a) $Ax+Bx\in M$ whenever*

$x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 3.4 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_1). In addition we assume that*

(H_2) $|f(t,x)| \le \varphi(t)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$, and $\varphi \in C([0,T], \mathbb{R}^+)$.

Then the problem (1)-(2) has at least one solution on $[0,T]$ provided

$$L\left\{ \frac{|\gamma|(|v_4| + T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha \Gamma(q+1)} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)} + \frac{(|v_2| + T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta \Gamma(q+1)} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)}\right)\right\} < 1. \tag{22}$$

Proof. Define the operators $\mathcal{Q}_1, \mathcal{Q}_2 : \mathcal{C} \to \mathcal{C}$ as follows

$$\begin{aligned} \mathcal{Q}_1 x(t) &= J^q f(s, x(s))(t), \quad t \in [0,T], \\ \mathcal{Q}_2 x(t) &= \frac{\gamma}{\Lambda}(v_4 - tv_2)\, {}^\rho I^\alpha J^q f(s,x(s))(\zeta) \\ &\quad + \frac{1}{\Lambda}(v_2 + tv_1)\Big(\delta\, {}^\rho I^\beta J^q f(s,x(s))(\xi) - J^q f(s,x(s))(T)\Big), \quad t \in [0,T]. \end{aligned}$$

Setting $\sup_{t\in[0,T]} \varphi(t) = \|\varphi\|$ and choosing $\rho \ge \|\varphi\|\Omega$, where Ω is defined by (18), we consider $B_\rho = \{x \in \mathcal{C} : \|x\| \le \rho\}$. For any $x, y \in B_\rho$, we have

$$\begin{aligned} |\mathcal{Q}_1 x(t) + \mathcal{Q}_2 y(t)| &\le \sup_{t\in[0,T]} \Bigg\{ J^q|f(s,x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\, {}^\rho I^\alpha J^q |f(s,x(s))|(\zeta) \\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\, {}^\rho I^\beta J^q|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big)\Bigg\} \\ &\le \|\varphi\|\Bigg\{ \frac{T^q}{\Gamma(q+1)} + \frac{|\gamma|(|v_4| + T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha\Gamma(q+1)} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)} \\ &\quad + \frac{(|v_2| + T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)} \frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)}\right)\Bigg\} \\ &= \|\varphi\|\Omega \le \rho. \end{aligned}$$

This shows that $\mathcal{Q}_1 x + \mathcal{Q}_2 y \in B_\rho$. Using (22), it ca easily be established that $\mathcal{Q}_2$ is a contraction.

Continuity of f implies that the operator $\mathcal{Q}_1$ is continuous. Also, $\mathcal{Q}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{Q}_1 x\| \le \frac{T^q}{\Gamma(q+1)}\|\varphi\|.$$

Now we prove the compactness of the operator $\mathcal{Q}_1$.

We define $\sup_{(t,x)\in[0,T]\times B_\rho} |f(t,x)| = \bar{f} < \infty$, and consequently, for $t_1, t_2 \in [0,T]$, $t_1 < t_2$, we have

$$\begin{aligned} |\mathcal{Q}_1 x(t_2) - \mathcal{Q}_1 x(t_1)| &= \Big| J^q f(s,x(s))(t_2) - J^q f(s,x(s))(t_1) \Big| \\ &\le \frac{\bar{f}}{\Gamma(q)} \left| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]ds + \int_{t_1}^{t_2} (\tau_2 - s)^{q-1} ds \right| \end{aligned}$$

$$\leq \frac{\bar{f}}{\Gamma(q+1)}[|t_2^q - t_1^q| + |t_2 - t_1|^q],$$

which tends to zero as $t_2 - t_1 \to 0$ is independent of x. Thus, $\mathcal{Q}_1$ is equicontinuous. So $\mathcal{Q}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{Q}_1$ is compact on B_ρ. Thus all the assumptions of Lemma 3.3 are satisfied. So the conclusion of Lemma 3.3 implies that problem (1)-(2) has at least one solution on $[0, T]$ □

Theorem 3.5 *Assume that* (H_1) *and* (H_2) *hold. Then the problem (1)-(3) has at least one solution on* $[0, T]$ *provided*

$$L\left\{\frac{|\gamma|(|u_4| + T|u_2|)\zeta^{\alpha+q}}{|\Lambda_1|\Gamma(\alpha+q+1)} + \frac{(|v_2| + T|v_1|)}{|\Lambda_1|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)}\right)\right\} < 1. \tag{23}$$

3.3 Existence and uniqueness result via nonlinear contractions

Definition 3.6 *Let E be a Banach space and let $\mathcal{F} : E \to E$ be a mapping. $\mathcal{F}$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property:*

$$\|\mathcal{F}x - \mathcal{F}y\| \leq \Theta(\|x - y\|), \qquad \forall x, y \in E.$$

Lemma 3.7 *(Boyd and Wong)[22]. Let E be a Banach space and let $\mathcal{F} : E \to E$ be a nonlinear contraction. Then $\mathcal{F}$ has a unique fixed point in E.*

Theorem 3.8 *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(H_3) $|f(t, x) - f(t, y)| \leq z(t)\dfrac{|x - y|}{A^* + |x - y|}$, *for $t \in [0, T]$, $x, y \geq 0$, where $z : [0, T] \to \mathbb{R}^+$ is continuous and A^* is the constant given by*

$$A^* := J^q z(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q z(\zeta) + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big\{|\delta|\ {}^\rho I^\beta J^q z(\xi) + J^q z(T)\Big\}.$$

Then the problem (1)-(2) has a unique solution on $[0, T]$.

Proof. Consider the operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ defined by (16) and a continuous nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Theta(\varepsilon) = \frac{A^*\varepsilon}{A^* + \varepsilon}, \qquad \forall \varepsilon \geq 0.$$

Note that the function Θ satisfies $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$.

For any $x, y \in \mathcal{C}$ and for each $t \in [0, T]$, we have

$$\begin{aligned}
&|\mathcal{Q}x(t) - \mathcal{Q}y(t)| \\
\leq\ & \sup_{t\in[0,T]}\Bigg\{J^q|f(s, x(s)) - f(s, y(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q|f(s, x(s)) - f(s, y(s))|(\zeta) \\
& + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q|f(s, x(s)) - f(s, y(s))|(\xi) + J^q|f(s, x(s)) - f(s, y(s))|(T)\Big)\Bigg\} \\
\leq\ & J^q\left(z(s)\frac{|x - y|}{A^* + |x - y|}\right)(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q\left(z(s)\frac{|x - y|}{A^* + |x - y|}\right)(\zeta)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\left\{|\delta|\ {}^{\rho}I^{\beta}J^{q}\left(z(s)\frac{|x-y|}{A^*+|x-y|}\right)(\xi)+J^{q}\left(z(s)\frac{|x-y|}{A^*+|x-y|}\right)(T)\right\}\\
\leq\ & \frac{\Theta(\|x-y\|)}{A^*}\Big[J^q z(T)+\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^{\rho}I^{\alpha}J^{q}z(\zeta)+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big\{|\delta|\ {}^{\rho}I^{\beta}J^{q}z(\xi)+J^{q}z(T)\Big\}\Big]\\
=\ & \Theta(\|x-y\|).
\end{aligned}
$$

This implies that $\|\mathcal{Q}x-\mathcal{Q}y\|\leq\Theta(\|x-y\|)$. Therefore $\mathcal{Q}$ is a nonlinear contraction. Hence, by Lemma 3.7 the operator $\mathcal{Q}$ has a unique fixed point which is the unique solution of the problem (1)-(2). This completes the proof. □

Theorem 3.9 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the assumption:*

$(H_3)'$ $|f(t,x)-f(t,y)|\leq z(t)\dfrac{|x-y|}{A_1^*+|x-y|}$, *for $t\in[0,T]$, $x,y\geq 0$, where $z:[0,T]\to\mathbb{R}^+$ is continuous and A_1^* is the constant given by*

$$
A_1^* := J^q z(T)+\frac{|\gamma|}{|\Lambda|}(|u_4|+T|u_2|)\ J^{\alpha+q}z(\zeta)+\frac{1}{|\Lambda|}(|u_2|+T|u_1|)\Big\{|\delta|\ {}^{\rho}I^{\beta}J^{q}z(\xi)+J^{q}z(T)\Big\}.
$$

Then the problem (1)-(3) has a unique solution on $[0,T]$.

3.4 Existence result via Schaefer fixed point theorem

Lemma 3.10 *[23] Let X be a Banach space. Assume that $T:X\to X$ is a completely continuous operator and the set $V=\{u\in X\mid u=\mu Tu, 0<\mu<1\}$ is bounded. Then T has a fixed point in X.*

Theorem 3.11 *Assume that there exists a positive constant L_1 such that $|f(t,x)|\leq L_1$ for $t\in[0,1]$, $x\in\mathbb{R}$. Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.*

Proof. As a first step, it will be shown that the operator $\mathcal{Q}$ defined by (16) is completely continuous. Observe that continuity of $\mathcal{Q}$ follows from the continuity of f. For a positive constant r, let $B_r=\{x\in\mathcal{C}:\|x\|\leq r\}$ be a bounded ball in $\mathcal{C}$. Then for $t\in[0,T]$ we have

$$
\begin{aligned}
|\mathcal{Q}x(t)| \leq\ & J^q|f(s,x(s))|(t)+\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^{\rho}I^{\alpha}J^{q}|f(s,x(s))|(\zeta)\\
&+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^{\rho}I^{\beta}J^{q}|f(s,x(s))|(\xi)+J^{q}|f(s,x(s))|(T)\Big)\\
\leq\ & L_1J^q(1)(T)+L_1\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^{\rho}I^{\alpha}J^{q}(1)(\zeta)\\
&+L_1\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^{\rho}I^{\beta}J^{q}(1)(\xi)+J^{q}(1)(T)\Big),\\
\leq\ & L_1\left\{\frac{T^q}{\Gamma(q+1)}+\frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)}\right.\\
&\left.+\frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^{\beta}\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)}-\frac{T^q}{\Gamma(q+1)}\right)\right\}\\
=\ & L_1\Omega.
\end{aligned}
$$

Now, for $\tau_1,\tau_2\in[0,1]$ with $\tau_1<\tau_2$, we get

$$
|\mathcal{Q}x(\tau_2)-\mathcal{Q}x(\tau_1)| \leq |J^qf(s,x(s))(\tau_2)-J^qf(s,x(s))(\tau_1)|+\frac{|\gamma||v_2||\tau_2-\tau_1|}{|\Lambda|}\ {}^{\rho}I^{\alpha}J^{q}|f(s,x(s))|(\zeta)
$$

$$
\begin{aligned}
& +\frac{|v_1||\tau_2-\tau_1|}{|\Lambda|}\Big(|\delta|\ {}^\rho I^\beta J^q|f(s,x(s))|(\xi)+J^q|f(s,x(s))|(T)\Big)\\
\le\ & \frac{L_1}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}]ds+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}ds\right|\\
& +\frac{L_1|\gamma||v_2||\tau_2-\tau_1|}{|\Lambda|}\ {}^\rho I^\alpha J^q(\zeta)+\frac{L_1|v_1||\tau_2-\tau_1|}{|\Lambda|}\Big(|\delta|\ {}^\rho I^\beta J^q(\xi)+J^q(T)\Big).
\end{aligned}
$$

As $\tau_2-\tau_1\to 0$, the right-hand side of the above inequality tends to zero independently of $x\in B_r$. Therefore by the Arzelá-Ascoli theorem the operator $\mathcal{Q}:\mathcal{C}\to\mathcal{C}$ is completely continuous.

Next, we consider the set $V=\{x\in\mathcal{C}: x=\mu\mathcal{Q}x,\ 0<\mu<1\}$. In order to show that V is bounded, let $x\in V$ and $t\in[0,T]$. Then

$$
\begin{aligned}
\|x\| \ \le\ & L_1\Bigg\{\frac{T^q}{\Gamma(q+1)}+\frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\alpha+\rho}{\rho}\Big)}\\
& +\frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\beta+\rho}{\rho}\Big)}-\frac{T^q}{\Gamma(q+1)}\right)\Bigg\}\\
=\ & L_1\Omega.
\end{aligned}
$$

Therefore, V is bounded. Hence, by Lemma 3.10, the boundary value problem (1)-(2) has at least one solution. □

Theorem 3.12 *Assume that there exists a positive constant L_1 such that $|f(t,x)|\le L_1$ for $t\in[0,1]$, $x\in\mathbb{R}$. Then the boundary value problem (1)-(3) has at least one solution on $[0,T]$.*

3.5 Existence result via Leray-Schauder's Degree Theory

Theorem 3.13 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Suppose that*

(H_4) *there exist constants $0\le\nu<\Omega^{-1}$, and $M>0$ such that*

$$|f(t,x)|\le\nu|x|+M \quad \textit{for all}\quad (t,x)\in[0,T]\times\mathbb{R},$$

where Ω is defined by (18).

Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.

Proof. In view of the fixed point problem

$$x=\mathcal{Q}x, \tag{24}$$

where the operator $\mathcal{Q}:\mathcal{C}\to\mathcal{C}$ is given by (16), we have to establish that there exists at least one solution $x\in C[0,T]$ satisfying (24). Set a ball $B_R\subset C[0,T]$ with a constant radius $R>0$ as

$$B_R=\{x\in\mathcal{C}:\max_{t\in[0,T]}|x(t)|<R\}.$$

Then we have to show that the operator $\mathcal{Q}:\overline{B}_R\to C[0,T]$ satisfies the condition

$$x\ne\theta\mathcal{Q}x,\quad \forall x\in\partial B_R,\quad \forall\theta\in[0,1]. \tag{25}$$

Next, we introduce

$$H(\theta,x)=\theta\mathcal{Q}x,\quad x\in\mathcal{C},\quad \theta\in[0,1].$$

As shown in Theorem 3.16 we have that the operator $\mathcal{Q}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map h_θ defined by $h_\theta(x)=x-H(\theta,x)=$

$x - \theta\mathcal{Q}x$ is completely continuous. If (25) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\theta, B_R, 0) &= \deg(I - \theta\mathcal{Q}, B_R, 0) = \deg(h_1, B_R, 0)\\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_R,\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, we have $h_1(x) = x - \mathcal{Q}x = 0$ for at least one $x \in B_R$. Let us assume that $x = \theta\mathcal{Q}x$ for some $\theta \in [0,1]$ and for all $t \in [0,T]$. Then

$$\begin{aligned}|x(t)| &= |\theta\mathcal{Q}x(t)|\\ &\leq J^q|f(s,x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q|f(s,x(s))|(\zeta)\\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big)\\ &\leq (\nu|x| + M)J^q p(s)(T) + (\nu|x| + M)\frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q(1)(\zeta)\\ &\quad + (\nu|x| + M)\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(1)(\xi) + J^q(1)(T)\Big)\\ &= (\nu|x| + M)\Omega,\end{aligned}$$

which, on taking the norm $\sup_{t\in[0,T]}|x(t)| = \|x\|$ and solving for $\|x\|$, yields

$$\|x\| \leq \frac{M\Omega}{1 - \nu\Omega}.$$

If $R = \dfrac{M\Omega}{1-\nu\Omega} + 1$, (25) holds. This completes the proof. $\square$

Theorem 3.14 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that*

$(H_4)'$ *there exist constants $0 \leq \nu < \Omega_1^{-1}$, and $M > 0$ such that*

$$|f(t,x)| \leq \nu|x| + M \quad \textit{for all} \quad (t,x) \in [0,T] \times \mathbb{R},$$

where Ω_1 is defined by (19).

Then the boundary value problem (1)-(3) has at least one solution on $[0,T]$.

3.6 Existence result via Leray-Schauder's nonlinear alternative

Lemma 3.15 *(Nonlinear alternative for single valued maps [24]). Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $\mathcal{A} : \bar{U} \to C$ is a continuous, compact (that is, $\mathcal{A}(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) $\mathcal{A}$ has a fixed point in $\bar{U}$, or

(ii) there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $x = \lambda\mathcal{A}(x)$.

Theorem 3.16 *Assume that*

(H_5) *there exists a continuous nondecreasing function $\Phi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,T], \mathbb{R}^+)$ such that*

$$|f(t,x)| \leq p(t)\Phi(\|x\|) \quad \textit{for each} \quad (t,x) \in [0,T] \times \mathbb{R};$$

(H_6) *there exists a constant* $N > 0$ *such that*

$$\frac{N}{\Phi(N)\Big\{J^q p(s)(T) + A_1 + A_2\Big\}} > 1,$$

where

$$\begin{aligned} A_1 &= \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta), \\ A_2 &= \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big). \end{aligned}$$

Then the boundary value problem (1)-(2) has at least one solution on $[0, T]$.

Proof. Let the operator $\mathcal{Q}$ be defined by (16). We first show that $\mathcal{Q}$ *maps bounded sets (balls) into bounded sets in* $C([0, T], \mathbb{R})$. For a positive constant r, let $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ be a bounded ball in $\mathcal{C}$. Then for $t \in [0, T]$ we have

$$\begin{aligned} |\mathcal{Q}x(t)| &\le J^q|f(s, x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q|f(s, x(s))|(\zeta) \\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q|f(s, x(s))|(\xi) + J^q|f(s, x(s))|(T)\Big) \\ &\le \Phi(\|x\|)J^q p(s)(T) + \Phi(\|x\|)\frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta) \\ &\quad + \Phi(\|x\|)\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big), \end{aligned}$$

and consequently,

$$\begin{aligned} \|\mathcal{Q}x\| &\le \Phi(r)\Big\{J^q p(s)(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta) \\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big)\Big\}. \end{aligned}$$

Next we will show that *the operator* $\mathcal{Q}$ *maps bounded sets into equicontinuous sets of* $\mathcal{C}$. Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$\begin{aligned} |\mathcal{Q}x(\tau_2) - \mathcal{Q}x(\tau_1)| &\le |J^q f(s, x(s))(\tau_2) - J^q f(s, x(s))(\tau_1)| + \frac{|\alpha||v_2||\tau_2 - \tau_1|}{|\Lambda|}\ {}^\rho I^\alpha J^q|f(s, x(s))|(\zeta) \\ &\quad + \frac{|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\delta|\ {}^\rho I^\beta J^q|f(s, x(s))|(\xi) + J^q|f(s, x(s))|(T)\Big) \\ &\le \frac{\Phi(r)}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]p(s)ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}p(s)ds\right| \\ &\quad + \frac{\Phi(r)|\gamma||v_2||\tau_2 - \tau_1|}{|\Lambda|}\ {}^\rho I^\alpha J^q p(s)(T) \\ &\quad + \frac{\Phi(r)|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big). \end{aligned}$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore by the Arzelá-Ascoli theorem the operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Finally, we show that there exists an open set $U \subset \mathcal{C}$ with $x \neq \theta\mathcal{P}x$ for $\theta \in (0, 1)$ and $x \in \partial U$.

Let x be a solution. Then, for $t \in [0, T]$, and following the similar computations as in the first step, we have

$$|x(t)| \le \Phi(\|x\|)\Big\{J^q p(s)(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta)$$

$$+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^{\rho}I^{\beta}J^{q}p(s)(\xi)+J^{q}p(s)(T)\Big)\Big\}$$

which leads to

$$\frac{\|x\|}{\Phi(\|x\|)\Big\{J^{q}p(s)(T)+A_1+A_2\Big\}}\leq 1.$$

In view of (H_6), there exists N such that $\|x\|\neq N$. Let us set

$$\mathcal{U}=\{x\in C([0,T],\mathbb{R})\ :\ \|x\|<N\}.$$

We see that the operator $\mathcal{Q}:\overline{\mathcal{U}}\to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathcal{U}$, there is no $x\in\partial\mathcal{U}$ such that $x=\theta\mathcal{Q}x$ for some $\theta\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{Q}$ has a fixed point $x\in\overline{\mathcal{U}}$ which is a solution of the boundary value problem (1)-(2). This completes the proof. □

Theorem 3.17 *Assume that (H_5) holds. In addition we suppose that:*

$(H_6)'$ *there exists a constant $N'>0$ such that*

$$\frac{N'}{\Phi_1(N')\Big\{J^{q}p(s)(T)+A_1'+A_2'\Big\}}>1, \tag{26}$$

where

$$\begin{aligned} A_1' &= \frac{|\gamma|}{|\Lambda_1|}(|u_4|+T|u_2|)J^{\alpha+q}p(s)(\zeta),\\ A_2' &= \frac{1}{|\Lambda_1|}(|u_2|+T|u_1|)\Big(|\delta|\ {}^{\rho}I^{\beta}J^{q}p(s)(\xi)+J^{q}p(s)(T)\Big). \end{aligned}$$

Then the boundary value problem (1)-(3) has at least one solution on $[0,T]$.

4 Examples

In this section, we present some examples to illustrate our results.

Example 4.1 *Consider the following nonlocal boundary value problem involving generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{3}{2}}x(t)=\dfrac{3}{25}\left(\dfrac{4x^2(t)+5|x(t)|}{3+4|x(t)|}\right)e^{-2t}+\dfrac{1}{2}\cos^2 t+1, & t\in\left[0,\dfrac{5}{3}\right],\\ x(0)=\dfrac{1}{2}\,{}^{\frac{\sqrt{3}}{2}}I^{\frac{4}{\sqrt{3}}}x\left(\dfrac{2}{3}\right),\quad x\left(\dfrac{5}{3}\right)=\dfrac{3}{4}\,{}^{\frac{\sqrt{3}}{2}}I^{\frac{\pi}{2}}x\left(\dfrac{4}{3}\right), \end{cases} \tag{27}$$

where $q=3/2$, $T=5/3$, $\gamma=1/2$, $\rho=\sqrt{3}/2$, $\alpha=4/\sqrt{3}$, $\zeta=2/3$, $\delta=3/4$, $\beta=\pi/2$, $\xi=4/3$ and $f(t,x)=(3/25)((4x^2+5|x|)/(3+4|x|))e^{-2t}+(1/2)\cos^2 t+1$. Using given information, we find that $v_1=0.8856776719$, $v_2=0.02007036728$, $v_3=0.0060494642$, $v_4=1.202612652$, $\Lambda=1.065248589$ and $\Omega=4.304419870$. Also $|f(t,x)-f(t,y)|\leq(1/5)|x-y|$. Thus the condition (H_1) is satisfied with $L=1/5$ and $L\Omega=0.8608839740<1$. Therefore, by Theorem 3.1, problem (27) has a unique solution on $[0,5/3]$.

Example 4.2 *Consider the following nonlocal boundary value problem*

$$\begin{cases} D^{\frac{5}{3}}x(t)=\dfrac{5}{48}(1+\sin^2 t)\dfrac{|x(t)|}{1+|x(t)|}+3t^2+\dfrac{2}{3}, & t\in\left[0,\dfrac{7}{4}\right],\\ x(0)=\dfrac{3}{2}\,{}^{\frac{5}{6}}I^{\frac{e}{\sqrt{2}}}x\left(\dfrac{5}{4}\right),\quad x\left(\dfrac{7}{4}\right)=\dfrac{4}{5}\,{}^{\frac{5}{6}}I^{\frac{11}{13}}x\left(\dfrac{3}{4}\right). \end{cases} \tag{28}$$

Here $q = 5/3$, $T = 7/4$, $\gamma = 3/2$, $\rho = 5/6$, $\alpha = e/\sqrt{2}$, $\zeta = 5/4$, $\delta = 4/5$, $\beta = 11/13$, $\xi = 3/4$ and $f(t, x) = (5(1 + \sin^2 t)/48)(|x|/(1 + |x|)) + 3t^2 + (2/3)$. Using the given data, we obtain $v_1 = -0.633695322$, $v_2 = 0.5982054854$, $v_3 = 0.1931118977$, $v_4 = 1.448388097$, $\Lambda = -0.8023161650 \neq 0$. As $|f(t, x) - f(t, y)| \leq (5/24)|x - y|$, we have that (H_1) is satisfied with $L = 5/24$. Further, we have $\Omega_2 = 0.9828570350 < 1$. Also

$$|f(t, x)| \leq \frac{5}{48}(1 + \sin^2 t) + 3t^2 + \frac{2}{3} := \varphi(t),$$

which implies that the condition (H_2) holds true. In consequence, the conclusion of Theorem 3.4 applies and problem (28) has at least one solution on $[0, 7/4]$.

Example 4.3 *Consider the following nonlocal boundary value problem*

$$\begin{cases} D^{\frac{4}{3}}x(t) = \dfrac{1}{4}(t^{\frac{1}{3}} + 1)\left(\dfrac{|x(t)|}{1 + |x(t)|}\right) + \dfrac{3}{2}t + \dfrac{1}{3}, & t \in \left[0, \dfrac{1}{2}\right], \\ x(0) = \dfrac{2}{\sqrt{\pi}}\, {}^{\frac{1}{\sqrt{3}}}I^{\frac{7}{4}}x\left(\dfrac{1}{4}\right), \quad x\left(\dfrac{1}{2}\right) = \dfrac{3}{e^2}\, {}^{\frac{1}{\sqrt{3}}}I^{\frac{8}{13}}x\left(\dfrac{1}{8}\right). \end{cases} \tag{29}$$

Here $q = 4/3$, $T = 1/2$, $\gamma = 2/\sqrt{\pi}$, $\rho = 1/\sqrt{3}$, $\alpha = 7/4$, $\zeta = 1/4$, $\delta = 3/e^2$, $\beta = 8/13$, $\xi = 1/8$ and $f(t, x) = ((t^{1/3} + 1)/4)(|x|/(1 + |x|)) + (3/2)t + (1/3)$. Using the previous information, we have $v_1 = 0.5478797820$, $v_2 = 0.02539640314$, $v_3 = 0.6962686485$, $v_4 = 0.4808910650$ and $\Lambda = 0.2811532112$. Choosing $z(t) = (t^{1/3} + 1)/4$, find that $A^* = 0.2768779852$ and also

$$|f(t, x) - f(t, y)| \leq \frac{1}{4}(t^{\frac{1}{3}} + 1)\frac{|x - y|}{0.2768779852 + |x - y|}.$$

Therefore, all assumptions of Theorem 3.8 are satisfied. Hence the problem (29) has at least one solution on $[0, 1/2]$.

Example 4.4 *Consider the following nonlocal boundary value problem with both Riemann-Liouville and generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{5}{4}}x(t) = \tan^{-1}\left(\dfrac{x^4(t) + 3x^2(t)}{1 + |x(t)|}\right)\left(e^{\frac{3}{2} - t} + 1\right) + 3\pi, & t \in \left[0, \dfrac{3}{2}\right], \\ x(0) = \dfrac{4}{\sqrt{7}} J^{\frac{5}{\sqrt{3}}}x\left(\dfrac{1}{2}\right), \quad x\left(\dfrac{3}{2}\right) = \dfrac{\pi}{2}\, {}^{\frac{2}{7}}I^{\frac{3}{8}}x\left(\dfrac{5}{4}\right). \end{cases} \tag{30}$$

Here $q = 5/4$, $T = 3/2$, $\gamma = 4/\sqrt{7}$, $\alpha = 5/\sqrt{3}$, $\zeta = 1/2$, $\delta = \pi/2$, $\rho = 2/7$, $\beta = 3/8$, $\xi = 5/4$ and $f(t, x) = \tan^{-1}((x^4 + 3x^2)/(1 + |x|))(e^{(3/2)-t} + 1) + 3\pi$. From the given constants, we have $u_1 = 0.9607949552$, $u_2 = 0.005043420754$, $u_3 = -1.895136694$, $u_4 = -0.378780447$ and $\Lambda_1 = -0.3734883143 \neq 0$. As $f(t, x) \leq 4\pi := L_1$ for all $x \in \mathbb{R}$, therefore from Theorem 3.11, the problem 30 has at least one solution on $[0, 3/2]$.

Example 4.5 *Consider the following nonlocal boundary value problem subjected to both Riemann-Liouville and generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{8}{5}}x(t) = \dfrac{1}{(t^{\frac{1}{2}} + 10)^2}\left(\dfrac{10x^2(t) + 1}{3 + |x(t)|}\right) + e^{-|x(t)|} + \dfrac{1}{3}, & t \in [0, \pi], \\ x(0) = \dfrac{\log 2}{\sqrt{3}} J^{\frac{3}{4}}x\left(\dfrac{\pi}{2}\right), \quad x(\pi) = \dfrac{\log 3}{\sqrt{8}}\, {}^{\frac{5}{\sqrt{7}}}I^{\frac{3}{\sqrt{e}}}x\left(\dfrac{\pi}{3}\right). \end{cases} \tag{31}$$

Here $q = 8/5$, $T = \pi$, $\gamma = \log 2/\sqrt{3}$, $\alpha = 3/4$, $\zeta = \pi/2$, $\delta = \log 3/\sqrt{8}$, $\rho = 5/\sqrt{7}$, $\beta = 3/\sqrt{e}$, $\xi = \pi/3$ and $f(t, x) = (1/(t^{1/2} + 10)^2)((10x^2 + 1)/(3 + |x|)) + e^{-|x|} + (1/3)$. By direct computation of given constants, we obtain $u_1 = 0.9607949552$, $u_2 = 0.2381638392$, $u_3 = 0.9635754531$, $u_4 = 3.121155944$ and $\Lambda_1 = 3.228279714 \neq 0$. In addition, we can find that $\Omega_1 = 8.997039531$. It is easy to see that

$$|f(t, x)| \leq \frac{1}{10}|x| + \frac{4}{3},$$

which leads to $\nu := 1/10 < \Omega_1^{-1} = 0.1111476721$ and $M := 4/3 > 0$. Applying the conclusion of Theorem 3.13, we get that the problem (31) has at least one solution on $[0, \pi]$.

Example 4.6 *Consider the following nonlocal boundary value problem supplemented with both Riemann-Liouville and generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{7}{4}}x(t) = \dfrac{(\sqrt{t}+1)}{12}\left(\dfrac{x^2(t)\sin^2 x(t)}{3(1+|x(t)|)} + e^{-t}\cos^2 t\right), & t \in \left[0, \dfrac{12}{5}\right], \\ x(0) = \dfrac{1}{\sqrt{3}} J^{\frac{7}{9}} x\left(\dfrac{8}{5}\right), \quad x\left(\dfrac{12}{5}\right) = \dfrac{3}{16}\, {}^{\frac{1}{\sqrt{\pi}}} I^{\frac{1}{\sqrt{e}}} x\left(\dfrac{11}{5}\right), \end{cases} \tag{32}$$

where $q = 7/4$, $T = 12/5$, $\gamma = 1/\sqrt{3}$, $\alpha = 7/9$, $\zeta = 8/5$, $\delta = 3/16$, $\rho = 1/\sqrt{\pi}$, $\beta = 1/\sqrt{e}$, $\xi = 11/15$ and $f(t, x) = ((\sqrt{t}+1)/12)((x^2 \sin^2 x)/(3(1+|x|)) + e^{-t}\cos^2 t)$. By the given values, we get $u_1 = 0.1010372543$, $u_2 = 0.8090664711$, $u_3 = 0.6114216572$, $u_4 = 1.970342759$, $\Lambda_1 = 0.6937587849 \neq 0$. Since

$$|f(t, x)| \leq \frac{(\sqrt{t}+1)}{12}\left(\frac{1}{3}|x| + 1\right) := p(t)\Phi_1(|x|),$$

the condition (H_4) is satisfied. Also $A_1' = 0.4202876316$, $A_2' = 0.7604168186$. Clearly condition (26) is satisfied for $N' > 3.560603169$. Therefore, by Theorem 3.17, problem (32) has at least one solution on $[0, 12/5]$.

Acknowledgement:

This research is partially supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

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On entire function sharing a small function CM with its high order forward difference operator

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August 31, 2016

Abstract: In this paper, we investigate the uniqueness of an entire function of finite order sharing a small entire function with its high order forward difference operator. The results obtained extend some known theorems and also show the exact solutions of some certain difference equations.

Key words and phrases: uniqueness; entire function; difference equation; differential equation; small function.

2000 Mathematics Subject Classification: 30D35; 34M10.

1 Introduction and main results

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane C. We assume that the reader is familiar with the standard notations in the Nevanlinna theory. We use the standard notations such as $T(r, f), m(r, f), N(r, f), \overline{N}(r, f)$ in value distribution theory (see [11, 18, 19]). And we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}, \quad \text{as } r \to \infty$, possibly outside of a set E with finite linear or logarithmic measure, not necessarily the same at each occurrence. A meromorphic function a is said to be a small function with respect to f if and only if $T(r, a) = S(r, f)$. We use $\lambda(f)$ and $\sigma(f)$ to denote the exponent of convergence of zeros of f and the order of f respectively. We say that two meromorphic functions f and g share a value a IM (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that they share the value a CM (counting multiplicities). We define the forward difference operator $\Delta f = f(z + 1) - f(z)$ and the high order forward difference operator $\Delta^n f = \Delta^{n-1}(\Delta f)$ by recurrence. Moreover, $\Delta^n f = \sum_{j=0}^{n} C_n^j (-1)^{n-j} f(z + j)$.

*Corresponding author. This research was supported by the Fundamental Research Funds for the Central Universities (No. 2015QNA52) and NSF of China (No. 11601507).

In 1976, L. Rubel and C.C. Yang [7] studied the uniqueness of an entire function sharing two values with its derivative and they proved the following classical result.

Theorem 1 *Let f be a nonconstant entire function. If f and f' share two distinct finite values CM, then $f \equiv f'$.*

In 1996, R. Brück [2] studied the uniqueness theory about an entire function sharing one value with its first derivative and posed the following interesting conjecture.

Conjecture 1 *Let f be nonconstant entire function satisfying that the super order $\sigma_2(f) < \infty$ is not a positive integer. If f and f' share one finite value a CM, then $f' - a = c(f - a)$ holds for some nonzero constant c.*

It is well known that Δf can be considered as the difference counterpart of f'. So regarding Theorem A and Conjecture, it is natural to ask that what can be said about the relationship between Δf and f if they share one or two values CM. The difference analogue of the lemma on the logarithmic derivative and Nevanlinna theory for the difference operator have been founded recently (see [3, 8, 9]), which brings about a number of papers focusing on such uniqueness problems. The authors in [17, 16, 20], for example, obtained the following results by considering the special case of entire functions of order less than 1 or 2 respectively.

Theorem 2 *[17] Let f be a transcendental entire function such that $\sigma(f) < 1$, n be a positive integer and η be a nonzero complex number. If f and $\Delta_\eta^n f$ share a finite value a CM, then $\Delta_\eta^n f - a = c(f - a)$ holds for some nonzero complex number c.*

Theorem 3 *[16] Let f be a transcendental entire function of order $\sigma(f) < 2$ and $\eta \neq 0$ be a complex number that is not a period of f. If f and $\Delta_\eta^n f$ share the value 0 CM, then $\Delta_\eta^n f/f$ reduces to a nonzero constant.*

Theorem 4 *[20] Let f be a transcendental entire function such that $\sigma(f) < 2$ and $\lambda(f) < \sigma(f)$. If f and $\Delta^n f$ share the value 0 CM, then f must be form of $f(z) = Ae^{\alpha z}$, where A and α are two nonzero constants.*

In this paper, we deal with the general case of entire function of finite order and obtain the following results which extend Theorem 2 and Theorem 4.

Theorem 5 *Let f be a transcendental entire function such that $\sigma(f) < \infty$, let $a \not\equiv 0$ be an entire function such that $\sigma(a) < 1$ and $\lambda(f - a) < \sigma(f)$. If f and $\Delta^n f$ share a CM, then a must reduce to a polynomial with degree at most $n - 1$ and f must be form of*

$$f(z) = a + bae^{\beta z},$$

where b and β are two nonzero constants such that $e^\beta = 1$.

2

Theorem 6 *Let f be a transcendental entire function such that $\lambda(f) < \sigma(f) < \infty$, let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$. If f and $\Delta^n f$ share a CM, then f must be form of $f(z) = be^{\beta z}$, where b and β are two nonzero constants such that $(e^{\beta} - 1)^n = 1$.*

Theorem 7 *Let f be a transcendental entire function such that $\lambda(f) < \max\{\sigma(f) - 1, 1\} < \infty$. If $f(z)$ and $\Delta^n f$ share the value 0 CM, then f must be form of $f(z) = he^{\beta z}$, where h and β are two nonzero constants.*

2 Some lemmas

Lemma 1 *(see[3]) Let f be a transcendental meromorphic function with finite order σ and η be a nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r),$$

$$i.e.,\ T(r, f(z+\eta)) = T(r, f) + S(r, f).$$

Lemma 2 *(see[3]) Let f be a transcendental meromorphic function with finite order σ. Then for each $\varepsilon > 0$, we have*

$$m\big(r, \frac{f(z+c)}{f(z)}\big) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 3 *(see[3]) Let η be a nonzero complex number and f be a meromorphic function of finite order σ. Let $\varepsilon > 0$ be given, then there exists a subset $E \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0, 1]$, we have*

$$e^{-r^{\sigma-1+\varepsilon}} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq e^{r^{\sigma-1+\varepsilon}}.$$

Lemma 4 *(see [4]) Let f be a nonconstant meromorphic function of order $\sigma < \infty$, and let λ' and λ'' be, respectively, the exponent of convergence of the zeros and poles of f. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, +\infty)$ of $|z| = r$ of finite logarithmic measure, so that*

$$2\pi i n_{z,\eta} + \log \frac{f(z+\eta)}{f(z)} = \eta \frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon}), \tag{1}$$

or equivalently,

$$\frac{f(z+\eta)}{f(z)} = e^{\eta \frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon})},$$

holds for $r \notin E \cup [0, 1]$, where $n_{z,\eta}$ in (1) is an integer depending on both z and η, $\beta = \max\{\sigma - 2, 2\lambda - 2\}$ if $\lambda < 1$ and $\beta = \max\{\sigma - 2, \lambda - 1\}$ if $\lambda \geq 1$ and $\lambda = \max\{\lambda', \lambda''\}$.

Lemma 5 *(see [5]) Suppose that $P(z) = (\alpha+i\beta)z^n+\cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geq 1$, that $A(z) \not\equiv 0$ is an entire function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}, z = re^{i\theta}, \delta(P, \theta) = \alpha\cos n\theta - \beta\sin n\theta$. Then for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$ that has the linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, there is $R > 0$ such that for $|z| = r > R$, we have*
(i) if $\delta(P, \theta) > 0$, then

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1+\varepsilon)\delta(P,\theta)r^n\};$$

(ii) if $\delta(P, \theta) < 0$, then

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1-\varepsilon)\delta(P,\theta)r^n\},$$

where $H_2 = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0\}$ is a finite set.

Lemma 6 *(see [1]) Let g be a transcendental function of order less than 1, and h be a positive constant. Then there exists an ε set E such that*

$$\frac{g'(z+\eta)}{g(z+\eta)} \to 0, \ \frac{g(z+\eta)}{g(z)} \to 1, \, as \ z \to \infty \ in \ C\backslash E$$

uniformly in η for $|\eta| \leq h$. Further, the set E may be chosen so that if $z \notin E$ and $|z|$ is sufficiently large, the function g has no zeroes or poles in $|\zeta - z| \leq h$.

Remark 1 *According to Hayman [12], an ε set is defined to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. Suppose E is an ε set, then the set of $r \geq 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.*

Lemma 7 *(see [18]) Suppose that $f_1, f_2, \cdots, f_n$ ($n \geq 2$) are meromorphic functions and $g_1, g_2, \cdots, g_n$ are entire functions satisfying the following conditions:*

(i) $\sum\limits_{j=1}^{n} f_j e^{g_j} \equiv 0;$
(ii) *$g_j - g_k$ are not constants for $1 \leq j < k \leq n$;*
(iii) *For $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}(r \to \infty, r \notin E)$.*

Then $f_j \equiv 0$ ($j = 1, 2, \cdots, n$).

Lemma 8 *(see [6]) Let w be a transcendental meromorphic function with $\sigma < \infty$. Let $\Gamma = \{(k_1, j_1), \cdots, (k_m, j_m)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geq 0$ for $i = 1, 2, \cdots, m$. Also let $\varepsilon > 0$ be a given constant, then there exists a set $E \subset (1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E \cup [0, 1]$ and for all $(k, j) \in \Gamma$, one has*

$$\left|\frac{w^{(k)}(z)}{w^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 9 *(see[18]) Let f be a nonconstant meromorphic function in the complex plane and $R(f) = p(f)/q(f)$, where $p(f) = \sum_{k=0}^{p} a_k f^k$ and $q(f) = \sum_{j=0}^{q} b_j f^j$ are two mutually prime polynomials in f. If the coefficients a_k, b_j are small functions of f and $a_k \not\equiv 0, b_j \not\equiv 0$, then*

$$T(r, R(f)) = \max\{p, q\}T(r, f) + S(r, f).$$

Lemma 10 *Let g be polynomial of degree at lest two. Then*

$$m\big(r, \sum_{j=0}^{n} a_j e^{g(z+j)-g(z)}\big) = m(r, e^{g(z+n)-g(z)}) + S(r, e^{g(z+n)-g(z)}),$$

where the coefficients a_j are small meromorphic functions of $e^{g(z+n)-g(z)}$.
Proof. Set $g(z) = a_l z^l + a_{l-1} z^{l-1} + \ldots + a_0$, $a_l \neq 0$, $l \geq 2$ and $H(z) = e^{l a_l z^{l-1}}$. Then we get $g(z+j) - g(z) = j l a_l z^{l-1} + \cdots$, and then $e^{g(z+j)-g(z)} = b_j e^{j l a_l z^{l-1}}$, where $\sigma(b_j) \leq l - 2$. So we have

$$\sum_{j=0}^{n} a_j e^{g(z+j)-g(z)} = \sum_{j=0}^{n} \tilde{a}_j e^{j l a_l z^{l-1}} = \sum_{j=0}^{n} \tilde{a}_j H^j,$$

where $\tilde{a}_j = a_j b_j$ are small function of H. Application Lemma 9 to the equation above gives our conclusion immediately.

Lemma 11 *Let f be a transcendental entire function such that $2 \leq \sigma(f) < \infty$, also let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$ and $\lambda(f - a) < \sigma(f)$. If the difference equation*

$$\Delta^n f - a = (f - a)e^Q \tag{2}$$

holds, where Q is a nonconstant entire function, then Q is a polynomial such that $\deg Q = \sigma(f) - 1$.

Proof. From our assumption and Lemma 1, it is obvious for us to get that Q is a polynomial and

$$F := f - a = he^g \tag{3}$$

holds, where g is a polynomial with degree l satisfying $l = \sigma(f) \geq 2$, and h is an entire function originated from the canonical product of $f - a$ satisfying $\lambda(h) = \sigma(h) < \sigma(f)$. Set $g(z) = a_l z^l + a_{l-1} z^{l-1} + \ldots + a_0$ and $Q(z) = b_s z^s + a_{s-1} z^{s-1} + \ldots + b_0$ respectively. Substitution (3) into (2) yields

$$e^Q = \frac{\Delta^n f - a}{f - a} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{F(z+j)}{F(z)} + \frac{\Delta^n a - a}{F(z)}. \tag{4}$$

First of all, we estimate the first term $\sum_{j=0}^{n} C_n^j (-1)^{n-j} F(z+j)/F(z)$ on the right side of (4). Employing the definition of F, it turns out that $\sigma(F) = \sigma(f) =$

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$l \geq 2$ and $\lambda(F) = \sigma(h) < \sigma(f)$. By applying Lemma 4 to F, for any given $\varepsilon > 0$ small enough, there exists a set E with finite logarithmic measure such that

$$\frac{F(z+j)}{F(z)} = e^{j\frac{F'(z)}{F(z)} + O(r^{\beta+\varepsilon})}, \text{ as } r \to \infty, \text{ not in } E \cup [0,1], \tag{5}$$

where $\beta = \sigma(f) - 2$ if $\sigma(h) < 1$ or $\beta = \max\{\sigma(f) - 2, \sigma(h) - 1\}$ if $\sigma(h) \geq 1$. Combining the fact $\sigma(h) < \sigma(f) = l$, we get $\beta < \sigma(f) - 1 = l - 1$. By Lemma 8, we see, for any given $\varepsilon > 0$ small enough, that

$$\left|\frac{h'(z)}{h(z)}\right| \leq r^{\sigma(h)-1+\varepsilon} = o(r^{l-1}) \tag{6}$$

holds for $|z| = r \notin E$. Thus from (3) and (6), we obtain

$$\frac{F'(z)}{F(z)} = g'(z) + \frac{h'(z)}{h(z)} = la_l z^{l-1}(1 + o(1)) \tag{7}$$

as $|z| = r \to \infty$ not in E. So from (5) and (7), we obtain

$$\frac{F(z+j)}{F(z)} = e^{jla_l z^{l-1}(1+o(1))}, \quad r \notin E. \tag{8}$$

Secondly, we estimate the second term $(\Delta^n a - a)/F$ on the right side of (4). It is easy to see $N := \sigma(\Delta^n a - a) \leq \sigma(a) < \sigma(f) = l$ in a similar way by Lemma 1, which gives, for any given $\varepsilon > 0$, that

$$M(r, \Delta^n a - a) < e^{r^{N+\varepsilon}} \tag{9}$$

holds for all r large sufficiently. Let $\delta(\theta) = \cos((l-1)\theta + \arg a_l)$, $\delta(g,\theta) = \cos(l\theta + \arg a_l)$ and $z = re^{i\theta}$. It follows Lemma 5 that for any given $\varepsilon > 0$, there exists a set $H \subset [0, 2\pi)$ that has the linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus H$, there is $R > 0$ such that for $|z| = r > R$, we have

$$\exp\{(1-\varepsilon)|a_l|\delta(g,\theta)r^l\} < |F(re^{i\theta})| \tag{10}$$

if $\delta(g,\theta) > 0$. So by (10) and (9), we see $(\Delta^n a - a)/F \to 0$, as $z = re^{i\theta} \to \infty$ such that $\delta(g,\theta) > 0$. By Lemma 3, for any for any given $\varepsilon > 0$ small enough, we have

$$e^{-r^{\sigma(h)-1+\varepsilon}} \leq \left|\frac{h(z+c)}{h(z)}\right| \leq e^{r^{\sigma(h)-1+\varepsilon}} \tag{11}$$

holds for all sufficient large $r \notin E$.

Lastly, we take such $z = re^{i\theta}$ that $\theta \in [0, 2\pi) \setminus H; \delta(g,\theta) > 0$ and consider three cases separately in the next section.

Case 1 If $\delta(\theta) < 0$, then

$$|e^{jla_l z^{l-1}(1+o(1))}| = e^{jl|a_l|r^{l-1}\delta(\theta)(1+o(1))} \to 0, \text{ as } r \to \infty.$$

By (4), (9), (11) and the equation above, we obtain $e^{Q(z)} = (-1)^n + o(1)$. It means Q is bounded on such θ and $r \notin E$, which implies Q is a constant. And then by (3) and (4), we obtain

$$k := e^Q = (-1)^n + \sum_{j=1}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)} + \frac{\Delta^n a - a}{h(z)e^{g(z)}}. \tag{12}$$

If $\Delta^n a - a \not\equiv 0$, then by (11), (12), and the fact $\sigma((\Delta^n a - a)/h) < \sigma(e^g)$, we see

$$\begin{aligned}
&\tfrac{|a_l|}{\pi} r^l (1+o(1)) + S(r,e^g) = m(r,e^{-g}) + S(r,e^g) = m(r, \tfrac{\Delta^n a - a}{he^g}) \\
&\leq \sum_{j=1}^{n} m(r, \tfrac{h(z+j)}{h(z)}) + \sum_{j=1}^{n} m(r, e^{g(z+j)-g(z)}) \\
&\leq r^{\sigma(h)-1+\varepsilon} + \tfrac{n(n+1)}{2} \tfrac{|a_l l|}{\pi} r^{l-1}(1+o(1)), r \notin E,
\end{aligned}$$

which is impossible. If $\Delta^n a - a \equiv 0$, then by (12), we see

$$k = (-1)^n + \sum_{j=1}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)}. \tag{13}$$

Employing representation $\sigma(h) < \deg g(z) = l$ and (11), we see

$$\left| \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)} \right| = e^{jl|a_l|r^{l-1}\delta(\theta)(1+o(1))}.$$

holds for $r \notin E$. And then in this situation, $(h(z+n)/h(z))e^{g(z+n)-g(z)}$ is the only maximal magnitude of module term in (13) by taking such z that $\delta(\theta) > 0$, which is also impossible.

Case 2 If $\delta(\theta) > 0$, then by (4), (8),(9) and (10), we obtain

$$e^{|b_s|r^s \cos(\arg b_s + s\theta)(1+o(1))} = |e^Q| = (1+o(1))e^{nl|a_l|r^{l-1}\delta(\theta)(1+o(1))} \to \infty.$$

It means $s = l-1$ on such θ and $r \notin E$, which yields $s = l-1$.

Case 3 $\delta(\theta) = 0$. Since the set $\{\theta : \delta(\theta) = 0\}$ is just a finite set and $\delta(g,\theta)$ is a continuous function of θ, so we can chose another $\tilde{\theta}$ near θ, possibly outside of a set with the linear measure zero, such that $\delta(g,\tilde{\theta}) > 0$ and $\delta(\tilde{\theta}) \neq 0$, and then this case can be transformed into case 1 or case 2.

Using the similar method in Lemma 11, we can prove the following lemma.

Lemma 12 *Let f be a transcendental entire function such that $2 \leq \sigma(f) < \infty$ and $\lambda(f) < \sigma(f)$, let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$. If the difference equation $\Delta^n f - a = (f-a)e^Q$ holds, where Q is a nonconstant entire function, then Q is a polynomial such that $\deg Q = \sigma(f) - 1$.*

Lemma 13 *Let a be an entire function of order less than 1. If a satisfies the difference equation $\Delta^n a - a = 0$, then $a \equiv 0$.*

Proof. Suppose on the contrary $a \not\equiv 0$. Then by Lemma 6, we see

$$1 = \frac{\Delta^n a}{a} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{a(z+j)}{a} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1-1)^n = 0$$

as $r \to +\infty, r \notin E_\varepsilon$, where E_ε is an ε set. It is impossible.

Lemma 14 *Let a be an entire function of order less than 1. Then a satisfies the difference equation $\Delta^n a = 0$ implies a is a polynomial of degree at most $n-1$.*

Proof. Set $H_i := \Delta^{n-i} a, \quad j = 0, 1, \ldots, n$. Then $H_1(z+1) - H_1(z) = \Delta H_1 = H_0 = \Delta^n a = 0$. If H_1 is a nonconstant entire function, then it is easy to see that $z_k = k \in Z$ are some different zeros of $H_1(z) - H_1(0)$, which implies

$$\overline{N}(r, \frac{1}{H_1(z) - H_1(0)}) \geq r(1 + o(1)).$$

So $\sigma(H_1) \geq 1$, which is a contradiction. Thus H_1 is a constant, and then $0 = H_1{}' = (\Delta H_2)' = \Delta H_2'$. By a similar discussion, we see H_2' is a constant and then $H_2'' = 0$. Repeating this process, we can obtain $a^{(n)} = H_n^{(n)} = 0$. Thus a is a polynomial whose degree is at most $n-1$.

3 The proofs of main theorems

1. Proof of theorem 5.

Since $\Delta^n f$ and f share the function a CM, so there exists a polynomial Q by Lemma 1 such that

$$\Delta^n f - a = (f - a)e^Q. \tag{14}$$

It follows $\lambda(f-a) < \sigma(f)$ that

$$f - a = he^g, \tag{15}$$

where g is a polynomial whose degree l satisfying $l = \sigma(f) \geq 1$, and h is an entire function originated from the canonical product of $f - a$ satisfying $\lambda(h) = \sigma(h) < \sigma(f) = l$. By substituting (15) into (14), we can obtain

$$[\Delta^n a - a] + \sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) e^{g(z+j)} = h(z) e^{g(z)+Q(z)}. \tag{16}$$

In what follows, we shall consider two cases separately to our discussion.

Case 1 $\sigma(f) \geq 2$. We rewrite (16) as the following form

$$[\Delta^n a - a] + [\sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) e^{g(z+j)-g(z)} - h(z)e^{Q(z)}] e^{g(z)} = 0. \tag{17}$$

By applying Lemma 11 to (14), we see $\deg Q = l-1$. Applying Lemma 7 to (17) and invoking the relation $\deg Q = l-1$, it turns out that $\Delta^n a - a = 0$, which mans $a \equiv 0$ by Lemma 13. Thus we get a contradiction with our assumption.
Case 2 $l = \deg g = \sigma(f) < 2$, in other worlds, $\sigma(f) = 1$. Thus without loss of generality, we can rewrite (15) as the form of $f - a = he^{\beta z}$, where β is a nonzero constant. By (14), we see $\deg(Q) \leq \sigma(f) = 1$, and then we shall consider two subcases in this case respectively as follows.
Case 2.1 Q is a constant. Then we can rewrite (17) as the following form

$$[\Delta^n a - a] + [H_n - he^{Q}]e^{\beta z} = 0, \tag{18}$$

where $H_n = \sum\limits_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j)k^j, \quad k = e^{\beta}$. It follows (18) and Lemma 7 that $\Delta^n a - a = 0$, which leads to a contradiction with our assumption similarly.
Case 2.2 $\deg(Q) = 1$. Set $Q(z) = \gamma z + d$, where γ is a nonzero constant. By substituting $Q(z) = \gamma z + d$ into (16), we see

$$[\Delta^n a - a] + H_n e^{\beta z} = e^d h e^{(\beta+\gamma)z}. \tag{19}$$

If $\beta + \gamma \neq 0$, then by (19) and Lemma 7, we get $h \equiv 0$, which is a contradiction. If $\beta + \gamma = 0$, then (19) reduces to

$$[\Delta^n a - a] + H_n e^{\beta z} = e^d h. \tag{20}$$

Then by (20) and Lemma 7, we see

$$H_n = \sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j)k^j = 0 \tag{21}$$

and

$$[\Delta^n a - a] = e^d h. \tag{22}$$

Employing representation (21) and Lemma 6, it turns out that

$$0 = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h} k^j \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j = (k-1)^n$$

as $z \to \infty$ not in an ε set. Thus we obtain $k = e^{\beta} = 1$ from the equation above. Substituting $k = 1$ into (21), we see $H_n = \sum\limits_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) = \Delta^n h = 0$. By Lemma 14 and the equation above, we see that h is a polynomial whose degree is at most $n-1$. If a is a transcendental function, and we take z such that $|z| = r$ and $|a(z)| = M(r,a)$, then we have

$$\lim_{z\to\infty} e^d \frac{h(z)}{a(z)} = 0.$$

However, we have by (22) that

$$e^d\frac{h}{a}=\frac{\Delta^n a}{a}-1=\sum_{j=0}^{n}C_n^j(-1)^{n-j}\frac{a(z+j)}{a}-1\to\sum_{j=0}^{n}C_n^j(-1)^{n-j}-1=-1$$

as $z\to\infty$ in $z\in\{z:|a(z)|=M(r,a)\}\backslash E_\varepsilon$, where E_ε is an ε set, which is impossible. Thus a is a polynomial and then $\deg(a)=\deg(\Delta^n a-a)=\deg e^d h=\deg h$, which leads to that a is a polynomial with degree at most $n-1$. Furthermore we get $\Delta^n a=0$ and $-a=e^d h$ from (22) and then f must be form of

$$f(z)=a(z)+ba(z)e^{\beta z},$$

where $b:=-e^{-d}$ and β are two nonzero constants such that $e^\beta=1$.

2. Proof of Theorem 6.

Using the same method as in Theorem 1, we see

$$\Delta^n f-a=(f-a)e^Q \tag{23}$$

and

$$f=he^g, \tag{24}$$

where g is a polynomial of degree l satisfying $l=\sigma(f)\geq 1$, h is an entire function originated from the canonical product of f satisfying $\lambda(h)=\sigma(h)<\sigma(f)=l$, and Q is a polynomial of degree at most l. From (23)-(24), we obtain

$$\sum_{j=0}^{n}C_n^j(-1)^{n-j}h(z+j)e^{g(z+j)}=h(z)e^{g(z)+Q(z)}+a(z)-a(z)e^{Q(z)}. \tag{25}$$

In the next section, we shall consider two cases separately.

Case 1 $\sigma(f)\geq 2$. We rewrite (25) as the following form

$$[\sum_{j=0}^{n}C_n^j(-1)^{n-j}h(z+j)e^{g(z+j)-g(z)}-h(z)e^{Q(z)}]e^{g(z)}=a(z)-a(z)e^{Q(z)}. \tag{26}$$

From Lemma 12, we see $\deg Q=l-1\geq 1$. Then by (26) and Lemma 7, we obtain $a-ae^Q=0$. Thus $e^Q\equiv 1$ or $a\equiv 0$, which is impossible.

Case 2 $l=\deg g=\sigma(f)<2$, in other words, $\sigma(f)=1$. Thus without loss of generality, we can rewrite (24) as the form of $f=he^{\beta z}$, where β is a nonzero constant. It is easy to see $\deg(Q)\leq 1$.We shall consider two subcases.

Case 2.1 Q is a constant. Then by (26), we see $e^Q=1$ and

$$\sum_{j=0}^{n}C_n^j(-1)^{n-j}h(z+j)k^j-h(z)=0, \tag{27}$$

where $k=e^\beta$. From (27), we see

$$1=\sum_{j=0}^{n}C_n^j(-1)^{n-j}k^j\frac{h(z+j)}{h}\to\sum_{j=0}^{n}C_n^j(-1)^{n-j}k^j=(k-1)^n \tag{28}$$

as $z \to \infty$ not in an ε set. It means $(k-1)^n = 1$ and then

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j = 1. \tag{29}$$

By (27) and (29), we see

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j [h(z+j) - h(z)] = 0. \tag{30}$$

Set $B(z) = \Delta h = h(z+1) - h(z)$, then from Lemma 1, it is easy for us to see $\sigma(B) \leq \sigma(h) < 1$. From the definition of $B(z)$. Using the same method in Theorem 4 [20], we can proof $B(z) \equiv 0$. That is $h(z+1) = h(z)$. So we get h is a nonzero constant using the same method as in Lemma 14, and then f must be form of $f(z) = be^{\beta z}$, where $b := h$ and β are two nonzero constants such that $(e^\beta - 1)^n = 1$.
Case 2.2 $\deg(Q) = 1$. Set $Q(z) = \gamma z + d$, where γ is a nonzero constant. Then (25) becomes

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j h(z+j) e^{\beta z} - a = e^d h(z) e^{(\beta+\gamma)z} - e^d a e^{\gamma z}. \tag{31}$$

If $\beta + \gamma \neq 0$ and $\beta - \gamma \neq 0$, then by (31) and Lemma 7, we get $a \equiv 0$ and $h \equiv 0$, which is a contradiction. If $\beta - \gamma = 0$, then (31) becomes

$$\{[\sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) k^j] + ae^d\} e^{\beta z} - a = e^d e^{2\beta z},$$

and we also get a contradiction by applying Lemma 7 to the equation above.
If $\beta + \gamma = 0$, then (31) becomes

$$\{\sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) k^j\} e^{2\beta z} = (e^d h(z) + a) e^{\beta z} - ae^d,$$

we can get a contradiction in a same way.

3. Proof of theorem 7.
We shall consider the following three cases separately to our discussion.
Case 1 $\sigma(f) < 1$. By Theorem 2, we get $\Delta^n f = cf$ holds for some nonzero complex number c. Then by Lemma 6, we get

$$c = \frac{\Delta^n f}{f} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{f(z+j)}{f(z)} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1-1)^n = 0$$

as $z \to \infty$, possibly outside of a ε set. Therefore $c = 0$, which is a contradiction.
Case 2 $1 \leq \sigma(f) < 2$ and $\lambda(f) < 1$. Then we can get our conclusion immediately by Theorem 4.

Case 3 $\sigma(f) \geq 2$ and $\lambda(f) < \sigma(f) - 1$. Using the same method as in Theorem 5, we see

$$\Delta^n f = fe^Q \tag{32}$$

and

$$f = he^g, \tag{33}$$

where $g(z) = a_l z^l + a_{l-1}z^{l-1} + \ldots + a_0$, $Q(z) = b_s z^s + a_{s-1}z^{s-1} + \ldots + b_0$, $l \geq 2$, $s \leq k$, are polynomials, h is an entire function originated from the canonical product of f satisfying $\lambda(h) = \sigma(h) < \sigma(f) - 1 = l - 1$. From (32)-(33), we obtain

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)} = e^{Q(z)}. \tag{34}$$

Recall $g(z+j) - g(z) = ja_l l z^{l-1}(1 + o(1))$. By (34), Lemma 1 and 10, we see

$$\begin{aligned} \tfrac{|b_s|}{\pi} r^s \sim m(r, e^Q) \qquad &= m\big(r, \sum_{j=0}^{n} C_n^j (-1)^{n-j} \tfrac{h(z+j)}{h(z)} e^{g(z+j)-g(z)}\big) \\ &= m\big(r, e^{g(z+n)-g(z)}\big) + S(r, e^{g(z+n)-g(z)}) \sim \tfrac{nl|a_l|}{\pi} r^{l-1}. \end{aligned}$$

It means $s = l - 1$ and $|b_s| = nl|a_l|$. We can rewrite (34) as the following form

$$\sum_{j=0}^{n-1} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{ja_l l z^{l-1}(1+o(1))} + \frac{h(z+n)}{h(z)} e^{A_n} e^{na_l l z^{l-1}} = e^B e^{b_{l-1}z^{l-1}}, \tag{35}$$

where A_n, B are two polynomials with degree at most $l - 2$. Recalling (11) and taking any θ such that $\delta(\theta) = \cos((l-1)\theta + \arg a_l) > 0$, then we get $\tilde{\delta}(\theta) = \cos((l-1)\theta + \arg b_{l-1}) > 0$ by (35), and then

$$e^{nl|a_l|r^{l-1}\delta(\theta)(1+o(1))} = e^{|b_{l-1}|r^{l-1}\tilde{\delta}(\theta)(1+o(1))}.$$

That means $\delta(\theta) = \tilde{\delta}(\theta)$. By the arbitrariness of θ, we see $\arg a_l = \arg b_{l-1}$. Thus we obtain $b_s = nla_l$, and then (35) becomes

$$\sum_{j=0}^{n-1} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{ja_l l z^{l-1}(1+o(1))} = e^B (1 - \frac{h(z+n)}{h(z)} e^{A_n - B}) e^{nla_l z^{l-1}}. \tag{36}$$

It is obvious $\sigma(e^B(1 - (h(z+n)/h)e^{A_n - B})) \leq \max\{\sigma(h), l-2\} < l - 1$. If $e^B - (h(z+n)/h)e^{A_n} \not\equiv 0$, then from (36) and Lemma 10, we see

$$\frac{nl|a_l|}{\pi} r^{l-1} \sim T(r, e^B(1 - \frac{h(z+n)}{h(z)} e^{A_n - B}) e^{nla_l z^{l-1}}) \sim \frac{(n-1)l|a_l|}{\pi} r^{l-1},$$

which is impossible. If $e^B - (h(z+n)/h(z))e^{A_n} \equiv 0$, then (36) becomes

$$\sum_{j=0}^{n-1} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{ja_l l z^{l-1}(1+o(1))} = 0, \tag{37}$$

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however $(h(z+n-1)/h(z))e^{(n-1)a_l l z^{l-1}}$ is the only maximal magnitude of module term in (37) when taking $\delta(\theta) > 0$, which is impossible.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Global Attractivity for Nonautonomous Difference Equation via Linearization

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Abstract. Consider the difference equation

$$\vec{x}_{n+1} = f(n, \vec{x}_n, ..., \vec{x}_{n-k}), \quad n = 0, 1, \ldots,$$

where $k \in \{0, 1, \ldots\}$ and the initial conditions are real vectors. We investigate the asymptotic behavior of the solutions of the considered equation. We give some effective conditions for the global stability and global asymptotic stability of the zero or positive equilibrium of this equation. Our results are based on application of the linearizations technique. We illustrate our results with many examples that include some equations from mathematical biology.

Keywords: attractivity, difference equations, discrete dynamical system, global, linear fractional, rational, stability

AMS 2000 Mathematics Subject Classification: 39A10, 39A20, 37B25, 37D10, 37M99.

1 Introduction and preliminaries

Consider the difference equation

$$\vec{x}_{n+1} = f(n, \vec{x}_n, ..., \vec{x}_{n-k}), \quad n = 0, 1, \ldots \tag{1}$$

where $k \in \{0, 1, \ldots\}$ and the initial conditions are real vectors in $\mathbb{R}^p, p \geq 2$. In many cases we investigate equation(1) by embedding equation(1) into a higher iteration of the form

$$\vec{x}_{n+l} = F(n, \vec{x}_{n+l-1}, ..., \vec{x}_{n-k}), \quad n = 0, 1, \ldots \tag{2}$$

where $l \in \{1, 2, \ldots\}$, see [4, 5, 8]. By linearizing equation (2) and bringing it to the form

$$\vec{x}_{n+1} = \sum_{i=1-l}^{k} g_i \vec{x}_{n-i}, \tag{3}$$

where g_i in general, depend on n and the state variables $\vec{x}_k$ we can prove very general attractivity and asymptotic stability results for both autonomous and nonautonomous difference equations. The functions g_i are in general matrices but they can also be the scalars as well, see Section 3. This approach was used to get effective and applicable global asymptotic and global attractivity results for linear fractional difference equation, see [2] and quadratic fractional difference equation, see [3] with both constant and nonconstant coefficients. Furthermore, this approach produced global asymptotic and global attractivity results for nonautonomous difference equations with very general coefficients which can be discontinuous functions of n or state variables, see [4, 5, 8]. See [1, 7, 10, 11] for the case of monotone systems, where more precise results were obtained.

In this paper we use method of linearization to extend some of the results about the global attractivity and asymptotic stability of scalar equation from [4] to the case of vector equation (2). We illustrate our results with many examples that include some transition functions from mathematical biology such as linear, Beverton-Holt, sigmoid Beverton-Holt, etc., see [6, 7, 9, 11, 12] for related results. The rest of this section contains some definitions and preliminary results. Second section contains our main results on global attractivity in the case when the sum of the norms of g_i is less than 1. The third section

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gives some results on global attractivity in the delicate case when the sum of the scalar functions g_i is 1. The fourth section provides several examples which illustrate our results.

Denote by $\|\vec{x}\|$ any norm in $\mathbb{R}^p$.

Definition 1 *The zero equilibrium of equation* (3) *is stable if for* $(\forall\epsilon > 0)(\exists\delta > 0, N)$*:*

$$\|\vec{x}_i\| < \delta, i = -k, \dots, 0 \Longrightarrow \|\vec{x}_n\| < \epsilon, \text{ for all } n \geq N.$$

The zero equilibrium is asymptotically stable if it is stable and

$$\lim_{n\to\infty} \vec{x}_n = \vec{0}.$$

Lemma 1 *Let* $\mathbf{I} - \sum_{i=0}^{k} g_i$ *be invertible for* $n = 1, 2, \dots,$ *where* $\mathbf{I}$ *is identity matrix.Then equation* (3) *has no nonzero equilibrium.*

Proof. Otherwise, equation (3) has the equilibrium $\bar{\mathbf{x}} \neq \vec{0}$. By pluging $\vec{x}_n = \bar{\mathbf{x}}$ in equation (3) we get

$$(\mathbf{I} - \sum_{i=0}^{k} g_i)\bar{\mathbf{x}} = \vec{0},$$

which implies $\bar{\mathbf{x}} = \vec{0}$, which is a contradiction. □

Remark 1 The matrix $\mathbf{I} - \sum_{i=0}^{k} g_i$ is invertible if the condition

$$\|\sum_{i=0}^{k} g_i\| < 1 \tag{4}$$

is satisfied in which case we have

$$(\mathbf{I} - \sum_{i=0}^{k} g_i)^{-1} = \sum_{k=0}^{\infty}\sum_{i=0}^{k} g_i. \tag{5}$$

The condition (4) is implied by more applicable condition

$$\sum_{i=0}^{k} \|g_i\| < 1. \tag{6}$$

Remark 2 Equation (1) admits the following generalized identity

$$\vec{x}_{n+1} - \sum_{i=0}^{k} g_i\vec{K} = \sum_{i=0}^{k} g_i(\vec{x}_{n-i} - \vec{K}), \tag{7}$$

where $\vec{K}$ is an arbitrary vector. Generalized identity (7) implies

$$\|\vec{x}_{n+1} - \sum_{i=0}^{k} g_i\vec{K}\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{n-i} - \vec{K}\|. \tag{8}$$

Furthermore by taking $\vec{K} = \vec{0}$ in equation (8), we obtain another useful inequality

$$\|\vec{x}_{n+1}\| - L\sum_{i=0}^{k} \|g_i\| \leq \sum_{i=0}^{k} \|g_i\|(\|\vec{x}_{n-i}\| - L), \tag{9}$$

where L is an arbitrary constant.

Lemma 2 *Suppose that equation (1) has the linearization (3) and the functions $g_i : R^{p+1} \to M_{p\times p}$, where $M_{p\times p}, p \geq 1$ is the set of all real $p \times p$ matrices, are such that*

$$\sum_{i=0}^{k} \|g_i\| \leq 1, \quad n = 0, 1, \ldots$$

Then if equation (1) has the zero equilibrium it is a stable fixed point.

Proof. Assume that equation (1) has the zero equilibrium and the linearization (3). By taking $\vec{K} = \vec{0}$ in equation (8) we have

$$\|\vec{x}_{n+1}\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{n-i}\|.$$

Assume that $\sum_{i=0}^{k} \|\vec{x}_{-i}\| < \delta$. Take $\delta = \epsilon$. Then $\|\vec{x}_{-i}\| < \delta$ for $i = 0, 1, \ldots$. Hence

$$\|\vec{x}_1\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{-i}\| < \delta \sum_{i=0}^{k} \|g_i\| \leq \delta = \epsilon,$$

$$\|\vec{x}_2\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{1-i}\| < \delta \sum_{i=0}^{k} \|g_i\| \leq \delta = \epsilon$$

and so by induction $\|\vec{x}_n\| < \epsilon$ for $n \geq -k$. □

2 Main results

In this section we present our main results on global attractivity and global asymptotic stability of the equilibrium solutions of equation (1) which has the linearization (3).

Theorem 1 *Let $l \in \{1, 2, \ldots\}$. Suppose that equation (1) has the linearization (3) subject to the condition*

$$\sum_{i=1-l}^{k} \|g_i\| \leq 1, n = 0, 1, \ldots \tag{10}$$

Let $M_0 = \max\{\|\vec{x}_{l-1}\|, \ldots, \|\vec{x}_{-k}\|\}$. Then every solution of equation (1) is bounded. In particular $\|\vec{x}_n\| \leq M_0$ for $n \geq -k$.

Proof. Let $L \in R$. Then equation (9) implies

$$\|\vec{x}_{n+l}\| - L \sum_{i=1-l}^{k} \|g_i\| \leq \sum_{i=1-l}^{k} \|g_i\|(\|\vec{x}_{n-i}\| - L), \quad n = 0, 1, \ldots \tag{11}$$

By taking $L = M_0$ and $n = 0$ in equation (11), we obtain

$$\|\vec{x}_l\| - M_0 \sum_{i=1-l}^{k} \|g_i\| \leq \|g_{1-l}\|(\|\vec{x}_{l-1}\| - M_0) + \ldots + \|g_k\|(\|\vec{x}_{-k}\| - M_0) \leq 0,$$

which in view of equation (10) implies $\|x_l\| \leq M_0$. By using induction, we obtain

$$\|\vec{x}_{n+l}\| - M_0 \sum_{i=1-l}^{k} \|g_i\| \leq \|g_{1-l}\|(\|\vec{x}_{n+l-1}\| - M_0) + \ldots + \|g_k\|(\|\vec{x}_{n-k}\| - M_0) \leq 0, \quad n = 0, 1, \ldots$$

and so

$$\|\vec{x}_{n+l}\| \leq M_0 \sum_{i=1-l}^{k} \|g_i\| \leq M_0, \quad n = 0, 1, \ldots$$

Thus $\|\vec{x}_{n+l}\| \leq M_0$ for $n \geq -k$. □

Theorem 2 *Let $l \in \{1, 2, \ldots\}$. Suppose that equation (1) has the linearization (3) where the functions $g_i : R^{k+1} \to M_{p\times p}$ are such that*

$$\sum_{i=1-l}^{k} \|g_i\| \leq a < 1, \quad n = 0, 1, \ldots \tag{12}$$

Then

$$\lim_{n\to\infty} \vec{x}_n = \vec{0}.$$

Proof. Let $L \in R$. Then every solution of equation (3) satisfies the inequality (11). Let $\gamma = l + k$. Define $M_N = \max\{\|\vec{x}_{\gamma N+l-1}\|, \ldots, \|\vec{x}_{\gamma N-k}\|\}$ for $N = 0, 1, \ldots$. Observe that if $\|\vec{x}_{\gamma N+l-1}\| = \ldots = \|\vec{x}_{\gamma N-k}\| = \vec{0}$ for some $N \geq 0$, then by (11) with $L = 0$ we get that

$$\|\vec{x}_{\gamma N+l+j}\| = \vec{0}, \quad j = 0, 1, \ldots$$

and so $\lim_{n\to\infty} \vec{x}_n = \vec{0}$.
Assume that $M_N > 0$ for all $N \geq 0$. By using (11) with $L = M_N$ and $n = \gamma N$ we obtain

$$\|\vec{x}_{\gamma N+l}\| - \sum_{i=1-l}^{k} \|g_i\| M_N \leq \|g_{1-l}\|(\|\vec{x}_{\gamma N+l-1}\| - M_N) + \ldots + \|g_k\|(\|\vec{x}_{\gamma N-k}\| - M_N) \leq 0$$

and so

$$\|\vec{x}_{\gamma N+l}\| \leq \sum_{i=1-l}^{k} \|g_i\| M_N \leq a M_N < M_N.$$

Similarly, by taking $n = \gamma N + 1$ in (11) we obtain

$$\|\vec{x}_{\gamma N+l+1}\| - \sum_{i=1-l}^{k} \|g_i\| M_N \leq \|g_{1-l}\|(\|\vec{x}_{\gamma N+l}\| - M_N) + \ldots + \|g_k\|(\|\vec{x}_{\gamma N-k+1}\| - M_N) \leq 0$$

and so

$$\|\vec{x}_{\gamma N+l+1}\| \leq \sum_{i=1-l}^{k} \|g_i\| M_N \leq a M_N < M_N.$$

Hence by induction we have that

$$\|\vec{x}_{\gamma N+l+j}\| \leq \sum_{i=1-l}^{k} \|g_i\| M_N \leq a M_N < M_N.$$

Thus

$$M_{N+1} \leq a M_N < M_N, \tag{13}$$

and so the sequence $\{M_N\}_{N=0}^{\infty}$ is decreasing sequence bounded below by zero. Furthermore (13) implies

$$M_N \leq a^{N+1} M_0 \to 0 \qquad as \qquad N \to \infty.$$

Hence

$$0 \leq \lim_{N\to\infty} \vec{x}_{\gamma N-j} \leq \lim_{N\to\infty} M_N = 0, \quad j = 1-l, \ldots, k.$$

Therefore

$$\lim_{n\to\infty} \vec{x}_n = \vec{0}.$$

□

Corollary 1 *Suppose that equation* (1) *has the linearization* (3), *where* $l = 1$ *and the functions* $g_i : R^{k+1} \to M_{p\times p}$ *are such that*

$$\sum_{i=0}^{k} \|g_i\| \leq a < 1, \quad n = 0, 1, \ldots.$$

Then if equation (1) *has a zero equilibrium it is globally asymptotically stable.*

Assuming that f is differentiable in some neighborhood of the equilibrium solution $\bar{x}$, by applying Theorem 2 and Lemma 2 to the standard linearization of equation (1) about the equilibrium solution $\bar{x}$

$$\vec{x}_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x})\vec{x}_{n-i}, \quad n = 0, 1, \ldots, \tag{14}$$

where $\frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x})$ is the Jacobian matrix evaluated at the equilibrium point, we obtain the following result, which is local in the nature because of the fact that the standard linearization is local.

Corollary 2 *Assume that* f *is differentiable in some neighborhood of the equilibrium solution* $\bar{x}$. *The equilibrium* $\bar{x}$ *of equation* (1) *is locally asymptotically stable if*

$$\sum_{i=0}^{k} \|\frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x})\| \leq a < 1.$$

3 The case when g_i are scalar functions

In this section we consider the case when all g_i are scalar functions. In this case the linearization (3) is equivalent to p scalar equations of the form

$$x_{n+1}^m = \sum_{i=1-l}^{k} g_i x_{n-i}^m, \quad n = 0, 1, \ldots; m = 1, \ldots, p. \tag{15}$$

For instance, in the case of second order difference equation in $\mathbb{R}^2$, we have that vector equation

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = g_0 \begin{bmatrix} x_n \\ y_n \end{bmatrix} + g_1 \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} \quad n = 0, 1, \ldots \qquad g_0, g_1 \geq 0 \tag{16}$$

is equivalent to the system

$$\begin{aligned} x_{n+1} &= g_0 x_n + g_1 x_{n-1} \\ y_{n+1} &= g_0 y_n + g_1 y_{n-1}. \end{aligned} \tag{17}$$

The next results apply to a special linearization (3) of equation (1), where all g_i are scalar functions.

Theorem 3 *Let $l \in \{1, 2, \ldots\}$. Suppose that equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that*

$$\sum_{i=1-l}^{k} g_i \geq a > 1, \quad n \geq 0.$$

Then if for some $n \geq 0$

(a) *$\vec{x}_{n+l-1}, \ldots, \vec{x}_{n-k} > 0$, then $\lim_{n\to\infty} \vec{x}_n = \infty$, componentwise;*

(b) *$\vec{x}_{n+l-1}, \ldots, \vec{x}_{n-k} < 0$, then $\lim_{n\to\infty} \vec{x}_n = -\infty$, componentwise.*

Proof. Proof follows from Theorem 2 in [4] applied to equation(15). □

A delicate case when

$$\sum_{i=1-l}^{k} g_i = 1, \quad n = 0, 1, \ldots. \tag{18}$$

is treated in the following three results.

Theorem 4 *Suppose that on some interval I equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that (18) is satisfied. Then there exists $A > 0$ such that for $n \geq 0$ every positive g_i satisfies*

$$A \leq g_i \leq 1, \quad n = 0, 1, \ldots. \tag{19}$$

Proof. Proof follows from Proposition 3 in [4] applied to equation (15). □

Theorem 5 *Suppose that on some interval I equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that (18) is satisfied. Assume that there exists $A > 0$ such that*

$$g_{1-l} \geq A, \quad n = 0, 1, \ldots. \tag{20}$$

Then if $\vec{x}_{-k}, \ldots, \vec{x}_0 \in I$

$$\lim_{n\to\infty} \vec{x}_n = \vec{L},$$

where $\vec{L} \in I^p$ is a constant vector

Proof. Proof follows from Theorem 4 in [4] applied to equation (15). □

Theorem 6 *Suppose that on some interval $I \subset \mathbb{R}$ equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that (18) is satisfied. Assume that there exists $A > 0$ such that for some $j \in \{2-l, \ldots, k-1\}$*

$$g_j \geq A, g_{j+1} \geq A, \quad n = 0, 1, \ldots. \tag{21}$$

If $\vec{x}_{l-1}, \ldots, \vec{x}_{-k} \in I$, then

$$\lim_{n\to\infty} \vec{x}_n = \vec{L},$$

where $\vec{L} \in I^p$ is a constant vector

Proof. Proof follows from Theorem 5 in [4] applied to equation (15). □

4 Examples

In this section we present some examples that illustrate our results.

Example 1 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & b_n \\ c_n & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \ldots,$$

where $a, d > 0, b_n, c_n \geq 0, x_0, y_0 \geq 0, n = 0, 1, \ldots$, converges to the zero equilibrium if $\max\{a + U_c, d + U_b\} < 1$ is satisfied, where U_b and U_c are upper bounds of sequences $\{b_n\}$ and $\{c_n\}$ respectively. Indeed, in this case if $\|x\|$ denotes the L_1 norm we have

$$\|g_0\| = \left\| \begin{bmatrix} \frac{a}{1+x_n} & b_n \\ c_n & \frac{d}{1+y_n} \end{bmatrix} \right\| = \max\left\{ \frac{a}{1+x_n} + c_n, \frac{d}{1+y_n} + b_n \right\} \leq \max\{a + U_c, d + U_b\} < 1,$$

that is $U_c < 1 - a, U_b < 1 - d$, and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable. If we use L_2 norm we have that the zero equilibrium is globally asymptotically stable if $\max\{a + U_b, d + U_c\} < 1$ is satisfied.

Example 2 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & b \\ c & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \ldots, \tag{22}$$

where $a, b, c, d > 0, x_0, y_0 \geq 0$, converges to the zero equilibrium if $\max\{a + c, b + d\} < 1$ is satisfied. Indeed, in this case if $\|x\|$ denotes the L_1 norm we have that

$$\|g_0\| = \left\| \begin{bmatrix} \frac{a}{1+x_n} & b \\ c & \frac{d}{1+y_n} \end{bmatrix} \right\| = \max\left\{ \frac{a}{1+x_n} + c, \frac{d}{1+y_n} + b \right\} \leq \max\{a + c, b + d\} < 1$$

and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable. If we use L_2 norm we have that $\max\{a + b, c + d\} < 1$ implies that the zero equilibrium is globally asymptotically stable.

Next, consider the positive equilibrium $E(\bar{x}, \bar{y})$. Then we have that the positive equilibrium $E(\bar{x}, \bar{y})$ of system (22) satisfies the system

$$\begin{array}{rcl} \bar{x} & = & a\frac{\bar{x}}{1+\bar{x}} + b\bar{y} \\ \bar{y} & = & c\bar{x} + d\frac{\bar{y}}{1+\bar{y}}. \end{array} \tag{23}$$

which implies

$$\begin{array}{rcl} \bar{x}\frac{1+\bar{x}-a}{1+\bar{x}} & = & b\bar{y} \\ \bar{y}\frac{1+\bar{y}-d}{1+\bar{y}} & = & c\bar{x}. \end{array}$$

Thus the positive equilibrium exists if

$$\bar{x} > a - 1, \bar{y} > d - 1. \tag{24}$$

Linearizing system (22) about the positive equilibrium E gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{(1+\bar{x})(1+x_n)} & b \\ c & \frac{d}{(1+\bar{y})(1+y_n)} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, n = 0, 1, \ldots, \tag{25}$$

where $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition for global asymptotic stability of $E(\bar{x}, \bar{y})$ to be

$$\bar{x} > \frac{a + c - 1}{1 - c} \quad \text{if} \quad c < 1 < a + c, \quad \bar{y} > \frac{b + d - 1}{1 - b} \quad \text{if} \quad b < 1 < b + d.$$

If we use L_2 norm we obtain sufficient condition for global asymptotic stability of $E(\bar{x},\bar{y})$ to be

$$\bar{x} > \frac{a+b-1}{1-b} \quad \text{if} \quad b<1<a+b, \quad \bar{y} > \frac{c+d-1}{1-c} \quad \text{if} \quad c<1<c+d.$$

Example 3 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n=0,1,\ldots, \tag{26}$$

where $a,b,c,d>0, x_0,y_0 \geq 0, n=0,1,\ldots$, converges to the zero equilibrium if $\max\{a+c,b+d\}<1$ is satisfied. Indeed, in this case if $\|x\|_1$ denotes the L_1 norm we have

$$\|g_0\|_1 = \left\| \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \right\|_1 = \max\left\{ \frac{a}{1+x_n} + \frac{c}{1+x_n}, \frac{b}{1+y_n} + \frac{d}{1+y_n} \right\} \leq \max\{a+c,b+d\} < 1$$

and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable.

In the case if $\|x\|_2$ denotes the L_2 norm we have

$$\|g_0\|_2 = \left\| \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \right\|_2 = \max\left\{ \frac{a}{1+x_n} + \frac{b}{1+y_n}, \frac{c}{1+x_n} + \frac{d}{1+y_n} \right\} \leq \max\{a+b,c+d\} < 1.$$

In this case the condition for global asymptotic stability of the zero equilibrium becomes $\max\{a+b,c+d\}<1$.

Now, consider global attractivity of the positive equilibrium $E(\bar{x},\bar{y})$ of system (26). The positive equilibrium of system (26) satisfies the system

$$\begin{array}{rcl} \bar{x} & = & a\frac{\bar{x}}{1+\bar{x}} + b\frac{\bar{y}}{1+\bar{y}} \\ \bar{y} & = & c\frac{\bar{x}}{1+\bar{x}} + d\frac{\bar{y}}{1+\bar{y}}. \end{array} \tag{27}$$

Adding two equations in (27) we obtain

$$\bar{x}+\bar{y} = (a+c)\frac{\bar{x}}{1+\bar{x}} + (b+d)\frac{\bar{y}}{1+\bar{y}},$$

which implies

$$\frac{\bar{x}}{1+\bar{x}}(1+\bar{x}-a-c) = \frac{\bar{y}}{1+\bar{y}}(b+d-1-\bar{y})$$

and so we obtain that the positive equilibrium satisfies

$$\bar{x} > a+c-1 \Leftrightarrow \bar{y} < b+d-1. \tag{28}$$

Linearizing system (26) about the positive equilibrium E gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{(1+\bar{x})(1+x_n)} & \frac{b}{(1+\bar{y})(1+y_n)} \\ \frac{c}{(1+\bar{x})(1+x_n)} & \frac{d}{(1+\bar{y})(1+y_n)} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, n=0,1,\ldots,$$

where $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition

$$\bar{x} > a+c-1, \bar{y} > b+d-1. \tag{29}$$

is sufficient for the global asymptotic stability of the positive equilibrium solution. The condition (29) contradicts condition (28). If we use L_2 norm we obtain sufficient condition for the global asymptotic stability of the positive equilibrium solution to be

$$\begin{array}{rcl} b\bar{x}+a\bar{y} & < & 1-a-b \\ d\bar{x}+c\bar{y} & < & 1-c-d. \end{array}$$

Example 4 Every solution of the vector equation in $\mathbb{R}^n$

$$\vec{x}_{n+1} = A_n \vec{x}_n \tag{30}$$

where

$$\vec{x}_n = \begin{bmatrix} x_n^1 \\ x_n^2 \\ \vdots \\ x_n^k \end{bmatrix}, \quad A_n = \begin{bmatrix} \frac{a_{11}}{1+x_n^1} & \frac{a_{12}}{1+x_n^2} & \cdots & \frac{a_{1k}}{1+x_n^k} \\ \frac{a_{21}}{1+x_n^1} & \frac{a_{22}}{1+x_n^2} & \cdots & \frac{a_{2k}}{1+x_n^k} \\ \vdots & & & \\ \frac{a_{k1}}{1+x_n^1} & \frac{a_{k2}}{1+x_n^2} & \cdots & \frac{a_{kk}}{1+x_n^k} \end{bmatrix}$$

where $a_{ij} > 0, i, j = 0, 1, \ldots \quad x_0, y_0 \geq 0, n = 0, 1, \ldots,$, converges to the zero equilibrium if

$$\|g_0\|_1 = \left\| \begin{bmatrix} \frac{a_{11}}{1+x_n^1} & \frac{a_{12}}{1+x_n^2} & \cdots & \frac{a_{1k}}{1+x_n^k} \\ \frac{a_{21}}{1+x_n^1} & \frac{a_{22}}{1+x_n^2} & \cdots & \frac{a_{2k}}{1+x_n^k} \\ \vdots & & & \\ \frac{a_{k1}}{1+x_n^1} & \frac{a_{k2}}{1+x_n^2} & \cdots & \frac{a_{kk}}{1+x_n^k} \end{bmatrix} \right\|_1$$

$$\begin{aligned} &= \max\left\{ \tfrac{a_{11}}{1+x_n^1} + \tfrac{a_{21}}{1+x_n^1} + \ldots + \tfrac{a_{k1}}{1+x_n^1}, \ldots, \tfrac{a_{1k}}{1+x_n^1} + \tfrac{a_{2k}}{1+x_n^1} + \ldots + \tfrac{a_{kk}}{1+x_n^1} \right\} \\ &\leq \max\{a_{11} + a_{21} + \ldots + a_{k1}, \ldots, a_{1k} + a_{2k} + \ldots + a_{kk}\} \\ &= \max_{1 \leq j \leq n} \{\sum_{i=1}^{k} a_{ij}\} < 1, \end{aligned}$$

which follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable.

Now, consider global attractivity of the positive equilibrium of system (30). The positive equilibrium satisfies the system

$$(A_n(\vec{\bar{x}}) - \mathbf{I})\vec{\bar{x}} = \vec{0},$$

where

$$A_n(\vec{\bar{x}}) = \begin{bmatrix} \frac{a_{11}}{1+\bar{x}^1} & \frac{a_{12}}{1+\bar{x}^2} & \cdots & \frac{a_{1k}}{1+\bar{x}^k} \\ \frac{a_{21}}{1+\bar{x}^1} & \frac{a_{22}}{1+\bar{x}^2} & \cdots & \frac{a_{2k}}{1+\bar{x}^k} \\ \vdots & & & \\ \frac{a_{k1}}{1+\bar{x}^1} & \frac{a_{k2}}{1+\bar{x}^2} & \cdots & \frac{a_{kk}}{1+\bar{x}^k} \end{bmatrix}.$$

Linearizing system (30) about the positive equilibrium E gives the following system

$$\vec{u}_{n+1} = \begin{bmatrix} \frac{a_{11}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{12}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{1k}}{(1+\bar{x})(1+x_n^k)} \\ \frac{a_{21}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{22}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{2k}}{(1+\bar{x})(1+x_n^k)} \\ \vdots & & & \\ \frac{a_{k1}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{k2}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{kk}}{(1+\bar{x})(1+x_n^k)} \end{bmatrix} \vec{u}_n, \quad n = 0, 1, \ldots,$$

where $\vec{u}_n = \vec{x}_n - \vec{\bar{x}}$. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition

$$\|g_0\|_1 = \left\| \begin{bmatrix} \frac{a_{11}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{12}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{1k}}{(1+\bar{x})(1+x_n^k)} \\ \frac{a_{21}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{22}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{2k}}{(1+\bar{x})(1+x_n^k)} \\ \vdots & & & \\ \frac{a_{k1}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{k2}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{kk}}{(1+\bar{x})(1+x_n^k)} \end{bmatrix} \right\|_1$$

$$\begin{aligned}
&= \max\left\{\tfrac{a_{11}}{(1+\bar{x})(1+x_n^1)} + \ldots + \tfrac{a_{k1}}{(1+\bar{x})(1+x_n^1)}, \ldots, \tfrac{a_{1k}}{(1+\bar{x})(1+x_n^k)} + \tfrac{a_{2k}}{(1+\bar{x})(1+x_n^k)} + \ldots + \tfrac{a_{kk}}{(1+\bar{x})(1+x_n^k)}\right\} \\
&\le \max\{\tfrac{1}{1+\bar{x}}(a_{11} + a_{21} + \ldots + a_{k1}, \ldots, a_{1k} + a_{2k} + \ldots + a_{kk})\} \\
&= \tfrac{1}{1+\bar{x}} \max_{1\le j\le n}\{\sum_{i=1}^{k} a_{ij}\} \\
&< 1
\end{aligned}$$

implies the global asymptotic stability of the positive equilibrium solution. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition for the global asymptotic stability of the positive equilibrium solution is

$$1 + \bar{x} > \sum_{i=1}^{k} a_{ij} \Longleftrightarrow \bar{x} > \sum_{i=1}^{k} a_{ij} - 1.$$

Example 5 The cooperative system

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \ldots, \tag{31}$$

where $a, b, c, d > 0$, $x_0, y_0 \ge 0$ was considered in [1]. The equilibrium solutions are the zero equilibrium $E_0(0,0)$ and when $a < 1, d < 1$ the unique positive equilibrium solution $E_+(\bar{x}, \bar{y})$, is given as

$$\bar{x} = \frac{b}{1-a}\frac{\bar{y}}{1+\bar{y}}, \quad \bar{y} = \frac{bc - (1-d)(1-a)}{(1-d)(b+1-a)},$$

when

$$(1-a)(1-d) < bc. \tag{32}$$

The local stability of system (31) is described with the following result, see [1]

Claim 1 *Consider system* (31).

1.) *The positive equilibrium $E_+(\bar{x}, \bar{y})$ of system* (31) *is locally asymptotically stable when* (32) *holds.*

2.) *The zero equilibrium $E_0(0,0)$ of system* (31) *is locally asymptotically stable if $bc < (1-a)(1-d)$; it is a saddle point if $bc > (1-a)(1-d)$; it is a nonhyperbolic equilibrium if $bc = (1-a)(1-d)$.*

The global dynamics of system (31) is described with the following result, see [1]:

Theorem 7 *Consider system* (31).

1.) *If $a \ge 1$ then $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n = \infty$ if $d \ge 1$ and $\lim_{n\to\infty} y_n = \frac{c}{1-d}$, if $d < 1$.*

2.) *If $d \ge 1$ then $\lim_{n\to\infty} y_n = \infty$ and $\lim_{n\to\infty} x_n = \infty$ if $a \ge 1$ and $\lim_{n\to\infty} x_n = \frac{b}{1-a}$, if $a < 1$.*

3.) *The positive equilibrium $E_+(\bar{x}, \bar{y})$ of system* (31) *is globally asymptotically stable when* (32) *holds.*

4.) *The zero equilibrium $E_+(\bar{x}, \bar{y})$ of system* (31) *is globally asymptotically stable when $a < 1, d < 1$ and*

$$bc \le (1-a)(1-d) \tag{33}$$

holds.

Theorem 2 and Corollary 1 implies that any of two conditions $\max\{a+c, b+d\} < 1$ or $\max\{a+b, c+d\} < 1$ provides the global asymptotic stability of the zero equilibrium. Both of these conditions imply (33) which is clearly the necessary and sufficient condition for the global asymptotic stability of the zero equilibrium..

Linearizing system (31) about the positive equilibrium $E(\bar{x}, \bar{y})$ gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} a & \frac{b}{(1+\bar{y})(1+y_n)} \\ \frac{c}{(1+\bar{x})(1+x_n)} & d \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \quad n = 0, 1, \dots,$$

where $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$. By using Theorem 2 and Corollary 1 with L_1 or L_2 norm, we obtain that the condition

$$\max\left\{a + \frac{c}{1+\bar{x}}, \frac{b}{1+\bar{y}} + d\right\} < 1 \quad \text{or} \quad \max\left\{a + \frac{b}{1+\bar{y}}, \frac{c}{1+\bar{x}} + d\right\} < 1 \tag{34}$$

implies that the positive equilibrium $E(\bar{x}, \bar{y})$ is globally asymptotically stable. Condition (34) implies condition (32) which is clearly the necessary and sufficient condition for the global asymptotic stability of the positive equilibrium.

Example 6 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{an}{1+n^2} & \frac{cn}{1+n^3} \\ \frac{bn}{1+n^2} & \frac{dn}{1+n^3} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} \frac{An}{1+n} & \frac{Cn}{1+n^2} \\ \frac{Bn}{1+n} & \frac{Dn}{1+n^2} \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, n = 0, 1, \dots,$$

where $a, b, c, d, A, B, C, D > 0, x_{-1}, y_{-1}, x_0, y_0 \geq 0, n = 0, 1, \dots,$ converges to the zero equilibrium if $\max\{\frac{a+b}{2}, \frac{2(c+d)}{32^{1/3}}\} + \max\{A + B, \frac{C+D}{2}\} < 1$ is satisfied. Indeed, in this case if $\|x\|$ denotes the L_1 norm we have

$$\|g_0\| = \left\| \begin{bmatrix} \frac{an}{1+n^2} & \frac{cn}{1+n^3} \\ \frac{bn}{1+n^2} & \frac{dn}{1+n^3} \end{bmatrix} \right\| = \max\left\{\frac{(a+b)n}{1+n^2}, \frac{(c+d)n}{1+n^3}\right\} \leq \max\left\{\frac{a+b}{2}, \frac{2(c+d)}{32^{1/3}}\right\}$$

and

$$\|g_1\| = \left\| \begin{bmatrix} \frac{An}{1+n} & \frac{Cn}{1+n^2} \\ \frac{Bn}{1+n} & \frac{Dn}{1+n^2} \end{bmatrix} \right\| = \max\left\{\frac{(A+B)n}{1+n}, \frac{(C+D)n}{1+n^2}\right\} \leq \max\left\{A + B, \frac{C+D}{2}\right\}$$

and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable.

Example 7 The vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{ax_n}{1+x_n} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \frac{a}{1+x_n} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, n = 0, 1, \dots \tag{35}$$

is equivalent to the system

$$\begin{aligned} x_{n+1} &= \tfrac{ax_n}{1+x_n} x_n + \tfrac{a}{1+x_n} x_{n-1} \\ y_{n+1} &= \tfrac{ax_n}{1+x_n} y_n + \tfrac{a}{1+x_n} y_{n-1}, \quad n = 0, 1, \dots, \end{aligned}$$

where $a > 0$. Since $g_0 + g_1 = a$ for all $n = 0, 1, \dots$ we have the following result which proof follows from Theorems 2, 3, 5 and Corollary 1.

Proposition 1 *The following trichotomy holds for equation* (35)*:*

(a) *if $a < 1$ then the zero equilibrium of* (35) *is globally asymptotically stable.*

(b) *if $a = 1$ then every nonnegative constant vector $\vec{L}$ is an equilibrium of* (35) *and every solution of* (35) *converges to some constant vector.*

(a) *if $a > 1$ then every set of positive (resp. negative) initial conditions generates the solution which component-wise tends to ∞ (resp. $-\infty$).*

Proposition 1 can be extended to the case of corresponding vector equation in $\mathbb{R}^p$.

Acknowledgements. M.R.S. Kulenović is supported in part by Maitland P. Simmons Foundation.

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Volume 23, Number 8 December 2017
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

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Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.8, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

The Naimark-Sacker bifurcation and symptotic approximation of the invariant curve of a certain difference equation

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September 11, 2016

Abstract

We compute the direction of the Naimark-Sacker bifurcation for the difference equation $x_{n+1} = p + \frac{x_n^2}{x_{n-1}^2}$ where p is a positive number and the initial conditions x_{-1} and x_0 are positive numbers. Furthermore, we give the asymptotic approximation of the invariant curve.

Keywords: difference equation, Naimark-Sacker bifurcation, normal form. invariant curve, stability.
AMS 2010 Mathematics Subject Classification: 39A10, 39A20, 65L20

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†Supported in part by FMON of Bosnia and Herzegovina, number 05-39-3935-1/15.

1 Introduction and Preliminaries

In this paper we consider the difference equation

$$x_{n+1} = p + \frac{x_n^2}{x_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{1}$$

where the parameter a is positive number and the initial conditions x_{-1} and x_0 are positive numbers. Clearly equation (1) has the unique equilibrium point $\bar{x} = p + 1$. Linear fractional version of equation (1)

$$x_{n+1} = p + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots, \tag{2}$$

was considered in [3], where we proved that the unique equilibrium $\bar{x} = p+1$ of equation (2) is globally asymptotically stable. Introduction of quadratic terms into equation (2) changes local stability analysis and consequently the global dynamics as well. In particular, quadratic terms introduces the possibility of Naimark-Sacker bifurcation and the existence of locally stable periodic solution, see [6] for several similar examples.

The linearized equation of equation (2) at the equilibrium point $\bar{x} = p + 1$ is

$$z_{n+1} = \frac{2}{p+1} z_n - \frac{2}{p+1}, \quad n = 0, 1, \ldots,$$

with the characteristic equation

$$\lambda^2 - \frac{2}{p+1}\lambda + \frac{2}{p+1} = 0,$$

and the characteristic roots

$$\lambda_{\pm} = \frac{1 \pm i\sqrt{2p+1}}{p+1}.$$

Since

$$|\lambda_{\pm}| = \sqrt{\frac{2}{p+1}}$$

it is clear that that the equilibrium point $\bar{x} = p + 1$ is asymptotically stable if $p > 1$, non-hyperbolic if $p = 1$ and unstable if $p < 1$. In all cases the eigenvalues are complex conjugate numbers which indicates the presence of the Naimark-Sacker bifurcation at $p = 1$. We will prove that indeed the equilibrium point $\bar{x} = p + 1$ is globally asymptotically stable if $p > \sqrt{2}$ and that the Naimark-Sacker bifurcation takes the place at $p = 1$. Our tool in proving global asymptotic stability of equation (2) is the result in [3, 5]. We conjecture that the equilibrium point $\bar{x} = p + 1$ is globally asymptotically stable if $a > 1$. Furthermore, we give some numeric values of parameter a with corresponding periodic solutions. Our bifurcation diagram indicates a complicated behavior and possible chaos for the values $p < 1$.

Now, for the sake of completness we give the basic facts about the Naimark-Sacker bifurcation.

The Hopf bifurcation is well known phenomenon for a system of ordinary differential equations in two or more dimension, whereby, when some parameter is varied, a pair of complex conjugate eigenvalues of the Jacobian matrix at a fixed point crosses the imaginary axis, so

that the fixed point changes its behavior from stable to unstable and a limit cycle appears. In the discrete setting, the Naimark-Sacker bifurcation is the discrete analogue of the Hopf bifurcation.

The Naimark-Sacker bifurcation occurs for a discrete system depending on a parameter, λ say, with a fixed point whose Jacobian has a pair of complex conjugate $\mu(\lambda)$, $\bar{\mu}(\lambda)$ which cross the unit transversally at $\lambda = \lambda_0$.

The following result is referred as the Neimark-Sacker bifurcation Theorem [1, 4, 7, 8, 11].

Theorem 1 (Naimark-Sacker bifurcation) *Let*

$$\mathbf{F} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to \mathbf{F}(\lambda, \mathbf{x})$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) $F(\lambda, \mathbf{0}) = 0$ *for λ near some fixed λ_0;*

(ii) $DF(\lambda, \mathbf{0})$ *has two non-real eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ for λ near λ_0 with $|\mu(\lambda_0)| = 1$;*

(iii) $\frac{d}{d\lambda}|\mu(\lambda)| = d(\lambda_0) < 0$ *at $\lambda = \lambda_0$ (transversality condition);*

(iv) $\mu^k(\lambda_0) \neq 1$ *for $k = 1, 2, 3, 4$. (nonresonance condition).*

Then there is a smooth λ-dependent change of coordinate bringing F into the form

$$F(\lambda, \mathbf{x}) = \mathcal{F}(\lambda, \mathbf{x}) + O(\| \mathbf{x} \|^5)$$

and there are smooth function $a(\lambda)$, $b(\lambda)$, and $\omega(\lambda)$ so that in polar coordinates the function $\mathcal{F}(\lambda, x)$ is given by

$$\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} |\mu(\lambda)|r + a(\lambda)r^3 \\ \theta + \omega(\lambda) + b(\lambda)r^2 \end{pmatrix}. \tag{3}$$

If $a(\lambda_0) < 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then ω-limit set of x_0 is the origin if $\lambda > \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda < \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.
If $a(\lambda_0) > 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then α-limit set of x_0 is the origin if $\lambda < \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda > \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.

Consider a general map $\mathbf{F}(\lambda_0, \mathbf{x})$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda_0) = \alpha(\lambda_0) + i\beta(\lambda_0)$ and $\bar{\mu}(\lambda_0) = \alpha(\lambda_0) - i\beta(\lambda_0)$ satisfying $\alpha(\lambda_0)^2 + \beta(\lambda_0)^2 = 1$ and $\beta(\lambda_0) \neq 0$. Assume that

$$\mathbf{F}(\lambda_0, \mathbf{x}) = \mathbf{A}(\lambda_0)\mathbf{x} + \mathbf{G}(\lambda_0, \mathbf{x}) \tag{4}$$

where $\mathbf{A}$ is Jacobian matrix of $\mathbf{F}$ evaluated at fixed point $(0, 0)$, and

$$\mathbf{G}(\lambda_0, \mathbf{x}) := \begin{pmatrix} g_1(\lambda_0, x_1, x_2) \\ g_2(\lambda_0, x_1, x_2) \end{pmatrix}.$$

Here we donate $\mu(\lambda_0) = \mu$, $\mathbf{A}(\lambda_0) = \mathbf{A}$ and $\mathbf{G}(\lambda_0, x) = \mathbf{G}(\mathbf{x})$. We let $\mathbf{p}$ and $\mathbf{q}$ be eigenvectors of A associated with μ satisfying

$$\mathbf{Aq} = \mu\mathbf{q}, \quad \mathbf{p}A = \mu\mathbf{p}, \quad \mathbf{pq} = 1$$

and $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$. Assume that

$$\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \frac{1}{2}(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2) + O(|z|^3)$$

and

$$\begin{array}{l} \mathbf{K}_{20} = (\mu^2 I - A)^{-1}\mathbf{g}_{20} \\ \mathbf{K}_{11} = (I - A)^{-1}\mathbf{g}_{11} \quad . \\ \mathbf{K}_{02} = (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02} \end{array} \tag{5}$$

Let

$$\begin{aligned} \mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}(\mathbf{K}_{20}\xi^2 + 2\mathbf{K}_{11}\xi\bar{\xi} + \mathbf{K}_{02}\bar{\xi}^2)\right) \\ = \frac{1}{2}(\mathbf{g}_{20}\xi^2 + 2\mathbf{g}_{11}\xi\bar{\xi} + \mathbf{g}_{02}\bar{\xi}^2) \\ + \frac{1}{6}(\mathbf{g}_{30}\xi^3 + 3\mathbf{g}_{21}\xi^2\bar{\xi} + 3\mathbf{g}_{12}\xi\bar{\xi}^2 + \mathbf{g}_{03}\bar{\xi}^3) + O(|\xi|^4), \end{aligned} \tag{6}$$

then

$$a(\lambda_0) = \frac{1}{2}Re(\mathbf{p}\mathbf{g}_{21}\bar{\mu}).$$

Corollary 1 ([9]) *Assume $a(\lambda_0) \neq 0$ and $\lambda = \lambda_0 + \eta$ where η is a sufficient small parameter. If $\bar{\mathbf{x}}$ is fixed point of F then invariant curve $\Gamma(\lambda)$ from Theorem 1 can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \bar{\mathbf{x}} + 2\rho_0 Re\left(\mathbf{q}e^{i\theta}\right) + \rho_0^2\left(Re\left(\mathbf{K}_{20}e^{2i\theta}\right) + \mathbf{K}_{11}\right),$$

where

$$d = \frac{d}{d\eta}|\mu(\lambda)|\Big|_{\lambda=\lambda_0}, \quad \rho_0 = \sqrt{-\frac{d}{a}\eta}, \quad \theta \in \mathbb{R}.$$

Here "Re" represents the real parts of those complex numbers.

The second section of the paper gives global asymptotic stability result for the values of parameter $p > \sqrt{2}$ and the third section gives the reduction to the normal form and computation of the coefficients of the Naimark-Sacker bifurcation and the asymptotic approximation of the invariant curve. Our computational method is based on the computational algorithm developed in [9] rather than more often used computational algorithm in [10]. The advantage of the computational algorithm of [9] lies in the fact that this algorithm computes also the approximate equation of the invariant curve in Naimark-Sacker theorem, which is not provided by Wan's algorithm. Here we give numeric and visual eveidence that the approximate equation of the invariant curve is accurate. See Figure 4.

2 Global Asymptotic Stability

We use the method of embedding [2]. By substituting

$$x_n = p + \left(\frac{x_{n-1}}{x_{n-2}}\right)^2$$

in equation (1) we get:

$$x_{n+1} = p + \left(\frac{p}{x_{n-1}} + \frac{x_{n-1}}{x_{n-2}^2}\right)^2.$$

Now by substituting for x_{n-1} in the term $\frac{x_{n-1}}{x_{n-2}^2}$ of the last equation we we obtain

$$x_{n+1} = p + \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2}\right)^2. \tag{7}$$

From equation (7) we observe that $p<x_n<p+(1+\frac{1}{p}+\frac{1}{p^2})^2$ for $n \geq 4$.

Also from (1) and (7) we have:

$$\begin{cases} x_{n+1} - p = \left(\frac{x_n}{x_{n-1}}\right)^2 \\ x_{n+1} - p = \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2}\right)^2 \end{cases}.$$

Consequently

$$\left(\frac{x_n}{x_{n-1}}\right)^2 = \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2}\right)^2,$$

which implies:

$$x_{n+1} = p + \frac{px_n}{x_{n-1}^2} + \frac{x_n}{x_{n-2}^2}. \tag{8}$$

Replacing x_n in (8) by $p + \left(\frac{x_{n-1}}{x_{n-2}}\right)^2$ we obtain the equation

$$x_{n+1} = p + \frac{a^2}{x_{n-1}^2} + \frac{p + x_n}{x_{n-2}^2}. \tag{9}$$

Observe now that every solution of equation (1) is also a solution of equation (9), with initial values x_{-2}, x_{-1} and $x_0 = p + (\frac{x_0}{x_{-1}})^2$.
Observe also that it is of the form $x_{n+1} = f(x_n, x_{n-1}, x_{n-2})$ where :

$$f(u,v,w) = p + \frac{p^2}{v^2} + \frac{p+u}{w^2}$$

.

Theorem 2 *If $p > \sqrt{2}$ then the equilibrium of equation (1) is globally asymptotically stable.*

Proof. First we show that every interval I of the form $[p, \mathcal{U}]$ where $\mathcal{U} \geq \frac{p(p^2+p+1)}{(p^2-1)}$ with $p > 1$ is invariant for the function f.

Let $\mathcal{U} > p$ then $I = [p, \mathcal{U}]$ is invariant if and only if for all $u, v, w \in I, f(u,v,w) \in I$ that is:

$$p \leq p + \frac{p^2}{v^2} + \frac{p+u}{w^2} \leq \mathcal{U}.$$

As $p \leq u, v, w \leq \mathcal{U}$ we have that: $p \leq f(u, v, w) \leq p + 1 + \frac{1}{p} + \frac{\mathcal{U}}{p^2}$. We also know that if $\mathcal{U}$ satisfies: $p + 1 + \frac{1}{p} + \frac{\mathcal{U}}{p^2} \leq \mathcal{U}$ then we have

$$f(u, v, w) \leq \mathcal{U}.$$

It follows that given $p > 1$ such $\mathcal{U}$ exists and therefore I is invariant for f where $\mathcal{U} \geq \frac{p(p^2+p+1)}{(p^2-1)}$. In the following we may assume $p > 1$ and $\mathcal{U} = \frac{p(p^2+p+1)}{(p^2-1)}$, so I is invariant by f.

Next, we prove that I is an attracting interval, that is every solution of equation (8) must enter the interval I. Observe that given the initial values x_{-2}, x_{-1} and x_0 for equation (8), we have $x_n > p$ for $n \geq 1$.

Now if $x_3 \leq \mathcal{U}$ then $x_n \in [p, \mathcal{U}]$ for all $n \geq 3$. Otherwise, from equation (4) given that x_{n-2} , $x_{n-3} > p$ we have

$$x_n < p + 1 + \frac{1}{p} + \frac{x_{n-1}}{p^2},$$

that is if we set $A = p + 1 + \frac{1}{p}$

$$x_n < A + \frac{x_{n-1}}{p^2}.$$

Thus by induction we can conclude that

$$x_n < A\frac{1 - (\frac{1}{p^2})^{n-3}}{1 - \frac{1}{p^2}} + \frac{x_3}{(p^2)^{n-3}}. \tag{10}$$

It is straightforward to check that when $x_3 > \mathcal{U}$ the right hand side of (10) is a decreasing sequence that converges to A $(\frac{1}{1-\frac{1}{p^2}})$. This limit is in fact $\mathcal{U} = \frac{p(p^2+p+1)}{(p^2-1)}$. It follows that there must exist $k > 3$ such that: $a < x_k < \mathcal{U}$ Otherwise x_n must converge to $\mathcal{U}$ which is impossible.

Thus we have $x_{k-1}, x_{k-2} > p$ and $x_k \leq \mathcal{U}$, hence $x_{k+1} \in [a, \mathcal{U}]$ it follows by induction that $x_n \in [p, \mathcal{U}]$ for $n \geq k$.

Consequently every solution of equation (8) must enter the interval $[p, \mathcal{U}]$.

Now that we have an invariant and attracting interval we check the conditions of Theorem A.0.5 [3]:

$$\begin{cases} f(M, m, m) = M \\ f(m, M, M) = m \end{cases} \Leftrightarrow \begin{cases} M = p + \frac{p^2+p+M}{m^2} \\ m = p + \frac{p^2+p+m}{M^2} \end{cases}.$$

From the second equation we get

$$M^2 = \frac{p^2 + p + m}{m - p}. \tag{11}$$

On the other hand the system is equivalent to:

$$\begin{cases} (M - p)m^2 = p^2 + p + M \\ (m - p)M^2 = p^2 + p + m \end{cases} \Leftrightarrow \begin{cases} Mm^2 = pm^2 + p^2 + p + M \\ mM^2 = pM^2 + p^2 + p + m \end{cases}$$

By subtracting the second equation from the first we obtain:

$$Mm(m-M) = p(m-M)(m+M) - (m-M)$$

and given that $m \neq M$ we have:

$$Mm = p(m+M) - 1$$

which implies:

$$M = \frac{pm-1}{m-p}. \tag{12}$$

Equations (11) and (12) yield

$$\frac{(pm-1)^2}{(m-p)^2} = \frac{p^2+p+m}{m-p},$$

which implies:

$$(pm-1)^2 = (p^2+p+m)(m-p).$$

This leads to the following quadratic equation:

$$m^2(p^2-1) - m(p^2+2p) + p^2(p+1) + 1 = 0,$$

which discriminant is

$$\Delta = (p^2+2p)^2 - 4(p^2-1)(p^2(p+1)+1)$$

and

$$\Delta = -4p^5 - 3p^4 + 8p^3 + 4p^2 + 4 = (\sqrt{2}-p)(4p^4 + (3+4\sqrt{2})p^3 + 3\sqrt{2}p^2 + 2p + 2\sqrt{2}).$$

It is clear that when $a > \sqrt{2}$ there is no real solutions. and when $p = \sqrt{2}$ there is one unique solution $m = p+1 = M$. Consequently if $a \geq \sqrt{2}$ the conditions of Theorem A.0.5 [3] or Theorem 1 [5] are fully satisfied and therefore every solution must converge to the unique equilibrium $(p+1)$ □

Conjecture 1 *The equilibrium point $\bar{x} = p+1$ of equation* (2) *is globally asymptotically stable if $p > 1$.*

Remark 1 It could have been easier to prove the fact if we restrict the set of solutions of equation (4) to the ones satisfied by equation (1) as the solutions must oscillate about the equilibrium $(p+1)$ that is there exist k such that: $p < x_k < p+1 < \mathcal{U}$.

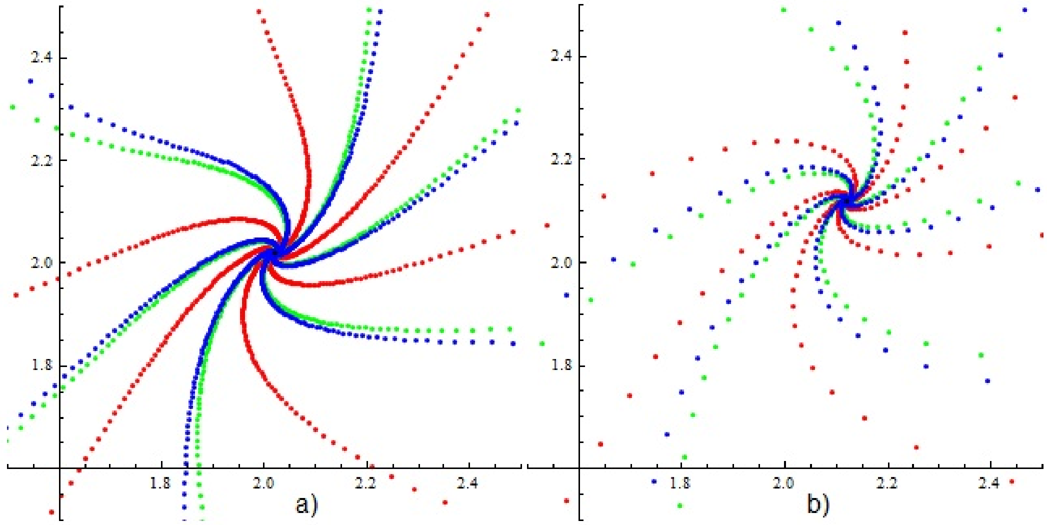

Figure 1: a) Phase diagrams when $n = 10,000$ and a) $p = 1.02$ b) $p = 1.12$

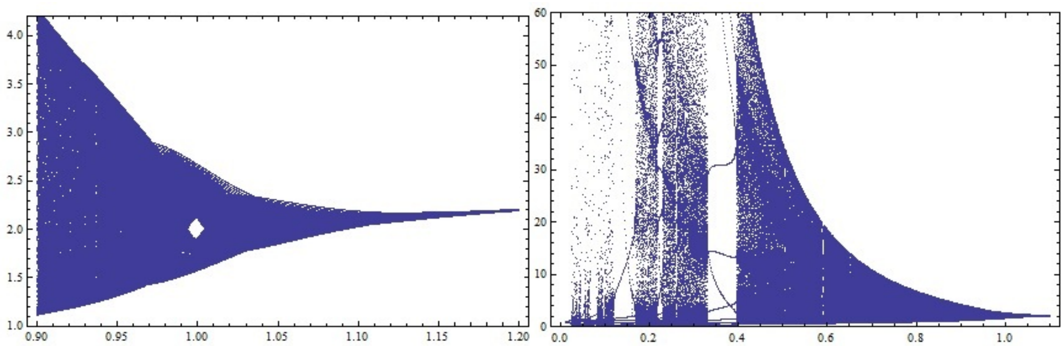

Figure 2: Bifurcation diagrams in $(p - x)$ plane.

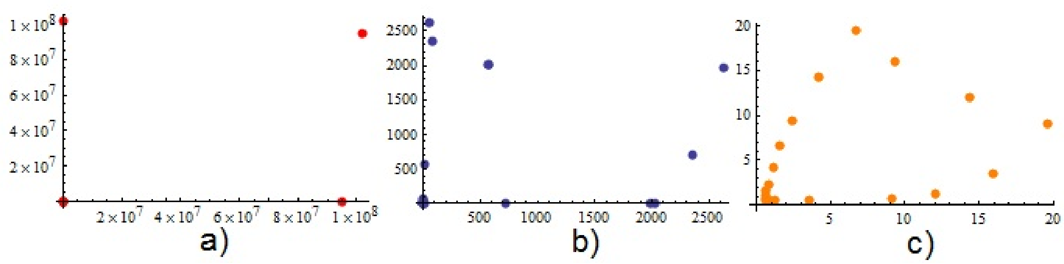

Figure 3: Periodic orbit for a) $p = 0.01$ b) $p = 0.15$ c) $p = 0.5901$ (See Table 2).

3 Reduction to the normal form

If we make a change of variable $y_n = x_n - \bar{x}$, then the transformed equation is given by

$$y_{n+1} = \frac{(p + y_n + 1)^2}{(p + y_{n-1} + 1)^2} - 1, \quad n = 0, 1, \ldots. \tag{13}$$

a	Period of the sol.	Solution
0.01	8	$\{0.877631, 0.01, 0.0101298, 1.03613, 10462.3, 1.01959 \times 10^8, 9.49713 \times 10^7, 0.877631\}$
0.15	20	$\{574.846, 2023.71, 12.5435, 0.150038, 0.150143, 1.1514, 58.9583, 2622.2, 1978.22, 0.719138, 0.15, 0.193507, 1.81422, 88.0493, 2355.59, 715.88, 0.242359, 0.15, 0.533058, 12.7789\}$
0.5901	19	$\{0.804816, 0.597988, 1.14217, 4.23826, 14.3595, 12.0691, 1.29653, 0.60164, 0.805431, 2.38228, 9.33854, 15.9565, 3.50965, 0.638479, 0.623195, 1.5428, 6.71883, 19.5558, 9.06166\}$

Table 1: Periodic solutions for some values of p.

Set

$$u_n = y_{n-1} \text{ and } v_n = y_n \text{ for } n = 0, 1, \ldots$$

and write Eq.(1) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{(p+v_n+1)^2}{(p+u_n+1)^2} - 1. \end{aligned} \tag{14}$$

Let F be the corresponding map defined by:

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{(p+v+1)^2}{(p+u+1)^2} - 1 \end{pmatrix}. \tag{15}$$

Then $\mathbf{F}$ has the unique fixed point $(0,0)$ and the Jacobian matrix of $\mathbf{F}$ at $(0,0)$ is given by

$$Jac_{\mathbf{F}}(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{2}{p+1} & \frac{2}{p+1} \end{pmatrix}$$

It is easy to see that

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{2}{p+1} & \frac{2}{p+1} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix}, \tag{16}$$

where

$$\mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{(p+v+1)^2}{(p+u+1)^2} + \frac{2u}{p+1} - \frac{2v}{p+1} - 1 \end{pmatrix}.$$

The eigenvalues of $Jac_{\mathbf{F}}(0,0)$ are $\mu(p)$ and $\overline{\mu(p)}$ where

$$\mu(p) = \frac{1 + i\sqrt{2p+1}}{p+1}, \quad |\mu(p)| = \sqrt{\frac{2}{p+1}}.$$

One can prove that for $p = p_0 = 1$ we obtain $\mu(p_0)| = 1$ and

$$\mu(p_0) = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \mu^2(p_0) = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \mu^3(p_0) = -1, \quad \mu^4(p_0) = -\frac{1}{2} - \frac{i\sqrt{3}}{2},$$

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from which follows that $\mu^k(p_0) \neq 1$ for $k = 1, 2, 3, 4$. Furthermore, we get

$$\frac{d}{dp}|\mu(p)| = -\frac{1}{\sqrt{2}}\left(\frac{1}{p+1}\right)^{3/2}, \quad \frac{d|\mu(p)|}{dp}\bigg|_{p=p_0} = -\frac{1}{4} < 0.$$

The eigenvectors of corresponding to $\mu(p)$ and $\overline{\mu(p)}$ are $\mathbf{q}(p)$ and $\overline{\mathbf{q}(p)}$, where

$$\mathbf{q}(p) = \left(\frac{1 - i\sqrt{2p+1}}{p+1}, 1\right)^T.$$

Substituting $p = p_0 = 1$ into (16) we get

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{17}$$

where

$$\mathbf{A} = Jac_{\mathbf{F}}(0,0)|_{p=1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 \\ \frac{(v+2)^2}{(u+2)^2} + u - v - 1 \end{pmatrix}.$$

Hence, for $p = p_0$ system (14) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \mathbf{G}\begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{18}$$

Define the basis of $\mathbb{R}^2$ by $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$, where $\mathbf{q} = \mathbf{q}(p_0)$, then we can represent (u, v) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (\mathbf{q}z + \bar{\mathbf{q}}\bar{z}) = \begin{pmatrix} \frac{1}{2}\left(1 + i\sqrt{3}\right)\bar{z} + \frac{1}{2}\left(1 - i\sqrt{3}\right)z \\ \bar{z} + z \end{pmatrix}.$$

By using this, we have

$$\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \frac{(\bar{z}+z+2)^2}{\left(\frac{1}{2}\left(1+i\sqrt{3}\right)\bar{z} + \frac{1}{2}\left(1-i\sqrt{3}\right)z+2\right)^2} + \frac{1}{2}\left(-1 + i\sqrt{3}\right)\bar{z} - \frac{1}{2}\left(1 + i\sqrt{3}\right)z - 1 \end{pmatrix} \tag{19}$$

Thus we obtain that

$$\begin{aligned} \mathbf{g}_{20} &= \frac{\partial^2}{\partial z^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ \frac{1}{4}i\left(\sqrt{3} + 5i\right) \end{pmatrix} \\ \mathbf{g}_{11} &= \frac{\partial^2}{\partial z \partial \bar{z}}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \mathbf{g}_{02} &= \frac{\partial^2}{\partial \bar{z}^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ -\frac{1}{4}i\left(\sqrt{3} - 5i\right) \end{pmatrix}, \end{aligned} \tag{20}$$

and

$$\begin{aligned} \mathbf{K}_{20} &= (\mu^2 I - A)^{-1}\mathbf{g}_{20} = \begin{pmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{4} \\ \frac{5}{8} - \frac{i\sqrt{3}}{8} \end{pmatrix} \\ \mathbf{K}_{11} &= (I - A)^{-1}\mathbf{g}_{11} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{K}_{02} &= (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02} = \overline{\mathbf{K}_{20}} \end{aligned} \tag{21}$$

By using $\mathbf{K}_{20}$, $\mathbf{K}_{11}$ and $\mathbf{K}_{02}$ we have that

$$\mathbf{g}_{21} = \frac{\partial^3}{\partial z^2 \partial \bar{z}} \mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}\mathbf{K}_{20}z^2 + \mathbf{K}_{11}z\bar{z} + \frac{1}{2}\mathbf{K}_{02}\bar{z}^2\right)\Big|_{z=0} = \begin{pmatrix} 0 \\ -\frac{i\sqrt{3}}{8} \end{pmatrix}. \tag{22}$$

It is easy to see that $\mathbf{pA} = \mu\mathbf{p}$ and $\mathbf{pq} = 1$ where

$$\mathbf{p} = \left(\frac{i}{\sqrt{3}}, \frac{1}{6}\left(3 - i\sqrt{3}\right)\right)$$

and

$$a(p_0) = \frac{1}{2}Re(\mathbf{pg}_{21}\bar{\mu}) = -\frac{1}{16} < 0.$$

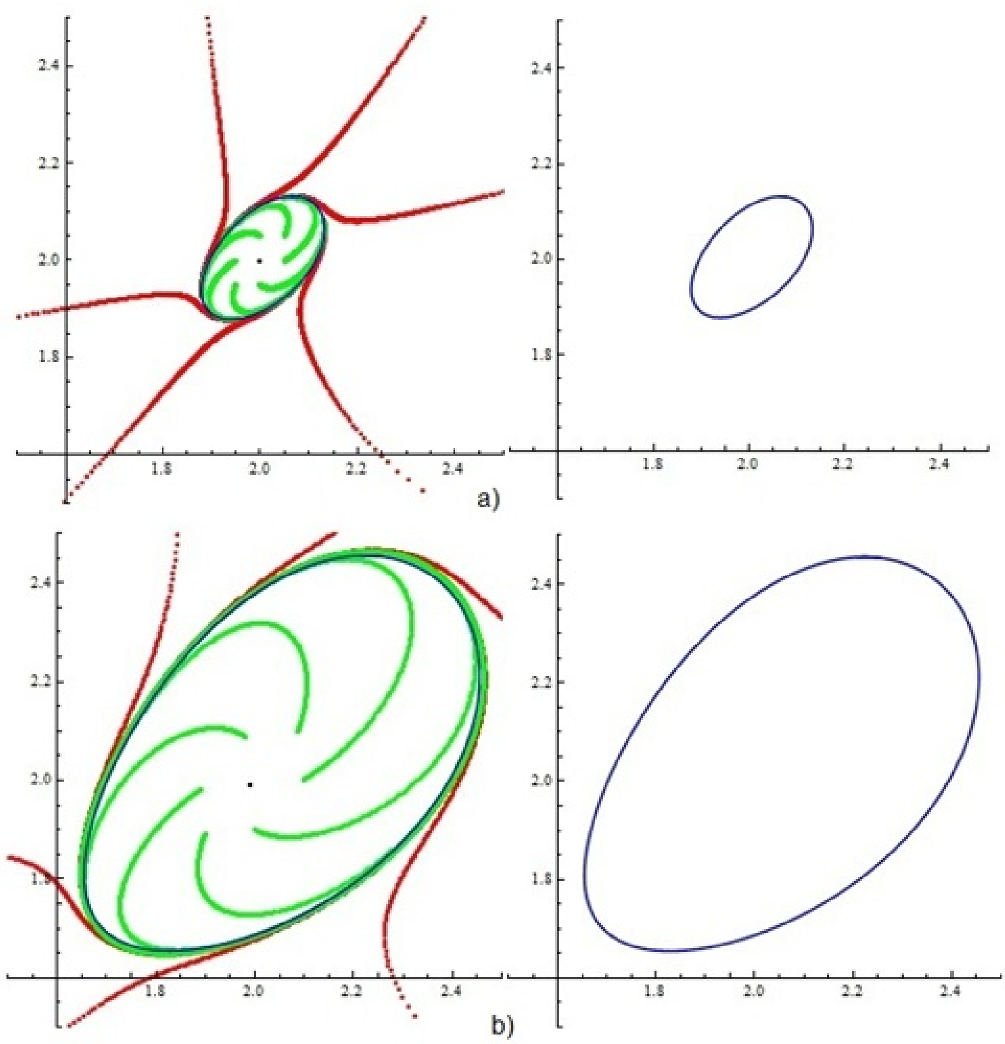

Figure 4: Trajectories and invariant curve for a) $p = 0.999$ b) $p = 0.99$.

Thus we prove the following result:

Theorem 3 *Let $\bar{x} = p + 1$. Then there is a neighborhood U of the equilibrium point $\bar{x}$ and a $\rho > 0$ such that for $|p - 1| < \rho$ and $x_0, x_{-1} \in U$, then ω-limit set of solution of $Eq(1)$, with initial condition x_0, x_{-1} is equilibrium point $\bar{x}$ if $p > 1$ and belongs to a closed invariant C^1 curve $\Gamma(p)$ encircling the equilibrium point $\bar{x}$ if $p < 1$. Furthermore, $\Gamma(1) = 0$ and invariant curve $\Gamma(p)$ can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} p + 1 + 2\sqrt{1-p}\left(\sqrt{3}\sin\theta + \cos\theta\right) - (p-1)\left(\sqrt{3}\sin 2\theta - 2\cos 2\theta + 4\right) \\ p + 1 + 4\sqrt{1-p}\cos\theta - \frac{1}{2}(p-1)\left(\sqrt{3}\sin 2\theta + 5\cos 2\theta + 8\right) \end{pmatrix}$$

Proof. The proof follows from above discussion and Theorem 1 and Corollary 1. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.8, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Triple reverse order law for Moore-Penrose inverse of operator product *

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April 14, 2016

Abstract

In this paper, we study the reverse order law for the Moore-Penrose inverse of an operator product $T_1T_2T_3$. In particular, using the matrix form of a bounded linear operator we derive some necessary and su cient conditions for the reverse order law $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$. Moreover, some nite dimensional results are extended to in nite dimensional settings.

Keywords: Moore-Penrose inverse; Reverse order law; Bounded linear operator; Operator product; Hilbert space.

AMS(MOS) Subject Classi cations: 47A05; 15A09; 15A24.

1 Introduction

Throughout this paper, "an operator" means "a bounded linear operator over Hilbert space". Let $\mathbb{H}$, $\mathbb{I}$, $\mathbb{J}$ and $\mathbb{K}$ denote arbitrary Hilbert spaces. We use $L(\mathbb{H},\mathbb{K})$ to denote the set of all bounded linear operators from $\mathbb{H}$ to $\mathbb{K}$. Especially, $L(\mathbb{H})$=$L(\mathbb{H},\mathbb{H})$. For an operator $T \in L(\mathbb{H},\mathbb{K})$, the symbols $R(T)$, $N(T)$ and T^* denote the range, the null-space and the adjoint of T, respectively. I denotes the unit operator over Hilbert space and O is the zero operator over Hilbert space. An operator $T \in L(\mathbb{H})$ is a Hermitian operator if and only if $T^* = T$. An operator $T \in L(\mathbb{H})$ is an invertible operator if and only if there is a operator $U \in L(\mathbb{H})$, such that $TU = UT = I$. If such operator U exists, we denotes it by T^{-1}.

Recall that an operator $X \in L(\mathbb{K},\mathbb{H})$ is called the Moore-Penrose inverse of $T \in L(\mathbb{H},\mathbb{K})$, if X satis es the following four operator equations [16],

$$(1)\ TXT = T, \quad (2)\ XTX = X, \quad (3)\ (TX)^* = TX, \quad (4)\ (XT)^* = XT.$$

*This work was supported by the NSFC (Grant No: 11301397) and the Guangdong Natural Science Fund of China (Grant No: 2014A030313625) and the Training plan for the Outstanding Young Teachers in Higher Education of Guangdong (Grant No: SYq2014002) and the Student Innovation Training Program of Guangdong province, P.R.China (No. 201511349071).

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If such operator X exists then it is unique and is denoted by $T^{\dagger}$. It is well known that the Moore-Penrose inverse of T exists if and only if $R(T)$ is closed [5, 8].

For a subset $\{i, j, \cdots, k\}$ of the set $\{1, 2, 3, 4\}$, the set of operators satisfying the equations (i), (j), $\cdots$, (k) from among equations (1)-(4) is denoted by $T\{i, j, \cdots, k\}$. An operator in $T\{i, j, \cdots, k\}$ is called an $\{i, j, \cdots, k\}$-inverse of T and is denoted by $T^{(i,j,\cdots,k)}$. For example, an operator X of the set $T\{1\}$ is called a $\{1\}$-inverse or a g-inverse of T and denoted by $X = T^{(1)}$. One usually denotes any $\{1, 3\}$-inverse of the set $T\{1, 3\}$ as $T^{(1,3)}$ which is also called a least squares g-inverse of T. Any $\{1, 4\}$-inverse of the set $T\{1, 4\}$ is denoted by $T^{(1,4)}$ which is also called a minimum norm g-inverse of T. The unique $\{1, 2, 3, 4\}$-inverse of T is the Moore-Penrose inverse of T. We refer the reader to [1, 14] for basic results on the generalized inverses of bounded linear operators.

If s is a semigroup with the unit 1 and if $a_i \in s$, $i = 1, 2, 3$, are invertible, then the equality $(a_1a_2a_3)^{-1} = a_3^{-1}a_2^{-1}a_1^{-1}$ is called the reverse order law for the ordinary inverse. Let T_i, $i = 1, 2, 3$, be three operators over Hilbert space such that the product $T_1T_2T_3$ is meaningful. If each of the three operators is invertible, then the product $T_1T_2T_3$ is invertible too, and the ordinary inverse of $T_1T_2T_3$ satis es the reverse order law $(T_1T_2T_3)^{-1} = T_3^{-1}T_2^{-1}T_1^{-1}$. However, this so-called reverse order law is not necessarily true for other kind generalized inverses. An interesting problem is, for given $\{i, j, \cdots, k\}$-inverses and operators T_i, $i = 1, 2, 3$, with $T_1T_2T_3$ is meaningful, when

$$(T_1T_2T_3)\{i, j, \cdots, k\} = T_3\{i, j, \cdots, k\}T_2\{i, j, \cdots, k\}T_1\{i, j, \cdots, k\}?$$

The reverse order laws for generalized inverses of operator product yield a class of interesting problems that are fundamental in the theory of generalized inverses of operator, see [1, 10, 21]. Theory and computations of the reverse order laws for generalized inverses of operator product are important subjects in many branches of applied science, such as nonlinear control theory, operator theory, operator algebra, global analysis and approximation theory, see [1, 6, 20, 21]. Suppose T_i, $i = 1, 2, 3$, and are bounded linear operators over Hilbert space. The least squares technique (LS):

$$\min_{Y} \|(T_1T_2T_3)Y \qquad \|_2,$$

is used in many practical scienti c problems. Any solution Y of the above LS problem can be expressed as $Y = (T_1T_2T_3)^{(1,3)}$. If the LS problem is consistent, then the minimum norm solution Y has the form $Y = (T_1T_2T_3)^{(1,4)}$. The unique minimal norm least square solution Y of the LS problem is $Y = (T_1T_2T_3)^{\dagger}$. One such problem concerned with the above LS problem is, under what conditions, $(T_1T_2T_3)^{(i,j,\cdots,k)} = T_3^{(i,j,\cdots,k)}T_2^{(i,j,\cdots,k)}T_1^{(i,j,\cdots,k)}$?

Since the middle 1960s, the reverse order law for generalized inverses have attracted considerable attention, and a signi cant number of paper treat the su cient or equivalent conditions such that the reverse order law holds in some sense. It is a classical result of Greville [10], that $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ if and only if $R(A^*AB) \subseteq R(B)$ and $R(BB^*A^*) \subseteq R(A^*)$, in this case when A and B are complex matrices. This result is extended to bounded linear operators on Hilbert space, by Bouldin [2] and Izumino [12]. In [13] the reverse order law for the Moore-Penrose

inverse is proved in rings with involutions. In [4] D.S.Cvetkovic-IIic studied this reverse order law in C^*-algebra. Then, in [7], the reverse order law for the Moore-Penrose inverse is obtained as a consequence of some set equalities. The reader can nd some interesting and related results in [7, 15, 17, 18, 19, 22].

In 1986, R.E.Hartwig [11] rst discussed the reverse order law for Moore-Penrose inverse of three matrices product. In the paper [9] D.S. Djordjevic et al., extended the results of [11] to the bounded linear operators on Hilbert space, using some algebraic method. In this paper, we revisit this reverse order law by applying the technique of matrix form of bounded linear operators [3]. Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$ such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. Then using the technique of matrix form of a bounded linear operator [3] and the solving operator equations, we will revisit the following reverse order law $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$. Some new simpler equivalent conditions for this reverse order law are obtained.

We rst mention the following results, which will be used in this paper.

Lemma 1.1. [3, 7, 8] *Let $T \in L(\mathbb{H}, \mathbb{K})$ have a closed range. Let H_1 and H_2 be closed and mutually orthogonal subspace of $\mathbb{H}$, such that $H_1 \bigoplus H_2 = \mathbb{H}$. Let K_1 and K_2 be closed and mutually orthogonal subspace of $\mathbb{K}$, such that $\mathbb{K} = K_1 \bigoplus K_2$. Then the operator T has the following matrix representations with respect to the orthogonal sums of subspaces $\mathbb{H} = H_1 \bigoplus H_2 = R(T^*) \bigoplus N(T)$ and $\mathbb{K} = K_1 \bigoplus K_2 = R(T) \bigoplus N(T^*)$:*

(1) $T = \begin{pmatrix} T_{11} & T_{12} \\ O & O \end{pmatrix} : \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \to \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix}$ *and* $T^\dagger = \begin{pmatrix} T_{11}^*E^{-1} & O \\ T_{12}^*E^{-1} & O \end{pmatrix} : \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \to \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$,
where $E = T_{11}T_{11}^* + T_{12}T_{12}^*$ is invertible on $R(T)$;

(2) $T = \begin{pmatrix} T_{11} & O \\ T_{21} & O \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ *and* $T^\dagger = \begin{pmatrix} F^{-1}T_{11}^* & F^{-1}T_{12}^* \\ O & O \end{pmatrix} : \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}$,
where $F = T_{11}^*T_{11} + T_{21}^*T_{21}$ is invertible on $R(T^*)$;

(3) $T = \begin{pmatrix} T_{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix}$ *and* $T^\dagger = \begin{pmatrix} T_{11}^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}$,
where T_{11} is invertible.

Lemma 1.2. [1] *Let $T \in L(\mathbb{H}, \mathbb{K})$ and $N \in L(\mathbb{K}, \mathbb{H})$ have closed ranges. Then,*

(1) $TT^\dagger N = N \Leftrightarrow R(N) \subseteq R(T)$;

(2) $NT^\dagger T = N \Leftrightarrow R(N^*) \subseteq R(T^*)$.

2 The triple reverse order law for Moore-Penrose inverse of operator product

Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$, such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. In this section, we will give necessary and su cient conditions for the triple reverse

order law of the Moore-Penrose inverse of the operator product $T_1T_2T_3$. First of all let us de ne

$$E = T_1^\dagger T_1, \quad F = T_3T_3^\dagger, \quad P = ET_2F, \quad Q = FT_2^\dagger E, \quad M = T_1T_2T_3, \quad G = T_3^\dagger T_2^\dagger T_1^\dagger. \tag{2.1}$$

In terms of these, we get the following results.

Theorem 2.1. *Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$, such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. Then the following statements are equivalent:*

(1) $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$;

(2) $Q \in P\{1, 2\}$, *and* $T_1^*T_1PQ$, $QPT_3T_3^*$ *are two Hermitian operators*;

(3) $MGM = G$, *and* $GMG = G$, *and* $(MG)^* = MG$, *and* $(GM)^* = GM$.

Proof. (1)$\Leftrightarrow$ (3): Obvious.

Next, we will prove (2)$\Leftrightarrow$ (3). From Lemma 1.1, we know that the operators T_1, T_2, T_3, $T_1T_2T_3$ and $T_3^\dagger T_2^\dagger T_1^\dagger$ have the following matrix form with respect to the orthogonal sum of subspaces:

$$T_1 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix}, \tag{2.2}$$

$$T_1^\dagger = \begin{pmatrix} (T_1^{11})^*D^{-1} & O \\ (T_1^{12})^*D^{-1} & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \tag{2.3}$$

where $D = T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*$ is invertible on $R(T_1)$.

$$T_2 = \begin{pmatrix} T_2^{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \tag{2.4}$$

$$T_2^\dagger = \begin{pmatrix} (T_2^{11})^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}, \tag{2.5}$$

where T_2^{11} is invertible.

$$T_3 = \begin{pmatrix} T_3^{11} & O \\ T_3^{21} & O \end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}, \tag{2.6}$$

$$T_3^\dagger = \begin{pmatrix} S^{-1}(T_3^{11})^* & S^{-1}(T_3^{21})^* \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \to \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix}, \tag{2.7}$$

where $S = (T_3^{11})^*T_3^{11} + (T_3^{21})^*T_3^{21}$ is invertible on $R(T_3^*)$.

Let $M = T_1T_2T_3$ and $G = T_3^\dagger T_2^\dagger T_1^\dagger$, then form (2.2)$\sim$(2.7), we have

$$M = T_1T_2T_3 = \begin{pmatrix} T_1^{11}T_2^{11}T_3^{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \to \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \tag{2.8}$$

and

$$G = T_3^\dagger T_2^\dagger T_1^\dagger = \begin{pmatrix} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix}. \tag{2.9}$$

According to the formulas (2.1)$\sim$(2.7), we have

$$Q = \begin{pmatrix} T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11} & T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12} \\ T_3^{21}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11} & T_3^{21}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12} \end{pmatrix} \tag{2.10}$$

and

$$P = \begin{pmatrix} (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^* & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^* \\ (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^* & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^* \end{pmatrix}. \tag{2.11}$$

From (2.2), (2.6), (2.10) and (2.11), we get

$$T_1^*T_1PQ = \begin{pmatrix} {}_{11} & {}_{12} \\ {}_{21} & {}_{22} \end{pmatrix}, \text{where} \tag{2.12}$$

$$\begin{aligned}
{}_{11} &= (T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}, \\
{}_{12} &= (T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}, \\
{}_{21} &= (T_1^{12})^*T_1^{11}T_2^{11}T_3^{21}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}, \\
{}_{22} &= (T_1^{12})^*T_1^{11}T_2^{11}T_3^{21}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12},
\end{aligned}$$

and

$$QPT_3T_3^* = \begin{pmatrix} {}_{11} & {}_{12} \\ {}_{21} & {}_{22} \end{pmatrix}, \text{where} \tag{2.13}$$

$$\begin{aligned}
{}_{11} &= T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}(T_3^{11})^*, \\
{}_{12} &= T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}(T_3^{21})^*, \\
{}_{21} &= T_3^{21}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}(T_3^{11})^*, \\
{}_{22} &= T_3^{21}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}(T_3^{21})^*.
\end{aligned}$$

Combining (2.8) with (2.9), we know that $G = M^\dagger$ (i.e. $T_3^\dagger T_2^\dagger T_1^\dagger = (T_1T_2T_3)^\dagger$), if and only if

$$(I)\ MGM = M, \quad (II)\ GMG = G, \quad (III)\ (MG)^* = MG, \quad (IV)\ (GM)^* = GM. \tag{2.14}$$

From the formulas (2.10)$\sim$(2.13), we know that the statement (2) of Theorem 2.1 can be rewrited as

$$(a)\ PQP = P,\ (b)\ QPQ = Q,\ (c)\ (T_1^*T_1PQ)^* = T_1^*T_1PQ,\ (d)\ (QPT_3T_3^*)^* = QPT_3T_3^*. \tag{2.15}$$

In the rest of this section, we will prove (2.14) is equivalent to (2.15). That is the conditions (2) in Theorem 2.1 is equal to the conditions (3) in Theorem 2.1.

(I)$\Leftrightarrow$(a): From (2.8) and (2.9), we have

$$\begin{aligned} MGM &= (T_1T_2T_3)(T_3^\dagger T_2^\dagger T_1^\dagger)(T_1T_2T_3) \\ &= \begin{pmatrix} T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11} & O \\ O & O \end{pmatrix}. \end{aligned} \tag{2.16}$$

Then from (2.8) and (2.16), we know that the inclusion $MGM = M$ is equivalent to

$$T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11} = T_1^{11}T_2^{11}T_3^{11}. \tag{2.17}$$

By the formulas (2.10) and (2.11), we have

$$PQP = \begin{pmatrix} _{11} & _{12} \\ _{21} & _{22} \end{pmatrix}, \text{where} \tag{2.18}$$

$$\begin{aligned} _{11} &= (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \\ _{12} &= (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*, \\ _{21} &= (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \\ _{22} &= (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*. \end{aligned}$$

From (2.11) and (2.18), we know that the inclusion $PQP = P$ is equivalent to

$$\begin{aligned} & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^* \\ = \; & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \end{aligned} \tag{2.19}$$

$$\begin{aligned} & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^* \\ = \; & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*, \end{aligned} \tag{2.20}$$

$$\begin{aligned} & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^* \\ = \; & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \end{aligned} \tag{2.21}$$

$$\begin{aligned} & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^* \\ = \; & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*. \end{aligned} \tag{2.22}$$

If the equation (2.17) holds, we have the equations (2.19)$\sim$(2.22) hold too. That is (I)$\Rightarrow$(a).

On the other hand, if the equations (2.19)$\sim$(2.22) hold, we have

$$\begin{aligned} & T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11} \\ = \; & T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11}, \end{aligned} \tag{2.23}$$

$$\begin{aligned}&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21}\\ &=T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21},\end{aligned}\tag{2.24}$$

$$\begin{aligned}&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11}\\ &=T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11},\end{aligned}\tag{2.25}$$

$$\begin{aligned}&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21}\\ &=T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21}.\end{aligned}\tag{2.26}$$

Combining (2.23), (2.24) with the de nition of S in (2.7), we have

$$\begin{aligned}&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}\\ =\;&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}.\end{aligned}\tag{2.27}$$

Combining (2.25), (2.26) with the de nition of D in (2.3), we have

$$\begin{aligned}&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}\\ =\;&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}.\end{aligned}\tag{2.28}$$

From the results in (2.27) and (2.28), we have

$$T_1^{11}T_2^{11}T_3^{11}=T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}.\tag{2.29}$$

That is (a)⇒(I).

(II)⇔(b): With the same method of the proof of (I)⇔(a), the condition $GMG = G$ is easily seen to be equivalent to $QPQ = Q$.

(III)⇔(c): From (2.8) and (2.9), we have

$$MG=(T_1T_2T_3)(T_3^\dagger T_2^\dagger T_1^\dagger)=\begin{pmatrix}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1} & O\\ O & O\end{pmatrix}.\tag{2.30}$$

Since S and D are Hermitian operators, then the inclusion $(MG)^* = MG$ is equivalent to

$$T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}=D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*.\tag{2.31}$$

By the formulas (2.12), we have that the inclusion $(T_1^*T_1PQ)^* = T_1^*T_1PQ$ is equivalent to

$$\begin{aligned}&(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}\\ =\;&(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11},\end{aligned}\tag{2.32}$$

$$\begin{aligned}&(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}\\ =\;&(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12},\end{aligned}\tag{2.33}$$

$$\begin{aligned}&(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}\\ =\;&(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11},\end{aligned} \tag{2.34}$$

$$\begin{aligned}&(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}\\ =\;&(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12}.\end{aligned} \tag{2.35}$$

If the equation (2.31) holds, we have the equations (2.32)$\sim$(2.35) hold too. That is (III)$\Rightarrow$(c).

On the other hand, if the equations (2.32)$\sim$(2.35) hold, we have

$$\begin{aligned}&T_1^{11}(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}(T_1^{11})^*\\ =\;&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11}(T_1^{11})^*,\end{aligned} \tag{2.36}$$

$$\begin{aligned}&T_1^{11}(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}(T_1^{12})^*\\ =\;&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12}(T_1^{12})^*,\end{aligned} \tag{2.37}$$

$$\begin{aligned}&T_1^{12}(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}(T_1^{11})^*\\ =\;&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11}(T_1^{11})^*,\end{aligned} \tag{2.38}$$

$$\begin{aligned}&T_1^{12}(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}(T_1^{12})^*\\ =\;&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12}(T_1^{12})^*.\end{aligned} \tag{2.39}$$

Combining (2.36), (2.37) with the de nition of $D = T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*$ in (2.3), we have

$$\begin{aligned}&T_1^{11}(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*\\ =\;&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*D.\end{aligned} \tag{2.40}$$

Combining (2.38), (2.39) with the de nition of D, we have

$$\begin{aligned}&T_1^{12}(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*\\ =\;&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*D.\end{aligned} \tag{2.41}$$

Finally, from (3.40), (3.41) and the de nition of D, we have

$$DT_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* = T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*D. \tag{2.42}$$

Since $D = (T_1^{11})(T_1^{11})^* + (T_1^{12})(T_1^{12})^*$ is invertible on $R(T_1)$, then (2.42) can be rewrited as

$$T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1} = D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*. \tag{2.43}$$

That is (c)$\Rightarrow$(III).

(IV)$\Leftrightarrow$(d): With the same method of the proof of (III)$\Leftrightarrow$(c), we can get the result that the condition $(GM)^* = GM$ is equivalent to $(QPT_3T_3^*)^* = QPT_3T_3^*$ without the proof.

From the above proof, the formulas (2.14) is equivalent to (2.15). We then complete the proof of the theorem. ■

Be the same as (2.1), $Q = FT_2^\dagger E$ and $P = ET_2F$, next we will derive some other equivalent conditions for the triple reverse order law $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$.

Theorem 2.2. *Let* $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ *and* $T_3 \in L(\mathbb{H}, \mathbb{I})$, *such that* T_1, T_2, T_3 *and* $T_1T_2T_3$ *have closed ranges. Then the following statements are equivalent:*

(1) $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$;

(2) $Q \in P\{1, 2\}$ and $T_1^*T_1PQ$, $QPT_3T_3^*$ are two Hermitian operators;

(3) $Q \in P\{1\}$ and $R(T_1^*T_1P) = R(Q^*)$ and $R(T_3T_3^*P^*) = R(Q)$;

(4) $(PQ)^2 = PQ$ and $R(T_1^*T_1P) = R(Q^*)$ and $R(T_3T_3^*P^*) = R(Q)$.

Proof. (1)⇔(2): By the results in Theorem 2.1, we know that (1)⇔(2).

(2)⇒(3): According to the de nitions of the generalized inverses of operators, we have

$$Q \in P\{1, 2\} \Rightarrow Q \in P\{1\}. \tag{2.44}$$

By the de nitions of the ranges of operators and the formula (2.44), we have

$$R(T_1^*T_1P) = R(T_1^*T_1PQP) \subseteq R(T_1^*T_1PQ) \subseteq R(T_1^*T_1P). \tag{2.45}$$

That is

$$R(T_1^*T_1P) = R(T_1^*T_1PQ). \tag{2.46}$$

If $T_1^*T_1PQ$ is a Hermitian operator, then

$$R(T_1^*T_1P) = R(T_1^*T_1PQ) = R(Q^*P^*T_1^*T_1) = R(Q^*P^*T_1^\dagger T_1). \tag{2.47}$$

Since $Q^*P^*T_1^\dagger T_1 = Q^*P^*$, then from (2.44) and (2.47), we have

$$R(T_1^*T_1P) = R(Q^*P^*T_1^\dagger T_1) = R(Q^*P^*) = R(Q^*). \tag{2.48}$$

Similarly, if $QPT_3T_3^*$ is a Hermitian operator, we have

$$R(T_3^*T_3P^*) = R(T_3^*T_3P^*Q^*) = R(QPT_3T_3^*) = R(QP) = R(Q). \tag{2.49}$$

Combining (2.44), (2.48) with (2.49), we have the result (2)⇒(3).

(3)⇒(4): Obvious.

(4)⇒(2): Firstly, we will prove that if the statement (4) in Theorem 2.2 is true, then $PQP = P$. Since $P = PT_3T_3^\dagger$ and $R(T_3T_3^*P^*) = R(Q)$, then we have

$$R(P) = R(PT_3) = R(PT_3T_3^*P^*) = R(PQ). \tag{2.50}$$

Combining (2.50) with $(PQ)^2 = PQ$, we have

$$PQP = P \text{ and } (QP)^2 = QP. \tag{2.51}$$

Secondly, we will prove that if the statement (4) in Theorem 2.2 is true, then $QPQ = Q$. From the statement (4) in Theorem 2.2 and the de nitions of Q and P, we have

$$\begin{aligned} R(Q^*) &= R(T_1^*T_1P) = R(T_1^*T_1PP^*T_1^*T_1) = R(T_1^*T_1PP^*T_1^\dagger T_1) \\ &= R(T_1^*T_1PP^*) = R(Q^*P^*). \end{aligned} \tag{2.52}$$

Combining (2.52) with $(Q^*P^*)^2 = Q^*P^*$, we have

$$Q^*P^*Q^* = Q^* \text{ i.e. } QPQ = Q. \tag{2.53}$$

Thirdly, we will prove that if the statement (4) in Theorem 2.2 is true, then $T_1^*T_1PQ$ is a Hermitian operator. Since $R(T_1^*T_1P) = R(Q^*)$ and $R(Q^*P^*) = R(Q^*)$, then we have

$$Q^*P^*T_1^*T_1P = T_1^*T_1P. \tag{2.54}$$

From (2.54), we have

$$Q^*P^*T_1^*T_1PQ = T_1^*T_1PQ = (T_1^*T_1PQ)^*. \tag{2.55}$$

Fourthly, we will prove that if the statement (4) in Theorem 2.2 is true, then $QPT_3T_3^*$ is a Hermitian operator. Since $R(T_3T_3^*P^*) = R(Q)$ and $QPQ = Q$, then we have

$$R(QP) = R(Q) \text{ and } QPT_3T_3^*P^* = T_3T_3^*P^*. \tag{2.56}$$

From (2.56), we have

$$QPT_3T_3^*P^*Q^* = T_3T_3^*P^*Q^* = (QPT_3T_3^*)^* = QPT_3T_3^*. \tag{2.57}$$

Combining the formulas (2.51), (2.53), (2.55) with (2.57), we immediately obtain the result (4)⇒(2). We then complete the proof of the theorem. ■

Let us now see how some of the special cases come out of the conditions of Theorem 2.2.

Corollary 2.1. *Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$, such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. If $R(T_2) \subseteq R(T_1^*)$ and $R(T_2^*) \subseteq R(T_3)$, then*

$$(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger \Leftrightarrow R(T_1^*T_1T_2) \subseteq R(T_2) \text{ and } R(T_3T_3^*T_2^*) \subseteq R(T_2^*).$$

Proof. According to the hypothesis $R(T_2) \subseteq R(T_1^*)$ and $R(T_2^*) \subseteq R(T_3)$ and the results in Lemma 1.2, we have

$$Q = FT_2^\dagger E = T_2^\dagger, \quad P = ET_2F = T_2. \tag{2.58}$$

$\Rightarrow$: If $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$, then from Theorem 2.1 and Theorem 2.2 , we have $(PQ)^2 = PQ$ and $R(T_1^*T_1P) = R(Q^*)$ and $R(T_3T_3^*P^*) = R(Q)$. So, we get

$$R(T_1^*T_1T_2) = R((T_2^\dagger)^*) \subseteq R(T_2) \text{ and } R(T_3T_3^*T_2^*) = R(T_2^\dagger) \subseteq R(T_2^*). \tag{2.59}$$

$\Leftarrow$: From (2.58), we have $PQP = P$ and $QPQ = Q$. That is

$$Q \in P\{1,2\}. \tag{2.60}$$

By (2.58), we also have

$$T_1^*T_1PQ = T_1^*T_1T_2T_2^\dagger \text{ and } QPT_3T_3^* = T_2^\dagger T_2T_3T_3^*. \tag{2.61}$$

Combining the hypothesis $R(T_1^*T_1T_2) \subseteq R(T_2)$ with results in Lemma 1.2, we have

$$T_2T_2^\dagger T_1T_1^*T_2T_2^\dagger = T_1T_1T_2^*T_2^\dagger = (T_1T_1T_2^*T_2^\dagger)^*. \tag{2.62}$$

Combining the hypothesis $R(T_3T_3^*T_2) \subseteq R(T_2^*)$ with results in Lemma 1.2, we have

$$T_2^\dagger T_2T_3T_3^*T_2^*(T_2^*)^\dagger = T_3T_3^*T_2^*(T_2^*)^\dagger = (T_3T_3^*T_2^*(T_2^*)^\dagger)^* = T_2^\dagger T_2T_3T_3^* = (T_2^\dagger T_2T_3T_3^*)^*. \tag{2.63}$$

According to the formulas (2.59), (2.60), (2.62), (2.63) and the statement (2) in Theorem 2.2, we immediately obtain the results of Corollary 2.1. ■

Corollary 2.2. *Let $T_1 \in L(\mathbb{J},\mathbb{K})$, $T_2 \in L(\mathbb{I},\mathbb{J})$ and $T_3 \in L(\mathbb{H},\mathbb{I})$, such that T_2 and $T_1T_2T_3$ have closed ranges. If $T_1^\dagger T_1 = I$ and $T_3T_3^\dagger = I$ (i.e. T_1 and T_3 are invertible operators), then*

$$(T_1T_2T_3)^\dagger = T_3^{-1}T_2^\dagger T_1^{-1} \Leftrightarrow R(T_1^*T_1T_2) \subseteq R(T_2) \text{ and } R(T_3T_3^*T_2^*) \subseteq R(T_2^*).$$

Corollary 2.3. *Let $T_1 \in L(\mathbb{J},\mathbb{K})$, $T_2 \in L(\mathbb{I},\mathbb{J})$ and $T_3 \in L(\mathbb{H},\mathbb{I})$, such that T_1, T_2, T_3, $T_1T_2T_3$ and $T_1^\dagger T_1T_2T_3T_3^\dagger$ have closed ranges. If $T_1^\dagger T_1 = T_1$ and $T_3T_3^\dagger = T_3$, then*

$$(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger \Leftrightarrow T_3T_3^\dagger T_2^\dagger T_1^\dagger T_1 = (T_1^\dagger T_1T_2T_3T_3^\dagger)^\dagger.$$

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DIFFERENTIAL EQUATIONS ARISING FROM CERTAIN SHEFFER SEQUENCE

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Abstract. In this paper, we study some differential equations arising from certain Sheffer sequence and investigate some identities for the Sheffer sequence of polynomials which is related to the theory of hyperbolic differential equations.

1. Introduction

A partial differential equation of the second-order

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0,$$

is called hyperbolic if the matrix is

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0, \text{ (see [6]).}$$

The wave equation is an example of a hyperbolic partial differential equation. A sequence $S_n(x)$ is called a Sheffer sequence if the generating function has the form

$$\sum_{k=0}^{\infty} S_k(x)\frac{t^k}{k!} = A(t)e^{xB(t)},$$

where

$$\begin{aligned} A(t) =& A_0 + A_1 t + A_2 t^2 + \cdots \\ B(t) =& B_1 t + B_2 t^2 + \cdots , \qquad \text{with} \quad A_0 \neq 0,\ B_0 \neq 0 \text{ (see [12]).} \end{aligned}$$

If $f(t)$ is a delta series and $g(t)$ is an invertible series, there exists a uniquen sequence $S_n(x)$ of Sheffer polynomials such that the orthogonality condition $< g(t)f(t)^k | S_n(x) >= \delta_{n,k}$ holds, where $\delta_{n,k}$ is the Kronecker delta (see [8-11]).

2010 *Mathematics Subject Classification.* 05A19; 11B83; 34A30.
Key words and phrases. Sheffer sequence, differential equations.
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In this paper, we consider the Sheffer sequence given by the pair $\left(\frac{1}{1+t}, 1-(1+t)^{-2}\right)$, namely

$$F(t,x)=\frac{1}{\sqrt{1-t}}e^{x\left(\frac{1}{\sqrt{1-t}}-1\right)}=\sum_{n=0}^{\infty}h_n(x)\frac{t^n}{n!}. \tag{1.1}$$

In [5], Erdélyi also considered a Sheffer sequence which is related to $h_n(x)$. Indeed, his sequence is given by $g_n(x)=\frac{1}{n!}h_n(x)$. Also, we note that

$$h_n(x)=xe^{-x}\left[\frac{d}{dx^2}\right]^n(x^{2n-1}e^x),\ \text{(see [5])}. \tag{1.2}$$

The polynomials $h_n(x)$ have applications to the theory of hyperbolic differential equations (see [1-4]). From (1.1), by replacing t by $1-e^{-2t}$, we can derive the following equation:

$$\begin{aligned} e^te^{x(e^t-1)}&=\sum_{n=0}^{\infty}(-1)^nh_n(x)\frac{1}{n!}(e^{-2t}-1)^n\\ &=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}(-1)^{n+m}h_n(x)2^mS_2(n,m)\right)\frac{t^m}{m!}, \end{aligned} \tag{1.3}$$

where $S_2(n,m)$ is the Stirling number of the second kind.
As is well known, the Bell polynomials are defined by the generating function

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}Bel_n(x)\frac{t^n}{n!},\ \text{(see [7])}. \tag{1.4}$$

By (1.3), we get

$$\begin{aligned} e^te^{x(e^t-1)}&=\left(\sum_{l=0}^{\infty}\frac{t^l}{l!}\right)\left(\sum_{n=0}^{\infty}Bel_n(x)\frac{t^n}{n!}\right)\\ &=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\binom{m}{n}Bel_n(x)\right)\frac{t^m}{m!}. \end{aligned} \tag{1.5}$$

From (1.3) and (1.5), we have

$$\sum_{n=0}^{m}\binom{m}{n}Bel_n(x)=\sum_{n=0}^{m}(-1)^{n+m}h_n(x)2^mS_2(n,m),\ (m\geq 0). \tag{1.6}$$

In this paper, we study some differential equations arising from certain sheffer sequence and investigate some identities for the Sheffer sequence of polynomials which is related to the theory of hyperbolic differential equations.

2. Differential equations arising from certain Sheffer sequence

Let

$$F = F(t,x) = (1-t)^{-\frac{1}{2}} e^{x\left((1-t)^{-\frac{1}{2}}-1\right)} \tag{2.1}$$

Then, we have

$$\begin{aligned} F^{(1)} =& \frac{dF(t,x)}{dt} = (1-t)^{-\frac{1}{2}} e^{x\left((1-t)^{-\frac{1}{2}}-1\right)} \left(\tfrac{1}{2}(1-t)^{-1} + \frac{1}{2}x(1-t)^{-\frac{3}{2}}\right) \\ =& \left(\frac{1}{2}(1-t)^{-1} + \frac{1}{2}x(1-t)^{-\frac{3}{2}}\right) F, \end{aligned} \tag{2.2}$$

$$F^{(2)} = \frac{dF^{(1)}}{dt} = \left(\frac{3}{4}(1-t)^{-2} + \frac{5}{4}x(1-t)^{-\frac{5}{2}} + \frac{1}{4}x^2(1-t)^{-3}\right) F, \tag{2.3}$$

and

$$F^{(3)} = \left(\frac{15}{8}(1-t)^{-3} + \frac{33}{8}x(1-t)^{-\frac{7}{2}} + \frac{12}{8}x^2(1-t)^{-4} + \frac{1}{8}x^3(1-t)^{-\frac{9}{2}}\right) F.$$

Thus, we are let to put

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) F, \tag{2.4}$$

where $N = 0, 1, 2, \cdots$.

Taking the derivative of (2.4) with respect to t, we have

$$\begin{aligned} F^{(N+1)} =& \frac{dF^{(N)}}{dt} = \left(\sum_{i=0}^{N} (N+\tfrac{1}{2}i) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i}\right) F \\ & \qquad + \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) F^{(1)} \\ =& \left(\sum_{i=0}^{N} (N+\tfrac{1}{2}i) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i}\right) F \\ & + \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) \left(\frac{1}{2}(1-t)^{-1} + \frac{1}{2}x(1-t)^{-\frac{3}{2}}\right) F \end{aligned} \tag{2.5}$$

$$\begin{aligned} =& \left(\sum_{i=0}^{N} \left(N+\tfrac{1}{2}i+\tfrac{1}{2}\right) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i} + \sum_{i=0}^{N} \frac{1}{2} a_i(N) x^{i+1} (1-t)^{-N-\frac{3}{2}-\frac{1}{2}i}\right) F \\ =& \left(\sum_{i=0}^{N} \left(N+\tfrac{1}{2}i+\tfrac{1}{2}\right) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i} + \sum_{i=1}^{N+1} \frac{1}{2} a_{i-1}(N) x^i (1-t)^{-N-1-\frac{1}{2}i}\right) F. \end{aligned}$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i(N+1)x^i(1-t)^{-N-1-\frac{1}{2}i} \right) F. \tag{2.6}$$

Comparing the coefficients on both sides of (2.5) and (2.6), we obtain the following recurrence relations:

$$a_0(N+1) = (N+\tfrac{1}{2})a_0(N), \;\; a_{N+1}(N+1) = \frac{1}{2}a_N(N), \tag{2.7}$$

and

$$a_i(N+1) = \frac{1}{2}a_{i-1}(N) + \left(N+\tfrac{1}{2}i+\tfrac{1}{2}\right) a_i(N), \;\; (1 \leq i \leq N). \tag{2.8}$$

In addition, we note that

$$F = F^{(0)} = a_0(0)F. \tag{2.9}$$

Thus, by (2.9), we easily get

$$a_0(0) = 1. \tag{2.10}$$

For $N = 1$ in (1.5) and (1.2), it is not difficult to show that

$$\begin{aligned} &\left(\frac{1}{2}(1-t)^{-1} + \frac{1}{2}x(1-t)^{-3/2} \right) F = F^{(1)} \\ &= \left(a_0(1)(1-t)^{-1} + a_1(x)x(1-t)^{-3/2} \right) F. \end{aligned} \tag{2.11}$$

By comparing the coefficients on both sides of (2.11), we easily get

$$a_0(1) = \frac{1}{2}, \quad a_1(1) = \frac{1}{2}. \tag{2.12}$$

From (2.7), we can easily derive the following equations:

$$\begin{aligned} &a_{N+1}(N+1) = \frac{1}{2}a_N(N) = \left(\frac{1}{2}\right)^2 a_{N-1}(N-1) = \cdots = \left(\frac{1}{2}\right)^{N+1}, \\ &a_0(0) = \left(\frac{1}{2}\right)^{N+1}, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} a_0(N+1) =&(N+\tfrac{1}{2})a_0(N) = (N+\tfrac{1}{2})(N-\tfrac{1}{2})a_0(N-1) = \cdots \\ =&(N+\tfrac{1}{2})(N-\tfrac{1}{2})\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}a_0(0) = (N+\tfrac{1}{2})_{N+1}, \end{aligned} \tag{2.14}$$

where

$$(x)_n = x(x-1)\cdots(x-n+1), \;\; (n \geq 1), \quad (x)_0 = 1.$$

The matrix $(a_i(j))$ $(0 \le i, j \le N)$ is given by

$$\big(a_i(j)\big) = \begin{pmatrix} 1 & \frac{1}{2} & \left(\frac{3}{2}\right)_2 & \left(\frac{5}{2}\right)_3 & \cdots & \left(N-\frac{1}{2}\right)_N \\ 0 & \frac{1}{2} & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & \left(\frac{1}{2}\right)^2 & \cdots & \cdots & \cdot \\ 0 & 0 & 0 & \left(\frac{1}{2}\right)^3 & \cdots & \cdot \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \left(\frac{1}{2}\right)^N \end{pmatrix}$$

For $i = 1, 2, 3$ in (2.8), we have

$$\begin{aligned} a_1(N+1) =& \frac{1}{2} a_0(N) + (N+1) a_1(N) \\ =& \frac{1}{2}\Big(a_0(N) + (N+1) a_0(N-1)\Big) + (N+1) N a_1(N-1) \\ =& \frac{1}{2}\Big(a_0(N) + (N+1) a_0(N-1) + (N+1) N a_0(N-2)\Big) \\ & + (N+1) N (N-1) a_1(N-2) \\ =& \cdots \\ =& \frac{1}{2} \sum_{k=0}^{N-1} (N+1)_k a_0(N-k) + (N+1)_N a_1(1) \\ =& \frac{1}{2} \sum_{k=0}^{N} (N+1)_k a_0(N-k), \end{aligned} \tag{2.15}$$

$$\begin{aligned} a_2(N+1) =& \frac{1}{2} \sum_{k=0}^{N-2} \left(N+\frac{3}{2}\right)_k a_1(N-k) + \left(N+\frac{3}{2}\right)_{N-1} a_2(2) \\ =& \frac{1}{2} \sum_{k=0}^{N-1} \left(N+\frac{3}{2}\right)_k a_1(N-k), \end{aligned}$$

and

$$\begin{aligned} a_3(N+1) =& \frac{1}{2} \sum_{k=0}^{N-3} (N+2)_k a_2(N-k) + (N+2)_{N-2} a_3(3) \\ =& \frac{1}{2} \sum_{k=0}^{N-2} (N+2)_k a_2(N-k). \end{aligned}$$

Continuing this process, we have

$$a_i(N+1) = \frac{1}{2}\sum_{k=0}^{N-i+1}\left(N+\tfrac{1}{2}i+\tfrac{1}{2}\right)_k a_{i-1}(N-k),\ (1 \le i \le N). \tag{2.16}$$

Now, we give explicit expressions for $a_i(N+1),\ (1 \le i \le N)$. From (2.16), we note that

$$a_1(N+1) = \frac{1}{2}\sum_{k_1=0}^{N}(N+1)_{k_1} a_0(N-k_1) = \frac{1}{2}\sum_{k_1=0}^{N}(N+1)_{k_1}(N-k_1-\tfrac{1}{2})_{N-k_1}, \tag{2.17}$$

$$\begin{aligned} a_2(N+1) &= \frac{1}{2}\sum_{k_2=0}^{N-1}\left(N+\tfrac{3}{2}\right)_{k_2} a_1(N-k_2) \\ &= \left(\frac{1}{2}\right)^2\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-k_2-1}\left(N+\tfrac{3}{2}\right)_{k_2}(N-k_2)_{k_1}(N-k_2-k_1-\tfrac{3}{2})_{N-k_2-k_1-1}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} a_3(N+1) = \left(\frac{1}{2}\right)^3 &\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-2-k_3}\sum_{k_1=0}^{N-2-k_3-k_2}(N+2)_{k_3}(N-k_3+\tfrac{1}{2})_{k_2} \\ &\times(N-k_3-k_2-1)_{k_1}(N-k_3-k_2-k_1-\tfrac{5}{2})_{N-k_3-k_2-k_1-2}, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} a_4(N+1) = \left(\frac{1}{2}\right)^4 &\sum_{k_4=0}^{N-3}\sum_{k_3=0}^{N-3-k_4}\sum_{k_2=0}^{N-3-k_4-k_3}\sum_{k_1=0}^{N-3-k_4-k_3-k_2}(N+\tfrac{5}{2})_{k_4} \\ &\times(N-k_4+1)_{k_3}(N-k_4-k_3-\tfrac{1}{2})_{k_2}(N-k_4-k_3-k_2-2)_{k_1} \\ &\times(N-k_4-k_3-k_2-k_1-\tfrac{7}{2})_{N-k_4-k_3-k_2-k_1-3}. \end{aligned} \tag{2.20}$$

So, we can deduce that, for $1 \le i \le N$,

$$\begin{aligned} &a_i(N+1) \\ &= \left(\frac{1}{2}\right)^i\sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-i+1-k_i}\cdots\sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2}\prod_{l=1}^{i}\left(N+\tfrac{3}{2}l+\tfrac{1}{2}-i-\sum_{j=l+1}^{i}k_j\right)_{k_l} \\ &\quad\times\left(N+\tfrac{1}{2}-i-\sum_{j=1}^{i}k_j\right)_{N+1-i-\sum_{j=1}^{i}k_j}. \end{aligned} \tag{2.21}$$

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Therefore, by (2.21), we obtain the following theorem.

Theorem 1. For $N = 0, 1, 2 \cdots$, the following family of differential equations

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \left(\sum_{i=0}^{N} a_i(N)x^i(1-t)^{-N-\frac{1}{2}i}\right) F$$

have a solution

$$F = F(t,x) = (1-t)^{-1/2} e^{x((1-t)^{-1/2}-1)},$$

where

$$\begin{aligned} a_0(N) &= \left(N - \tfrac{1}{2}\right)_N, \\ a_i(N) &= \left(\frac{1}{2}\right)^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \prod_{l=1}^{i} \left(N + \tfrac{3}{2}l - \tfrac{1}{2} - i - \sum_{j=l+1}^{i} k_j\right)_{k_l} \\ &\quad \times \left(N - \tfrac{1}{2} - i - \sum_{j=1}^{i} k_j\right)_{N-i-\sum_{j=1}^{i} k_j}. \end{aligned}$$

From (1.1), we note that

$$\begin{aligned} \sum_{k=0}^{\infty} h_{k+N}(x)\frac{t^k}{k!} &= F^{(N)} = \left(\sum_{i=0}^{N} a_i(N)x^i(1-t)^{-N-\frac{1}{2}i}\right) F \\ &= \sum_{i=0}^{N} a_i(N)x^i \sum_{l=0}^{\infty} \left(N + \tfrac{1}{2}i + l - 1\right)_l \frac{t^l}{l!} \sum_{m=0}^{\infty} h_m(x)\frac{t^m}{m!} \\ &= \sum_{i=0}^{N} a_i(N)x^i \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} \binom{k}{l} \left(N + \tfrac{1}{2}i + l - 1\right)_l h_{k-l}(x)\right) \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} \sum_{l=0}^{k} \binom{k}{l} \left(N + \tfrac{1}{2}i + l - 1\right)_l a_i(N)x^i h_{k-l}(x)\right) \frac{t^k}{k!}. \end{aligned} \tag{2.22}$$

Thus, by comparing the coefficients on both sides of (2.22), we obtain the following theorem.

Theorem 2. For $k, N = 0, 1, 2, \cdots$, we have

$$h_{k+N}(x) = \sum_{i=0}^{N} \sum_{l=0}^{k} \binom{k}{l} \left(N + \tfrac{1}{2}i + l - 1\right)_l a_i(N)x^i h_{k-l}(x) \tag{2.23}$$

Letting $k = 0$ in (2.23), we obtain the following corollary.

Corollary 3. For $N = 0, 1, 2, \cdots$, we have

$$h_N(x) = \sum_{i=0}^{N} a_i(N)x^i. \tag{2.24}$$

ACKNOWLEDGEMENTS. This paper is supported by grant NO 14-11-00022 of Russian Scientific Fund.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.8, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Hyers-Ulam stability of the first order inhomogeneous matrix difference equation

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Abstract

We prove Hyers-Ulam stability of the first order linear inhomogeneous matrix difference equation $\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i)$ for all integers $i \in \mathbb{Z}$. Moreover, we show Hyers-Ulam stability of the nth order linear difference equation as a corollary.

1 Introduction

Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ the set of all complex numbers, of all positive integers, of all nonnegative integers, and the set of all integers, respectively. Given a fixed positive integer n, let $(\mathbb{C}^n, \|\cdot\|_n)$ be a complex normed space, each of whose elements is a column vector, and let $\mathbb{C}^{n\times n}$ be a vector space consisting of all $(n \times n)$ complex matrices. We choose a norm $\|\cdot\|_{n\times n}$ on $\mathbb{C}^{n\times n}$ which is compatible with $\|\cdot\|_n$, *i.e.*, both norms obey

$$\|\mathbf{AB}\|_{n\times n} \leq \|\mathbf{A}\|_{n\times n}\|\mathbf{B}\|_{n\times n} \quad \text{and} \quad \|\mathbf{A}\vec{x}\|_n \leq \|\mathbf{A}\|_{n\times n}\|\vec{x}\|_n \tag{1.1}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ and $\vec{x} \in \mathbb{C}^n$.

A matrix difference equation is a difference equation with matrix coefficients in which the value of vector at one point depends on the values of preceding (succeeding) points.

In this paper, we prove Hyers-Ulam stability of the first order linear inhomogeneous matrix difference equation

$$\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i) \tag{1.2}$$

for all integers $i \in \mathbb{Z}$, where the transition matrices $\mathbf{A}(i)$ are nonsingular. More precisely, we prove that if a vector sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ satisfies the inequality

$$\|\vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i)\|_n \leq \varepsilon_{i+1}$$

for all $i \in \mathbb{Z}$, then there exists a solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the first order matrix difference equation (1.2) such that the bound for $\|\vec{y}_i - \vec{x}_i\|_n$ depends on the sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ and the transition

[0]Key words and phrases: difference equation; matrix difference equation; Hyers-Ulam stability; Fibonacci difference equation; extended Fibonacci number; approximation.

[0]2010 Mathematics Subject Classification: Primary 39A45, 39B82; Secondary 39A06, 39B42.

matrices $\mathbf{A}(i)$ only. Moreover, we investigate Hyers-Ulam stability of the nth order linear inhomogeneous difference equation of the form

$$a(i+1) = p_1(i)a(i) + p_2(i)a(i-1) + \cdots + p_n(i)a(i-n+1) + r(i), \tag{1.3}$$

where $p_j, r : \mathbb{Z} \to \mathbb{C}$ are given functions with $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$. We refer the reader to [7, 8, 9, 12, 20] for the exact definition of Hyers-Ulam stability.

2 Preliminaries

In this section, we investigate the general solution to the first order linear inhomogeneous matrix difference equation (1.2) for all integers $i \in \mathbb{Z}$, where

$$\vec{x}_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{pmatrix} \in \mathbb{C}^n \quad \text{and} \quad \mathbf{A}(i) = \begin{pmatrix} a_{11}(i) & a_{12}(i) & \cdots & a_{1n}(i) \\ a_{21}(i) & a_{22}(i) & \cdots & a_{2n}(i) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(i) & a_{n2}(i) & \cdots & a_{nn}(i) \end{pmatrix} \in \mathbb{C}^{n\times n}.$$

Throughout this paper, we use the following abbreviation.

$$\Phi(n,m) := \begin{cases} \displaystyle\prod_{k=m}^{n-1} \mathbf{A}(k) = \mathbf{A}(n-1)\mathbf{A}(n-2)\cdots\mathbf{A}(m) & (\text{for } n > m), \\ \mathbf{I} & (\text{for } n = m), \end{cases} \tag{2.1}$$

where we set $\Phi(n,m) := \big(\Phi(m,n)\big)^{-1} = \mathbf{A}(n)^{-1}\mathbf{A}(n+1)^{-1}\cdots\mathbf{A}(m-1)^{-1}$ for $n < m$ and $\mathbf{I}$ denotes the identity matrix. Sometimes, we use $\Phi(n)$ and $\Phi^{-1}(m,n)$ instead of $\Phi(n,0)$ and $\big(\Phi(m,n)\big)^{-1}$, respectively.

In the following lemma, we introduce some properties of $\Phi(n,m)$ without proof.

Lemma 2.1 *Given a fixed positive integer n, assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular. It holds that*

(i) $\Phi(i+1,k) = \mathbf{A}(i)\Phi(i,k)$;

(ii) $\Phi^{-1}(i,k+1) = \mathbf{A}(k)\Phi^{-1}(i,k)$;

(iii) $\mathbf{A}(k-1)^{-1}\Phi^{-1}(i,k) = \Phi^{-1}(i,k-1)$

for all integers $i, k \in \mathbb{Z}$.

In the following lemma, we give the general solution to the first order linear inhomogeneous matrix difference equation (1.2).

Lemma 2.2 *Given a fixed positive integer n, assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular and the vectors $\vec{g}(i) \in \mathbb{C}^n$ are given. A vector sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$*

is a solution to the first order linear inhomogeneous matrix difference equation (1.2) *if and only if the sequence* $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ *is given in the form of*

$$\vec{x}_i := \begin{cases} \Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) & (\textit{for } i \geq 0), \\ \Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i} \Phi^{-1}(i+k,i)\vec{g}(i+k-1) & (\textit{for } i < 0), \end{cases} \tag{2.2}$$

where $\vec{x}_0 \in \mathbb{C}^n$ *is an arbitrarily given vector.*

Proof. First, we assume that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ is given in the form of (2.2) and we prove that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ is a solution to the first order linear inhomogeneous matrix difference equation (1.2).

If i is a nonnegative integer, then it follows from the first formula of (2.2) and Lemma 2.1 (i) that

$$\begin{aligned} \vec{x}_{i+1} &= \Phi(i+1,0)\vec{x}_0 + \sum_{k=0}^{i} \Phi(i+1,k+1)\vec{g}(k) \\ &= \mathbf{A}(i)\Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i} \mathbf{A}(i)\Phi(i,k+1)\vec{g}(k) \\ &= \mathbf{A}(i)\left(\Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k)\right) + \vec{g}(i) \\ &= \mathbf{A}(i)\vec{x}_i + \vec{g}(i) \end{aligned}$$

for any integer $i \geq 0$.

If $i = -1$, then we use (2.2) to get

$$\vec{x}_{i+1} = \vec{x}_0$$

and

$$\vec{x}_i = \vec{x}_{-1} = \Phi^{-1}(0,-1)\vec{x}_0 - \Phi^{-1}(0,-1)\vec{g}(-1) = \mathbf{A}(-1)^{-1}\vec{x}_0 - \mathbf{A}(-1)^{-1}\vec{g}(-1).$$

Hence, we have

$$\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i)$$

for $i = -1$.

If i is an integer less than -1, then it follows from the second formula of (2.2) and Lemma

2.1 (ii) that

$$\begin{aligned}
\vec{x}_{i+1} &= \Phi^{-1}(0,i+1)\vec{x}_0 - \sum_{k=1}^{-i-1} \Phi^{-1}(i+1+k,i+1)\vec{g}(i+k) \\
&= \mathbf{A}(i)\Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i-1} \mathbf{A}(i)\Phi^{-1}(i+k+1,i)\vec{g}(i+k) \\
&= \mathbf{A}(i)\Phi^{-1}(0,i)\vec{x}_0 - \sum_{j=2}^{-i} \mathbf{A}(i)\Phi^{-1}(i+j,i)\vec{g}(i+j-1) \\
&= \mathbf{A}(i)\Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i} \mathbf{A}(i)\Phi^{-1}(i+k,i)\vec{g}(i+k-1) + \mathbf{A}(i)\Phi^{-1}(i+1,i)\vec{g}(i) \\
&= \mathbf{A}(i)\vec{x}_i + \vec{g}(i)
\end{aligned}$$

for all integers $i < -1$.

Now, we assume that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ is a solution to the first order linear inhomogeneous matrix difference equation (1.2) and we prove that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ has the form of (2.2). We can easily show that the first formula of (2.2) holds for $i = 0$. We now assume that the first formula of (2.2) holds for some nonnegative integer i. Then, by using Lemma 2.1 (i), we obtain

$$\begin{aligned}
\vec{x}_{i+1} &= \mathbf{A}(i)\vec{x}_i + \vec{g}(i) \\
&= \mathbf{A}(i)\left(\Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k)\right) + \vec{g}(i) \\
&= \Phi(i+1,0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i+1,k+1)\vec{g}(k) + \vec{g}(i) \\
&= \Phi(i+1,0)\vec{x}_0 + \sum_{k=0}^{i} \Phi(i+1,k+1)\vec{g}(k)
\end{aligned}$$

by replacing i with $i+1$ in the first formula of (2.2).

Finally, we assume that the sequence $\{\vec{x}_i\}$ is a solution to (1.2) and we will prove that $\vec{x}_i$ is expressed by the second formula of (2.2) for every negative integer i. If we set $i = -1$ in (1.2), then we get

$$\vec{x}_0 = \mathbf{A}(-1)\vec{x}_{-1} + \vec{g}(-1) \quad \text{or} \quad \vec{x}_{-1} = \mathbf{A}(-1)^{-1}\vec{x}_0 - \mathbf{A}(-1)^{-1}\vec{g}(-1),$$

which we obtain from the second formula of (2.2) by setting $i = -1$. We now assume that $\vec{x}_i$ is expressed as the second formula of (2.2) for some negative integer i. Then, it follows from (1.2), the second formula of (2.2), and Lemma 2.1 (iii) that

$$\vec{x}_i = \mathbf{A}(i-1)\vec{x}_{i-1} + \vec{g}(i-1)$$

or

$$\begin{aligned}
\vec{x}_{i-1} &= \mathbf{A}(i-1)^{-1}\vec{x}_i - \mathbf{A}(i-1)^{-1}\vec{g}(i-1) \\
&= \mathbf{A}(i-1)^{-1}\left(\Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i}\Phi^{-1}(i+k,i)\vec{g}(i+k-1)\right) - \mathbf{A}(i-1)^{-1}\vec{g}(i-1) \\
&= \Phi^{-1}(0,i-1)\vec{x}_0 - \sum_{k=0}^{-i}\Phi^{-1}(i+k,i-1)\vec{g}(i+k-1) \\
&= \Phi^{-1}(0,i-1)\vec{x}_0 - \sum_{k=1}^{-i+1}\Phi^{-1}(i+k-1,i-1)\vec{g}(i+k-2),
\end{aligned}$$

which is a consequence of the second formula of (2.2) provided we replace i with $i-1$. □

Remark 2.3 *Given a fixed positive integer n, assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular and the vectors $\vec{g}(i) \in \mathbb{C}^n$ are given. If vector sequences $\{\vec{x}_{i,h}\}_{i\in\mathbb{Z}}$ and $\{\vec{x}_{i,p}\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ are defined by*

$$\vec{x}_{i,h} := \begin{cases} \Phi(i,0)\vec{x}_0 & (\text{for } i \geq 0), \\ \Phi^{-1}(0,i)\vec{x}_0 & (\text{for } i < 0) \end{cases}$$

resp.

$$\vec{x}_{i,p} := \begin{cases} \displaystyle\sum_{k=0}^{i-1}\Phi(i,k+1)\vec{g}(k) & (\text{for } i \geq 0), \\ \displaystyle -\sum_{k=1}^{-i}\Phi^{-1}(i+k,i)\vec{g}(i+k-1) & (\text{for } i < 0), \end{cases}$$

then then the sequence $\{\vec{x}_{i,h}\}_{i\in\mathbb{Z}}$ is a solution to the homogeneous difference equation $\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i$ corresponding to (1.2) and the sequence $\{\vec{x}_{i,p}\}_{i\in\mathbb{Z}}$ is a particular solution to the first order linear inhomogeneous matrix difference equation (1.2).

3 Hyers-Ulam stability of $\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i)$

We now prove our main theorem concerning the Hyers-Ulam stability of the first order linear inhomogeneous matrix difference equation (1.2). Obviously, our theorem is a generalization and an improvement of [13, Theorem 2.1].

Theorem 3.1 *Given a fixed positive integer n, let $(\mathbb{C}^n, \|\cdot\|_n)$ and $(\mathbb{C}^{n\times n}, \|\cdot\|_{n\times n})$ be complex normed spaces, whose elements are column vectors resp. $(n\times n)$ complex matrices, with the property (1.1). Assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular, the vectors $\vec{g}(i) \in \mathbb{C}^n$ are given, and that $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ is a sequence of nonnegative real numbers. If a vector sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ satisfies the inequality*

$$\|\vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i)\|_n \leq \varepsilon_{i+1} \tag{3.1}$$

for all $i \in \mathbb{Z}$, then there exists a solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the first order linear inhomogeneous matrix difference equation (1.2) *such that*

$$\|\vec{y}_i - \vec{x}_i\|_n \leq \begin{cases} \sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n} + \|\Phi(i,0)\|_{n\times n}\|\vec{y}_0 - \vec{x}_0\|_n & (\text{for } i \geq 0), \\ \sum_{k=1}^{-i} \varepsilon_{i+k} \left\|\Phi^{-1}(i+k,i)\right\|_{n\times n} + \left\|\Phi^{-1}(0,i)\right\|_{n\times n}\|\vec{y}_0 - \vec{x}_0\|_n & (\text{for } i < 0). \end{cases}$$

Proof. First, we assume that $i \geq 0$. In view of Lemma 2.2, the vector sequence $\{\vec{x}_i\}_{i=0,1,\ldots}$ defined by

$$\vec{x}_i = \Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) \tag{3.2}$$

satisfies the first order linear inhomogeneous matrix difference equation (1.2) for $i \geq 0$.

We now apply the mathematical induction to prove that

$$\vec{y}_i - \Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) = \sum_{k=1}^{i} \Phi(i,k)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big) \tag{3.3}$$

for all integers $i \geq 0$. It is obvious that the equality (3.3) holds for $i = 0$. We assume that the equality (3.3) holds for some integer $i \geq 0$. Then, it follows from Lemma 2.1 (i) and (3.3) that

$$\begin{aligned} &\vec{y}_{i+1} - \Phi(i+1,0)\vec{y}_0 - \sum_{k=0}^{i} \Phi(i+1,k+1)\vec{g}(k) \\ &= \vec{y}_{i+1} - \mathbf{A}(i)\Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i} \mathbf{A}(i)\Phi(i,k+1)\vec{g}(k) \\ &= \vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i) + \mathbf{A}(i)\left(\vec{y}_i - \Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k)\right) \\ &= \sum_{k=1}^{i} \mathbf{A}(i)\Phi(i,k)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big) + \vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i) \\ &= \sum_{k=1}^{i+1} \Phi(i+1,k)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big), \end{aligned}$$

which can be obtained from the equality (3.3) by replacing i with $i+1$. Thus, we conclude by induction that the equality (3.3) holds for all integers $i \geq 0$.

Hence, it follows from (3.1) and (3.3) that

$$\begin{aligned}
&\left\| \vec{y}_i - \Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) \right\|_n \\
&\quad \leq \sum_{k=1}^{i} \|\Phi(i,k)\|_{n\times n} \big\| \vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1) \big\|_n \\
&\quad \leq \sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n}
\end{aligned} \tag{3.4}$$

for $i \geq 0$. In view of (3.2) and (3.4), we have

$$\|\vec{y}_i - \Phi(i,0)\vec{y}_0 + \Phi(i,0)\vec{x}_0 - \vec{x}_i\|_n \leq \sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n}$$

or

$$\|\vec{y}_i - \vec{x}_i\|_n \leq \sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n} + \|\Phi(i,0)\|_{n\times n} \|\vec{y}_0 - \vec{x}_0\|_n$$

for all integers $i \geq 0$.

Now, assume that $i < 0$. By Lemma 2.2, the sequence $\{\vec{x}_i\}_{i=-1,-2,\ldots}$ defined by

$$\vec{x}_i = \Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i} \Phi^{-1}(i+k,i)\vec{g}(i+k-1) \tag{3.5}$$

satisfies the first order linear inhomogeneous matrix difference equation (1.2) for $i < 0$. Using the mathematical induction, we prove that

$$\begin{aligned}
&\vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i} \Phi^{-1}(i+k,i)\vec{g}(i+k-1) \\
&\quad = -\sum_{k=i+1}^{0} \Phi^{-1}(k,i)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big)
\end{aligned} \tag{3.6}$$

for all integers $i < 0$. It is obvious that the equality (3.6) holds for $i = -1$. We assume that the equality (3.6) holds for some integer $i < 0$. Then, it follows from Lemma 2.1 (*ii*), (*iii*), and (3.6) that

$$\begin{aligned}
&\vec{y}_{i-1} - \Phi^{-1}(0,i-1)\vec{y}_0 + \sum_{k=1}^{-i+1} \Phi^{-1}(i+k-1,i-1)\vec{g}(i+k-2) \\
&= \vec{y}_{i-1} - \mathbf{A}(i-1)^{-1}\Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i+1} \mathbf{A}(i-1)^{-1}\Phi^{-1}(i+k-1,i)\vec{g}(i+k-2) \\
&= \mathbf{A}(i-1)^{-1}\left(\mathbf{A}(i-1)\vec{y}_{i-1} - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i+1} \Phi^{-1}(i+k-1,i)\vec{g}(i+k-2) \right) \\
&= -\mathbf{A}(i-1)^{-1}\big(\vec{y}_i - \mathbf{A}(i-1)\vec{y}_{i-1} - \vec{g}(i-1)\big)
\end{aligned}$$

$$
\begin{aligned}
& + \mathbf{A}(i-1)^{-1}\left(\vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=2}^{-i+1} \Phi^{-1}(i+k-1,i)\vec{g}(i+k-2)\right) \\
= & -\mathbf{A}(i-1)^{-1}\big(\vec{y}_i - \mathbf{A}(i-1)\vec{y}_{i-1} - \vec{g}(i-1)\big) \\
& - \mathbf{A}(i-1)^{-1} \sum_{k=i+1}^{0} \Phi^{-1}(k,i)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big) \\
= & -\sum_{k=i}^{0} \mathbf{A}(i-1)^{-1}\Phi^{-1}(k,i)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big) \\
= & -\sum_{k=i}^{0} \Phi^{-1}(k,i-1)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big),
\end{aligned}
$$

which can be obtained from the equality (3.6) by replacing i with $i-1$. By induction, we conclude that the equality (3.6) holds for any integer $i < 0$.

Therefore, by (3.1) and (3.6), we get

$$
\begin{aligned}
& \left\| \vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i} \Phi^{-1}(i+k,i)\vec{g}(i+k-1) \right\|_n \\
& \quad \le \sum_{k=i+1}^{0} \|\Phi^{-1}(k,i)\|_{n\times n} \big\|\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big\|_n \\
& \quad \le \sum_{k=i+1}^{0} \varepsilon_k \|\Phi^{-1}(k,i)\|_{n\times n}
\end{aligned}
\tag{3.7}
$$

for any integer $i < 0$. Taking (3.5) and (3.7) into account, we get

$$
\|\vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \Phi^{-1}(0,i)\vec{x}_0 - \vec{x}_i\|_n \le \sum_{k=i+1}^{0} \varepsilon_k \|\Phi^{-1}(k,i)\|_{n\times n}
$$

or

$$
\begin{aligned}
\|\vec{y}_i - \vec{x}_i\|_n & \le \sum_{k=i+1}^{0} \varepsilon_k \big\|\Phi^{-1}(k,i)\big\|_{n\times n} + \big\|\Phi^{-1}(0,i)\big\|_{n\times n} \|\vec{y}_0 - \vec{x}_0\|_n \\
& = \sum_{k=1}^{-i} \varepsilon_{i+k} \big\|\Phi^{-1}(i+k,i)\big\|_{n\times n} + \big\|\Phi^{-1}(0,i)\big\|_{n\times n} \|\vec{y}_0 - \vec{x}_0\|_n
\end{aligned}
$$

for all integers $i < 0$. □

4 Applications

In this section, let n be a fixed positive integer. We assume that the nth order linear inhomogeneous difference equation of the form (1.3) is given, where $p_j, r : \mathbb{Z} \to \mathbb{C}$ are given functions with $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$.

If we set

$$
\|\mathbf{A}\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| \quad \text{and} \quad \|\vec{x}\|_\infty = \max_{1\le j\le n} |x_j|
$$

for all $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\vec{x} \in \mathbb{C}^n$, then these norms satisfy the conditions in (1.1).

We now prove Hyers-Ulam stability of the nth order linear inhomogeneous difference equation (1.3).

Theorem 4.1 *Let n be a fixed positive integer and $p_1, \ldots, p_n, r : \mathbb{Z} \to \mathbb{C}$ be given functions with $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$. Assume that a sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ of nonnegative numbers is given. If a sequence $\{a(i)\}_{i\in\mathbb{Z}}$ of complex numbers satisfies the inequality*

$$\big|a(i+1) - p_1(i)a(i) - p_2(i)a(i-1) - \cdots - p_n(i)a(i-n+1) - r(i)\big| \leq \varepsilon_{i+1} \tag{4.1}$$

for all $i \in \mathbb{Z}$, then there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the nth order linear inhomogeneous difference equation (1.3) such that

$$|a(i) - c(i)| \leq \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_\infty + \|\Phi(i,0)\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\textit{for } i \geq 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} \big\|\Phi^{-1}(i+k,i)\big\|_\infty + \big\|\Phi^{-1}(0,i)\big\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\textit{for } i < 0), \end{cases}$$

where $\Phi(i,k)$ and $\Phi^{-1}(i,k)$ are defined in (2.1) and (4.2), and where $\vec{y}_0$ and $\vec{x}_0$ are defined in (4.7).

Proof. For any $k \in \{1, 2, \ldots, n-1\}$, we define the complex numbers $b_k(i)$ by

$$\begin{aligned} b_1(i) &= a(i-1), \\ b_2(i) &= b_1(i-1), \\ b_3(i) &= b_2(i-1), \\ &\vdots \\ b_{n-1}(i) &= b_{n-2}(i-1) \end{aligned}$$

for all $i \in \mathbb{Z}$. We further define

$$\mathbf{A}(i) := \begin{pmatrix} p_1(i) & p_2(i) & p_3(i) & \cdots & p_{n-1}(i) & p_n(i) \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{4.2}$$

$$\vec{y}_i := \begin{pmatrix} a(i) \\ b_1(i) \\ b_2(i) \\ \vdots \\ b_{n-1}(i) \end{pmatrix} \quad \text{and} \quad \vec{g}(i) := \begin{pmatrix} r(i) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.3}$$

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for all $i \in \mathbb{Z}$, where $\mathbf{A}(i)$ is an $n \times n$ matrix and $\vec{y}_i, \vec{g}(i)$ are $n \times 1$ vectors.

Using these notations and considering (4.1), the sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ satisfies the inequality

$$\|\vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i)\|_\infty \le \varepsilon_{i+1}$$

for all $i \in \mathbb{Z}$. Moreover, by the assumption that $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$, we can see that every $\mathbf{A}(i)$ is nonsingular.

According to Theorem 3.1, there exists a solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the first order linear inhomogeneous matrix difference equation (1.2) such that

$$\|\vec{y}_i - \vec{x}_i\|_\infty \le \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_\infty + \|\Phi(i,0)\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \ge 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} \left\|\Phi^{-1}(i+k,i)\right\|_\infty + \left\|\Phi^{-1}(0,i)\right\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0). \end{cases} \tag{4.4}$$

If we set

$$\vec{x}_i := \begin{pmatrix} x_1(i) \\ x_2(i) \\ \vdots \\ x_n(i) \end{pmatrix}, \tag{4.5}$$

then it follows from (1.2) that

$$\begin{aligned} x_1(i+1) &= p_1(i)x_1(i) + p_2(i)x_2(i) + p_3(i)x_3(i) + \cdots + p_n(i)x_n(i) + r(i), \qquad (4.6)\\ x_2(i+1) &= x_1(i), \\ x_3(i+1) &= x_2(i), \\ &\vdots \\ x_n(i+1) &= x_{n-1}(i) \end{aligned}$$

for all $i \in \mathbb{Z}$. Moreover, if we define $c(i) := x_1(i)$ for all integers i, then we have

$$\begin{aligned} x_1(i+1) &= c(i+1), \\ x_1(i) &= c(i), \\ x_2(i) &= x_1(i-1) = c(i-1), \\ &\vdots \\ x_n(i) &= x_{n-1}(i-1) = \cdots = x_1(i-n+1) = c(i-n+1). \end{aligned}$$

Hence, by (4.6), the sequence $\{c(i)\}_{i\in\mathbb{Z}}$ is a solution to the nth order linear inhomogeneous difference equation (1.3).

Since

$$\vec{y}_i = \begin{pmatrix} a(i) \\ a(i-1) \\ a(i-2) \\ \vdots \\ a(i-n+1) \end{pmatrix} \quad \text{and} \quad \vec{x}_i = \begin{pmatrix} c(i) \\ c(i-1) \\ c(i-2) \\ \vdots \\ c(i-n+1) \end{pmatrix} \tag{4.7}$$

for all $i \in \mathbb{Z}$, we get

$$|a(i) - c(i)| \le \|\vec{y}_i - \vec{x}_i\|_\infty$$

for all $i \in \mathbb{Z}$. In view of (4.4), we complete the proof of this theorem. $\square$

We now consider the second order linear homogeneous difference equation of the form

$$a(i+1) = p_1(i)a(i) + p_2(i)a(i-1) \tag{4.8}$$

for all $i \in \mathbb{Z}$. The solution of (4.8) is called the (extended) Fibonacci numbers when $p_1(i) = p_2(i) \equiv 1$, $a(0) = 1$, and $a(1) = 1$.

If we substitute $n = 2$, $p_1(i) = 1$, $p_2(i) = 1$, and $r(i) = 0$ for all $i \in \mathbb{Z}$ in Theorem 4.1, then we prove the following corollary concerning Hyers-Ulam stability of the Fibonacci difference equation. However, this corollary shows that Theorem 4.1 is not efficient when the transition matrices $\mathbf{A}(i)$ are constant, *i.e.*, $\mathbf{A}(i) = \mathbf{A}$ for all $i \in \mathbb{Z}$. Nevertheless, we introduce this corollary because its proof includes some new properties of the extended Fibonacci numbers. (In general, it is reasonable to apply [21, Theorem 5] when the transition matrices $\mathbf{A}(i)$ are constant.)

Corollary 4.2 *Assume that a sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ of nonnegative numbers is given. If a sequence $\{a(i)\}_{i\in\mathbb{Z}}$ of complex numbers satisfies the inequality*

$$|a(i+1) - a(i) - a(i-1)| \le \varepsilon_{i+1} \tag{4.9}$$

for all $i \in \mathbb{Z}$, then there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the Fibonacci difference equation, i.e., the difference equation (4.8) with $p_1(i) = p_2(i) \equiv 1$ such that

$$|a(i) - c(i)| \le \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k F(i-k+1) + F(i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \ge 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} F(k+1) + F(-i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases}$$

where $F(i)$ denotes the ith extended Fibonacci number and

$$\|\vec{y}_0 - \vec{x}_0\|_\infty = \max\big\{|a(0) - c(0)|, |a(-1) - c(-1)|\big\}.$$

Proof. If we set

$$\mathbf{A} := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \vec{y}_i := \begin{pmatrix} a(i) \\ a(i-1) \end{pmatrix},$$

then it follows from (4.9) that

$$\|\vec{y}_{i+1} - \mathbf{A}\vec{y}_i\|_\infty \le \varepsilon_{i+1}$$

for all $i \in \mathbb{Z}$.

According to Theorem 4.1, there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the Fibonacci difference equation (4.8) with $p_1(i) = p_2(i) \equiv 1$ such that

$$|a(i) - c(i)| \leq \begin{cases} \sum_{k=1}^{i} \varepsilon_k \left\|\mathbf{A}^{i-k}\right\|_\infty + \left\|\mathbf{A}^i\right\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \geq 0), \\ \sum_{k=1}^{-i} \varepsilon_{i+k} \left\|\mathbf{A}^{-k}\right\|_\infty + \left\|\mathbf{A}^i\right\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases} \tag{4.10}$$

where $\vec{y}_i$ and $\vec{x}_i$ are defined in (4.7) for all $i \in \mathbb{Z}$.

Here, we introduce some (extended) Fibonacci numbers explicitly.

$$\begin{aligned} \ldots,\ F(-4) = 2,\ F(-3) = -1,\ F(-2) = 1,\ F(-1) = 0, \\ F(0) = 1,\ F(1) = 1,\ F(2) = 2,\ F(3) = 3,\ F(4) = 5,\ \ldots \end{aligned} \tag{4.11}$$

and we prove that

$$F(i)F(i-1) < 0 \tag{4.12}$$

for any integer $i \leq -2$. If the relation (4.12) were not true, then there would exist an integer $i_0 \leq -2$ such that $F(i_0)F(i_0 - 1) \geq 0$. Then we would have

$$\begin{aligned} -1 &= F(-2)F(-3) \\ &= F(-3)^2 + F(-3)F(-4) \\ &= F(-3)^2 + F(-4)^2 + F(-4)F(-5) \\ &\ \,\vdots \\ &= F(-3)^2 + F(-4)^2 + \cdots + F(i_0)^2 + F(i_0)F(i_0 - 1) \\ &\geq 0, \end{aligned}$$

which is a contradiction.

We now prove that

$$|F(i)| = |F(-i-2)| \tag{4.13}$$

for any $i \in \mathbb{Z}$. First, we apply the induction to prove that the equality (4.13) holds for all integers $i \geq 0$. In view of (4.11), it is obvious that the equality (4.13) holds for $i \in \{0, 1, 2\}$. Assume that (4.13) holds for all integers $1 \leq i \leq i_0$, where i_0 is an integer not less than 2. In view of (4.11) and (4.12), we further have

$$\begin{aligned} |F(i_0 + 1)| &= |F(i_0) + F(i_0 - 1)| \\ &= |F(i_0)| + |F(i_0 - 1)| \\ &= |F(-i_0 - 2)| + |F(-i_0 - 1)| \\ &= | - F(-i_0 - 2) + F(-i_0 - 1)| \\ &= |F(-i_0 - 3)|, \end{aligned}$$

which can be obtained from (4.13) by replacing i with $i_0 + 1$. Hence, we conclude that the equality (4.13) holds for all integers $i \geq 0$.

Now, we apply an induction to prove that the equality (4.13) holds for all integers $i < 0$. In view of (4.11), we easily see that the equality (4.13) holds for $i \in \{-1, -2\}$. Assume that (4.13) holds for all integers $i_0 \le i \le -3$, where i_0 is an integer less than -2. Then, by (4.12) and (4.13), we have

$$\begin{aligned} |F(i_0 - 1)| &= |F(i_0 + 1) - F(i_0)| \\ &= |F(i_0 + 1)| + |F(i_0)| \\ &= |F(-i_0 - 3)| + |F(-i_0 - 2)| \\ &= |F(-i_0 - 3) + F(-i_0 - 2)| \\ &= |F(-i_0 - 1)|, \end{aligned}$$

which we can obtain from (4.13) by replacing i with $i_0 - 1$. Thus, the equality (4.13) holds for all integers $i < 0$.

Moreover, we apply the mathematical induction to prove

$$\mathbf{A}^i = \begin{pmatrix} F(i) & F(i-1) \\ F(i-1) & F(i-2) \end{pmatrix} \tag{4.14}$$

for any $i \in \mathbb{Z}$. Obviously, the equality (4.14) holds for $i \in \{0, 1\}$. Assume that (4.14) holds for some integer $i \ge 0$. Then, we get

$$\begin{aligned} \mathbf{A}^{i+1} = \mathbf{A}^i \mathbf{A} &= \begin{pmatrix} F(i) & F(i-1) \\ F(i-1) & F(i-2) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} F(i) + F(i-1) & F(i) \\ F(i-1) + F(i-2) & F(i-1) \end{pmatrix} \\ &= \begin{pmatrix} F(i+1) & F(i) \\ F(i) & F(i-1) \end{pmatrix}, \end{aligned}$$

which can be obtained from (4.14) by replacing i with $i + 1$. Similarly, we prove that the equality (4.14) holds for all negative integers i.

Using (4.13) and (4.14), we prove that

$$\left\| \mathbf{A}^i \right\|_\infty = \begin{cases} F(i+1) & (\text{for } i \ge 0), \\ F(-i+1) & (\text{for } i < 0). \end{cases} \tag{4.15}$$

It is obvious that the first equality of (4.15) is true for $i \in \{0, 1\}$. Assume that $i \ge 2$. Then, considering (4.14) and the fact that $i - 2 \ge 0$, we have

$$\begin{aligned} \left\| \mathbf{A}^i \right\|_\infty &= \max\left\{ |F(i)| + |F(i-1)|, |F(i-1)| + |F(i-2)| \right\} \\ &= \max\left\{ F(i) + F(i-1), F(i-1) + F(i-2) \right\} \\ &= \max\left\{ F(i+1), F(i) \right\} \\ &= F(i+1) \end{aligned}$$

for any integer $i \ge 2$.

Now, we prove the equality (4.15) for $i < 0$. It follows from (4.13) and (4.14) that

$$\begin{aligned} \left\| \mathbf{A}^i \right\|_\infty &= \max\left\{ |F(i)| + |F(i-1)|, |F(i-1)| + |F(i-2)| \right\} \\ &= \max\left\{ |F(-i-2)| + |F(-i-1)|, |F(-i-1)| + |F(-i)| \right\} \\ &= \max\left\{ F(-i-2) + F(-i-1), F(-i-1) + F(-i) \right\} \\ &= \max\left\{ F(-i), F(-i+1) \right\} \\ &= F(-i+1) \end{aligned}$$

for any integer $i < 0$.

Finally, by (4.10) and (4.15), we have

$$|a(i) - c(i)| \leq \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k F(i-k+1) + F(i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \geq 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} F(k+1) + F(-i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases}$$

which completes our proof. □

According to [16, Theorem 5.1], the following formula is true:

$$\sum_{k=1}^{i} F(k) = F(i+2) - 2 \tag{4.16}$$

for all $i \in \mathbb{N}_0$, where $F(i)$ denotes the ith extended Fibonacci number with the initial values, $F(-1) = 0$, $F(0) = 1$, and $F(1) = 1$.

Remark 4.3 Let ε be an arbitrarily given positive number. Assume that a sequence $\{a(i)\}_{i\in\mathbb{Z}}$ of complex numbers satisfies the inequality

$$|a(i+1) - a(i) - a(i-1)| \leq \varepsilon$$

for all $i \in \mathbb{Z}$. According to Corollary 4.2 and (4.16), there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the Fibonacci difference equation such that

$$|a(i) - c(i)| \leq \begin{cases} F(i+2)\varepsilon - 2\varepsilon + F(i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i > 0), \\ \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i = 0), \\ F(-i+3)\varepsilon - 3\varepsilon + F(-i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases}$$

where $F(i)$ denotes the ith extended Fibonacci number with the initial values, $F(-1) = 0$, $F(0) = 1$, and $F(1) = 1$, and

$$\|\vec{y}_0 - \vec{x}_0\|_\infty = \max\big\{|a(0) - c(0)|, |a(-1) - c(-1)|\big\}.$$

In particular, under strong additional conditions that $a(-1) = c(-1)$ and $a(0) = c(0)$, the last inequality reduces into

$$|a(i) - c(i)| \leq \begin{cases} F(i+2)\varepsilon - 2\varepsilon & (\text{for } i > 0), \\ 0 & (\text{for } i = 0), \\ F(-i+3)\varepsilon - 3\varepsilon & (\text{for } i < 0). \end{cases}$$

Remark 4.4 The Hyers-Ulam stability of the Fibonacci functional equation has been investigated in [1, 10, 11, 14, 15], while Hyers-Ulam stability of the linear difference equations has been investigated in [1, 2, 3, 5, 17, 18, 19]. It should be remarked that many interesting theorems have been proved in [4, 6] concerning the linear (or nonlinear) recurrences. Especially, Hyers-Ulam stability of the first order matrix difference equations with constant matrix has been proved in [21] in the domain $\mathbb{N}_0$.

Acknowledgment. The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2013R1A1A2005557 and 2015R1D1A1A02061826).

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Self Adjoint Operator Ostrowski type Inequalities

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Abstract

We present here several self adjoint operator Ostrowski type inequalities to all directions. These are based in the operator order over a Hilbert space.

2010 AMS Subject Classification: 26D10, 26D20, 47A60, 47A67.
Key Words and Phrases: Self adjoint operator, Hilbert space, Ostrowski inequality.

1 Motivation

In 1938, A. Ostrowski [12] proved the following important inequality:

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty ,$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

In this article we present self adjoint operator Ostrowski type inequalities on a Hilbert space in the operator order.

2 Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set

1

$C(Sp(A))$ of all continuous functions defind on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [10, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f), \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued continuous functions on $Sp(A)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$ (the Banach algebra of all bounded linear operators from H into itself).

Equivalently, we use (see [8], pp. 7-8):

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y \rangle),$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle,$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

We have also the formula

$$\langle f(U)x, x \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x \rangle), \quad \forall\, x \in H.$$

As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda.$$

Above, $m = \min\{\lambda | \lambda \in Sp(U)\} := \min Sp(U)$, $M = \max\{\lambda | \lambda \in Sp(U)\} := \max Sp(U)$. The projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_\lambda := \varphi_\lambda(U), \ \forall\, \lambda \in \mathbb{R},$$

is a projection which reduces U, with

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

The spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ determines uniquely the self-adjoint operator U and vice versa.

For more on the topic see [11], pp. 256-266, and for more detalis see there pp. 157-266. See also [7].

Some more basics are given (we follow [8], pp. 1-5):

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x\rangle \in \mathbb{R}$, $\forall\, x \in H$, and if A is selfadjoint, then

$$\|A\| = \sup_{x \in H: \|x\|=1} |\langle Ax, x\rangle|.$$

Let A, B be selfadjoint operators on H. Then $A \leq B$ iff $\langle Ax, x\rangle \leq \langle Bx, x\rangle$, $\forall$ $x \in H$.

In particular, A is called positive if $A \geq 0$.

Denote by

$$\mathcal{P} := \left\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k | n \geq 0,\ \alpha_k \in \mathbb{C},\ 0 \leq k \leq n \right\}.$$

If $A \in \mathcal{B}(H)$ is selfadjoint, and $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A := \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

3

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [8], p. 4, Theorem 7).

Hence it holds

$$\||\varphi(A)|\| = \||\varphi|\|_A = \sup\{||\varphi(\lambda)||, \lambda \in Sp(A)\}$$

$$= \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\} = \|\varphi\|_A = \|\varphi(A)\|,$$

that is

$$\||\varphi(A)|\| = \|\varphi(A)\|.$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)| \, dE_\lambda =$$

$$\int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}.$$

3 Main Results

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$, $m < M$; $m, M \in \mathbb{R}$.

In the next we obtain Ostrowski type inequalities in the operator order of $\mathcal{B}(H)$ (the Banach algebra of all bounded linear operators from H into itself).

We mention

Theorem 1 *([2], p. 498) Let $f \in C^1([m, M])$, $m < M$, $s \in [m, M]$. Then*

$$\left| \frac{1}{M-m} \int_m^M f(t) \, dt - f(x) \right| \leq \left(\frac{(s-m)^2 + (M-s)^2}{2(M-m)} \right) \|f'\|_\infty . \qquad (1)$$

By applying property (P) to (1), we obtain in the operator order the following inequality:

Theorem 2 *Let $f \in C^1([m,M])$. Then*

$$\left| \left(\frac{1}{M-m} \int_m^M f(t)\,dt \right) 1_H - f(A) \right| \leq \left(\frac{(A - m1_H)^2 + (M1_H - A)^2}{2(M-m)} \right) \|f'\|_\infty . \tag{2}$$

We mention

Theorem 3 *([1], p. 191, Cerone-Dragomir) Let $f : [m,M] \to \mathbb{R}$ be a continuous on $[m,M]$ and twice differentiable function on (m,M), whose second derivative $f'' : (m,M) \to \mathbb{R}$ is bounded on (m,M). Then*

$$\left| f(s) - \frac{1}{M-m} \int_m^M f(t)\,dt - \left(\frac{f(M) - f(m)}{M-m} \right) \left(s - \frac{m+M}{2} \right) \right| \leq \tag{3}$$

$$\frac{1}{2} \left\{ \left[\frac{\left(s - \frac{m+M}{2}\right)^2}{(M-m)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (M-m)^2 \|f''\|_\infty \leq \frac{\|f''\|_\infty}{6} (M-m)^2 ,$$

$\forall\ s \in [m,M]$.

By applying property (P) to (3), we obtain in the operator order the following inequality:

Theorem 4 *All as in Theorem 3. Then*

$$\left| f(A) - \left(\frac{1}{M-m} \int_m^M f(t)\,dt \right) 1_H - \left(\frac{f(M) - f(m)}{M-m} \right) \left(A - \left(\frac{m+M}{2} \right) 1_H \right) \right| \tag{4}$$

$$\leq \frac{1}{2} \left\{ \left[\frac{\left(A - \left(\frac{m+M}{2}\right) 1_H\right)^2}{(M-m)^2} + \frac{1}{4} 1_H \right]^2 + \frac{1}{12} 1_H \right\} (M-m)^2 \|f''\|_\infty$$

$$\leq \left(\frac{\|f''\|_\infty}{6} (M-m)^2 \right) 1_H .$$

We mention

Theorem 5 *([3], p. 14) Let $f : [m,M] \to \mathbb{R}$ be 3-times differentiable on $[m,M]$. Assume that f''' is bounded on $[m,M]$. Let any $s \in [m,M]$. Then*

$$\left| f(s) - \frac{1}{M-m} \int_m^M f(t)\,dt - \left(\frac{f(M) - f(m)}{M-m} \right) \left(s - \left(\frac{m+M}{2} \right) \right) - \right.$$

$$\left(\frac{f'(M)-f'(m)}{2(M-m)}\right)\left[s^2-(m+M)s+\left(\frac{m^2+M^2+4mM}{6}\right)\right]\Bigg| \tag{5}$$
$$\leq \frac{\|f'''\|_\infty}{(M-m)^3} Z(s),$$

where

$$Z(s)=\Big[mMs^4-\frac{1}{3}m^2M^3s+\frac{1}{3}m^3Ms^2-mM^2s^3-\frac{1}{3}m^3M^2s+\frac{1}{3}mM^3s^2$$
$$+m^2M^2s^2-m^2Ms^3-\frac{1}{2}ms^5-\frac{1}{2}Ms^5+\frac{1}{6}s^6+\frac{3}{4}m^2s^4+\frac{3}{4}M^2s^4+\frac{1}{3}M^2m^4-$$
$$\frac{2}{3}m^3s^3-\frac{2}{3}M^3s^3-\frac{1}{3}M^3m^3+\frac{5}{12}m^4s^2+\frac{5}{12}M^4s^2+\frac{1}{3}M^4m^2-$$
$$\frac{2}{15}Mm^5-\frac{2}{15}mM^5-\frac{1}{6}m^5s-\frac{1}{6}M^5s+\frac{m^6}{20}+\frac{M^6}{20}\Big]. \tag{6}$$

Using (P) property and (5), (6) we derive

Theorem 6 *Let $f:[m,M]\to\mathbb{R}$ be 3-times differentiable on $[m,M]$. Assume that f''' is bounded on $[m,M]$. Then*

$$\Bigg|f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)1_H-\left(\frac{f(M)-f(m)}{M-m}\right)\left(A-\left(\frac{m+M}{2}\right)1_H\right)$$
$$-\left(\frac{f'(M)-f'(m)}{2(M-m)}\right)\left[A^2-(m+M)A+\left(\frac{m^2+M^2+4mM}{6}\right)1_H\right]\Bigg| \tag{7}$$
$$\leq \frac{\|f'''\|_\infty}{(M-m)^3} Z(A),$$

where

$$Z(A)=\Big[mMA^4-\frac{1}{3}m^2M^3A+\frac{1}{3}m^3MA^2-mM^2A^3-\frac{1}{3}m^3M^2A+$$
$$\frac{1}{3}mM^3A^2+m^2M^2A^2-m^2MA^3-\frac{1}{2}mA^5-\frac{1}{2}MA^5+\frac{1}{6}A^6+\frac{3}{4}m^2A^4+$$
$$\frac{3}{4}M^2A^4+\left(\frac{1}{3}M^2m^4\right)1_H-\frac{2}{3}m^3A^3-\frac{2}{3}M^3A^3-\left(\frac{1}{3}M^3m^3\right)1_H+$$
$$\frac{5}{12}m^4A^2+\frac{5}{12}M^4A^2+\left(\frac{1}{3}M^4m^2\right)1_H-$$
$$\left(\frac{2}{15}Mm^5\right)1_H-\left(\frac{2}{15}mM^5\right)1_H-\frac{1}{6}m^5A-\frac{1}{6}M^5A+\left(\frac{m^6+M^6}{20}\right)1_H\Big]. \tag{8}$$

6

Let $f \in AC([m,M])$ (absolutely continuous functions on $[m,M]$), $0 < \alpha < 1$. Denote the right Caputo fractional derivative by $D_{t-}^{\alpha} f$ (see [4], p. 22) and the left Caputo fractional derivative by $D_{*t}^{\alpha} f$ (see [4], p. 78), $\forall\, t \in [m,M]$.

We need

Theorem 7 *([4], p. 44) Let $0 < \alpha < 1$, $f \in AC([m,M])$, and $\left\|D_{t-}^{\alpha} f\right\|_{\infty,[m,t]}$, $\left\|D_{*t}^{\alpha} f\right\|_{\infty,[t,M]} < \infty$, $\forall\, t \in [m,M]$. Then*

$$\left|\frac{1}{M-m}\int_m^M f(z)\,dz - f(t)\right| \leq$$

$$\frac{1}{(M-m)\Gamma(\alpha+2)}\left\{\left\|D_{t-}^{\alpha} f\right\|_{\infty,[m,t]}(t-m)^{\alpha+1} + \left\|D_{*t}^{\alpha} f\right\|_{\infty,[t,M]}(M-t)^{\alpha+1}\right\} \leq \tag{9}$$

$$\frac{1}{\Gamma(\alpha+2)}\max\left\{\left\|D_{t-}^{\alpha} f\right\|_{\infty,[m,t]}, \left\|D_{*t}^{\alpha} f\right\|_{\infty,[t,M]}\right\}(M-m)^{\alpha}, \tag{10}$$

$\forall\, t \in [m,M]$.

By property (P) and Theorem 7 we derive

Theorem 8 *Let $0 < \alpha < 1$, $f \in AC([m,M])$, and there exists $K > 0$, such that*

$$\left\|D_{t-}^{\alpha} f\right\|_{\infty,[m,t]}, \left\|D_{*t}^{\alpha} f\right\|_{\infty,[t,M]} \leq K, \quad \forall\, t \in [m,M]. \tag{11}$$

Then

$$\left|\left(\frac{1}{M-m}\int_m^M f(z)\,dz\right)1_H - f(A)\right| \leq$$

$$\frac{K}{(M-m)\Gamma(\alpha+2)}\left\{(A-m1_H)^{\alpha+1} + (M1_H - A)^{\alpha+1}\right\} \leq \tag{12}$$

$$\left(\frac{K}{\Gamma(\alpha+2)}(M-m)^{\alpha}\right)1_H. \tag{13}$$

We mention the Fink ([9]) inequality

Theorem 9 *Let $f^{(n-1)}$ be absolutely continuous on $[m,M]$ and $f^{(n)} \in L_{\infty}(m,M)$, $n \in \mathbb{N}$. Then*

$$\left|f(s) + \sum_{k=1}^{n-1} F_k(s) - \frac{n}{M-m}\int_m^M f(t)\,dt\right| \leq$$

$$\frac{\left\|f^{(n)}\right\|_{\infty}}{(n+1)!(M-m)}\left[(M-s)^{n+1} + (s-m)^{n+1}\right], \quad \forall\, s \in [m,M], \tag{14}$$

where

$$F_k(s) := \left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(m)(s-m)^k - f^{(k-1)}(M)(s-M)^k}{M-m}\right). \tag{15}$$

7

If $n = 1$, *then* $\sum_{k=1}^{n-1} = 0$.

Inequality (14) is sharp, in the sense that is attained by an optimal f *for any* $s \in [m, M]$.

By property (P) and Theorem 9 we obtain

Theorem 10 *Let* $f^{(n-1)}$ *be absolutely continuous on* $[m, M]$ *and* $f^{(n)} \in L_\infty(m, M)$, $n \in \mathbb{N}$. *Then*

$$\left| f(A) + \sum_{k=1}^{n-1} F_k(A) - \left(\frac{n}{M-m} \int_m^M f(t)\,dt \right) 1_H \right| \leq \tag{16}$$

$$\frac{\left\| f^{(n)} \right\|_\infty}{(n+1)!\,(M-m)} \left[(M1_H - A)^{n+1} + (A - m1_H)^{n+1} \right],$$

where

$$F_k(A) := \left(\frac{n-k}{k!} \right) \left(\frac{f^{(k-1)}(m)(A - m1_H)^k - f^{(k-1)}(M)(A - M1_H)^k}{M-m} \right). \tag{17}$$

If $n = 1$, *then* $\sum_{k=1}^{n-1} F_k(A) = 0_H$.

We use here the sequence $\{B_k(t),\ k \geq 0\}$ of Bernoulli polynomials which is uniquely determined by the following identities:

$$\begin{array}{l} B_k'(t) = kB_{k-1}(t),\ \ k \geq 1,\ \ B_0(t) = 1 \\ \text{and} \\ B_k(t+1) - B_k(t) = kt^{k-1},\ \ k \geq 0. \end{array} \tag{18}$$

The values $B_k = B_k(0)$, $k \geq 0$ are the known Bernoulli numbers.

We mention

Theorem 11 *([3], p. 23) (see also [5]) Let* $f : [m, M] \to \mathbb{R}$ *be such that* $f^{(n-1)}$, $n \in \mathbb{N}$, *is a continuous function and* $f^{(n)}(t)$ *exists and is finite for all but a countable set of* t *in* (m, M) *and that* $f^{(n)} \in L_\infty([m, M])$.

Dentote by

$$\Delta_n(s) := f(s) - \frac{1}{M-m} \int_m^M f(t)\,dt -$$

$$\sum_{k=1}^{n-1} \frac{(M-m)^{k-1}}{k!} B_k\left(\frac{s-m}{M-m} \right) \left[f^{(k-1)}(M) - f^{(k-1)}(m) \right], \tag{19}$$

$\forall\ s \in [m, M]$.

8

Then

$$|\Delta_n(s)| \leq \frac{(M-m)^n}{n!} \left(\sqrt{\frac{(n!)^2}{(2n)!} |B_{2n}| + B_n^2 \left(\frac{s-m}{M-m} \right)} \right) \left\| f^{(n)} \right\|_\infty , \tag{20}$$

$\forall\ n \in \mathbb{N}$; $\forall\ s \in [m, M]$.

Using the (P) property and Theorem 11 we derive:

Theorem 12 *All terms and assumptions as in Theorem 11. Denote by*

$$\Delta_n(A) := f(A) - \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) 1_H -$$

$$\sum_{k=1}^{n-1} \frac{(M-m)^{k-1}}{k!} B_k \left(\frac{A - m1_H}{M-m} \right) \left[f^{(k-1)}(M) - f^{(k-1)}(m) \right]. \tag{21}$$

Then

$$|\Delta_n(A)| \leq \frac{(M-m)^n}{n!} \left(\sqrt{\left(\frac{(n!)^2}{(2n)!} |B_{2n}| \right) 1_H + B_n^2 \left(\frac{A - m1_H}{M-m} \right)} \right) \left\| f^{(n)} \right\|_\infty , \tag{22}$$

$\forall\ n \in \mathbb{N}$.

Denote by (see [3], p. 24)

$$I_4(\lambda) := \begin{cases} \frac{16\lambda^5}{5} - 7\lambda^4 + \frac{14}{3}\lambda^3 - \lambda^2 + \frac{1}{30}, & 0 \leq \lambda \leq \frac{1}{2}, \\ -\frac{16\lambda^5}{5} + 9\lambda^4 - \frac{26\lambda^3}{3} + 3\lambda^2 - \frac{1}{10}, & \frac{1}{2} \leq \lambda \leq 1, \end{cases} \tag{23}$$

which is continuous in $\lambda \in [0,1]$.

Also denote by

$$B := \left(\frac{A - m1_H}{M-m} \right)$$

and

$$I_4 \left(\frac{A - m1_H}{M-m} \right) = I_4(B) =$$

$$\begin{cases} \frac{16}{5}B^5 - 7B^4 + \frac{14}{3}B^3 - B^2 + \frac{1}{30}1_H, & 0_H \leq B \leq \frac{1}{2}1_H, \\ -\frac{16}{5}B^5 + 9B^4 - \frac{26B^3}{3} + 3B^2 - \frac{1}{10}1_H, & \frac{1}{2}1_H \leq B \leq 1_H. \end{cases} \tag{24}$$

We mention

Theorem 13 *([3], p. 25) All terms and assumptions as in Theorem 11, case of $n = 4$. For every $s \in [m, M]$ it holds*

$$|\Delta_4(s)| \leq \frac{(M-m)^4}{24} I_4(\lambda) \left\| f^{(4)} \right\|_\infty ,$$

where $I_4(\lambda)$ *is given by (23) with*

$$\lambda = \frac{s-m}{M-m}. \tag{25}$$

Furthermore we have that

$$|\Delta_4(s)| \leq \frac{(M-m)^4}{720} \left\| f^{(4)} \right\|_\infty , \tag{26}$$

$\forall\ s \in [m, M]$.

Using property (P) and Theorem 13 we find

Theorem 14 *All terms and assumptions are according to Theorem 11-13. Then*

$$|\Delta_4(A)| \leq \frac{(M-m)^4}{24} I_4\left(\frac{A - m1_H}{M-m}\right) \left\| f^{(4)} \right\|_\infty , \tag{27}$$

where $I_4\left(\frac{A-m1_H}{M-m}\right)$ *is given by (24).*

Furthermore we have that

$$|\Delta_4(A)| \leq \left(\frac{(M-m)^4}{720} \left\| f^{(4)} \right\|_\infty\right) 1_H. \tag{28}$$

Next we follow [6].

Let $(P_n)_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $P_0 = 1$. Let $f : [m, M] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \in \mathbb{N}$. Setting

$$\overline{F_k} = \frac{(-1)^k (n-k)}{M-m} \left[P_k(m) f^{(k-1)}(m) - P_k(M) f^{(k-1)}(M) \right], \quad k = 1, ..., n-1, \tag{29}$$

and

$$k(t,s) = \begin{cases} t-m, & \text{if } t \in [m,s] \\ t-M, & \text{if } t \in (s,M], \end{cases} \tag{30}$$

we get that

$$\frac{1}{n}\left[f(s) + \sum_{k=1}^{n-1} (-1)^k P_k(s) f^{(k)}(s) + \sum_{k=1}^{n-1} \overline{F_k} \right] - \frac{1}{M-m}\int_m^M f(t)\,dt = \tag{31}$$

$$\frac{(-1)^{n-1}}{n(M-m)} \int_m^M P_{n-1}(t)\, k(t,s)\, f^{(n)}(t)\,dt,$$

$\forall\ s \in [m, M]$. The above sums are defined to be zero for $n = 1$.

For the harmonic sequence of polynomials

$$P_k(t) = \frac{(t-s)^k}{k!}, \quad k \geq 0 \tag{32}$$

identity (31) collapses to the Fink identity, see [9].

We may rewrite generalized Fink identity (31) as follows:

$$f(s)=\sum_{k=1}^{n-1}(-1)^{k+1}P_k(s)f^{(k)}(s)+ \tag{33}$$

$$\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)f^{(k-1)}(M)-P_k(m)f^{(k-1)}(m)\right]+$$

$$\frac{n}{M-m}\int_m^M f(t)\,dt+\frac{(-1)^{n+1}}{M-m}\int_m^M P_{n-1}(t)\,k(t,s)\,f^{(n)}(t)\,dt,$$

$\forall\ s\in[m,M]$, $n\in\mathbb{N}$, when $n=1$ the above sums are zero.

Next we integrate the representation formula (33) against projections E_s to derive the operator representation formula:

$$f(A)=\sum_{k=1}^{n-1}(-1)^{k+1}P_k(A)f^{(k)}(A)+ \tag{34}$$

$$\left[\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)f^{(k-1)}(M)-P_k(m)f^{(k-1)}(m)\right]+\right.$$

$$\left.\frac{n}{M-m}\int_m^M f(t)\,dt\right]1_H+\frac{(-1)^{n+1}}{M-m}\int_{m-0}^M\left(\int_m^M P_{n-1}(t)\,k(t,s)\,f^{(n)}(t)\,dt\right)dE_s.$$

The sequence of polynomials

$$P_k(t)=\frac{1}{k!}\left(t-\frac{m+M}{2}\right)^k,\ \ k\geq 0, \tag{35}$$

is also harmonic.

We mention

Theorem 15 *([6]) Let $f:[m,M]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\in\mathbb{N}$ and $f^{(n)}\in L_p([m,M])$, $1\leq p\leq\infty$. Then*

$$\left|\left[f(s)+\sum_{k=1}^{n-1}(-1)^k P_k(s)f^{(k)}(s)+\sum_{k=1}^{n-1}\overline{F_k}\right]-\frac{n}{M-m}\int_m^M f(t)\,dt\right|\leq \tag{36}$$

$$\frac{1}{M-m}\left\|P_{n-1}(\cdot)\,k(\cdot,s)\right\|_{p',[m,M]}\left\|f^{(n)}\right\|_p,$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

We observe that

$$\int_m^M\left|P_{n-1}(t)\,k(t,s)\right|^{p'}dt\leq\left\|P_{n-1}\right\|_{\infty,[m,M]}^{p'}\int_m^M\left|k(t,s)\right|^{p'}dt= \tag{37}$$

$$\|P_{n-1}\|_{\infty,[m,M]}^{p'} \left[\int_m^s (t-m)^{p'} dt + \int_s^M (M-t)^{p'} dt \right] =$$

$$\|P_{n-1}\|_{\infty,[m,M]}^{p'} \left[\frac{(s-m)^{p'+1} + (M-s)^{p'+1}}{p'+1} \right].$$

Therefore we obtain

$$\|P_{n-1}(\cdot) k(\cdot, s)\|_{p',[m,M]} \leq \|P_{n-1}\|_{\infty,[m,M]} \left[\frac{(M-s)^{p'+1} + (s-m)^{p'+1}}{p'+1} \right]^{\frac{1}{p'}}. \tag{38}$$

Hence we have

Theorem 16 *Let $f : [m, M] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \in \mathbb{N}$ and $f^{(n)} \in L_p([m, M])$, $1 \leq p \leq \infty$. Then*

$$\left| \left(f(s) + \sum_{k=1}^{n-1} (-1)^k P_k(s) f^{(k)}(s) \right) + \left(\sum_{k=1}^{n-1} \overline{F_k} \right) - \left(\frac{n}{M-m} \int_m^M f(t) dt \right) \right| \leq$$

$$\left(\frac{\|f^{(n)}\|_p}{M-m} \|P_{n-1}\|_{\infty,[m,M]} \right) \left[\frac{(M-s)^{p'+1} + (s-m)^{p'+1}}{p'+1} \right]^{\frac{1}{p'}}, \tag{39}$$

$\forall\ s \in [m, M]$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

We get the following operator inequality:

Theorem 17 *Let $f : [m, M] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \in \mathbb{N}$ and $f^{(n)} \in L_p([m, M])$, $1 \leq p \leq \infty$. Then*

$$\left| \left(f(A) + \sum_{k=1}^{n-1} (-1)^k P_k(A) f^{(k)}(A) \right) + \left(\sum_{k=1}^{n-1} \overline{F_k} \right) 1_H - \right.$$

$$\left. \left(\frac{n}{M-m} \int_m^M f(t) dt \right) 1_H \right| \leq$$

$$\left(\frac{\|f^{(n)}\|_p}{M-m} \|P_{n-1}\|_{\infty,[m,M]} \right) \left[\frac{(M1_H - A)^{p'+1} + (A - m1_H)^{p'+1}}{p'+1} \right]^{\frac{1}{p'}}, \tag{40}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. By (P) property and (39). ■

We give

Corollary 18 *(to Theorem 16) (see also [6]) We have*

$$\left\| \left[f(s) + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \left(s - \frac{m+M}{2} \right)^k f^{(k)}(s) + \right. \right.$$

$$\left. \sum_{k=1}^{n-1} \frac{(M-m)^{k-1}(n-k)}{k!2^k} \left[f^{(k-1)}(m) - (-1)^k f^{(k-1)}(M) \right] \right]$$

$$\left. - \frac{n}{M-m} \int_m^M f(t)\,dt \right| \leq$$

$$\left(\frac{\left\| f^{(n)} \right\|_p (M-m)^{n-2}}{2^{n-1}(n-1)!} \right) \left[\frac{(M-s)^{p'+1} + (s-m)^{p'+1}}{p'+1} \right]^{\frac{1}{p'}}, \tag{41}$$

$\forall\ s \in [m, M]$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Set $P_k(t) = \frac{1}{k!}\left(t - \frac{m+M}{2}\right)^k$, $k \geq 0$, in Theorem 16. ■

We finish with the operator inequality:

Corollary 19 *(to Theorem 17) We have*

$$\left\| \left[f(A) + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \left(A - \left(\frac{m+M}{2} \right) 1_H \right)^k f^{(k)}(A) + \right. \right.$$

$$\left. \left(\sum_{k=1}^{n-1} \frac{(M-m)^{k-1}(n-k)}{k!2^k} \left[f^{(k-1)}(m) - (-1)^k f^{(k-1)}(M) \right] \right) 1_H \right]$$

$$\left. - \left(\frac{n}{M-m} \int_m^M f(t)\,dt \right) 1_H \right| \leq$$

$$\left(\frac{\left\| f^{(n)} \right\|_p (M-m)^{n-2}}{2^{n-1}(n-1)!} \right) \left[\frac{(M1_H - A)^{p'+1} + (A - m1_H)^{p'+1}}{p'+1} \right]^{\frac{1}{p'}}, \tag{42}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. By Corollary 18 and (P) property. ■

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.8, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Integer and Fractional Self Adjoint Operator Opial type Inequalities

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Abstract

We present here several integer and fractional self adjoint operator Opial type inequalities to many directions. These are based in the operator order over a Hilbert space.

2010 AMS Subject Classification: 26A33, 26D10, 26D20, 47A60, 47A67.
Key Words and Phrases: Self adjoint operator, Hilbert space, Opial inequality, fractional derivative.

1 Motivation

In 1960, Z. Opial ([9]) proved the following famous inequality that motivates our work here.

Let $f \in C^1([0,h])$ be such that $f(0) = f(h) = 0$, and $f(t) > 0$ in $(0,h)$. Then

$$\int_0^h |f(t) f'(t)| \, dt \leq \frac{h}{4} \int_0^h (f'(t))^2 \, dt.$$

The constant $\frac{h}{4}$ is the best.

In this article we present integer and fractional self adjoint operator Opial type inequalities on a Hilbert space in the operator order.

2 Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted

$Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [6, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f), \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued continuous functions on $Sp(A)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$. (the Banach algebra of all bounded linear operators from H into itself).

Equivalently, we use (see [5], pp. 7-8):

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda) \, d(\langle E_\lambda x, y \rangle),$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle,$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

We have also the formula

$$\langle f(U)x, x \rangle = \int_{m-0}^{M} f(\lambda) \, d(\langle E_\lambda x, x \rangle), \quad \forall\, x \in H.$$

As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda.$$

Above, $m = \min\{\lambda|\lambda \in Sp(U)\} := \min Sp(U)$, $M = \max\{\lambda|\lambda \in Sp(U)\} := \max Sp(U)$. The projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_\lambda := \varphi_\lambda(U), \ \ \forall\, \lambda \in \mathbb{R},$$

is a projection which reduces U, with

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

The spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ determines uniquely the self-adjoint operator U and vice versa.

For more on the topic see [8], pp. 256-266, and for more detalis see there pp. 157-266. See also [4].

Some more basics are given (we follow [5], pp. 1-5):

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x\rangle \in \mathbb{R}$, $\forall\, x \in H$, and if A is selfadjoint, then

$$\|A\| = \sup_{x \in H: \|x\|=1} |\langle Ax, x\rangle|.$$

Let A, B be selfadjoint operators on H. Then $A \leq B$ iff $\langle Ax, x\rangle \leq \langle Bx, x\rangle$, $\forall$ $x \in H$.

In particular, A is called positive if $A \geq 0$.

Denote by

$$\mathcal{P} := \left\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k | n \geq 0,\ \alpha_k \in \mathbb{C},\ 0 \leq k \leq n \right\}.$$

If $A \in \mathcal{B}(H)$ is selfadjoint, and $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A := \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\}.$$

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [5], p. 4, Theorem 7).

Hence it holds

$$\||\varphi(A)|\| = \||\varphi|\|_A = \sup\{||\varphi(\lambda)||, \lambda \in Sp(A)\}$$

$$= \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\} = \|\varphi\|_A = \|\varphi(A)\|,$$

that is

$$\||\varphi(A)|\| = \|\varphi(A)\|.$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)| \, dE_\lambda =$$

$$\int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}.$$

3 Main Results

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$, $m < M$; $m, M \in \mathbb{R}$.

In the next we obtain Opial type inequalities, both integer and fractional cases, in the operator order of $\mathcal{B}(H)$ (the Banach algebra of all bounded linear operators from H into itself).

Let the real valued function $f \in C([m, M])$, and we consider

$$g(t) = \int_m^t f(z) \, dz, \ \ \forall \ t \in [m, M], \tag{1}$$

then $g \in C([m, M])$.

We denote by

$$\int_{m1_H}^{A} f := \Phi(g) = g(A). \tag{2}$$

We understand and write that $(r > 0)$

$$g^r(A) = \Phi(g^r) =: \left(\int_{m1_H}^{A} f\right)^r.$$

Clearly $\left(\int_{m1_H}^{A} f\right)^r$ is a self adjoint operator on H, for any $r > 0$.

All of our functions in this article will be real valued. From [3] we mention the following basic version of Opial inequality:

Theorem 1 *Let $f \in C^1([m, M])$ with $f(m) = 0$. Then*

$$\int_m^\lambda |f(t)| |f'(t)| dt \le \left(\frac{\lambda - m}{2}\right) \int_m^\lambda (f'(t))^2 dt, \quad \forall\ \lambda \in [m, M]. \tag{3}$$

When $f(t) = t - m$, $t \in [m, M]$, inequality (3) becomes equality.

By applying properties (P) and (ii) to (3) we obtain

Theorem 2 *Let $f \in C^1([m, M])$ with $f(m) = 0$. Then*

$$\int_{m1_H}^{A} |ff'| \le \frac{1}{2}(A - m1_H)\left(\int_{m1_H}^{A} (f')^2\right). \tag{4}$$

We mention

Theorem 3 *([3]) Let $f \in C^1([m, M])$ with $f(m) = 0$, and $1 \le p \le 2$. Then*

$$\int_m^\lambda |f(t)|^p |f'(t)|^p dt \le K(p)(\lambda - m)\left(\int_m^\lambda (f'(t))^2 dt\right)^p, \quad \forall\ \lambda \in [m, M], \tag{5}$$

where

$$K(p) = \begin{cases} \frac{1}{2}, & p = 1, \\ \frac{4}{\pi^2}, & p = 2, \\ \frac{2-p}{2p}\left(\frac{1}{p}\right)^{2p-2} I^{-p}, & 1 < p < 2, \end{cases} \tag{6}$$

with

$$I = \int_0^1 \left\{1 + \frac{2(p-1)}{2-p} z\right\}^{-2} \{1 + (p-1) z\}^{\frac{1}{p}-1} dz.$$

For $p = 1$, equality holds in (5) only for f linear.

By applying properties (P) and (ii) to (5) we derive

Theorem 4 *Here all are as in Theorem 3. It holds*

$$\int_{m1_H}^{A} |ff'|^p \leq K(p)(A - m1_H)\left(\int_{m1_H}^{A} (f')^2\right)^p. \tag{7}$$

We mention

Theorem 5 *([7]) Let $f \in C^1([m,M])$ with $f(m) = 0$, and $p, q \geq 1$. Then*

$$\int_m^{\lambda} |f(t)|^p |f'(t)|^q dt \leq \left(\frac{q}{p+q}\right)(\lambda - m)^p \int_m^{\lambda} |f'(t)|^{p+q} dt, \quad \forall\ \lambda \in [m,M]. \tag{8}$$

By applying properties (P) and (ii) to (8) we find

Theorem 6 *Let $f \in C^1([m,M])$ with $f(m) = 0$, and $p, q \geq 1$. Then*

$$\int_{m1_H}^{A} |f|^p |f'|^q \leq \left(\frac{q}{p+q}\right)(A - m1_H)^p \left(\int_{m1_H}^{A} |f'|^{p+q}\right). \tag{9}$$

We mention

Theorem 7 *([11]) Let $p > -1$. Let $f \in C^1([m,M])$, and $f(m) = 0$. Then*

$$\int_m^{\lambda} t^p |f(t) f'(t)| dt \leq \frac{1}{2\sqrt{p+1}} \int_m^{\lambda} \left(\lambda^{p+1} - mt^p\right)(f'(t))^2 dt \tag{10}$$

$$\leq \frac{1}{2\sqrt{p+1}} \int_m^{\lambda} \left(M^{p+1} - mt^p\right)(f'(t))^2 dt, \quad \forall\ \lambda \in [m,M]. \tag{11}$$

(inequality (11) is our derivation).

By applying properties (P) and (ii) to (10), (11) we obtain

Theorem 8 *Let $p > -1$. Let $f \in C^1([m,M])$ and $f(m) = 0$. Then*

$$\int_{m1_H}^{A} (id)^p |ff'| \leq \frac{1}{2\sqrt{p+1}} \left(\int_{m1_H}^{A} \left(M^{p+1} - m(id)^p\right)(f')^2\right). \tag{12}$$

We mention

Theorem 9 *([1], p. 20) Let $q(t)$ be positive continuous and non-increasing function on $[m,M]$. Further, let $f \in C^1([m,M])$, and $f(m) = 0$. Let $l \geq 0$, $w \geq 1$. Then*

$$\int_m^{\lambda} q(t) |f(t)|^l |f'(t)|^w dt \leq \left(\frac{w}{l+w}\right)(\lambda - m)^l \int_m^{\lambda} q(t) |f'(t)|^{l+w} dt, \tag{13}$$

$\forall\ \lambda \in [m,M]$.

By applying property (P) and (ii) to (13) we obtain

Theorem 10 *All as in Theorem 9. Then*

$$\int_{m1_H}^{A} q\left|f\right|^{l}\left|f'\right|^{w} \leq \left(\frac{w}{l+w}\right)\left(A-m1_H\right)^{l}\int_{m1_H}^{A} q\left|f'\right|^{l+w}. \tag{14}$$

We mention

Theorem 11 *(see [1], p. 68) Let $q(t)$ positive, continuous and non-increasing on $[m,M]$. Further let $f_1, f_2 \in C^1([m,M])$ with $f_1(m) = f_2(m) = 0$. Let $l \geq 0$, $w \geq 1$. Then*

$$\int_{m}^{\lambda} q(t)\left|f_1(t) f_2(t)\right|^{l}\left[\left|f_1(t) f_2'(t)\right|^{w} + \left|f_1'(t) f_2(t)\right|^{w}\right] dt \leq$$

$$\frac{w}{2(l+w)}(\lambda-m)^{2l+w}\int_{m}^{\lambda} q(t)\left[\left(f_1'(t)\right)^{2(l+w)} + \left(f_2'(t)\right)^{2(l+w)}\right] dt, \tag{15}$$

$\forall\ \lambda \in [m,M]$.

By applying property (P) and (ii) to (15) we obtain

Theorem 12 *All as in Theorem 11. Then*

$$\int_{m1_H}^{A} q\left|f_1 f_2\right|^{l}\left[\left|f_1 f_2'\right|^{w} + \left|f_1' f_2\right|^{w}\right] \leq \tag{16}$$

$$\frac{w}{2(l+w)}\left(A-m1_H\right)^{2l+w}\int_{m1_H}^{A} q\left[\left(f_1'\right)^{2(l+w)} + \left(f_2'\right)^{2(l+w)}\right].$$

We mention

Theorem 13 *([10], p. 308) Let $f \in C^n([m,M])$, $n \in \mathbb{N}$, $f^{(i)}(m) = 0$, for $i = 0,1,2,...,n-1$. Then*

$$\int_{m}^{\lambda}\left|f(t) f^{(n)}(t)\right| dt \leq \frac{(\lambda-m)^n}{2}\int_{m}^{\lambda}\left(f^{(n)}(t)\right)^2 dt, \quad \forall\ \lambda \in [m,M]. \tag{17}$$

Using properties (P) and (ii) on (17) we derive

Theorem 14 *All as in Theorem 13. Then*

$$\int_{m1_H}^{A}\left|f \cdot f^{(n)}\right| \leq \frac{\left(A-m1_H\right)^n}{2}\left(\int_{m1_H}^{A}\left(f^{(n)}\right)^2\right). \tag{18}$$

We mention from [10], p. 309

Theorem 15 *Let $f_1, f_2 \in C^n([m, M])$ such that $f_1^{(k)}(m) = f_2^{(k)}(m) = 0$, for $k = 0, 1, ..., n-1$, $n \in \mathbb{N}$. Then*

$$\int_m^{\lambda} \left[\left| f_1(t) f_2^{(n)}(t) \right| + \left| f_2(t) f_1^{(n)}(t) \right| \right] dt \leq$$

$$B(\lambda - m)^n \int_m^{\lambda} \left[\left(f_1^{(n)}(t) \right)^2 + \left(f_2^{(n)}(t) \right)^2 \right] dt, \quad \forall\ \lambda \in [m, M], \tag{19}$$

where

$$B = \frac{1}{2n!} \left(\frac{n}{2n-1} \right)^{\frac{1}{2}}. \tag{20}$$

Using (19) and properties (P) and (ii) we obtain

Theorem 16 *All as in Theorem 15. Then*

$$\int_{m1_H}^{A} \left[\left| f_1 f_2^{(n)} \right| + \left| f_2 f_1^{(n)} \right| \right] \leq$$

$$B(A - m1_H)^n \left(\int_{m1_H}^{A} \left(\left(f_1^{(n)} \right)^2 + \left(f_2^{(n)} \right)^2 \right) \right). \tag{21}$$

Here we follow [2], p. 8.

Definition 17 *Let $\nu > 0$, $n := [\nu]$ (integral part), and $\alpha := \nu - n$ $(0 < \alpha < 1)$. Let $f \in C([m, M])$ and define*

$$(J_\nu^m f)(z) = \frac{1}{\Gamma(\nu)} \int_m^z (z-t)^{\nu-1} f(t)\, dt, \tag{22}$$

all $m \leq z \leq M$, where Γ is the gamma function, the generalized Riemann-Liouville integral. We define the subspace $C_m^\nu([m, M])$ of $C^n([m, M])$:

$$C_m^\nu([m, M]) := \left\{ f \in C^n([m, M]) : J_{1-\alpha}^m f^{(n)} \in C^1([m, M]) \right\}. \tag{23}$$

So let $f \in C_m^\nu([m, M])$; we define the generalized ν-fractional derivative (of Canavati type) of f over $[m, M]$ as

$$D_m^\nu f := \left(J_{1-\alpha}^m f^{(n)} \right)'. \tag{24}$$

Notice that

$$\left(J_{1-\alpha}^m f^{(n)} \right)(z) = \frac{1}{\Gamma(1-\alpha)} \int_m^z (z-t)^{-\alpha} f^{(n)}(t)\, dt \tag{25}$$

exists for $f \in C_m^\nu([m, M])$, all $m \leq z \leq M$.

Also notice that $D_m^\nu f \in C([m, M])$.

8

We need

Theorem 18 *([2], p. 15) Let* $f \in C_m^\nu([m,M])$, $\nu \geq 1$ *and* $f^{(i)}(m) = 0$, $i = 0,1,...,n-1$, $n := [\nu]$. *Here* $\lambda \in [m,M]$, *and* $l = 1,...,n-1$. *Let* $p,q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_m^\lambda \left| f^{(l)}(w) \right| \left| \left(D_m^\nu f \right)(w) \right| dw \leq$$

$$\frac{2^{-\frac{1}{q}} (\lambda - m)^{\frac{(\nu p - lp - p + 2)}{p}}}{\Gamma(\nu - l) \left((\nu p - lp - p + 1)(\nu p - lp - p + 2) \right)^{\frac{1}{p}}} \left(\int_m^\lambda \left| \left(D_m^\nu f \right)(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{26}$$

Using (26), properties (P) and (ii) we get

Theorem 19 *All as in Theorem 18. Then*

$$\int_{m1_H}^A \left| f^{(l)} \right| \left| \left(D_m^\nu f \right) \right| \leq$$

$$\frac{2^{-\frac{1}{q}} (A - m1_H)^{\frac{(\nu p - lp - p + 2)}{p}}}{\Gamma(\nu - l) \left((\nu p - lp - p + 1)(\nu p - lp - p + 2) \right)^{\frac{1}{p}}} \left(\int_{m1_H}^A \left| \left(D_m^\nu f \right) \right|^q \right)^{\frac{2}{q}}. \tag{27}$$

We need

Theorem 20 *([2], p. 26) Let* $\gamma_1, \gamma_2 \geq 0$, $\nu \geq 1$ *be such that* $\nu - \gamma_1, \nu - \gamma_2 \geq 1$ *and* $f \in C_m^\nu([m,M])$ *with* $f^{(i)}(m) = 0$, $i = 0,1,...,n-1$, $n := [\nu]$. *Here* $\lambda \in [m,M]$. *Let* q *be a nonnegative continuous functions on* $[m,M]$. *Denote*

$$Q(\lambda) := \left(\int_m^\lambda (q(w))^2 dw \right)^{\frac{1}{2}}, \quad \forall\ \lambda \in [m,M]. \tag{28}$$

Then

$$\int_m^\lambda q(w) \left| D_m^{\gamma_1}(f)(w) \right| \left| D_m^{\gamma_2}(f)(w) \right| dw \leq$$

$$K(q, \gamma_1, \gamma_2, \nu, \lambda, m) \left(\int_m^\lambda \left(D_m^\nu f(w) \right)^2 dw \right), \tag{29}$$

where

$$K(q, \gamma_1, \gamma_2, \nu, \lambda, m) := \frac{Q(\lambda)}{\sqrt[3]{6}} \frac{1}{\Gamma(\nu - \gamma_1)\Gamma(\nu - \gamma_2)} \cdot$$

$$\frac{(\lambda - m)^{2\nu - \gamma_1 - \gamma_2 - \frac{1}{2}}}{\left(\nu - \gamma_1 - \frac{5}{6} \right)^{\frac{1}{6}} \left(\nu - \gamma_2 - \frac{5}{6} \right)^{\frac{1}{6}} \left(4\nu - 2\gamma_1 - 2\gamma_2 - \frac{7}{3} \right)^{\frac{1}{2}}}. \tag{30}$$

Using (30) and Remark 3.4 of [2], p. 26, and properties (P) and (ii) to obtain

Theorem 21 *All terms and assumptions as in Theorem 20. Then*

$$\int_{m1_H}^{A} q\left|D_m^{\gamma_1}(f)\right|\left|D_m^{\gamma_2}(f)\right| \leq$$

$$K(q,\gamma_1,\gamma_2,\nu,A,m)\left(\int_{m1_H}^{A}\left(D_m^{\nu}f\right)^2\right), \tag{31}$$

where

$$K(q,\gamma_1,\gamma_2,\nu,A,m) := \frac{Q(A)}{\sqrt[3]{6}}\frac{1}{\Gamma(\nu-\gamma_1)\Gamma(\nu-\gamma_2)}\cdot$$

$$\frac{(A-m1_H)^{2\nu-\gamma_1-\gamma_2-\frac{1}{2}}}{\left(\nu-\gamma_1-\frac{5}{6}\right)^{\frac{1}{6}}\left(\nu-\gamma_2-\frac{5}{6}\right)^{\frac{1}{6}}\left(4\nu-2\gamma_1-2\gamma_2-\frac{7}{3}\right)^{\frac{1}{2}}}. \tag{32}$$

We need

Theorem 22 *([2], p. 30) Let $\gamma \geq 0$, $\nu \geq 1$, $\nu-\gamma \geq 1$, let q be a nonnegative continuous function on $[m,M]$. Let $f \in C_m^{\nu}([m,M])$ with $f^{(i)}(m) = 0$, $i = 0,1,...,n-1$, $n := [\nu]$. Let $\lambda \in [m,M]$. Call*

$$Q(\lambda) := \left(\int_m^{\lambda}(q(w))^2(w-m)^{2\nu-2\gamma-1}dw\right)^{\frac{1}{2}}, \tag{33}$$

and

$$K(q,\gamma,\nu,\lambda,m) := \frac{Q(\lambda)}{\sqrt{2(2\nu-2\gamma-1)}\Gamma(\nu-\gamma)}. \tag{34}$$

Then

$$\int_m^{\lambda} q(w)\left|D_m^{\gamma}f(w)\right|\left|D_m^{\nu}f(w)\right|dw \leq K(q,\gamma,\nu,\lambda,m)\left(\int_m^{\lambda}\left(\left(D_m^{\nu}f\right)(w)\right)^2dw\right). \tag{35}$$

Using (33)-(35) and properties (P) and (ii) we derive

Theorem 23 *All as in Theorem 22. Denote by*

$$K(q,\gamma,\nu,A,m) := \frac{Q(A)}{\sqrt{2(2\nu-2\gamma-1)}\Gamma(\nu-\gamma)}. \tag{36}$$

Then

$$\int_{m1_H}^{A} q\left|D_m^{\gamma}f\right|\left|D_m^{\nu}f\right| \leq K(q,\gamma,\nu,A,m)\left(\int_{m1_H}^{A}\left(\left(D_m^{\nu}f\right)\right)^2\right). \tag{37}$$

We need

Theorem 24 *([2], p. 92) Let $\nu \geq 1$, $\gamma_1, \gamma_2 \geq 0$, such that $\nu - \gamma_1 \geq 1$, $\nu - \gamma_2 \geq 1$, and $f_1, f_2 \in C_m^{\nu}([m, M])$ with $f_1^{(i)}(m) = f_2^{(i)}(m) = 0$, $i = 0, 1, ..., n-1$, $n := [\nu]$. Here $\lambda \in [m, M]$. Let $\lambda_\alpha, \lambda_\beta, \lambda_\nu \geq 0$. Set*

$$\rho(\lambda) := \frac{(\lambda - m)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)}}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}}. \tag{38}$$

Then

$$\int_m^\lambda \Big[\left|(D_m^{\gamma_1} f_1)(w)\right|^{\lambda_\alpha} \left|(D_m^{\gamma_2} f_2)(w)\right|^{\lambda_\beta} \left|(D_m^{\nu} f_1)(w)\right|^{\lambda_\nu} +$$

$$\left|(D_m^{\gamma_2} f_1)(w)\right|^{\lambda_\beta} \left|(D_m^{\gamma_1} f_2)(w)\right|^{\lambda_\alpha} \left|(D_m^{\nu} f_2)(w)\right|^{\lambda_\nu} \Big] dw \leq$$

$$\frac{\rho(\lambda)}{2} \left[\|D_m^\nu f_1\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} + \|D_m^\nu f_1\|_\infty^{2\lambda_\beta} + \|D_m^\nu f_2\|_\infty^{2\lambda_\beta} + \|D_m^\nu f_2\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} \right], \tag{39}$$

all $m \leq \lambda \leq M$.

Using (39) and properties (P) and (ii) we derive

Theorem 25 *All here as in Theorem 24. Set*

$$\rho(A) := \frac{(A - m1_H)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)}}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}}. \tag{40}$$

Then

$$\int_{m1_H}^A \Big[\left|(D_m^{\gamma_1} f_1)\right|^{\lambda_\alpha} \left|(D_m^{\gamma_2} f_2)\right|^{\lambda_\beta} \left|(D_m^{\nu} f_1)\right|^{\lambda_\nu} +$$

$$\left|(D_m^{\gamma_2} f_1)\right|^{\lambda_\beta} \left|(D_m^{\gamma_1} f_2)\right|^{\lambda_\alpha} \left|(D_m^{\nu} f_2)\right|^{\lambda_\nu} \Big] \leq$$

$$\frac{\rho(A)}{2} \left[\|D_m^\nu f_1\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} + \|D_m^\nu f_1\|_\infty^{2\lambda_\beta} + \|D_m^\nu f_2\|_\infty^{2\lambda_\beta} + \|D_m^\nu f_2\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} \right]. \tag{41}$$

We give

Definition 26 *([2], p. 270) Let $\nu > 0$, $n := \lceil \nu \rceil$ (ceiling of ν), $f \in AC^n([m, M])$ (i.e. $f^{(n-1)}$ is absolutely continuous on $[m, M]$, that is in $AC([m, M])$). We define the Caputo fractional derivative*

$$(D_{*m}^\nu f)(z) := \frac{1}{\Gamma(n - \nu)} \int_m^z (z - t)^{n - \nu - 1} f^{(n)}(t)\, dt, \tag{42}$$

which exists almost everywhere for $z \in [m, M]$.

Notice that $D_{*m}^0 f = f$, and $D_{*m}^n f = f^{(n)}$.

We mention

Theorem 27 *([2], p. 397) Let $\nu \geq \gamma+1$, $\gamma \geq 0$. Call $n := \lceil \nu \rceil$ and assume $f \in C^n([m,M])$ such that $f^{(k)}(m) = 0$, $k = 0,1,...,n-1$. Let $p, q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $m \leq \lambda \leq M$. Then*

$$\int_m^{\lambda} |(D_{*m}^{\gamma} f)(w)| \, |(D_{*m}^{\nu} f)(w)| \, dw \leq$$

$$\frac{(\lambda - m)^{\frac{(p\nu - p\gamma - p + 2)}{p}}}{\left(\sqrt[q]{2}\right) \Gamma(\nu - \gamma) \left((p\nu - p\gamma - p + 1)(p\nu - p\gamma - p + 2)\right)^{\frac{1}{p}}} \left(\int_m^{\lambda} |D_{*m}^{\nu} f(w)|^q \, dw \right)^{\frac{2}{q}}. \tag{43}$$

Note: By Proposition 15.114 ([2], p. 388) we have that $D_{*m}^{\nu} f, D_{*m}^{\gamma} f \in C([m,M])$.

Using (43) and Properties (P) and (ii) we give

Theorem 28 *All as in Theorem 27. Then*

$$\int_{m1_H}^{A} |(D_{*m}^{\gamma} f)| \, |(D_{*m}^{\nu} f)| \leq$$

$$\frac{(A - m1_H)^{\frac{(p\nu - p\gamma - p + 2)}{p}}}{\left(\sqrt[q]{2}\right) \Gamma(\nu - \gamma) \left((p\nu - p\gamma - p + 1)(p\nu - p\gamma - p + 2)\right)^{\frac{1}{p}}} \left(\int_{m1_H}^{A} |D_{*m}^{\nu} f|^q \right)^{\frac{2}{q}}. \tag{44}$$

We need

Theorem 29 *([2], p. 398) Let $\nu \geq 2$, $k \geq 0$, $\nu \geq k+2$. Call $n := \lceil \nu \rceil$ and $f \in C^n([m,M]) : f^{(j)}(m) = 0$, $j = 0,1,...,n-1$. Let $p, q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $m \leq \lambda \leq M$. Then*

$$\int_m^{\lambda} \left|\left(D_{*m}^{k} f\right)(w)\right| \left|\left(D_{*m}^{k+1} f\right)(w)\right| dw \leq$$

$$\frac{(\lambda - m)^{\frac{2(p\nu - pk - p + 1)}{p}}}{2\left(\Gamma(\nu - k)\right)^2 (p\nu - pk - p + 1)^{\frac{2}{p}}} \left(\int_m^{\lambda} |D_{*m}^{\nu} f(w)|^q \, dw \right)^{\frac{2}{q}}. \tag{45}$$

Using (45) and Properties (P) and (ii) we find

Theorem 30 *All as in Theorem 29. Then*

$$\int_{m1_H}^{A} \left|\left(D_{*m}^{k} f\right)\right| \left|\left(D_{*m}^{k+1} f\right)\right| \leq$$

$$\frac{(A - m1_H)^{\frac{2(p\nu - pk - p + 1)}{p}}}{2\left(\Gamma(\nu - k)\right)^2 (p\nu - pk - p + 1)^{\frac{2}{p}}} \left(\int_{m1_H}^{A} |D_{*m}^{\nu} f|^q \right)^{\frac{2}{q}}. \tag{46}$$

We need

Theorem 31 *([2], p. 399) Let* $\gamma_i \geq 0$, $\nu \geq 1$, $\nu - \gamma_i \geq 1$; $i = 1, ..., l$, $n := \lceil \nu \rceil$, *and* $f \in C^n([m, M])$ *such that* $f^{(k)}(m) = 0$, $k = 0, 1, ..., n-1$. *Here* $m \leq \lambda \leq M$; $q_1(\lambda), q_2(\lambda)$ *continuous functions on* $[m, M]$ *such that* $q_1(\lambda) \geq 0$, $q_2(\lambda) > 0$ *on* $[m, M]$, *and* $r_i > 0 : \sum_{i=1}^{l} r_i = r$. *Let* $s_1, s_1' > 1 : \frac{1}{s_1} + \frac{1}{s_1'} = 1$ *and* $s_2, s_2' > 1 : \frac{1}{s_2} + \frac{1}{s_2'} = 1$, *and* $p > s_2$.

Denote by

$$Q_1(\lambda) := \left(\int_m^\lambda (q_1(w))^{s_1'} dw \right)^{\frac{1}{s_1'}} \tag{47}$$

and

$$Q_2(\lambda) := \left(\int_m^\lambda (q_2(w))^{\frac{-s_2'}{p}} dw \right)^{\frac{r}{s_2'}}, \tag{48}$$

$$\sigma := \frac{p - s_2}{p s_2}. \tag{49}$$

Then

$$\int_m^\lambda q_1(w) \prod_{i=1}^{l} \left| D_{*m}^{\gamma_i} f(w) \right|^{r_i} dw \leq$$

$$Q_1(\lambda) Q_2(\lambda) \prod_{i=1}^{l} \left\{ \frac{\sigma^{r_i \sigma}}{(\Gamma(\nu - \gamma_i))^{r_i} (\nu - \gamma_i - 1 + \sigma)^{r_i \sigma}} \right\} \cdot$$

$$\frac{(\lambda - m)^{\left(\sum_{i=1}^{l} (\nu - \gamma_i - 1) r_i + \sigma r\right) + \frac{1}{s_1}}}{\left(\left(\sum_{i=1}^{l} (\nu - \gamma_i - 1) r_i s_1 \right) + r s_1 \sigma + 1 \right)^{\frac{1}{s_1}}} \left(\int_m^\lambda q_2(w) \left| D_{*m}^{\nu} f(w) \right|^p dw \right)^{\frac{r}{p}}. \tag{50}$$

Using (50) and properties (P) and (ii) we obtain

Theorem 32 *All here as in Theorem 31. Set*

$$Q_1(A) := \left(\int_{m1_H}^{A} (q_1)^{s_1'} \right)^{\frac{1}{s_1'}} \tag{51}$$

and

$$Q_2(A) := \left(\int_{m1_H}^{A} (q_2)^{\frac{-s_2'}{p}} \right)^{\frac{r}{s_2'}}. \tag{52}$$

Then

$$\int_{m1_H}^{A} q_1 \prod_{i=1}^{l} \left| D_{*m}^{\gamma_i} f \right|^{r_i} \leq$$

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$$Q_1(A)\,Q_2(A)\prod_{i=1}^{l}\left\{\frac{\sigma^{r_i\sigma}}{\left(\Gamma(\nu-\gamma_i)\right)^{r_i}(\nu-\gamma_i-1+\sigma)^{r_i\sigma}}\right\}\cdot$$

$$\frac{(A-m1_H)^{\left(\sum_{i=1}^{l}(\nu-\gamma_i-1)r_i+\sigma r\right)+\frac{1}{s_1}}}{\left(\left(\sum_{i=1}^{l}(\nu-\gamma_i-1)r_is_1\right)+rs_1\sigma+1\right)^{\frac{1}{s_1}}}\left(\int_{m1_H}^{A}q_2\left|D_{*m}^{\nu}f\right|^p\right)^{\frac{r}{p}}. \tag{53}$$

One can give many more operator Opial type (both integer and fractional) inequalities.

We choose to stop here.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.8, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Numerical solution of the generalized Hirota-Satsuma coupled Korteweg-de Vries equation by Fourier Pseudospectral method

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Abstract

In this paper, an approximate solution of the generalized Hirota-Satsuma (HS) coupled Korteweg-de Vries (KdV) equation by the use of Fourier pseudospectral method is presented. A time discrete scheme is constructed by approximating the time derivative using forward difference formula, while the pseudospectral method is used in the space direction. The stability and convergence of the scheme are investigated using the energy method. The numerical results reveal that the Fourier pseudospectral method is a convenient, effective and accurate method to solve the generalized HS coupled KdV equation.

Key words: Generalized Hirota-Satsuma coupled Korteweg-de Vries equation, Fourier pseudospectral method, Stability, Convergence.

1 Introduction

The generalized HS coupled KdV equations are as follows [1, 2]:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3u\frac{\partial u}{\partial x} + 3\frac{\partial}{\partial x}(vw), \qquad x \in \Omega, t \in [0,T], \tag{1.1}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial^3 v}{\partial x^3} + 3u\frac{\partial v}{\partial x}, \qquad x \in \Omega, t \in [0,T], \tag{1.2}$$

$$\frac{\partial w}{\partial t} = -\frac{\partial^3 w}{\partial x^3} + 3u\frac{\partial w}{\partial x}, \qquad x \in \Omega, t \in [0,T] \tag{1.3}$$

with initial conditions

$$u(x,0) = f(x), \qquad v(x,0) = g(x), \qquad w(x,0) = h(x), \qquad x \in \Omega, \tag{1.4}$$

and boundary conditions

$$u(-L,t) = u(L,t) = 0,\ v(-L,t) = v(L,t) = 0,\ w(-L,t) = w(L,t) = 0,\ t \in [0,T], \tag{1.5}$$

where $\Omega = [-L, L]$. Hirota-Satsuma [1] introduced generalized the HS coupled KdV equations in 1976 and these equations are models of shallow water waves. The equations (1.1)–(1.5) have travelling wave solutions and multiple soliton solutions.

The equations (1.1)–(1.5) have attracted the attention of many researchers and a lot of work has already been carried out on solution methods. For example, the homotopy perturbation method (HPM) by Ganji and Rafei [3], homotopy analysis method (HAM) and Adomian's decomposition method (ADM) by Abbasbandy [4], modified extended tanh function method by Ali [5], direct algebraic method by Zhang Huiqun [6]. Rong Jihong et al. [7] used bifurcation theory technique. The auxiliary function method was used by Yang Feng and Hong-Qing [8], analytical technique by Ganji et al. [9], homogenous balance

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method by Adel Raly et al. [10]. Jacobi elliptic functions expansion method by Baojin Hong [11]. Travelling wave solutions of the above equations investigated by Zuo and Zhang [12], Xie and Ding [13], Feng and Li [14]. A differential transform method (DTM) and reduced differential transform method (RDTM) was used by Reze and Malek [15], Hirota's bilinear method and pfaffian techniques by Junchao Chen et al. [16], while the Lie group method was applied by Mina B. et al. [17].

1.1 A brief review of Fourier pseudospectral method

In the last two decades spectral methods have been extensively used in the field of numerical solution of nonlinear partial differential equations. The use of spectral methods for solving partial differential and integro-differential equations have the advantage that its accuracy is higher than other standard numerical methods. Spectral methods retain the exponential rate of convergence when the solutions of the problems is sufficiently smooth. Spectral methods have three different categories namely Galerkin method, collocation method and tau method. The pseudospectral method is a type of spectral method which is easy to apply for nonlinear partial differential equations with periodic boundary value problems. For a more detailed discussion of spectral methods, please see ([18, 19, 20, 21, 22]).

The Fourier pseudospectral method involves two steps. First, the discrete representation of the solution is constructed by using trigonometric polynomial to interpolate the solution at collocation points. Second, the equations for the discrete values of the solution are obtained from the original equations. This second step involves finding an approximation for the differential operator in terms of the discrete values of the solution at collocation points. For detailed, please see ([18, 19, 23, 26]).

1.2 The main aim of the paper

In this paper, a Fourier pseudospectral method is applied to solve the generalized HS coupled KdV equation. A finite difference method is used in the time direction and Fourier pseudospectral method in the space direction. The stability of the time discrete scheme and convergence of the approximate solution is investigated by the energy method [29]. Numerical results are shown to demonstrate the efficiency of the method. It should be noted that Darvishi et al. [27] solved the same equation by pseudospectral method and transformed the partial differential equation to ordinary differential equations. They found the numerical solution by using classical fourth-order Runge-Kutta method. There is no proof of stability and convergence. In our paper, we follow the approach of [23, 28].

The outline of the paper is as follows. In section 2 we present some preliminaries which will be used in next two sections. Section 3 is related to stability of the scheme for generalized Hirota-Satsuma (HS) coupled Korteweg-de Vries (KdV) equation. Convergence of the approximate solution is proved in section 4. Numerical results are presented for the applicability of the method section 5. Finally the conclusion is given in section 6.

2 Preliminaries

The inner product and norm are defined by $(u,v)=\int_\Omega u(x)v(x)dx$ and $\|u\|^2=(u,u)$ respectively. The maximum norm is denoted by $\|u\|_\infty$. The periodic Sobolev space is defined by [23]:

$$H^1=\left\{u\in L^2(R):\frac{du}{dx}\in L^2(R)\right\},\qquad H_p^1=\{u\in H^1(R):u(x-L)=u(x+L)\}.$$

The Sobolev norm and semi-norms are defined by [23]:

$$\|u\|=(u,u)^{1/2},\quad \|u\|_{H^1}=(\|u\|^2+\|\frac{\partial u}{\partial x}\|^2)^{1/2},\quad |u|_k=|u|_{H^k}=\sum_{|\beta|=k}\Big(\int_\Omega (D^\beta u)^2dx\Big)^{1/2}.$$

We define $t_n=n\tau,\quad n=0,1,...,N$, where $\tau=T/N$ is the step size in time direction. The equation (1.1)–(1.3) is evaluated at the point (x,t_n), $n=0,1,\ldots,N$. We denote $u^n=u(x,t_n)$, $v^n=v(x,t_n)$ and

$w^n = w(x, t_n)$, then equation (1.1), (1.2) and (1.3) can be written as:

$$u^{n+1} = u^n + \tau \left(\frac{1}{2} \frac{\partial^3}{\partial x^3} u^n - 3u^n \frac{\partial u^n}{\partial x} + 3 \frac{\partial}{\partial x}(v^n w^n) \right) + \tau R_1^n, \tag{2.1}$$

$$v^{n+1} = v^n + \tau \left(-\frac{\partial^3}{\partial x^3} v^n + 3u^n \frac{\partial v^n}{\partial x} \right) + \tau R_2^n, \tag{2.2}$$

$$w^{n+1} = w^n + \tau \left(-\frac{\partial^3}{\partial x^3} w^n + 3u^n \frac{\partial w^n}{\partial x} \right) + \tau R_3^n, \tag{2.3}$$

where R_1^n, R_2^n, and R_3^n are residual of the equation (2.1), (2.2) and (2.3) respectively. Furthermore $|R_1^n| < C_1\tau$, $|R_2^n| < C_2\tau$ and $|R_3^n| < C_3\tau$ for some positive constants C_1, C_2 and C_3. By ignoring the small terms R_1^n, R_2^n and R_3^n in the above equations, the time discrete scheme for the equation (2.1), (2.2) and (2.3) can be obtained as:

$$U^{n+1} = U^n + \tau \left(\frac{1}{2} \frac{\partial^3}{\partial x^3} U^n - 3U^n \frac{\partial U^n}{\partial x} + 3 \frac{\partial}{\partial x}(V^n W^n) \right), \tag{2.4}$$

$$V^{n+1} = V^n + \tau \left(-\frac{\partial^3}{\partial x^3} V^n + 3U^n \frac{\partial V^n}{\partial x} \right), \tag{2.5}$$

$$W^{n+1} = W^n + \tau \left(-\frac{\partial^3}{\partial x^3} W^n + 3U^n \frac{\partial W^n}{\partial x} \right), \tag{2.6}$$

where $U^n = U(x, t_n)$, $V^n = V(x, t_n)$ and $W^n = W(x, t_n)$. We present a lemma, which will be useful for the proof of stability and convergence.

Lemma 2.1 ([24]). *If $m \geq 1$, and $u, v \in H^m(\Omega)$, there exists a constant C independent of u, v and N, such that*

$$\|uv\|_m \leq C \|u\|_m \|v\|_m .$$

3 Stability

Assume $U^n(x,t)$ to be the approximate solution of $u^n(x,t)$, $V^n(x,t)$ to be the approximate solution of $v^n(x,t)$ and $W^n(x,t)$ be the approximate solution of $w^n(x,t)$. For simplicity we denote $u^n = u^n(x,t)$ and similarly for other variables. Let

$$\widetilde{u}^n = u^n - U^n, \qquad \widetilde{v}^n = v^n - V^n, \qquad \widetilde{w}^n = w^n - W^n.$$

Subtracting (2.4) from (2.1), (2.5) from (2.2) and (2.6) from (2.3) results in

$$\widetilde{u}^{n+1} = \widetilde{u}^n + \frac{\tau}{2} \frac{\partial^3}{\partial x^3} \widetilde{u}^n - 3\tau \left(u^n \frac{\partial u^n}{\partial x} - U^n \frac{\partial U^n}{\partial x} \right) + 3\tau \frac{\partial}{\partial x} \left(v^n w^n - V^n W^n \right), \tag{3.1}$$

$$\widetilde{v}^{n+1} = \widetilde{v}^n + \tau \left(-\frac{\partial^3}{\partial x^3} \widetilde{v}^n \right) + 3\tau \left(u^n \frac{\partial v^n}{\partial x} - U^n \frac{\partial V^n}{\partial x} \right), \tag{3.2}$$

$$\widetilde{w}^{n+1} = \widetilde{w}^n + \tau \left(-\frac{\partial^3}{\partial x^3} \widetilde{w}^n \right) + 3\tau \left(u^n \frac{\partial w^n}{\partial x} - U^n \frac{\partial W^n}{\partial x} \right). \tag{3.3}$$

Taking the inner product of (3.1), (3.2) and (3.3) with $\widetilde{u}^{n+1}$, $\widetilde{v}^{n+1}$ and $\widetilde{w}^{n+1}$ respectively. By applying Cauchy-Schwartz inequality, algebraic and Young's inequalities, we have

$$\begin{aligned} (1+3\tau)\|\widetilde{u}^{n+1}\|^2 + \tau \left\| \frac{\partial \widetilde{u}^{n+1}}{\partial x} \right\|^2 \leq \|\widetilde{u}^n\|^2 + \tau \left\| \frac{\partial^2 \widetilde{u}^n}{\partial x^2} \right\|^2 - 3\tau \left\| u^n \frac{\partial u^n}{\partial x} - U^n \frac{\partial U^n}{\partial x} \right\|^2 \\ + 3\tau \left\| v^n w^n - V^n W^n \right\|^2 , \end{aligned} \tag{3.4}$$

$$(1+3\tau)\|\widetilde{v}^{n+1}\|^2 + \tau \left\| \frac{\partial \widetilde{v}^{n+1}}{\partial x} \right\|^2 \leq \|\widetilde{v}^n\|^2 + \tau \left\| \frac{\partial^2 \widetilde{v}^n}{\partial x^2} \right\|^2 + 3\tau \left\| u^n \frac{\partial v^n}{\partial x} - U^n \frac{\partial V^n}{\partial x} \right\|^2 , \tag{3.5}$$

$$(1+3\tau)\|\widetilde{w}^{n+1}\|^2+\tau\left\|\frac{\partial\widetilde{w}^{n+1}}{\partial x}\right\|^2\leq\|\widetilde{w}^n\|^2+\tau\left\|\frac{\partial^2\widetilde{w}^n}{\partial x^2}\right\|^2+3\tau\left\|u^n\frac{\partial w^n}{\partial x}-U^n\frac{\partial W^n}{\partial x}\right\|^2, \tag{3.6}$$

Now we are going to estimate nonlinear terms of (3.4), (3.5) and (3.6). Again we apply Cauchy-Schwartz inequality and lemma 2.1, we get

$$\begin{aligned}
\left\|u^n\frac{\partial u^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\| &= \left\|u^n\frac{\partial u^n}{\partial x}-u^n\frac{\partial U^n}{\partial x}+u^n\frac{\partial U^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\| \\
&= \left\|u^n\left(\frac{\partial u^n}{\partial x}-\frac{\partial U^n}{\partial x}\right)+\frac{\partial U^n}{\partial x}(u^n-U^n)\right\| \\
&\leq \|u^n\|_\infty\left\|\frac{\partial u^n}{\partial x}-\frac{\partial U^n}{\partial x}\right\|+\left\|\frac{\partial U^n}{\partial x}\right\|_\infty\|u^n-U^n\| \\
&\leq C_4\left(\left\|\frac{\partial u^n}{\partial x}-\frac{\partial U^n}{\partial x}\right\|+\|u^n-U^n\|\right)
\end{aligned}$$

where $C_4=\left(\|\frac{\partial U^n}{\partial x}\|_\infty,\,\|u^n\|_\infty\right)$, we obtain

$$\left\|u^n\frac{\partial u^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\|^2\leq C_4\left(\left\|\frac{\partial\widetilde{u}^n}{\partial x}\right\|^2+\|\widetilde{u}^n\|^2\right)$$

Similarly we can apply Cauchy-Schwartz inequality and lemma 2.1, we get the estimation of nonlinear terms of (3.4), (3.5) and (3.6), we have

$$\begin{aligned}
&\|v^nw^n-V^nW^n\|^2\leq C_5\left(\|\widetilde{v}^n\|^2+\|\widetilde{w}^n\|^2\right)\\
&\left\|u^n\frac{\partial v^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\|^2\leq C_6\left(\|\widetilde{u}^n\|^2+\left\|\frac{\partial\widetilde{v}^n}{\partial x}\right\|^2\right),\\
&\left\|u^n\frac{\partial w^n}{\partial x}-U^n\frac{\partial W^n}{\partial x}\right\|^2\leq C_7\left(\|\widetilde{u}^n\|^2+\left\|\frac{\partial\widetilde{w}^n}{\partial x}\right\|^2\right).
\end{aligned}$$

where $C_5=\left(\|\frac{\partial V^n}{\partial x}\|_\infty,\,\|u^n\|_\infty\right)$, $C_6=\left(\|\frac{\partial W^n}{\partial x}\|_\infty,\,\|u^n\|_\infty\right)$, where $C_7=(\|v^n\|_\infty,\,\|W^n\|_\infty)$. Substituting the value of above values into (3.4), (3.5) and (3.6). Further more $\widetilde{C}=\max(C_4,C_5,C_6,C_8)$. We get

$$\begin{aligned}
&(1-3\tau)\left(\|\widetilde{u}^{n+1}\|^2+\left\|\frac{\partial\widetilde{u}^{n+1}}{\partial x}\right\|^2+\|\widetilde{v}^{n+1}\|^2+\left\|\frac{\partial\widetilde{v}^{n+1}}{\partial x}\right\|^2+\|\widetilde{w}^{n+1}\|^2+\left\|\frac{\partial\widetilde{w}^{n+1}}{\partial x}\right\|^2\right)\\
&\qquad\leq(1+3\tau)\widetilde{C}\left(\|\widetilde{u}^n\|^2+\left\|\frac{\partial\widetilde{u}^n}{\partial x}\right\|^2+\|\widetilde{v}^n\|^2+\left\|\frac{\partial\widetilde{v}^n}{\partial x}\right\|^2+\|\widetilde{w}^n\|^2+\left\|\frac{\partial\widetilde{w}^n}{\partial x}\right\|^2\right)
\end{aligned}\tag{3.7}$$

$$\begin{aligned}
\|\widetilde{u}^{n+1}\|_{H^1}^2+\|\widetilde{v}^{n+1}\|_{H^1}^2+\|\widetilde{w}^{n+1}\|_{H^1}^2 &\leq\left(\frac{(1+3\tau)\widetilde{C}}{1-3\tau}\right)\left(\|\widetilde{u}^n\|_{H^1}^2+\|\widetilde{v}^n\|_{H^1}^2+\|\widetilde{w}^n\|_{H^1}^2\right)\\
&\leq\left(\frac{(1+3\tau)\widetilde{C}}{1-3\tau}\right)^2\left(\|\widetilde{u}^{n-1}\|_{H^1}^2+\|\widetilde{v}^{n-1}\|_{H^1}^2+\|\widetilde{w}^{n-1}\|_{H^1}^2\right)\\
&\ \vdots\\
&\leq\left(\frac{(1+3\tau)\widetilde{C}}{1-3\tau}\right)^{n+1}\left(\|\widetilde{u}^0\|_{H^1}^2+\|\widetilde{v}^0\|_{H^1}^2+\|\widetilde{w}^0\|_{H^1}^2\right)
\end{aligned}$$

Let

$$\lim_{n\longrightarrow\infty}\left(\frac{\widetilde{C}(1+3\tau)}{1-3\tau}\right)^{n+1}=\lim_{n\longrightarrow\infty}\left(\frac{\widetilde{C}(1+\frac{3\tau}{n+1})}{1-\frac{3\tau}{n+1}}\right)^{n+1}=\frac{\widetilde{C}e^{3\tau}}{e^{-3\tau}}=e^{6\widetilde{C}\tau} \tag{3.8}$$

Therefore

$$\|\widetilde{u}^{n+1}\|^2_{H^1} + \|\widetilde{v}^{n+1}\|^2_{H^1} + \|\widetilde{w}^{n+1}\|^2_{H^1} \leq \sqrt{e^{6\widetilde{C}\tau}}\left(\|\widetilde{u}^0\|^2_{H^1} + \|\widetilde{v}^0\|^2_{H^1} + \|\widetilde{w}^0\|^2_{H^1}\right)$$

Theorem 1. *Let u_0, v_0 and w_0 belong to $H^1(\Omega)$. Further, let u^n, v^n and w^n be the solution for initial boundary value problem (1.1)–(1.5) and U^n, V^n and W^n be the solution of the time discrete scheme (2.4)–(2.6). If $\tau < 1/3$ then solution of the discrete scheme is stable in H^1 norm*

4 Convergence

In this section we consider the convergence of of approximate solution of generalized HS coupled KdV equation. Define

$$\widetilde{U}^n = u^n - U^n, \qquad \widetilde{V}^n = v^n - V^n, \qquad \widetilde{W}^n = w^n - W^n.$$

From equations (2.1)–(2.3) and (2.4)–(2.6), we obtain

$$\widetilde{U}^{n+1} = \widetilde{U}^n + \frac{\tau}{2}\frac{\partial^3 \widetilde{U}^n}{\partial x^3} + 3\tau\left(u^n\frac{\partial u^n}{\partial x} - U^n\frac{\partial U^n}{\partial x}\right) - 3\tau\frac{\partial}{\partial x}\left(v^n w^n - V^n W^n\right) + \tau R_1^n, \tag{4.1}$$

$$\widetilde{V}^{n+1} = \widetilde{V}^n + \tau\left(-\frac{\partial^3 \widetilde{V}^n}{\partial x^3}\right) + 3\tau\left(u^n\frac{\partial v^n}{\partial x} - U^n\frac{\partial V^n}{\partial x}\right) + \tau R_2^n, \tag{4.2}$$

$$\widetilde{W}^{n+1} = \widetilde{W}^n + \tau\left(-\frac{\partial^3 \widetilde{W}^n}{\partial x^3}\right) + 3\tau\left(u^n\frac{\partial w^n}{\partial x} - U^n\frac{\partial W^n}{\partial x}\right) + \tau R_3^n. \tag{4.3}$$

Taking the inner product of (4.1), (4.2) and (4.3) with $\widetilde{U}^{n+1}$, $\widetilde{V}^{n+1}$ and $\widetilde{W}^{n+1}$ respectively, yields

$$\|\widetilde{U}^{n+1}\|^2 \leq \frac{1}{2}\|\widetilde{U}^n\|^2 - \frac{\tau}{2}\left(\left\|\frac{\partial^2 \widetilde{U}^n}{\partial x^2}\right\|^2 + \left\|\frac{\partial \widetilde{U}^{n+1}}{\partial x}\right\|^2\right) + \tau|R_1^n|\|\widetilde{U}^{n+1}\| + G_1 + G_2, \tag{4.4}$$

$$\|\widetilde{V}^{n+1}\|^2 \leq \frac{1}{2}\|\widetilde{V}^n\|^2 + \frac{\tau}{2}\left(\left\|\frac{\partial^2 \widetilde{V}^n}{\partial x^2}\right\|^2 + \left\|\frac{\partial \widetilde{V}^{n+1}}{\partial x}\right\|^2\right) + \tau|R_2^n|\|\widetilde{V}^{n+1}\| + G_3, \tag{4.5}$$

$$\|\widetilde{W}^{n+1}\|^2 \leq \frac{1}{2}\|\widetilde{W}^n\|^2 + \frac{\tau}{2}\left(\left\|\frac{\partial^2 \widetilde{W}^n}{\partial x^2}\right\|^2 + \left\|\frac{\partial \widetilde{W}^{n+1}}{\partial x}\right\|^2\right) + \tau|R_3^n|\|\widetilde{W}^{n+1}\| + G_4, \tag{4.6}$$

where

$$G_1 = -3\tau\left(u^n\frac{\partial u^n}{\partial x} - U^n\frac{\partial U^n}{\partial x}, \widetilde{U}^{n+1}\right), \qquad G_2 = 3\tau\frac{\partial}{\partial x}\left(v^n w^n - V^n W^n, \widetilde{U}^{n+1}\right),$$

$$G_3 = \tau\left(u^n\frac{\partial v^n}{\partial x} - U^n\frac{\partial V^n}{\partial x}, \widetilde{V}^{n+1}\right), \qquad G_4 = 3\tau\left(u^n\frac{\partial w^n}{\partial x} - U^n\frac{\partial W^n}{\partial x}, \widetilde{W}^{n+1}\right).$$

By using the algebraic inequality and lemma 2.1, we get

$$|G_1| \leq 3\tau\left\|u^n\frac{\partial u^n}{\partial x} - U^n\frac{\partial U^n}{\partial x}\right\|^2 + \|\widetilde{U}^{n+1}\|^2 \leq C_8\left(\left\|\frac{\partial \widetilde{u}^n}{\partial x}\right\|^2 + \|\widetilde{u}^n\|^2\right) + \|\widetilde{U}^{n+1}\|^2, \tag{4.7}$$

$$|G_2| \leq 3\tau\left\|v^n w^n - V^n W^n\right\|^2 + \|\widetilde{U}^{n+1}\|^2 \leq C_9\left(\|\widetilde{v}^n\|^2 + \|\widetilde{w}^n\|^2\right) + \|\widetilde{U}^{n+1}\|^2, \tag{4.8}$$

$$|G_3| \leq 3\tau\left\|u^n\frac{\partial v^n}{\partial x} - U^n\frac{\partial U^n}{\partial x}\right\|^2 + \|\widetilde{V}^{n+1}\|^2 \leq C_{10}\left(\|\widetilde{u}^n\|^2 + \left\|\frac{\partial \widetilde{v}^n}{\partial x}\right\|^2\right) + \|\widetilde{V}^{n+1}\|^2, \tag{4.9}$$

$$|G_4| \leq 3\tau\left\|u^n\frac{\partial w^n}{\partial x} - U^n\frac{\partial W^n}{\partial x}\right\|^2 + \|\widetilde{W}^{n+1}\|^2 \leq C_{11}\left(\|\widetilde{u}^n\|^2 + \left\|\frac{\partial \widetilde{w}^n}{\partial x}\right\|^2\right) + \|\widetilde{W}^{n+1}\|^2, \tag{4.10}$$

where C_8, C_9, C_{10} and C_{11} are constants independent of τ and N. Let $\widetilde{M} = \max(C_8, C_9, C_{10}, C_{11})$ Putting the values of (4.7) and (4.8) in to (4.4). Also substituting the values of (4.9) and (4.10) in to

(4.5) and (4.6) respectively. By using the same technique as in the previous section, we can obtain a equation similar to (3.7).

$$\begin{aligned}&(1-3\tau)\left(\|\widetilde{U}^{n+1}\|^2+\left\|\frac{\partial\widetilde{U}^{n+1}}{\partial x}\right\|^2+\|\widetilde{V}^{n+1}\|^2+\left\|\frac{\partial\widetilde{V}^{n+1}}{\partial x}\right\|^2+\|\widetilde{W}^{n+1}\|^2+\left\|\frac{\partial\widetilde{W}^{n+1}}{\partial x}\right\|^2\right)\\&\le(1+3\tau)\widetilde{M}\left(\|\widetilde{U}^{n}\|^2+\left\|\frac{\partial\widetilde{U}^{n}}{\partial x}\right\|^2+\|\widetilde{V}^{n}\|^2+\left\|\frac{\partial\widetilde{V}^{n}}{\partial x}\right\|^2+\|\widetilde{W}^{n}\|^2+\left\|\frac{\partial\widetilde{W}^{n}}{\partial x}\right\|^2\right)\\&+\tau\vartheta^2|R_1^n|^2+\tau\vartheta^2|R_2^n|^2+\tau\vartheta^2|R_3^n|^2.\end{aligned}\tag{4.11}$$

$$\begin{aligned}\|\widetilde{U}^{n+1}\|_{H^1}^2+\|\widetilde{V}^{n+1}\|_{H^1}^2+\|\widetilde{W}^{n+1}\|_{H^1}^2\le\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)\Big[\Big(\|\widetilde{U}^{n}\|_{H^1}^2+\|\widetilde{V}^{n}\|_{H^1}^2+\|\widetilde{W}^{n}\|_{H^1}^2\Big)\\+\left(\tau\vartheta^2|R_1^n|^2+\tau\vartheta^2|R_2^n|^2+\tau\vartheta^2|R_3^n|^2\right)\Big]\end{aligned}$$

Let

$$\begin{aligned}\widetilde{E}^{n+1}&=\|\widetilde{U}^{n+1}\|_{H^1}^2+\|\widetilde{V}^{n+1}\|_{H^1}^2+\|\widetilde{W}^{n+1}\|_{H^1}^2\\\widetilde{R}^{n}&=\tau\vartheta^2(|R_1^n|^2+|R_2^n|^2+|R_3^n|^2)\end{aligned}$$

Then equation (4.11) is written as

$$\begin{aligned}\widetilde{E}^{n+1}&\le\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)\left[\widetilde{E}^{n}+\tau\vartheta^2\widetilde{R}^{n}\right]\\&\le\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^2\widetilde{E}^{n-1}+\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)\tau\vartheta^2\widetilde{R}^{n-1}+\tau\vartheta^2\widetilde{R}^{n}\\&\vdots\\&\le\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^n\widetilde{E}^{0}+\tau\vartheta^2\sum_{j=0}^{n}\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^j\widetilde{R}^{n-j}\end{aligned}$$

Since $\widetilde{E}^0=0$, we obtain

$$\widetilde{E}^{n+1}\le(n+1)\tau\vartheta^2\sum_{j=0}^{n}\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^j\widetilde{R}^{n-j}$$

Finally, using the result of (3.8) we get

$$\|u^n-U^n\|+\|v^n-V^n\|+\|w^n-W^n\|\le(n+1)\tau\vartheta^2e^{6\widetilde{M}t}|R^n|\le\widetilde{M}\sqrt{\vartheta^2e^{6\widetilde{M}t}}\tau$$

Theorem 2. *Let u^n, v^n and w^n be the solution for initial boundary value problem for (1.1)–(1.5) and let U^n, V^n and W^n be the solution of (2.4)–(2.6) time discrete scheme. If the conditions of Theorem 1 holds. Then the time discrete solution is convergent in H^1 and the convergence rate is $O(\tau)$.*

5 Numerical Results

In this section, we present numerical results to show the efficiency and accuracy of the method, mentioned in previous section. We define maximum error $\|E(u)\|_\infty$, $\|E(v)\|_\infty$ and $\|E(w)\|_\infty$ as follows

$$\begin{aligned}\|E(u)\|_\infty&=\max_{0\le j\le N}|u(x_j,t)-U(x_j,t)|,\\\|E(v)\|_\infty&=\max_{0\le j\le N}|v(x_j,t)-V(x_j,t)|,\\\|E(w)\|_\infty&=\max_{0\le j\le N}|w(x_j,t)-W(x_j,t)|,\end{aligned}$$

where u,v,w are the exact solutions of (1.1)–(1.5) and U,V,W are the approximate solutions.

5.1 Example 1

Consider the generalized HS coupled KdV equations (1.1)–(1.5) with the initial conditions [25]:

$$\begin{aligned} u(x,0) &= \frac{\beta - 2\alpha^2}{3} + 2\alpha^2 \tanh^2(\alpha x), \\ v(x,0) &= \frac{4\alpha^2(\beta+\alpha^2)}{3c_1}\left(\frac{c_0}{c_1} - \tanh(\alpha x)\right), \\ w(x,0) &= c_0 + c_1 \tanh(\alpha x) \end{aligned}$$

where c_0, c_1, α and β are arbitrary constants. For practical computation we choose the parameters as $c_0 = 1.5$, $c_1 = 0.1$, $\alpha = 0.1$, $\beta = 1.5$ and $N = 64$.

The absolute error of the U, V and W are given in Table-1, Table-2 and Table-3 respectively. The results of the present method are compared with the results of methods already available in the literature i.e., Reza and Malik [15], Xie and Ding [13] for the variable U, V and W at different values of t. We observe that the absolute error is less than 0.2×10^{-6}. The numerical results of the present method are better than the results obtained by Reza and Malik [15], Xie and Ding [13]. The space-time graphs of U, V and W are given in Figure-1, Figure-2 and Figure-3 respectively. The graph of exact and approximate solution are plotted in Figure-1 to Figure-3 at different values of t.

Table 1: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable U at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	3.290e-06	6.719e-10	6.739e-10	2.541e-06
0.4	5.252e-05	1.711e-07	1.719e-07	3.345e-07
0.7	1.597e-04	1.593e-06	1.603e-06	6.144e-07
1.0	3.227e-04	6.574e-06	6.625e-06	8.363e-07

Table 2: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable V at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	8.559e-11	3.320e-13	8.828e-11	1.430e-08
0.4	1.698e-10	8.490e-11	3.818e-08	2.234e-08
0.7	8.793e-10	7.951e-10	5.028e-07	5.933e-08
1.0	3.389e-09	3.306e-09	2.689e-06	7.474e-08

Table 3: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable W at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	5.349e-08	2.075e-10	4.385e-11	6.095e-08
0.4	1.061e-07	5.306e-08	1.896e-08	7.780e-08
0.7	5.496e-07	4.969e-07	2.497e-07	9.188e-08
1.0	2.118e-06	2.066e-06	1.335e-06	8.989e-08

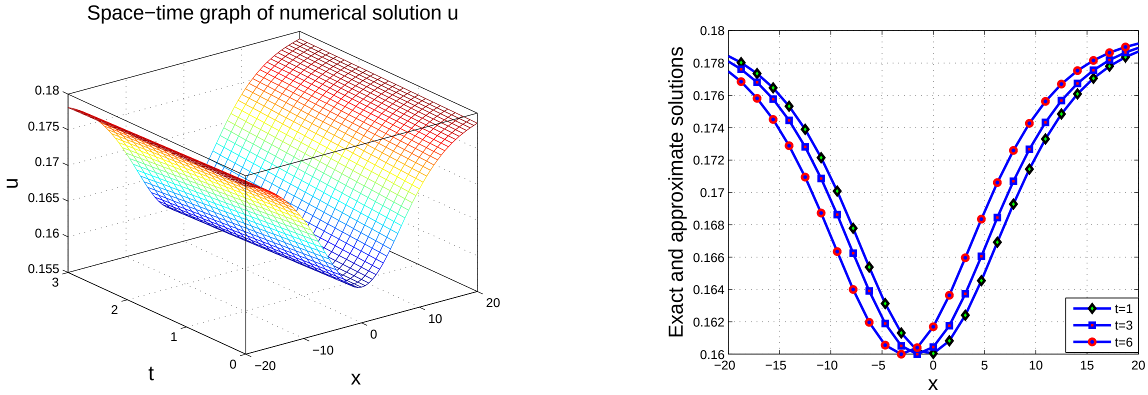

Figure 1: The left figure shows the space-time graphs of U, while the right figure shows the graph of U for different values of t.

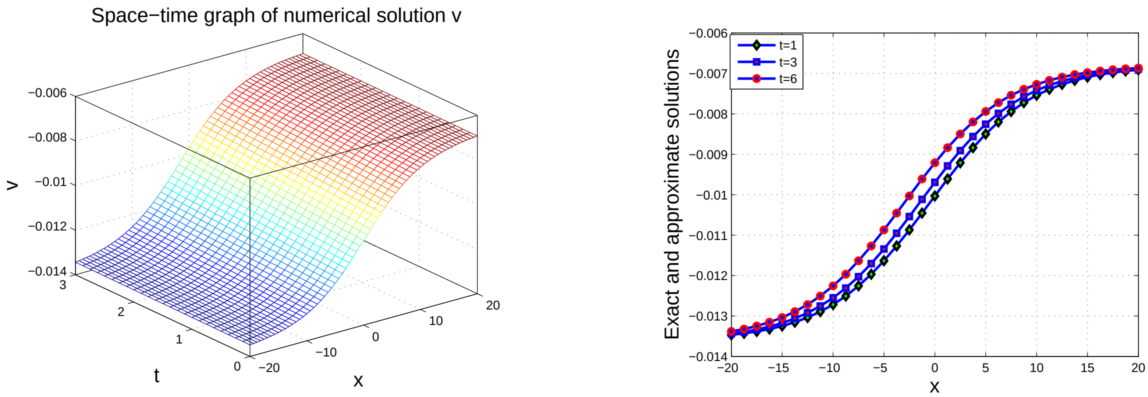

Figure 2: The left figure shows the space-time graphs of V, while the right figure shows the graph of V for different values of t.

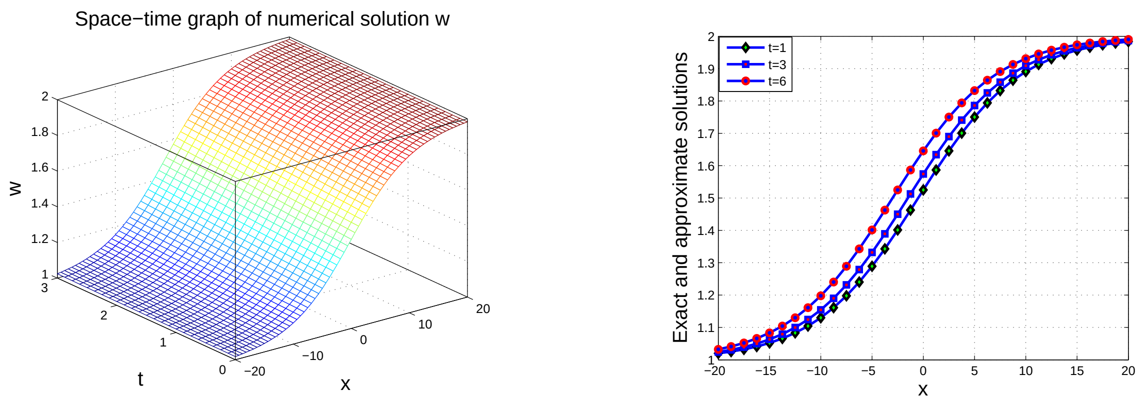

Figure 3: The left figure shows the space-time graphs of W, while the right figure shows the graph of W for different values of t.

5.2 Example 2

We consider the generalized HS coupled KdV equations (1.1)–(1.5) with the initial conditions [25]:

$$
\begin{aligned}
u(x,0) &= \frac{\beta-8\alpha^2}{3}+4\alpha^2\tanh^2(\alpha x),\\
v(x,0) &= -\frac{4}{3}\frac{\alpha^2(3\alpha^2c_0-2\beta c_2+4\alpha^2c_2)}{c_2^2}+\left(\frac{4\alpha^2}{c_2}\tanh^2(\alpha x)\right),\\
w(x,0) &= c_0+c_2\tanh^2(\alpha x)
\end{aligned}
$$

where c_0, c_1, c_2, α and β are arbitrary constants. We choose the arbitrary constants for practical computation as, $c_0 = 1.5$, $c_1 = 0.1$, $c_2 = 0.5$, $\alpha = 0.1$, $\beta = 1.5$ and $N = 64$.

The absolute error of U, V and W are given in Table-4, Table-5 and Table-6 respectively. we compare the results of the present method with Reza and Malik [15], Xie and Ding [13] for the variable U, V and W at different value of t. The results are already available in the literature. We observe that the absolute error is less than 0.2×10^{-6}. The numerical results of the present method are comparatively better than the results obtained from Reza and Malik [15], Xie and Ding [13]. The space-time graphs of U, V and W are given in Figure-4, Figure-5 and Figure-6 respectively. The graph of exact and approximate solution are shown in Figure-4 to Figure-6 at different value of t.

Table 4: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable U at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	4.279e-09	1.660e-11	2.495e-05	3.762e-09
0.4	8.490e-09	4.245e-09	1.146e-04	4.677e-09
0.7	4.396e-08	3.975e-08	2.293e-04	5.366e-09
1.0	1.694e-07	1.653e-07	3.744e-04	7.595e-09

Table 5: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable V at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	8.559e-11	3.320e-13	8.828e-11	1.430e-08
0.4	1.698e-10	8.490e-11	3.818e-08	2.234e-08
0.7	8.793e-10	7.951e-10	5.028e-07	5.933e-08
1.0	3.389e-09	3.306e-09	2.689e-06	7.474e-08

Table 6: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable W at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	5.349e-08	2.075e-10	4.385e-11	6.095e-08
0.4	1.061e-07	5.306e-08	1.896e-08	7.780e-08
0.7	5.496e-07	4.969e-07	2.497e-07	9.188e-08
1.0	2.118e-06	2.066e-06	1.335e-06	8.989e-08

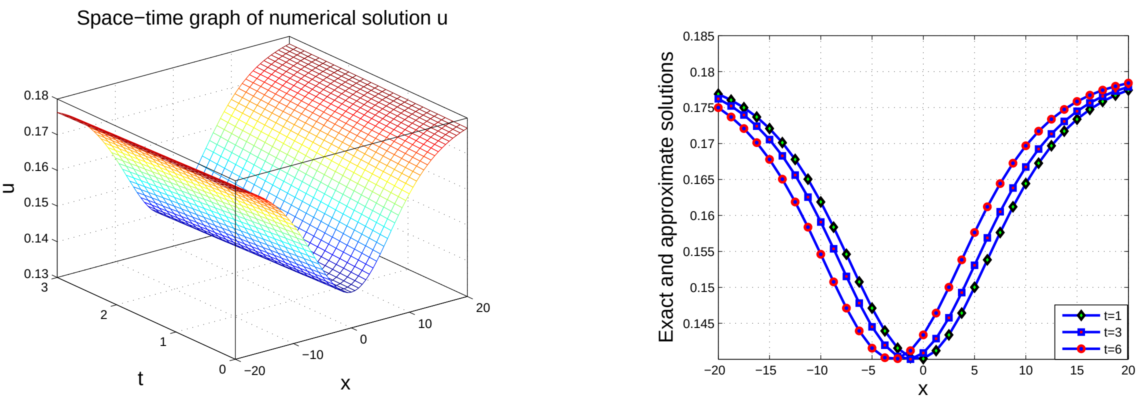

Figure 4: The left figure shows the space-time graphs of U, while the right figure shows the graph of U for different values of t.

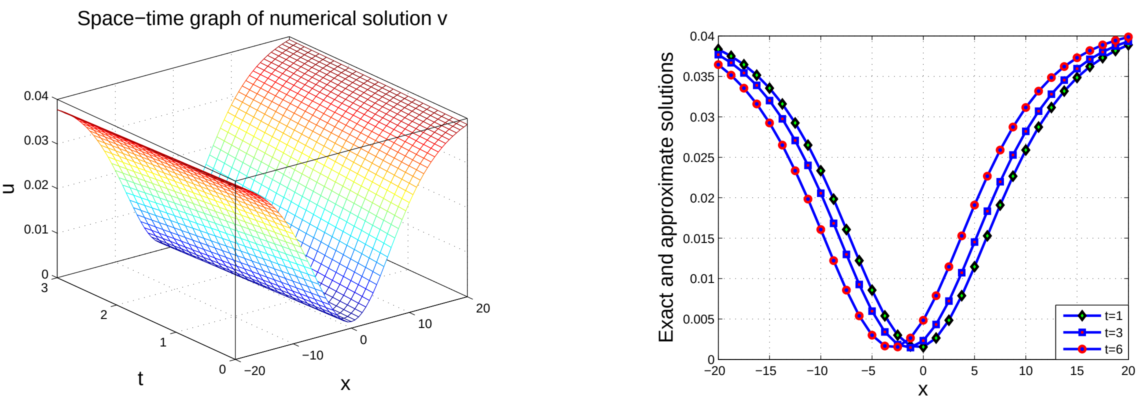

Figure 5: The left figure shows the space-time graphs of V, while the right figure shows the graph of V for different values of t.

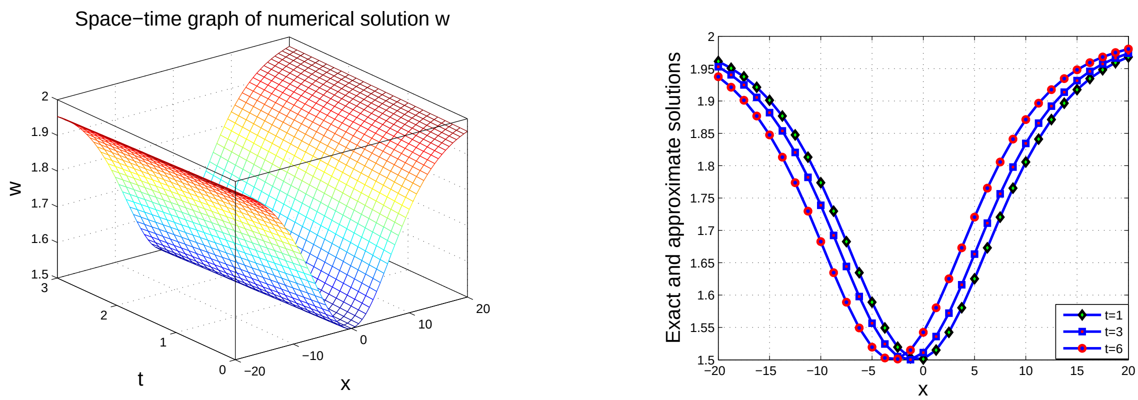

Figure 6: The left figure shows the space-time graphs of W, while the right figure shows the graph of W for different values of t.

6 Conclusion

In this paper, the generalized Hirota-Satsuma (HS) coupled Korteweg-de Vries (KdV) equation is solved numerically using the Fourier pseudospectral method. The time derivative of discrete scheme is approximated by the forward finite difference formula while the pseudospectral method is used in the space direction. The stability and convergence of the discrete scheme are proved by energy estimation method. The obtained solution is presented graphically at various time levels. The numerical results reveal that the Fourier pseudospectral method is convenient, effective and accurate to solve the generalized HS coupled KdV equations.

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